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Research article

Weak solutions of semilinear elliptic equations with Leray-Hardy potentials and measure data

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Abstract: We study existence and stability of solutions of

$$-\Delta u + \frac{\mu}{|x|^2}u + g(u) = \nu \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded, smooth domain of $\mathbb{R}^N$, $N \geq 2$, containing the origin, $\mu \geq -\frac{(N-2)^2}{4}$ is a constant, g is a nondecreasing function satisfying some integral growth assumption and the weak Δ_2-condition, and ν is a Radon measure in Ω. We show that the situation differs depending on whether the measure is diffuse or concentrated at the origin. When g is a power function, we introduce a capacity framework to find necessary and sufficient conditions for solvability.

Keywords: Leray-Hardy potential; Radon measure; capacity; weak solution

1. Introduction

Schrödinger operators with singular potentials under the form

$$u \mapsto H(u) := -\Delta u + V(x)u, \qquad x \in \mathbb{R}^3 \tag{1.1}$$

are at the core of the description of many aspects of nuclear physics. The associated energy, the sum of the momentum energy and the potential energy, endows the form

$$\mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx. \tag{1.2}$$

In classical physics $V(x) = -\kappa|x|^{-1}$ ($\kappa > 0$) is the Coulombian potential and $\mathcal{H}$ is not bounded from below and there is no ground state. In quantum physics there are reasons arising from its mathematical formulation which leads, at least in the case of the hydrogen atom, to $V(x) = -\kappa|x|^{-2}$ ($\kappa > 0$) and $\mathcal{H}$ is bounded from below provided $\kappa \geq -\frac{1}{4}$. Furthermore, a form of the *uncertainty principle* is *Hardy's inequality*

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} dx \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^3). \tag{1.3}$$

The meaning of this inequality is that if u is localized close to a point 0 (i.e., the right side term is large), then its momentum has to be large (i.e., the left side term is large), and the power $|x|^{-2}$ is the consequence of a dimensional analysis (see [19, 20]). Such potential is often called a Leray-Hardy potential. The study of the mathematical properties of generalisations of the operator H in particular in N-dimensional domains generated hundred of publications in the last thirty years. In this article we define the Schrödinger operator $\mathcal{L}$ in $\mathbb{R}^N$ by

$$\mathcal{L}_\mu := -\Delta + \frac{\mu}{|x|^2}, \tag{1.4}$$

where μ is a real number satisfying

$$\mu \geq \mu_0 := -\frac{(N-2)^2}{4}. \tag{1.5}$$

Note that $\frac{(N-2)^2}{4}$ achieves the value $\frac{1}{4}$ when $N = 3$. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded, smooth domain containing the origin and $g : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $g(0) \geq 0$, we are interested in the nonlinear Poisson equation

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where ν is a Radon measure in Ω. The reason for a measure framework is that the problem is essentially trivial if $\nu \in L^2(\Omega)$, more complicated if $\nu \in L^1(\Omega)$ and very rich if ν is a measure.

When $\mu = 0$, problem (1.6) reduces to

$$\begin{cases} -\Delta u + g(u) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

which has been extensively studied by numerous authors in the last 30 years. A fundamental contribution is due to Brezis [6], Benilan and Brezis [2], where ν is bounded and the function $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, positive on $(0, +\infty)$ and satisfies the *subcritical assumption* in dimension $N \geq 3$:

$$\int_1^{+\infty} (g(s) - g(-s)) s^{-1-\frac{N}{N-2}} ds < +\infty. \tag{1.8}$$

They obtained the existence, uniqueness and stability of weak solutions for the problem. When $N = 2$, Vàzquez [26] introduced the exponential orders of growth of g defined by

$$\begin{aligned} \beta_+(g) &= \inf \left\{ b > 0 : \int_1^\infty g(t) e^{-bt} dt < \infty \right\}, \\ \beta_-(g) &= sup \left\{ b < 0 : \int_{-\infty}^{-1} g(t) e^{bt} dt > -\infty \right\}, \end{aligned} \tag{1.9}$$

and proved that if ν is any bounded measure in Ω with Lebesgue decomposition

$$\nu = \nu_r + \sum_{j\in\mathbb{N}} \alpha_j \delta_{a_j},$$

where ν_r is part of ν with no atom, $a_j \in \Omega$ and $\alpha_j \in \mathbb{R}$ satisfies

$$\frac{4\pi}{\beta_-(g)} \le \alpha_j \le \frac{4\pi}{\beta_+(g)}, \tag{1.10}$$

then (1.7) admits a (unique) weak solution. Later on, Baras and Pierre [1] studied (1.7) when $g(u) = |u|^{p-1}u$ for $p > 1$ and they discovered that if $p \ge \frac{N}{N-2}$ the problem is well posed if and only if ν is absolutely continuous with respect to the Bessel capacity $c_{2,p'}$ with $p' = \frac{p}{p-1}$.

It is a well established fact that, by the improved Hardy inequality in [9] and Lax-Milgram Theorem, the non-homogeneous problem

$$\mathcal{L}_\mu u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.11}$$

with $f \in L^2(\Omega)$, has a unique solution in $H_0^1(\Omega)$ if $\mu > \mu_0$, or in a weaker space $H(\Omega)$ if $\mu = \mu_0$, see [18]. When $f \notin L^2(\Omega)$, a natural question is to find sharp conditions on f for the existence or nonexistence of solutions of (1.11) and the difficulty comes from the fact that the Hardy term $|x|^{-2}u$ may not be locally integrable in Ω. An attempt done by Dupaigne in [18] is to consider problem (1.11) when $\mu \in [\mu_0, 0)$ and $N \ge 3$ in the sense of distributions

$$\int_\Omega u\mathcal{L}_\mu \xi \, dx = \int_\Omega f\xi \, dx, \quad \forall\, \xi \in C_c^\infty(\Omega). \tag{1.12}$$

The corresponding semi-linear problem is studied in [5] with this approach.

We adopt here a different point of view in using a different notion of weak solutions. It is known that the equation $\mathcal{L}_\mu u = 0$ in $\mathbb{R}^N \setminus \{0\}$ has two distinct radial solutions:

$$\Phi_\mu(x) = \begin{cases} |x|^{\tau_-(\mu)} & \text{if} \quad \mu > \mu_0, \\ |x|^{-\frac{N-2}{2}} \ln\left(\frac{1}{|x|}\right) & \text{if} \quad \mu = \mu_0, \end{cases} \qquad \text{and} \qquad \Gamma_\mu(x) = |x|^{\tau_+(\mu)},$$

with

$$\tau_-(\mu) = -\frac{N-2}{2} - \sqrt{\frac{(N-2)^2}{4} + \mu} \quad \text{and} \quad \tau_+(\mu) = -\frac{N-2}{2} + \sqrt{\frac{(N-2)^2}{4} + \mu}.$$

In the remaining of the paper and when there is no ambiguity, we put $\tau_+ = \tau_+(\mu)$, $\tau_+^0 = \tau_+(\mu_0)$, $\tau_- = \tau_-(\mu)$ and $\tau_-^0 = \tau_-(\mu_0)$. It is noticeable that identity (1.12) cannot be used to express that Φ_μ is a fundamental solution, i.e., $f = \delta_0$, since Φ_μ is not locally integrable if $\mu \ge 2N$. Recently, Chen, Quaas and Zhou found in [12] that the function Φ_μ is the fundamental solution in the sense that

$$\int_{\mathbb{R}^N} \Phi_\mu \mathcal{L}_\mu^* \xi \, d\gamma_\mu(x) = c_\mu \xi(0) \quad \text{for all } \xi \in C_0^{1,1}(\mathbb{R}^N), \tag{1.13}$$

where

$$d\gamma_\mu(x) = \Gamma_\mu(x)dx, \qquad \mathcal{L}_\mu^* \xi = -\Delta\xi - 2\frac{\tau_+}{|x|^2}\langle x, \nabla\xi\rangle, \tag{1.14}$$

and

$$c_\mu = \begin{cases} 2\sqrt{\mu-\mu_0}\,|\,S^{N-1}\,| & \text{if } \mu > \mu_0, \\ \left|S^{N-1}\right| & \text{if } \mu = \mu_0. \end{cases} \tag{1.15}$$

With the power-absorption nonlinearity in $\Omega^* = \Omega \setminus \{0\}$, the precise behaviour near 0 of any positive solution of

$$\mathcal{L}_\mu u + u^p = 0 \qquad \text{in } D'(\Omega^*) \tag{1.16}$$

is given in [22] when $p > 1$. In this paper it appears a critical exponent

$$p^*_\mu = 1 - \frac{2}{\tau_-} \tag{1.17}$$

with the following properties: if $p \geq p^*_\mu$ any solution of (1.16) can be extended by continuity as a solution in $D'(\Omega)$. If $1 < p < p^*_\mu$ any positive solution of (1.16) either satisfies

$$\lim_{x\to 0} |x|^{\frac{2}{p-1}} u(x) = \ell, \tag{1.18}$$

where $\ell = \ell_{N,p,\mu} > 0$, or there exists $k \geq 0$ such that

$$\lim_{x\to 0} \frac{u(x)}{\Phi_\mu(x)} = k, \tag{1.19}$$

and in that case $u \in L^p_{loc}(\Omega; d\gamma_\mu)$. In view of [12], it implies that u satisfies

$$\int_{\mathbb{R}^N} \left(u\mathcal{L}^*_\mu \xi + u^p \xi\right) d\gamma_\mu(x) = c_\mu k \xi(0), \quad \forall\, \xi \in C^{1,1}_0(\mathbb{R}^N). \tag{1.20}$$

Note the threshold p^*_μ and its role is put into light by the existence or non-existence of explicit solutions of (1.16) under the form $x \mapsto a|x|^b$, where necessarily $b = -\frac{2}{p-1}$ and $a = \ell$. It is also proved in [22] that when $\mu > \mu_0$ and $g : \mathbb{R} \to \mathbb{R}_+$ is a continuous nondecreasing function satisfying

$$\int_1^\infty (g(s) - g(-s))\, s^{-1-p^*_\mu} ds < \infty, \tag{1.21}$$

then for any $k > 0$ there exists a radial solution of

$$\mathcal{L}_\mu u + g(u) = 0 \qquad \text{in } D'(B^*_1) \tag{1.22}$$

satisfying (1.19), where $B^*_1 := B_1(0) \setminus \{0\}$. When $\mu = \mu_0$ and $N \geq 3$ it is proved in [22] that if there exists $b > 0$ such that

$$\int_0^1 g\left(-bs^{-\frac{N-2}{N+2}} \ln s\right) ds < \infty, \tag{1.23}$$

then there exists a radial solution of (1.22) satisfying (1.19) with $\gamma = \frac{(N+2)b}{2}$. In fact this condition is independent of $b > 0$, by contrast to the case $N = 2$ and $\mu = 0$ where the introduction of the exponential order of growth of g is a necessity. Moreover, it is easy to see that u satisfies

$$\int_{\mathbb{R}^N} \left(u\mathcal{L}^*_\mu \xi + g(u)\xi\right) d\gamma_\mu(x) = c_\mu \gamma \xi(0), \quad \forall \xi \in C^{1,1}_0(\mathbb{R}^N). \tag{1.24}$$

In view of these results and identity (1.13), we introduce a definition of weak solutions adapted to the operator $\mathcal{L}_\mu$ in a measure framework. Since Γ_μ is singular at 0 if $\mu < 0$, there is need of defining specific set of measures and *we denote by* $\mathfrak{M}(\Omega^*;\Gamma_\mu)$*, the set of Radon measures* ν *in* Ω^* *such that*

$$\int_{\Omega^*}\Gamma_\mu d|\nu| := \sup\left\{\int_{\Omega^*}\zeta d|\nu| \;:\; \zeta\in C_0(\Omega^*),\; 0\le\zeta\le\Gamma_\mu\right\} < \infty. \tag{1.25}$$

If $\nu\in\mathfrak{M}_+(\Omega^*)$, we define its natural extension, with the same notation since there is no ambiguity, as a measure in Ω by

$$\int_{\Omega}\zeta d\nu = \sup\left\{\int_{\Omega^*}\eta d\nu \;:\; \eta\in C_0(\Omega^*)\,,\; 0\le\eta\le\zeta\right\}\quad \text{for all } \zeta\in C_0(\Omega)\,,\; \zeta\ge 0, \tag{1.26}$$

a definition which is easily extended if $\nu=\nu_+-\nu_-\in\mathfrak{M}(\Omega^*)$. Since the idea is to use the weight Γ_μ in the expression of the weak solution, the expression $\Gamma_\mu\nu$ has to be defined properly if $\tau_+ < 0$. *We denote by* $\mathfrak{M}(\Omega;\Gamma_\mu)$ *the set of measures* ν *on* Ω *which coincide with the above natural extension of* $\nu\lfloor_{\Omega^*}\in\mathfrak{M}_+(\Omega^*;\Gamma_\mu)$. If $\nu\in\mathfrak{M}_+(\Omega;\Gamma_\mu)$ we define the measure $\Gamma_\mu\nu$ in the following way

$$\int_{\Omega}\zeta d(\Gamma_\mu\nu) = \sup\left\{\int_{\Omega^*}\eta\Gamma_\mu d\nu \;:\; \eta\in C_0(\Omega^*)\,,\; 0\le\eta\le\zeta\right\}\quad \text{for all } \zeta\in C_0(\Omega)\,,\; \zeta\ge 0. \tag{1.27}$$

If $\nu=\nu_+-\nu_-$, $\Gamma_\mu\nu$ is defined accordingly. Notice that the Dirac mass at 0 does not belong to $\mathfrak{M}(\Omega;\Gamma_\mu)$ although it is a limit of $\{\nu_n\}\subset\mathfrak{M}(\Omega;\Gamma_\mu)$. We denote by $\overline{\mathfrak{M}}(\Omega;\Gamma_\mu)$ the set of measures which can be written under the form

$$\nu = \nu\lfloor_{\Omega^*} + k\delta_0, \tag{1.28}$$

where $\nu\lfloor_{\Omega^*}\in\mathfrak{M}(\Omega;\Gamma_\mu)$ and $k\in\mathbb{R}$. Before stating our main theorem we make precise the notion of weak solution used in this article. We denote $\overline{\Omega}^* := \overline{\Omega}\setminus\{0\}$, $\rho(x)=\operatorname{dist}(x,\partial\Omega)$ and

$$\mathbb{X}_\mu(\Omega) = \left\{\xi\in C_0(\overline{\Omega})\cap C^1(\overline{\Omega}^*) : |x|\mathcal{L}^*_\mu\xi\in L^\infty(\Omega)\right\}. \tag{1.29}$$

Clearly $C^{1,1}_0(\overline{\Omega})\subset\mathbb{X}_\mu(\Omega)$.

Definition 1.1. *We say that* u *is a weak solution of (1.6) with* $\nu\in\overline{\mathfrak{M}}(\Omega;\Gamma_\mu)$ *such that* $\nu=\nu\lfloor_{\Omega^*}+k\delta_0$ *if* $u\in L^1(\Omega,\,|x|^{-1}d\gamma_\mu)$*,* $g(u)\in L^1(\Omega,\rho d\gamma_\mu)$ *and*

$$\int_\Omega\left[u\mathcal{L}^*_\mu\xi + g(u)\xi\right]d\gamma_\mu(x) = \int_\Omega \xi d(\Gamma_\mu\nu) + k\xi(0)\quad \textit{for all}\;\; \xi\in\mathbb{X}_\mu(\Omega), \tag{1.30}$$

where $\mathcal{L}^*_\mu$ *is given by (1.13) and* c_μ *is defined in (1.15).*

A measure for which problem (1.6) admits a solution is *a g-good measure*. In the regular case we prove the following

Theorem A. *Let* $\mu\ge 0$ *if* $N=2$*,* $\mu\ge\mu_0$ *if* $N\ge 3$ *and* $g:\mathbb{R}\to\mathbb{R}$ *be a Hölder continuous nondecreasing function such that* $g(r)r\ge 0$ *for any* $r\in\mathbb{R}$*. Then for any* $\nu\in L^1(\Omega,d\gamma_\mu)$*, problem (1.6) has a unique weak solution* u_ν *such that for some* $c_1>0$*,*

$$\|u_\nu\|_{L^1(\Omega,|x|^{-1}d\gamma_\mu)}\le c_1\|\nu\|_{L^1(\Omega,d\gamma_\mu)}.$$

Furthermore, if $u_{\nu'}$ is the solution of (1.6) with right-hand side $\nu' \in L^1(\Omega, d\gamma_\mu)$, there holds

$$\int_\Omega \Big[|u_\nu - u_{\nu'}|\mathcal{L}_\mu^*\xi + |g(u_\nu) - g(u_{\nu'})|\xi\Big] d\gamma_\mu(x) \le \int_\Omega (\nu - \nu')\mathrm{sgn}(u - u')\xi d\gamma_\mu(x), \tag{1.31}$$

and

$$\int_\Omega \Big[(u_\nu - u_{\nu'})_+\mathcal{L}_\mu^*\xi + (g(u_\nu) - g(u_{\nu'}))_+\xi\Big] d\gamma_\mu(x) \le \int_\Omega (\nu - \nu')\mathrm{sgn}_+(u - u')\xi d\gamma_\mu(x), \tag{1.32}$$

for all $\xi \in \mathbb{X}_\mu(\Omega)$, $\xi \ge 0$.

Definition 1.2. *A continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $rg(r) \ge 0$ for all $r \in \mathbb{R}$ satisfies the weak Δ_2-condition if there exists a positive nondecreasing function $t \in \mathbb{R} \mapsto K(t)$ such that*

$$|g(s + t)| \le K(t)\,(|g(s)| + |g(t)|) \quad \text{for all } (s, t) \in \mathbb{R}\times\mathbb{R} \text{ s.t. } st \ge 0. \tag{1.33}$$

It satisfies the Δ_2-condition if the above function K is constant.

The Δ_2-condition has been intruduced in the study of Birnbaum-Orlicz spaces [4, 23] and it is satisfied by power function $r \mapsto |r|^{p-1}r$, $p > 0$, but not by exponential functions $r \mapsto e^{ar}$. It plays a key role in the study of semilinear equation with a power type reaction term (see eg., [29, 30]). The new weak Δ_2-condition is more general and it is also satisfied by exponential functions.

Theorem B. *Let $\mu > 0$ if $N = 2$ or $\mu > \mu_0$ if $N \ge 3$ and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function such that $g(r)r \ge 0$ for any $r \in \mathbb{R}$. If g satisfies the weak Δ_2-condition and*

$$\int_1^\infty (g(s) - g(-s))s^{-1-\min\{p_\mu^*, p_0^*\}} ds < \infty, \tag{1.34}$$

where p_μ^* is given by (1.17), then for any $\nu \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ problem (1.6) admits a unique weak solution u_ν.

Note that $\min\{p_\mu^*, p_0^*\} = p_\mu^*$ for $\mu > 0$ and $\min\{p_\mu^*, p_0^*\} = p_0^*$ if $\mu < 0$. Furthermore, the mapping: $\nu \mapsto u_\nu$ is increasing. In the case $N \ge 3$ and $\mu = \mu_0$ we have a more precise result.

Theorem C. *Assume that $N \ge 3$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $g(r)r \ge 0$ for any $r \in \mathbb{R}$ satisfying the weak Δ_2-condition and (1.8). Then for any $\nu = \nu\lfloor_{\Omega^*} + c_\mu k\delta_0 \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ problem (1.6) admits a unique weak solution u_ν.*

Furthermore, if $\nu\lfloor_{\Omega^*} = 0$, condition (1.8) can be replaced by the following weaker one

$$\int_1^\infty (g(t) - g(-t))\,(\ln t)^{\frac{N+2}{N-2}}\, t^{-\frac{2N}{N-2}} dt < \infty. \tag{1.35}$$

The optimality of these conditions depends whether the measure is concentrated at 0 or not. When the measure is of the form $k\delta_0$ the condition proved to be optimal in [22] and when it is of the type $k\delta_a$ with $a \neq 0$ optimality is shown in [28]. Normally, the estimates on the Green kernel plays an essential role for approximating the solution of elliptic problems with absorption and Radon measure data. However, we have avoided to use the estimates on the Green kernel for Hardy operators which are not easily tractable when $0 > \mu \geq \mu_0$, and our main idea is to separate the measure ν^* in $\mathfrak{M}(\Omega;\Gamma_\mu)$ and the Dirac mass at the origin, and then to glue the solutions with above measures respectively. This technique requires this new weak Δ_2-condition.

In the previous result, it is noticeable that if $k = 0$ (resp. $\nu\lfloor_{\Omega^*} = 0$) only condition (1.8) (resp. condition (1.35)) is needed. In the two cases the weak Δ_2-condition is unnecessary. In the power case where $g(u) = |u|^{p-1}u := g_p(u)$,

$$\begin{cases} \mathcal{L}_\mu u + g_p(u) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.36}$$

the following result follows from Theorem B and C.

Corollary D. *Let $\mu \geq \mu_0$ if $N \geq 3$ and $\mu > 0$ if $N = 2$. Any nonzero measure $\nu = \nu\lfloor_{\Omega^*} + c_\mu k\delta_0 \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$ is g_p-good if one of the following holds:*

(i) $1 < p < p^*_\mu$ in the case $\nu\lfloor_{\Omega^*} = 0$;

(ii) $1 < p < p^*_0$ in the case $k = 0$;

(iii) $1 < p < \min\left\{p^*_\mu, p^*_0\right\}$ in the case $k \neq 0$ and $\nu\lfloor_{\Omega^*} \neq 0$.

We remark that p^*_μ is the sharp exponent for the existence of (1.36) when $\nu\lfloor_{\Omega^*} = 0$, while the critical exponent becomes p^*_0 when $k = 0$ and ν has atom in $\Omega \setminus \{0\}$.

The supercritical case of equation (1.36) corresponds to the fact that not all measures are g_p-good and the case where $k \neq 0$ is already treated.

Theorem E. *Assume that $N \geq 3$. Then $\nu = \nu\lfloor_{\Omega^*} \in \mathfrak{M}(\Omega;\Gamma_\mu)$ is g_p-good if and only if for any $\epsilon > 0$, $\nu_\epsilon = \nu\chi_{B^c_\epsilon}$ is absolutely continuous with respect to the $c_{2,p'}$-Bessel capacity.*

Finally we characterize the compact removable sets in Ω.

Theorem F. *Assume that $N \geq 3$, $p > 1$ and K is a compact set of Ω. Then any weak solution of*

$$\mathcal{L}_\mu u + g_p(u) = 0 \quad \text{in } \Omega \setminus K \tag{1.37}$$

can be extended a weak solution of the same equation in whole Ω if and only if

(i) $c_{2,p'}(K) = 0$ if $0 \notin K$;

(ii) $p \geq p_{\mu^*}$ if $K = \{0\}$;

(iii) $c_{2,p'}(K) = 0$ if $\mu \geq 0$, $0 \in K$ and $K \setminus \{0\} \neq \emptyset$;

(iv) $c_{2,p'}(K) = 0$ and $p \geq p^*_\mu$ if $\mu < 0$, $0 \in K$ and $K \setminus \{0\} \neq \emptyset$.

The case (i) is already proved in [22, Theorem 1.2]. Notice also that if $A \neq \emptyset$ necessarily $c_{2,p'}(A) = 0$ holds only if $p \geq p_0$. Therefore, if $\mu \geq 0$ there holds $p \geq p^*_0 \geq p^*_\mu$, while if $\mu < 0$, then $p_0 < p^*_\mu$.

The rest of this paper is organized as follows. In Section 2, we build the framework for weak solutions of (1.6) involving L^1 data. Section 3 is devoted to solve existence and uniqueness of weak solution of (1.6), where the absorption is subcritical and ν is a related Radon measure. Finally, we deal with the super critical case in Section 4 by characterized by Bessel Capacity.

2. L^1 data

Throughout this section we assume $N \geq 2$ and $\mu \geq \mu_0$ and in what follows, we denote by c_i with $i \in \mathbb{N}$ a generic positive constant. We first recall some classical comparison results for Hardy operator $\mathcal{L}_\mu$. The next lemma is proved in [12, Lemma 2.1], and in [15, Lemma 2.1] if $h(s) = s^p$.

Lemma 2.1. *Let G be a bounded domain in $\mathbb{R}^N$ such that $0 \notin \bar{G}$, $L : G \times [0,+\infty) \mapsto [0,+\infty)$ be a continuous function satisfying for any $x \in G$,*

$$h(x, s_1) \geq h(x, s_2) \quad \text{if} \quad s_1 \geq s_2,$$

and functions u, $v \in C^{1,1}(G) \cap C(\overline{G})$ satisfy

$$\begin{cases} \mathcal{L}_\mu u + h(x,u) \geq \mathcal{L}_\mu v + h(x,v) & \text{in } G, \\ u \geq v & \text{on } \partial G, \end{cases}$$

then

$$u \geq v \quad \text{in} \quad G.$$

As an immediate consequence we have

Lemma 2.2. *Assume that Ω is a bounded C^2 domain containing 0. If L is a continuous function as in Lemma 2.1 verifying that $L(x,0) = 0$ for all $x \in \Omega$, and $u \in C^{1,1}(\Omega^*) \cap C(\overline{\Omega}^*)$ satisfies*

$$\begin{cases} \mathcal{L}_\mu u + L(x,u) = 0 & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial\Omega, \\ \lim_{x\to 0} u(x)\Phi_\mu^{-1}(x) = 0. \end{cases} \tag{2.1}$$

Then $u = 0$.

We recall that if $u \in L^1(\Omega, |x|^{-1} d\gamma_\mu)$ is a weak solution of

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

in the sense of Definition 1.1, then it satisfies that

$$\int_\Omega u \mathcal{L}_\mu^*(\xi)\, d\gamma_\mu(x) = \int_\Omega f\xi\, d\gamma_\mu(x) \quad \text{for all } \xi \in \mathbb{X}_\mu(\Omega). \tag{2.3}$$

If u is a weak solution of (2.2), there holds

$$\mathcal{L}_\mu u = f \quad \text{in } \mathcal{D}'(\Omega^*), \tag{2.4}$$

and $v = \Gamma_\mu^{-1} u$ verifies

$$\mathcal{L}_\mu^* v = \Gamma_\mu^{-1} f \quad \text{in } \mathcal{D}'(\Omega^*), \tag{2.5}$$

a fact which is expressed by the commutating formula

$$\Gamma_\mu \mathcal{L}_\mu^* v = \mathcal{L}_\mu(\Gamma_\mu v). \tag{2.6}$$

The following form of Kato's inequality, proved in [12, Proposition 2.1], plays an essential role in the obtention a priori estimates and uniqueness of weak solution of (1.6).

Proposition 2.1. *If $f \in L^1(\Omega, \rho d\gamma_\mu)$, then there exists a unique weak solution $u \in L^1(\Omega, |x|^{-1}d\gamma_\mu)$ of (2.2). Furthermore, for any $\xi \in \mathbb{X}_\mu(\Omega)$, $\xi \geq 0$, we have*

$$\int_\Omega |u|\mathcal{L}_\mu^*(\xi)\, d\gamma_\mu(x) \leq \int_\Omega \mathrm{sign}(u) f\xi\, d\gamma_\mu(x) \tag{2.7}$$

and

$$\int_\Omega u_+\mathcal{L}_\mu^*(\xi)\, d\gamma_\mu(x) \leq \int_\Omega \mathrm{sign}_+(u) f\xi\, d\gamma_\mu(x). \tag{2.8}$$

The proof is done if $\xi \in C_0^{1,1}(\Omega)$, but it is valid if $\xi \in \mathbb{X}_\mu(\Omega)$. The next result is proved in [13, Lemma 2.3].

Lemma 2.3. *Assume that $\mu > \mu_0$ and $f \in C^1(\Omega^*)$ verifies*

$$0 \leq f(x) \leq c_2|x|^{\tau-2}, \tag{2.9}$$

for some $\tau > \tau_-$. Let u_f be the solution of

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial\Omega, \\ \lim\limits_{x\to 0} \dfrac{u(x)}{\Phi_\mu(x)} = 0. \end{cases} \tag{2.10}$$

Then there holds:

(i) if $\tau_- < \tau < \tau_+$,

$$0 \leq u_f(x) \leq c_3|x|^\tau \quad \text{in} \quad \Omega^*; \tag{2.11}$$

(ii) if $\tau = \tau_+$,

$$0 \leq u_f(x) \leq c_4|x|^\tau(1 + (-\ln|x|)_+) \quad \text{in} \quad \Omega^*; \tag{2.12}$$

(iii) if $\tau > \tau_+$,

$$0 \leq u_f(x) \leq c_5|x|^{\tau_+} \quad \text{in} \quad \Omega^*. \tag{2.13}$$

Proof of Theorem A. Let $\mathbb{H}^1_{\mu,0}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ under the norm of

$$\|u\|_{\mathbb{H}^1_{\mu,0}(\Omega)} = \sqrt{\int_\Omega |\nabla u|^2 dx + \mu \int_\Omega \frac{u^2}{|x|^2} dx.} \tag{2.14}$$

Then $\mathbb{H}^1_{\mu,0}(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle_{\mathbb{H}^1_{\mu,0}(\Omega)} = \int_\Omega \langle \nabla u, \nabla v\rangle dx + \mu \int_\Omega \frac{uv}{|x|^2} dx \tag{2.15}$$

and the embedding $\mathbb{H}^1_{\mu,0}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous and compact for $p \in [2, 2^*)$ with $2^* = \frac{2N}{N-2}$ when $N \geq 3$ and any $p \in [2, \infty)$ if $N = 2$. Furthermore, if $\eta \in C_0^1(\overline{\Omega})$ has the value 1 in a neighborhood of 0, then $\eta\Gamma_\mu \in \mathbb{H}^1_{\mu,0}(\Omega)$. We put

$$G(v) = \int_0^v g(s)ds,$$

then G is a convex nonnegative function. If $\rho\nu \in L^2(\Omega)$ we define the functional J_ν in the space $\mathbb{H}^1_{\mu,0}(\Omega)$ by

$$J_\nu(v) = \begin{cases} \frac{1}{2}\|v\|^2_{\mathbb{H}^1_{\mu,0}(\Omega)} + \int_\Omega G(v)dx - \int_\Omega \nu v dx & \text{if } G(v) \in L^1(\Omega, d\gamma_\mu), \\ \infty & \text{if } G(v) \notin L^1(\Omega, d\gamma_\mu). \end{cases} \tag{2.16}$$

The functional J is strictly convex, lower semicontinuous and coercive in $\mathbb{H}^1_{\mu,0}(\Omega)$, hence it admits a unique minimum u which satisfies

$$\langle u, v\rangle_{\mathbb{H}^1_{\mu,0}(\Omega)} + \int_\Omega g(u)vdx = \int_\Omega \nu v dx \quad \text{for all } v \in \mathbb{H}^1_{\mu,0}(\Omega).$$

If $\xi \in C^{1,1}_0(\Omega)$ then $v = \xi\Gamma_\mu \in \mathbb{H}^1_{\mu,0}(\Omega)$, then

$$\langle u, \xi\Gamma_\mu\rangle_{\mathbb{H}^1_{\mu,0}(\Omega)} = \int_\Omega \langle\nabla u, \nabla\xi\rangle d\gamma_\mu(x) + \int_\Omega \left(\langle\nabla u, \nabla\Gamma_\mu\rangle + \frac{\mu\Gamma_\mu}{|x|^2}\right)\xi dx, \tag{2.17}$$

and

$$\int_\Omega \langle\nabla u, \nabla\Gamma_\mu\rangle\xi dx = -\int_\Omega \langle\nabla\xi, \nabla\Gamma_\mu\rangle u dx - \int_\Omega u\xi\Delta\Gamma_\mu dx,$$

since $C^\infty_0(\Omega)$ is dense in $\mathbb{H}^1_{\mu,0}(\Omega)$. Furthermore, since $u \in L^p(\Omega)$ for any $p < 2^*$, $|x|^{-1}u \in L^1(\Omega, d\gamma_\mu)$, hence $u\mathcal{L}^*_\mu\xi \in L^1(\Omega, d\gamma_\mu)$. Therefore

$$\int_\Omega \left(u\mathcal{L}^*_\mu\xi + g(u)\xi\right) d\gamma_\mu = \int_\Omega \nu\xi d\gamma_\mu. \tag{2.18}$$

Next, if $\nu \in L^1(\Omega, \rho d\gamma_\mu)$ we consider a sequence $\{\nu_n\} \subset C^\infty_0(\Omega)$ converging to ν in $L^1(\Omega, \rho d\gamma_\mu)$ and denote by $\{u_n\}$ the sequence of the corresponding minimizing problem in $\mathbb{H}^1_{\mu,0}(\Omega)$. By Proposition 2.1 we have that, for any $\xi \in \mathbb{X}_\mu(\Omega)$,

$$\int_\Omega \left(|u_n - u_m|\mathcal{L}^*_\mu\xi + (g(u_n) - g(u_m))\text{sgn}(u_n - u_m)\xi\right) d\gamma_\mu \leq \int_\Omega (\nu_n - \nu_m)\text{sgn}(u_n - u_m)\xi d\gamma_\mu. \tag{2.19}$$

We denote by η_0 the solution of

$$\mathcal{L}^*_\mu\eta = 1 \quad \text{in } \Omega, \qquad \eta = 0 \quad \text{on } \partial\Omega. \tag{2.20}$$

Its existence is proved in [12, Lemma 2.2], as well as the estimate $0 \leq \eta_0 \leq c_6\rho$ for some $c_6 > 0$. Since g is monotone, we obtain from (2.19)

$$\int_\Omega (|u_n - u_m| + |g(u_n) - g(u_m)|\eta_0)\, d\gamma_\mu \leq \int_\Omega |\nu_n - \nu_m|\eta_0 d\gamma_\mu. \tag{2.21}$$

Hence $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega, d\gamma_\mu)$. Next we construct a solution η_1 to

$$\mathcal{L}^*_\mu\eta = |x|^{-1} \quad \text{in } \Omega^*, \quad \eta = 0 \quad \text{on } \partial\Omega. \tag{2.22}$$

For this aim, we consider for $0 < \theta < 1$, the function $y_\theta(x) = \frac{1-|x|^{2-\theta}}{N-\theta+2\tau_+(\mu)}$ which verifies

$$\mathcal{L}^*_\mu y_\theta = |x|^{-\theta} \quad \text{in } B_1, \quad y_\theta = 0 \quad \text{on } \partial B_1$$

(we can always assume that $\Omega \subset B_1$). As in the proof of [12, Lemma 2.2], for any $x_0 \in \Omega$ there exists $r_0 > 0$ such that $B_{r_0}(x_0) \subset \Omega$ and for $t > 0$ small enough $w_{t,x_0}(x) = t(r_0^2 - |x - x_0|^2)$ is a subsolution of (2.20), hence of (2.22). Therefore there exists η_θ such that

$$\mathcal{L}_\mu^* \eta_\theta = |x|^{-\theta} \quad \text{in } \Omega^*, \quad \eta_\theta = 0 \quad \text{on } \partial\Omega. \tag{2.23}$$

Furthermore $\theta \mapsto \eta_\theta$ is increasing and bounded from above by y_1, hence it converges to a function η_1 which satisfies (2.23). Then

$$\int_\Omega \left(|u_n - u_m||x|^{-\theta} + |g(u_n) - g(u_m)|\eta_\theta \right) d\gamma_\mu \leq \int_\Omega |\nu_n - \nu_m|\eta_\theta d\gamma_\mu. \tag{2.24}$$

Letting $\theta \to 1$, we obtain as a complement of (2.21) that

$$\int_\Omega \left(\frac{|u_n - u_m|}{|x|} + |g(u_n) - g(u_m)|\eta_1 \right) d\gamma_\mu \leq \int_\Omega |\nu_n - \nu_m|\eta_1 d\gamma_\mu. \tag{2.25}$$

Hence $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ with limit u and $\{g(u_n)\}$ is a Cauchy sequence in $L^1(\Omega, \rho d\gamma_\mu)$ with limit $g(u)$. Then (2.18) holds. As for (1.31) it is a consequence of (2.19) and (1.32) is proved similarly. □

3. The subcritical case

In this section as well as in the next one we always assume that $N \geq 3$ and $\mu \geq \mu_0$, or $N = 2$ and $\mu > 0$, since the case $N = 2$, $\mu = 0$, which necessitates specific tools, has already been completely treated in [26].

We recall that the set $\mathfrak{M}(\Omega^*; \Gamma_\mu)$ of Radon measures is defined in the introduction as the set of measures in Ω^* satisfying (1.25), and any positive measure $\nu \in \mathfrak{M}(\Omega^*; \Gamma_\mu)$ is naturaly extended by formula (1.26) as a positive measure in Ω. The space $\overline{\mathfrak{M}}(\Omega; \Gamma_\mu)$ is the space of measures ν on $C_0(\Omega)$ such that

$$\nu = \nu\lfloor_{\Omega^*} + k\delta_0, \tag{3.1}$$

where $\nu\lfloor_{\Omega^*} \in \mathfrak{M}(\Omega^*; \Gamma_\mu)$.

3.1. The linear equation

Lemma 3.1. *If $\nu \in \overline{\mathfrak{M}}(\Omega; \Gamma_\mu)$, then there exists a unique weak solution $u \in L^1(\Omega, |x|^{-1}d\gamma_\mu)$ to*

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

This solution is denoted by $\mathbb{G}_\mu[\nu]$, and this defines the Green operator of $\mathcal{L}_\mu$ in Ω with homogeneous Dirichlet conditions.

Proof. By linearity and using the result of [12] on fundamental solution, we can assume that $k = 0$ and $\nu \geq 0$. Let $\{\nu_n\} \subset L^1(\Omega, \rho d\gamma_\mu)$ be a sequence such that $\nu_n \geq 0$ and

$$\int_\Omega \xi \Gamma_\mu \nu_n dx \to \int_\Omega \xi d(\Gamma_\mu \nu) \quad \text{for all } \xi \in \mathbb{X}_\mu(\Omega),$$

and by Proposition 2.1, we may let u_n be the unique, nonnegative weak solution of

$$\begin{cases} \mathcal{L}_\mu u_n = \nu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

with $n \in \mathbb{N}$. There holds

$$\int_\Omega u_n \mathcal{L}_\mu^* \xi d\gamma_\mu(x) = \int_\Omega \xi \nu_n \Gamma_\mu dx \quad \text{for all } \xi \in \mathbb{X}_\mu(\Omega). \tag{3.4}$$

Then $u_n \geq 0$ and using the function η_1 defined in the proof of Theorem A for test function, we have

$$c_6 \int_\Omega \frac{u_n}{|x|} d\gamma_\mu = \int_\Omega \eta_1 \Gamma_\mu \nu_n dx \leq c_7 \|\nu\|_{\mathfrak{M}(\Omega,\Gamma_\mu)}, \tag{3.5}$$

which implies that $\{u_n\}$ is bounded in $L^1(\Omega, \frac{1}{|x|} d\gamma_\mu(x))$.

For any $\epsilon > 0$ sufficiently small, set the test function ξ in $\left\{\zeta \in \mathbb{X}_\mu(\Omega) : \zeta = 0 \text{ in } B_\epsilon\right\}$, then we have that

$$\int_{\Omega \setminus B_\epsilon(0)} u_n \mathcal{L}_\mu^* \xi d\gamma_\mu(x) = \int_{\Omega \setminus B_\epsilon(0)} \xi \nu_n \Gamma_\mu dx \quad \text{for all } \xi \in \mathbb{X}_\mu(\Omega). \tag{3.6}$$

Therefore, for any open sets O and O' verifying $\bar{O} \subset O' \subset \bar{O}' \subset \Omega \setminus B_\epsilon(0)$, there exists $c_8 > 0$ independent of n such that

$$\|u_n\|_{L^1(O')} \leq c_8 \|\nu\|_{\mathfrak{M}(\Omega,\Gamma_\mu)}.$$

Note that in $\Omega \setminus B_\epsilon$, the operator $\mathcal{L}_\mu^*$ is uniformly elliptic and the measure $d\gamma_\mu$ is equivalent to the N-dimensional Lebesgue measure dx, then [30, Corollary 2.8] could be applied to obtain that for some $c_9, c_{10} > 0$ independent of n but dependent of O',

$$\begin{aligned} \|u_n\|_{W^{1,q}(O)} &\leq c_9 \|u_n\|_{L^1(O')} + \|\tilde{\nu}_n\|_{L^1(\Omega, d\gamma_\mu)} \\ &\leq c_{10} \|\nu\|_{\mathfrak{M}(\Omega,\Gamma_\mu)}. \end{aligned}$$

That is, $\{u_n\}$ is uniformly bounded in $W^{1,q}_{loc}(\Omega \setminus \{0\})$.

As a consequence, since ϵ is arbitrary, there exist a subsequence, still denoted by $\{u_n\}_n$ and a function u such that

$$u_n \to u \quad \text{a.e. in } \Omega.$$

Meanwhile, we deduce from Fatou's lemma,

$$\int_\Omega \frac{u}{|x|} d\gamma_\mu \leq c_{11} \int_\Omega \eta_1 \Gamma_\mu d\nu. \tag{3.7}$$

Next we claim that $u_n \to u$ in $L^1(\Omega, |x|^{-1} d\gamma_\mu)$. Let $\omega \subset \Omega$ be a Borel set and ψ_ω be the solution of

$$\begin{cases} \mathcal{L}_\mu^* \psi_\omega = |x|^{-1} \chi_\omega & \text{in } \Omega, \\ \psi_\omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Then $\psi_\omega \leq \eta_1$, thus it is uniformly bounded. Assuming that $\Omega \subset B_1$, clearly ψ_ω is bounded from above by the solution Ψ_ω of

$$\begin{cases} \mathcal{L}_\mu^* \Psi_\omega = |x|^{-1} \chi_\omega & \text{in } B_1, \\ \Psi_\omega = 0 & \text{on } \partial B_1, \end{cases} \tag{3.9}$$

and by standard rearrangement, $\sup_{B_1} \Psi_\omega \leq \sup_{B_1} \Psi^r_\omega$, where Ψ^r_ω solves

$$\begin{cases} \mathcal{L}^*_\mu \Psi^r_\omega = |x|^{-1} B_{\epsilon(|\omega|)} & \text{in } B_1, \\ \quad \Psi^r_\omega = 0 & \text{on } \partial B_1, \end{cases} \tag{3.10}$$

where $\epsilon(|\omega|) = \left(\frac{|\omega|}{|B_1|}\right)^{\frac{1}{N}}$. Then Ψ^r_ω is radially decreasing and $\lim_{|\omega|\to 0} \Psi^r_\omega = 0$, uniformly on B_1. This implies

$$\lim_{|\omega|\to 0} \psi_\omega(x) = 0 \quad \text{uniformly in } B_1. \tag{3.11}$$

Using (3.4) with $\xi = \psi_\omega$,

$$\int_\omega \frac{u_n}{|x|} d\gamma_\mu(x) = \int_\omega \nu_n \Gamma_\mu \psi_\omega dx \leq \sup_\Omega \psi_\omega \int_\omega \nu_n \Gamma_\mu dx \to 0 \text{ as } |\omega| \to 0.$$

Therefore $\{u_n\}$ is uniformly integrable for the measure $|x|^{-1} d\gamma_\mu$. Letting $n \to \infty$ in (3.4) implies the claim. □

3.2. Dirac masses

We assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $rg(r) \geq 0$ for all $r \in \mathbb{R}$. The next lemma dealing with problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = k\delta_0 & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.12}$$

is an extension of [22, Theorem 3.1, Theorem 3.2]. Actually it was quoted without demonstration in this article as Remark 3.1 and Remark 3.2 and we give here their proof. Notice also that when $N \geq 3$ and $\mu = \mu_0$ we give a more complete result than [22, Theorem 3.2].

Lemma 3.2. *Let $k \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $rg(r) \geq 0$ for all $r \in \mathbb{R}$. Then problem (3.12) admits a unique solution $u := u_{k\delta_0}$ if one of the following conditions is satisfied:*

(i) $N \geq 2$, $\mu > \mu_0$ and g satisfies (1.21);

(ii) $N \geq 3$, $\mu = \mu_0$ and g satisfies (1.35).

Proof. Without loss of generality we assume $B_R \subset \Omega \subset B_1$ for some $R \in (0, 1)$.
(i) *The case $\mu > \mu_0$*. It follows from [22, Theorem 3.1] that for any $k \in \mathbb{R}$ there exists a radial function $v_{k,1}$ (resp. $v_{k,R}$) defined in B^*_1 (resp. B^*_R) satisfying

$$\mathcal{L}_\mu v + g(v) = 0 \qquad \text{in } B^*_1 \quad (\text{resp. in } B^*_R), \tag{3.13}$$

vanishing respectively on ∂B_1 and ∂B_R and satisfying

$$\lim_{x\to 0} \frac{v_{k,1}(x)}{\Phi_\mu(x)} = \lim_{x\to 0} \frac{v_{k,R}(x)}{\Phi_\mu(x)} = \frac{k}{c_\mu}. \tag{3.14}$$

Furthermore $g(v_{k,1}) \in L^1(B_1, d\gamma_\mu)$ (resp. $g(v_{k,R}) \in L^1(B_R, d\gamma_\mu)$). Assume that $k > 0$, then $0 \le v_{k,R} \le v_{k,1}$ in B_R^* and the extension of $\tilde{v}_{k,R}$ by 0 in Ω^* is a subsolution of (3.13) in Ω^* and it is still smaller than $v_{k,1}$ in Ω^*. By the well known method on super and subsolutions (see e.g., [32, Theorem 1.4.6]), there exists a function u in Ω^* satisfying $\tilde{v}_{k,R} \le u \le v_{k,1}$ in Ω^* and

$$\begin{cases} \mathcal{L}_\mu u + g(u) = 0 & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial\Omega, \\ \lim\limits_{x\to 0} \dfrac{u(x)}{\Phi_\mu(x)} = \dfrac{k}{c_\mu}. \end{cases} \tag{3.15}$$

By standard methods in the study of isolated singularities (see e.g., [16, 17, 22, 29] for various extensions)

$$\lim_{x\to 0} |x|^{1-\tau_-} \nabla u(x) = \tau_- \frac{k}{c_\mu} \frac{x}{|x|}. \tag{3.16}$$

For any $\epsilon > 0$ and $\xi \in \mathbb{X}_\mu(\Omega)$,

$$\begin{aligned} 0 &= \int_{\Omega\setminus B_\epsilon} (\mathcal{L}_\mu u + g(u))\Gamma_\mu \xi dx \\ &= \int_{\Omega\setminus B_\epsilon} u\mathcal{L}^*_\mu \xi d\gamma_\mu(x) + (\tau_- - \tau_+)\frac{k}{c_\mu}|S^{N-1}|\xi(0)(1+o(1)). \end{aligned}$$

Using (1.15), we obtain

$$\int_\Omega u\mathcal{L}^*_\mu \xi d\gamma_\mu(x) = k\xi(0). \tag{3.17}$$

(ii) *The case* $\mu = \mu_0$. In [22, Theorem 3.2] it is proved that if for some $b > 0$ there holds

$$I := \int_1^\infty g\left(bt^{\frac{N-2}{N+2}} \ln t\right) t^{-2} dt < \infty, \tag{3.18}$$

then there exists a solution of (1.22) satisfying (1.19) with $\gamma = \frac{(N+2)b}{2}$. Actually we claim that *the finiteness of this integral is independent of the value of b*. To see that, set $s = t^{\frac{N-2}{N+2}}$, then

$$I = \frac{N+2}{N-2} \int_1^\infty g(\beta s \ln s)\, s^{-\frac{2N}{N-2}} ds,$$

with $\beta = \frac{N+2}{N-2} b$. Set $\tau = \beta s \ln s$, then

$$\ln s \left(1 + \frac{\ln\ln s}{\ln s} + \frac{\ln\beta}{\ln s}\right) \Longrightarrow \ln s = \ln\tau(1+o(1)) \quad \text{as } s \to \infty.$$

We infer that for $\epsilon > 0$ there exists $s_\epsilon > 2$ and $\tau_\epsilon = s_\epsilon \ln s_\epsilon$ such that

$$(1-\epsilon)\beta^{\frac{N+2}{N-2}} \le \frac{\displaystyle\int_{s_\epsilon}^\infty g(\beta s \ln s)\, s^{-\frac{2N}{N-2}} ds}{\displaystyle\int_{\tau_\epsilon}^\infty g(\tau)(\ln\tau)^{\frac{N+2}{N-2}}\, \tau^{-\frac{2N}{N-2}} d\tau} \le (1+\epsilon)\beta^{\frac{N+2}{N-2}}, \tag{3.19}$$

which implies the claim. Next we prove as in case (i) the existence of $v_{k,1}$ (resp. $v_{k,R}$) defined in B_1^* (resp. B_R^*) satisfying

$$\mathcal{L}_{\mu_0} v + g(v) = 0 \qquad \text{in } B_1^* \quad (\text{resp. in } B_R^*), \tag{3.20}$$

vanishing respectively on ∂B_1 and ∂B_R and satisfying

$$\lim_{x\to 0} \frac{v_{k,1}(x)}{\Phi_\mu(x)} = \lim_{x\to 0} \frac{v_{k,R}(x)}{\Phi_\mu(x)} = \frac{k}{c_{\mu_0}}. \tag{3.21}$$

We end the proof as above. □

Remark. It is important to notice that conditions (1.21) and (1.35) (or equivalently (1.23)) are also necessary for the existence of radial solutions in a ball, hence their are also necessary for the existence of non radial solutions of the Dirichlet problem (3.12).

3.3. Measures in Ω^*

We consider now the problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.22}$$

where $\nu \in \mathfrak{M}(\Omega^*; \Gamma_\mu)$.

Lemma 3.1. *Let $\mu \geq \mu_0$. Assume that g satisfies (1.8) if $N \geq 3$ or the $\beta_\pm(g)$ defined by (1.9) satisfy $\beta_-(g) < 0 < \beta_+(g)$ if $N = 2$, and let $\nu \in \mathfrak{M}(\Omega^*; \Gamma_\mu)$. If $N = 2$, we assume that ν can be decomposed as $\nu = \nu_r + \sum_j \alpha_j \delta_{a_j}$ where ν_r has no atom, the α_j satisfy (1.10) and $\{a_j\} \subset \Omega^*$. Then problem (3.22) admits a unique weak solution.*

Proof. We assume first that $\nu \geq 0$ and let $r_0 = \operatorname{dist}(x, \partial\Omega)$. For $0 < \sigma < r_0$, we set $\Omega^\sigma = \Omega \setminus \{\overline{B}_\sigma\}$ and $\nu_\sigma = \nu\chi_{\Omega^\sigma}$ and for $0 < \epsilon < \sigma$ we consider the following problem in Ω^ϵ

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\sigma & \text{in } \Omega^\epsilon, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \\ \qquad\qquad u = 0 & \text{on } \partial \mathrm{B}_\epsilon. \end{cases} \tag{3.23}$$

Since $0 \notin \Omega^\epsilon$ problem (3.23) admits a unique solution $u_{\nu_\sigma,\epsilon}$ which is smaller than $\mathbb{G}_\mu[\nu]$ and satisfies

$$0 \leq u_{\nu_\sigma,\epsilon} \leq u_{\nu_{\sigma'},\epsilon'} \quad \text{in } \Omega^\epsilon \ \text{ for all } \ 0 < \epsilon' \leq \epsilon \ \text{ and } \ 0 < \sigma' \leq \sigma.$$

For any $\xi \in \mathbb{C}_c^{1,1}(\Omega^*)$ and ϵ small enough so that $\operatorname{supp}(\xi) \subset \Omega^\epsilon$, there holds

$$\int_\Omega \left(u_{\nu_\sigma,\epsilon} \mathcal{L}_\mu^* \xi + g(u_{\nu_\sigma,\epsilon})\xi \right) d\gamma_\mu = \int_\Omega \xi \Gamma_\mu d\nu_\sigma.$$

There exists $u_{\nu_\sigma} = \lim_{\epsilon\to 0} u_{\nu_\sigma,\epsilon}$ and it satisfies the identity

$$\int_\Omega \left(u_{\nu_\sigma} \mathcal{L}_\mu^* \xi + g(u_{\nu_\sigma})\xi \right) d\gamma_\mu = \int_\Omega \xi \Gamma_\mu d\nu_\sigma \quad \text{for all } \ \xi \in \mathbb{C}_c^{1,1}(\Omega^*). \tag{3.24}$$

As a consequence of the maximum principle and Lemma 3.1, there holds

$$0 \leq u_{\nu_\sigma} \leq \mathbb{G}_\mu[\nu_\sigma] \leq \mathbb{G}_\mu[\nu]. \tag{3.25}$$

Since ν_σ vanishes in B_σ, $\mathbb{G}_\mu[\nu_\sigma](x) \leq c_{12}\Phi_\mu(x)$ in a neighborhood of 0, and u_{ν_σ} is also bounded by $c_{12}\Phi_\mu$ in this neighborhood. This implies that $\Phi_\mu^{-1}(x)u_{\nu_\sigma}(x) \to c'$ as $x \to 0$ for some $c' \geq 0$. Next let $\xi \in \mathbb{C}_c^{1,1}(\Omega)$,

$$\ell_n(r) = \begin{cases} 2^{-1}\left(1+\cos\left(\frac{2\pi|x|}{\sigma}\right)\right) & \text{if } \ |x| \leq \frac{\sigma}{2}, \\ 0 & \text{if } \ |x| > \frac{\sigma}{2}, \end{cases}$$

and $\xi_n = \xi\ell_n$. Then

$$\int_\Omega \left(u_{\nu_\sigma}\mathcal{L}_\mu^*\xi_n + g(u_{\nu_\sigma})\xi_n\right)d\gamma_\mu = \int_\Omega \xi_n\Gamma_\mu d\nu_\sigma. \tag{3.26}$$

When $n \to \infty$,

$$\int_\Omega \xi_n\Gamma_\mu d\nu_\sigma \to \int_\Omega \xi\Gamma_\mu d\nu_\sigma$$

and

$$\int_\Omega g(u_\sigma)\xi_n d\gamma_\mu \to \int_\Omega g(u_\sigma)\xi d\gamma_\mu.$$

Now for the first inegral term in (3.26), we have

$$\int_\Omega u_{\nu_\sigma}\mathcal{L}_\mu^*\xi_n d\gamma_\mu = \int_\Omega \ell_n u_\sigma \mathcal{L}_\mu^*\xi d\gamma_\mu + I_n + II_n + III_n,$$

where

$$I_n = -\int_{B_{\frac{\sigma}{2}}} u_\sigma \xi \Delta\ell_n d\gamma_\mu,$$

$$II_n = -2\int_{B_{\frac{\sigma}{2}}} u_\sigma \langle\nabla\xi, \nabla\ell_n\rangle d\gamma_\mu$$

and

$$III_n = -\tau_+\int_{B_{\frac{\sigma}{2}}} u_\sigma \langle \frac{x}{|x|^2}, \nabla\ell_n\rangle d\gamma_\mu.$$

Using the fact that $\xi(x) \to \xi(0)$ and $\nabla\xi(x) \to \nabla\xi(0)$ we easily infer that I_n, II_n and III_n converge to 0 when $n \to \infty$, the most complicated case being the case when $\mu = \mu_0$, which is the justification of introducing the explicit cut-off function ℓ_n. Therefore (3.24) is still valid if it is assumed that $\xi \in C_c^{1,1}(\Omega)$. This means that u_{ν_σ} is a weak solution of

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\sigma & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

Furthermore u_{ν_σ} is unique and u_{ν_σ} is a decreasing function of σ with limit u when $\sigma \to 0$. Taking η_1 as test function, we have

$$\int_\Omega \left(c|x|^{-1}u_{\nu_\sigma} + \eta_1 g(u_{\nu_\sigma})\right)d\gamma_\mu = \int_\Omega \eta_1 d\left(\gamma_\mu\nu_\sigma\right) \leq \int_\Omega \eta_1 d\left(\gamma_\mu\nu\right).$$

By using the monotone convergence theorem we infer that $u_{\nu_\sigma} \to u$ in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ and $g(u_{\nu_\sigma}) \to g(u_\nu)$ in $L^1(\Omega, d\gamma_\mu)$. Hence $u = u_\nu$ is the weak solution of (3.22).

Next we consider a signed measure $\nu = \nu_+ - \nu_-$. We denote by $u_{\nu_+^\sigma,\epsilon}$, $u_{-\nu_-^\sigma,\epsilon}$ and $u_{\nu^\sigma,\epsilon}$ the solutions of (3.23) in Ω^ϵ corresponding to ν_+^σ, $-\nu_-^\sigma$ and ν^σ, ϵ respectively. Then

$$u_{-\nu_-^\sigma,\epsilon} \le u_{\nu^\sigma,\epsilon} \le u_{\nu_+^\sigma,\epsilon}. \tag{3.28}$$

The correspondence $\epsilon \mapsto u_{\nu_+^\sigma,\epsilon}$ and $\epsilon \mapsto u_{-\nu_-^\sigma,\epsilon}$ are respectively increasing and decreasing. Furthermore $u_{\nu^\sigma,\epsilon}$ is locally bounded, hence by local compactness and up to a subsequence $u_{\nu^\sigma,\epsilon}$ converges a.e. in B_ϵ to some function u_{ν^σ}. Since $u_{-\nu_-^\sigma,\epsilon} \to u_{-\nu_-^\sigma}$ and $u_{\nu_+^\sigma,\epsilon} \to u_{\nu_+^\sigma}$ in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$, it follows by Vitali's theorem that $u_{\nu^\sigma,\epsilon} \to u_{\nu^\sigma}$ in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$. Similarly, using the monotonicity of g, $g(u_{\nu^\sigma,\epsilon}) \to g(u_{\nu^\sigma})$ in $L^1(\Omega, d\gamma_\mu)$. By local compactness, $u_{\nu^\sigma} \to u$ a.e. in Ω. Using the same argument of uniform integrability, we have that $u_{\nu^\sigma} \to u$ in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ and $g(u_{\nu^\sigma}) \to g(u)$ in $L^1(\Omega, d\gamma_\mu)$ when $\sigma \to 0$ and u satisfies

$$\int_\Omega \left(u\mathcal{L}_\mu^*\xi + g(u)\xi\right) d\gamma_\mu = \int_\Omega \xi d(d\gamma_\mu \nu) \quad \text{for any } \xi \in \mathbb{C}_c^{1,1}(\Omega^*). \tag{3.29}$$

Finally the singularity at 0 is removable by the same argument as above which implies that u solves (3.29) and thus $u = u_\nu$ is the weak solution of (3.22). □

3.4. Proof of Theorem B

The idea is to glue altogether two solutions one with the Dirac mass and the other with the measure in Ω^*, this is the reason why the weak Δ_2 condition is introduced.

Lemma 3.3. *Let $\nu = \nu\lfloor_{\Omega^*} + k\delta_0 \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ and $\sigma > 0$. We assume that $\nu\lfloor_{\Omega^*}(\overline{B}_\sigma) = 0$. Then there exists a unique weak solution to (1.6).*

Proof. Set $\nu_\sigma = \nu\lfloor_{\Omega^*}$. It has support in $\Omega_\sigma = \Omega \setminus \overline{B}_\sigma$. For $0 < \epsilon < \sigma$ we consider the approximate problem in $\Omega^\epsilon = \Omega \setminus \overline{B}_\epsilon$,

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\sigma & \text{in } \Omega^\epsilon, \\ u = 0 & \text{on } \partial\Omega, \\ u = u_{k\delta_0} & \text{on } \partial B_\epsilon, \end{cases} \tag{3.30}$$

where $u_{k\delta_0}$ is the solution of problem (3.12) obtained in Lemma 3.2. It follows from [30, Theorem 3.7] that problem (3.30) admits a unique weak solution denoted by $U_{\nu_\sigma,\epsilon}$, thanks to the fact that the operator is not singular in Ω^ϵ. We recall that $u_{\nu_\sigma,\epsilon}$ is the solution of (3.23) and $\mathbb{G}_\mu[\delta_0]$ the fundamental solution in Ω. Then

$$\max\{u_{k\delta_0}, u_{\nu_\sigma,\epsilon}\} \le U_{\nu_\sigma,\epsilon} \le u_{\nu_\sigma} + k\mathbb{G}_\mu[\delta_0] \quad \text{in } \Omega^\epsilon. \tag{3.31}$$

Furthermore one has $U_{\nu_\sigma,\epsilon} \le U_{\nu_\sigma,\epsilon'}$ in Ω^ϵ, for $0 < \epsilon' < \epsilon$. Since $u_{\nu_\sigma} \le u_\nu$ and both $k\mathbb{G}_\mu[\delta_0]$ and u_ν belong to $L^1(\Omega, |x|^{-1}d\gamma_\mu)$, then it follows by the monotone convergence theorem that $U_{\nu_\sigma,\epsilon}$ converges in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ and almost everywhere to some function $U_{\nu_\sigma} \in L^1(\Omega, |x|^{-1}d\gamma_\mu)$. Since Γ_μ is a supersolution for equation $\mathcal{L}_\mu u + g(u) = 0$ in B_σ, for $0 < \epsilon_0 < \sigma$ there exists $c_{13} := c_{13}(\epsilon_0, \sigma) > 0$ such that

$$u_{\nu_\sigma}(x) \le c_{13}|x|^{\tau_+} \quad \text{for all } x \in B_{\epsilon_0}.$$

For any $\delta > 0$, there exists ϵ_0 such that $u_{\nu_\sigma}(x) \leq \delta\mathbb{G}_\mu[\delta_0](x)$ in B_{ϵ_0}. Hence $u_{\nu_\sigma} + k\mathbb{G}_\mu[\delta_0] \leq (k+\delta)\mathbb{G}_\mu[\delta_0]$ in B_{ϵ_0}, which implies

$$g(U_{\nu_\sigma,\epsilon}) \leq g((k+\delta)\mathbb{G}_\mu[\delta_0]) \quad \text{in } B_{\epsilon_0} \setminus \overline{B}_\epsilon, \tag{3.32}$$

and

$$\begin{aligned}\int_\Omega g((k+\delta)\mathbb{G}_\mu[\delta_0])d\gamma_\mu(x) &\leq \int_{B_1} g(\tfrac{k+\delta}{c_\mu}|x|^{\tau_-})|x|^{\tau_+}dx = |S^{N-1}|\int_0^1 g(\tfrac{k+\delta}{c_\mu}r^{\tau_-})r^{\tau_++N-1}dr\\ &= c_{14}\int_{\frac{k+\delta}{c_\mu}}^\infty g(t)t^{-2+\frac{2}{\tau_-}} = c_{14}\int_{\frac{k+\delta}{c_\mu}}^\infty g(t)t^{-1-p^*_\mu}dt\\ &< \infty.\end{aligned}$$

Now, using the local Δ_2-condition,with $a' = \frac{k}{c_\mu}\epsilon_0^{\tau_-}$, we see that

$$g(U_{\nu_\sigma,\epsilon}) \leq g(u_{\nu_\sigma} + \tfrac{k}{c_\mu}\epsilon_0^{\tau_-}) \leq K(a')\left(g(u_{\nu_\sigma}) + g(a')\right) \quad \text{in } \Omega^{\epsilon_0}. \tag{3.33}$$

From (3.32) and (3.33) we infer that $g(U_{\nu_\sigma,\epsilon})$ is bounded in $L^1(\Omega^\epsilon, d\gamma_\mu)$ independently of ϵ. If $\xi \in C_0^{1,1}(\Omega^*)$, we have for $\epsilon > 0$ small enough so that $\operatorname{supp}(\xi) \subset \Omega^\epsilon$

$$\int_\Omega \left(U_{\nu_\sigma,\epsilon}\mathcal{L}^*_\mu\xi + g(U_{\nu_\sigma,\epsilon})\xi\right)d\gamma_\mu = \int_\Omega \xi\Gamma_\mu d\nu_\sigma.$$

Letting $\epsilon \to 0$ we obtain that

$$\int_\Omega \left(U_{\nu_\sigma}\mathcal{L}^*_\mu\xi + g(U_{\nu_\sigma})\xi\right)d\gamma_\mu = \int_\Omega \xi\Gamma_\mu d\nu_\sigma. \tag{3.34}$$

Let $\xi \in C_0^{1,1}(\overline{\Omega})$ and $\eta_n \in C^{1,1}(\mathbb{R}^N)$ be a nonnegative cut-off function such that $0 \leq \eta_n \leq 1$, $\eta_n \equiv 1$ in $B^c_{\frac{2}{n}}$, $\eta_n \equiv 0$ in $B_{\frac{1}{n}}$, and choose $\xi\eta_n$ for test function. Then

$$\int_\Omega \left(\eta_n U_{\nu_\sigma}\mathcal{L}^*_\mu\xi + g(U_{\nu_\sigma})\eta_n\xi\right)d\gamma_\mu - \int_\Omega U_{\nu_\sigma}A_n d\gamma_\mu = \int_\Omega \xi\eta_n\Gamma_\mu d\nu_\sigma, \tag{3.35}$$

with

$$A_n = \xi\Delta\eta_n + 2\langle\nabla\eta_n, \nabla\xi\rangle + 2\tau_+\xi\langle\nabla\eta_n, \tfrac{x}{|x|^2}\rangle. \tag{3.36}$$

Clearly

$$\lim_{n\to\infty}\int_\Omega \left(\eta_n U_{\nu_\sigma}\mathcal{L}^*_\mu\xi + g(U_{\nu_\sigma})\eta_n\xi\right)d\gamma_\mu = \int_\Omega \left(U_{\nu_\sigma}\mathcal{L}^*_\mu\xi + g(U_{\nu_\sigma})\xi\right)d\gamma_\mu,$$

and

$$\lim_{n\to\infty}\int_\Omega \xi\eta_n\Gamma_\mu d\nu_\sigma = \int_\Omega \xi\Gamma_\mu d\nu_\sigma.$$

We take

$$\eta_n(r) = \begin{cases} \frac{1}{2} - \frac{1}{2}\cos\left(n\pi\left(r - \frac{1}{n}\right)\right) & \text{if } \frac{1}{n} \leq r \leq \frac{2}{n},\\ 0 & \text{if } r < \frac{1}{n},\\ 1 & \text{if } r > \frac{2}{n}.\end{cases}$$

Then

$$A_n = \frac{n^2\pi^2}{2}\cos\left(n\pi\left(r - \frac{1}{n}\right)\right) + \frac{n\pi}{2}\frac{N-1+2\tau_+}{r}\sin\left(n\pi\left(r - \frac{1}{n}\right)\right).$$

Letting $\epsilon \to 0$ in (3.31), we have

$$U_{\nu_\sigma}(x) = k\mathbb{G}_\mu[\delta_0](x)(1 + o(1)) = \frac{k}{c_\mu}|x|^{\tau_-}(1 + o(1)) \qquad \text{as } x \to 0.$$

Hence

$$\lim_{n\to\infty} \int_\Omega U_{\nu_\sigma} A_n d\gamma_\mu = \frac{2k|S^{N-1}|\sqrt{\mu-\mu_0}}{c_\mu} = k. \tag{3.37}$$

This implies that U_{ν_σ} is the solution of (1.6) with ν replaced by $\nu_\sigma + k\delta_0$. □

Lemma 3.4. *Let $\nu = \nu\lfloor_{\Omega^*} + k\delta_0 \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$. Then there exists a unique weak solution to (1.6).*

Proof. Following the notations of Lemma 3.3, we set $\nu_\sigma = \chi_{B_\sigma}\nu\lfloor_{\Omega^*}$ and denote by U_{ν_σ} the solution of

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\sigma + k\delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.38}$$

It is a positive function and there holds

$$\max\{u_{k\delta_0}, u_{\nu_\sigma}\} \le U_{\nu_\sigma} \le u_{\nu_\sigma} + k\mathbb{G}_\mu[\delta_0] \qquad \text{in } \Omega. \tag{3.39}$$

Since the mapping $\sigma \mapsto U_{\nu_\sigma}$ is decreasing, then there exists $U = \lim_{\sigma\to 0} U_{\nu_\sigma}$ and U satisfies (3.39). As a consequence $U_{\nu_\sigma} \to U$ in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ as $\sigma \to 0$. We take η_1 for test function in the weak formulation of (3.39), then

$$\int_\Omega \left(|x|^{-1}U_{\nu_\sigma} + \eta_1 g(U_{\nu_\sigma})\right) d\gamma_\mu = \int_\Omega \eta_1 \Gamma_\mu d\nu_\sigma + k\eta_1(0).$$

By the monotone convergence theorem we obtain the identity

$$\int_\Omega \left(|x|^{-1}U + \eta_1 g(U)\right) d\gamma_\mu = \int_\Omega \eta_1 d(\gamma_\mu \nu\lfloor_{\Omega^*}) + k\eta_1(0) = \int_\Omega \eta_1 d(\gamma_\mu \nu),$$

and the fact that $g(U_{\nu_\sigma}) \to g(U)$ in $L^1(\Omega, \rho d\gamma_\mu)$. Going to the limit as $\sigma \to 0$ in the weak formulation of (3.38), we infer that $U = u_\nu$ is the solution of (1.6). □

Proof of Theorem B. Assume $\nu = \nu\lfloor_{\Omega^*} + k\delta_0 \in \overline{\mathfrak{M}}(\Omega; \Gamma_\mu)$ satisfies $k > 0$ and let $\nu_+ = \nu_+\lfloor_{\Omega^*} + k\delta_0$ and $\nu_- = \nu_-\lfloor_{\Omega^*}$ the positive and the negative part of ν. We denote by u_{ν_+} and $u_{-\nu_-}$ the weak solutions of (1.6) with respective data ν_+ and $-\nu_-$. For $0 < \epsilon < \sigma$ such that $\overline{B}_\sigma \subset \Omega$, we set $\nu_\sigma = \chi_{B_\sigma}\nu\lfloor_{\Omega^*}$, with positive and negative part $\nu_{\sigma+}$ and $\nu_{\sigma-}$ and denote by $U_{\nu_{\sigma+},\epsilon}$, $U_{-\nu_{\sigma-},\epsilon}$ and $U_{\nu_\sigma,\epsilon}$ the respective solutions of

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_{\sigma+} & \text{in } \Omega^\epsilon, \\ u = 0 & \text{on } \partial\Omega, \\ u = u_{k\delta_0} & \text{on } \partial B_\epsilon, \end{cases} \tag{3.40}$$

$$\begin{cases} \mathcal{L}_\mu u + g(u) = -\nu_{\sigma-} & \text{in } \Omega^\epsilon, \\ u = 0 & \text{on } \partial\Omega \cup \partial B_\epsilon, \end{cases} \tag{3.41}$$

and

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\sigma & \text{in } \Omega^\epsilon, \\ u = 0 & \text{on } \partial\Omega, \\ u = u_{k\delta_0} & \text{on } \partial B_\epsilon, \end{cases} \tag{3.42}$$

then

$$U_{-\nu_{\sigma-},\epsilon} \le U_{\nu_\sigma,\epsilon} \le U_{\nu_{\sigma+},\epsilon}. \tag{3.43}$$

Furthermore $U_{\nu_{\sigma+},\epsilon}$ satisfies (3.31) and, in coherence with the notations of Lemma 3.1 with ν_σ replaced by $-\nu_{\sigma-}$,

$$u_{-\nu_{\sigma-}} \le U_{-\nu_{\sigma-},\epsilon} = u_{-\nu_{\sigma-},\epsilon}. \tag{3.44}$$

By compactness, $\{U_{\nu_\sigma,\epsilon_j}\}_{\epsilon_j}$ converges almost everywhere in Ω to some function U for some sequence $\{\epsilon_j\}$ converging to 0. Moreover $U_{\nu_\sigma,\epsilon_j}$ converges to U_{ν_σ} in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ because $U_{\nu_{\sigma+},\epsilon} \to u_{\nu_{\sigma+}+k\delta_0}$ and $u_{-\nu_{\sigma-},\epsilon} \to u_{-\nu_{\sigma-}}$ in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ by Lemma 3.1 and (3.43) holds. Similarly $g(U_{\nu_\sigma,\epsilon_j})$ converges to $g(U)$ in $L^1(\Omega, \rho d\gamma_\mu)$. This implies that U satisfies

$$\int_\Omega \left(U\mathcal{L}_\mu^* \xi + g(U)\xi \right) d\gamma_\mu = \int_\Omega \xi \Gamma_\mu d\nu_\sigma \quad \text{for all } \xi \in C_0^{1,1}(\Omega^*).$$

In order to use test functions in $C_0^{1,1}(\overline{\Omega})$, we proceed as in the proof of Lemma 3.3, using the inequality (derived from (3.43)) and the

$$u_{-\nu_{\sigma-}} \le U_{\nu_\sigma} \le u_{\nu_{\sigma+}+k\delta_0}. \tag{3.45}$$

By (3.33), $u_{\nu_{\sigma+}+k\delta_0}(x) = k\mathbb{G}_\mu[\delta_0](x)(1 + o(1))$ when $x \to 0$ and $u_{-\nu_{\sigma-}} = o(\mathbb{G}_\mu[\delta_0])$ near 0. This implies $U_{\nu_\sigma}(x) = k\mathbb{G}_\mu[\delta_0](x)(1 + o(1))$ as $x \to 0$ and we conclude as in the proof of Lemma 3.3 that $u = u_{\nu_\sigma+k\delta_0}$.

At end we let $\sigma \to 0$. Up to a sequence $\{\sigma_j\}$ converging to 0 such that $u_{\nu_{\sigma_j}+k\delta_0} \to U$ almost everywhere and

$$u_{-\nu_-} \le U \le u_{\nu_++k\delta_0}. \tag{3.46}$$

Since by Lemma 3.4, $u_{\nu_{\sigma+}+k\delta_0} \to u_{\nu_++k\delta_0}$ in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ and $g(u_{\nu_{\sigma+}+k\delta_0}) \to g(u_{\nu_++k\delta_0})$ in $L^1(\Omega, \rho d\gamma_\mu)$, we infer that the convergences of $u_{\nu_{\sigma_j}+k\delta_0} \to U$ and $g(u_{\nu_{\sigma_j}+k\delta_0}) \to g(U)$ occur respectively in the same space, therefore $U = u_{\nu+k\delta_0}$, it is the weak solution of (1.6). □

Remark. In the course of the proof we have used the following result which is independent of any assumption on g except for the monotonicity: If $\{\nu_n\} \subset \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ is an increasing sequence of g-good measures converging to a measure $\nu \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$, then ν is a g-good measure, $\{u_{\nu_n}\}$ converges to u_ν in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$ and $\{g(u_{\nu_n})\}$ converges to $g(u_\nu)$ in $L^1(\Omega, \rho d\gamma_\mu)$.

3.5. Proof of Theorem C

The construction of a solution is essentially similar to the one of Theorem B, the only modifications lies in Lemma 3.3. Estimate (3.31) remains valid with

$$u_{k\delta_0}(x) = \tfrac{k}{|S^{N-1}|}|x|^{\frac{2-N}{2}} \ln |x|^{-1}(1 + o(1)) = k\mathbb{G}_\mu[\delta_0](x)(1 + o(1)) \quad \text{as } x \to 0. \tag{3.47}$$

Since $u_{\nu_\sigma}(x) \leq c_{15}|x|^{\frac{2-N}{2}}$, (3.32) holds with $\delta > 0$ arbitrarily small. Next

$$\begin{aligned}\int_\Omega g((k+\delta)\mathbb{G}_\mu[\delta_0])d\gamma_\mu(x) &\leq \int_{B_1} g\left(\tfrac{k+\delta}{|S^{N-1}|}|x|^{\frac{2-N}{2}}\ln|x|^{-1}\right)|x|^{\frac{2-N}{2}}dx\\ &= |S^{N-1}|\int_0^1 g\left(\tfrac{k+\delta}{|S^{N-1}|}r^{\frac{2-N}{2}}\ln r^{-1}\right)r^{\frac{N}{2}}dr\\ &= c_{16}\int_{c'}^\infty g(t\ln t)t^{-\frac{2N}{N-2}} < \infty,\end{aligned}$$

by (3.19) and (1.35). The end of the proof for Theorem C is similar to the one of Theorem B.
□

Proof of Corollary D. If $g(r) = g_p(r) = |r|^{p-1}r$, $p > 1$, the existence of a solution with $\nu = k\delta_0$ is a direct consequence of conditions (1.34) and (1.35). If $k = 0$ and $\nu\lfloor_{\Omega^*} \neq 0$, the existence is ensured if (1.8) holds, hence $p < \frac{N}{N-2}$. Assertion (iii) follows. □

4. The supercritical case

The notion of reduced measures introduced by Brezis, Marcus and Ponce [8] turned out to be a useful tool in the construction of solutions in a measure framework. We will develop only the aspect needed for the proof of Theorem E. If $k \in \mathbb{N}^*$, we set

$$g_k(r) = \begin{cases} \min\{g(r), g(k)\} & \text{if } r \geq 0,\\ \max\{g(r), g(-k)\} & \text{if } r > 0.\end{cases} \tag{4.1}$$

Since g_k satisfies (1.34) and (1.35), for any $\nu \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$ there exists a unique weak solution $u = u_{\nu,k}$ of

$$\begin{cases} \mathcal{L}_\mu u + g_k(u) = \nu & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{4.2}$$

Furthermore, from the proof of Lemma 3.4 and Kato's type estimates Proposition 2.1 we have that

$$0 \leq u_{\nu_+,k'} \leq u_{\nu_+,k} \qquad \text{for all } k' \geq k > 0. \tag{4.3}$$

Proposition 4.1. *Let* $\nu \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$. *Then the sequence of weak solutions* $\{u_{\nu,k}\}$ *of*

$$\begin{cases} \mathcal{L}_\mu u + g_k(u) = \nu & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{4.4}$$

decreases and converges, when $k \to \infty$, *to some nonnegative function u, and there exists a measure* $\nu^* \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$ *such that* $0 \leq \nu^* \leq \nu$ *and* $u = u_{\nu^*}$.

Proof. The proof is similar to the one of [8, Prop. 4.1]. Observe that $u_{\nu,k} \downarrow u^*$ and the sequence $\{u_{\nu,k}\}$ is uniformly integrable in $L^1(\Omega, |x|^{-1}d\gamma_\mu)$. By Fatou's lemma u satisfies

$$\int_\Omega \left(u^*\mathcal{L}_\mu^*\xi + g(u^*)\xi\right)d\gamma_\mu(x) \leq \int_\Omega \xi d(\Gamma_\mu\nu) \quad \text{for all } \xi \in \mathbb{X}\mu(\Omega),\ \xi \geq 0. \tag{4.5}$$

Hence u^* is a subsolution of (1.6) and by construction it is the largest of all nonnegative subsolutions. The mapping

$$\xi \mapsto \int_\Omega \left(u^* \mathcal{L}_\mu^* \xi + g(u^*)\xi\right) d\gamma_\mu(x) \qquad \text{for all } \xi \in C_c^\infty(\Omega),$$

is a positive distribution, hence a measure ν^*, called *the reduced measure of* ν. It satisfies $0 \le \nu^* \le \nu$ and $u^* = u_{\nu^*}$. □

Lemma 4.2. *Let $\nu, \nu' \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$. If $\nu' \le \nu$ and $\nu = \nu^*$, then $\nu' = \nu'^*$.*

Proof. Let $u_{\nu',k}$ be the weak solution of the truncated equation

$$\begin{cases} \mathcal{L}_\mu u + g_k(u) = \nu' & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Then $0 \le u_{\nu',k} \le u_{\nu,k}$. By Proposition 4.1, we know that $u_{\nu,k} \downarrow u_{\nu^*} = u_\nu$ and $u_{\nu',k} \downarrow u'^*$ a.e. in $L^1(\Omega, |x|^{-1} d\gamma_\mu)$ and then

$$\mathcal{L}_\mu(u_{\nu,k} - u_\nu) + g_k(u_{\nu,k}) - g_k(u_\nu) = g(u_\nu) - g_k(u_\nu),$$

from what follows, by Proposition 2.1,

$$\int_\Omega (u_{\nu,k} - u_\nu))|x|^{-1} d\gamma_\mu + \int_\Omega |g_k(u_{\nu,k}) - g_k(u_\nu)|\eta_1 d\gamma_\mu \le \int_\Omega |g(u_\nu) - g_k(u_\nu)|\eta_1 d\gamma_\mu.$$

By the increasing monotonicity of mapping $k \mapsto g_k(u_\nu)$, we have $g_k(u_\nu) \to g(u_\nu)$ in $L^1(\Omega, \rho d\gamma_\mu)$ as $k \to +\infty$, hence

$$\int_\Omega |g_k(u_{\nu,k}) - g(u_\nu)|\eta_1 d\gamma_\mu \le 2\int_\Omega |g(u_\nu) - g_k(u_\nu)|\eta_1 d\gamma_\mu \to 0 \ \text{ as } \ n \to \infty.$$

Because $g_k(u_{\nu',k}) \le g_k(u_{\nu,k})$ it follows by Vitali's convergence theorem that $g_k(u_{\nu',k}) \to g(u'^*)$ in $L^1(\Omega, \rho d\gamma_\mu)$. Using the weak formulation of (4.6), we infer that u'^* verifies

$$\int_\Omega \left(u'^* \mathcal{L}_\mu^* \xi + g(u'^*)\xi\right) d\gamma_\mu = \int_\Omega \xi d(\gamma_\mu \nu') \quad \text{ for all } \xi \in \mathbb{X}_\mu(\Omega).$$

This yields $u'^* = u_{\nu'}$. □

The next result follows from Lemma 4.2.

Lemma 4.3. *Assume that $\nu = \nu\lfloor_{\Omega^*} + k\delta_0 \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$, then $\nu^* = \nu^*\lfloor_{\Omega^*} + k^*\delta_0 \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$ with $\nu^*\lfloor_{\Omega^*} \le \nu\lfloor_{\Omega^*}$ and $k^* \le k$. More precisely,*

(i) If $\mu > \mu_0$ and g satisfies (1.34), then $k = k^*$.

(ii) If $\mu = \mu_0$ and g satisfies (1.35), then $k = k^*$.

(ii) If $\mu > \mu_0$ (resp. $\mu = \mu_0$) and g does not satisfy (1.21) (resp. (1.35)), then $k^* = 0$.

The next result is useful in applications.

Corollary 4.1. *If $\nu \in \overline{\mathfrak{M}}_+(\Omega; \Gamma_\mu)$, then ν^* is the largest g-good measure smaller or equal to ν.*

Proof. Let $\lambda \in \overline{\mathfrak{M}}_+(\Omega;\Gamma_\mu)$ be a g-good measure, $\lambda \leq \nu$. Then $\lambda^* = \lambda \leq \nu^*$. Since ν^* is a g-good measure the result follows. □

Proof of Theorem E. Assume that $\nu \geq 0$. By Lemma 4.2 and Remark at the end of Section 3.5 the following assertions are equivalent:
(i) ν is g_p-good.
(ii) For any $\sigma > 0$, $\nu_\sigma = \chi_{B^c_\sigma}\nu$ is g_p-good.

If ν_σ is good, then u_{ν_σ} satisfies

$$-\Delta u_{\nu_\sigma} + u^p_{\nu_\sigma} = \nu_\sigma - \frac{\mu}{|x|^2}u_{\nu_\sigma} \quad \text{in } \mathcal{D}'(\Omega^*) \tag{4.7}$$

and since $u_{\nu_\sigma}(x) \leq c|x|^{\tau_+}$ if $|x| \leq \frac{\sigma}{2}$ (4.7) holds in $\mathcal{D}'(\Omega)$. This implies that $u \in L^p(\Omega)$ and $|x|^{-2}u_{\nu_\sigma} \in L^\alpha(B_{\frac{\sigma}{2}})$ for any $\alpha < \frac{N}{(2-\tau_+)_+}$. Using [1] the measure ν_σ is absolutely continuous with respect to the $c_{2,p'}$-Bessel capacity. If $E \subset \Omega$ is a Borel set such that $c_{2,p'}(E) = 0$, then $c_{2,p'}(E \cap B^c_\sigma) = 0$, hence $\nu(E \cap B^c_\sigma) = \nu_\sigma(E \cap B^c_\sigma) = 0$. By the monotone convergence theorem $\nu(E) = 0$.

Conversely, if ν is nonnegative and absolutely continuous with respect to the $c_{2,p'}$-Bessel capacity, then so is $\nu_\sigma = \chi_{B^c_\sigma}\nu$. For $0 \leq \epsilon \leq \frac{\sigma}{2}$ we consider the problem

$$\begin{cases} -\Delta u + \dfrac{\mu}{|x|^2}u + u^p = \nu_\sigma & \text{in } \Omega^\epsilon := \Omega \setminus B_\epsilon, \\ u = 0 & \text{on } \partial B_\epsilon, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

Since $\frac{\mu}{|x|^2}$ is bounded in Ω^ϵ and ν_σ is absolutely continuous with respect to the $c_{2,p'}$ capacity there exists a solution $u_{\nu_\sigma,\epsilon}$ thanks to [1], unique by monotonicity. Now the mapping $\epsilon \mapsto u_{\nu_\sigma,\epsilon}$ is decreasing. We use the method developed in Lemma 3.1, when $\epsilon \to 0$, we know that $u_{\nu_\sigma,\epsilon}$ increase to some u_σ which is dominated by $\mathbb{G}[\nu_\sigma]$ and satisfies

$$\begin{cases} -\Delta u + \dfrac{\mu}{|x|^2}u + u^p = \nu_\sigma & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Because $u_\sigma \leq \mathbb{G}[\nu_\sigma]$ and $\nu_\sigma = 0$ in B_σ, there holds $u(x) \leq c'_{11}\Gamma_\mu(x)$ in $B_{\frac{\sigma}{2}}$, and then u_σ is a solution in Ω and $u = u_{\nu_\sigma}$. Letting $\sigma \to 0$, we conclude as in Lemma 3.1 that u_{ν_σ} converges to u_ν which is the weak solution of

$$\begin{cases} -\Delta u + \dfrac{\mu}{|x|^2}u + u^p = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

If ν is a signed measure absolutely continuous with respect to the $c_{2,p'}$-capacity, so are ν_+ and ν_-. Hence there exists solutions u_{ν_+} and u_{ν_-}. For $0 < \epsilon < \frac{\sigma}{2}$ we construct $u_{\nu_\sigma,\epsilon}$ with the property that $-u_{-\nu_{-\sigma},\epsilon} \leq u_{\nu_\sigma,\epsilon} \leq u_{\nu_{+\sigma},\epsilon}$, we let $\epsilon \to 0$ and deduce the existence of u_{ν_σ} which is eventually the weak solution of

$$\begin{cases} -\Delta u + \dfrac{\mu}{|x|^2}u + |u|^{p-1}u = \nu_\sigma & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.11}$$

and satisfies $-u_{-\nu_{-\sigma}} \leq u_{\nu_\sigma} \leq u_{\nu_{+\sigma}}$. Letting $\sigma \to 0$ we obtain that $u = \lim_{\sigma\to 0} u_{\nu_\sigma}$ satisfies

$$\begin{cases} -\Delta u + \dfrac{\mu}{|x|^2}u + |u|^{p-1}u = \nu & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

Hence $u = u_\nu$ and ν is a good solution. □

Proof of Theorem F. **Part 1.** Without loss of generality we can assume that Ω is a bounded smooth domain. Let $K \subset \Omega$ be compact. If $0 \in K$ and $p < p^*_\mu$ there exists a solution $u_{k\delta_0}$, hence K is not removable. If $0 \notin K$ and $c_{2,p'}(K) > 0$, there exists a capacitary measure $\nu_K \in W^{-2,p}(\Omega) \cap \mathfrak{M}_+(\Omega)$ with support in K. This measure is g_p-good by Theorem E, hence K is not removable.

Part 2. Conversely we first assume that $0 \notin K$. Then there exists a subdomain $D \subset \Omega$ such that $0 \notin \bar{D}$ and $K \subset D$. Hence a solution u of (1.37) is also a solution of

$$-\Delta u + \frac{\mu}{|x|^2}u + |u|^{p-1}u = 0 \quad \text{in } D \setminus K,$$

and the coefficient $\frac{\mu}{|x|^2}$ is uniformly bounded in $\bar{D}$. By [1, Theorem 3.1] it can be extended as a C^2 solution of the same equation in Ω'. Hence, if $c_{2,p'}(K) = 0$ the set K is removable.

If $0 \in K$ we have to assume at least $p \geq p^*_\mu$ in order that 0 is removable and $p \geq p_0$ in order there exists non-empty set with zero $c_{2,p'}$-capacity. Let $\zeta \in C^{1,1}_0(\Omega)$ with $0 \leq \zeta \leq 1$, vanishing in a compact neighborhood D of K. Since $0 \notin \Omega \setminus D$, we first consider the case where u is nonnegative and satisfies in the usual sense

$$-\Delta u + \frac{\mu}{|x|^2}u + u^p = 0 \quad \text{in } \Omega \setminus D.$$

Taking $\zeta^{2p'}$ for test function, we get

$$-2p'\int_\Omega u\zeta^{2p'-1}\Delta\zeta dx - 2p'(2p'-1)\int_\Omega u\zeta^{2p'-2}|\nabla\zeta|^2 dx + \mu\int_\Omega \frac{u\zeta^{2p'}}{|x|^2}dx + \int_\Omega \zeta^{2p'}u^p dx = 0.$$

There holds

$$\left|\int_\Omega u\zeta^{2p'-1}\Delta\zeta dx\right| \leq \left(\int_\Omega \zeta^{2p'}u^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |\Delta\zeta|^{p'}\zeta^{p'}dx\right)^{\frac{1}{p'}},$$

$$0 \leq \int_\Omega u\zeta^{2p'-2}|\nabla\zeta|^2 dx \leq \left(\int_\Omega \zeta^{2p'}u^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |\nabla\zeta|^{2p'}dx\right)^{\frac{1}{p'}},$$

and

$$0 \leq \int_\Omega \frac{u\zeta^{2p'}}{|x|^2}dx \leq \left(\int_\Omega \zeta^{2p'}u^p dx\right)^{\frac{1}{p}} \left(\int_\Omega \frac{\zeta^{2p'}}{|x|^{2p'}}dx\right)^{\frac{1}{p'}}.$$

By standard elliptic equations regularity estimates and Gagliardo-Nirenberg inequality [21] (and since $0 \leq \zeta \leq 1$),

$$\left(\int_\Omega |\Delta\zeta|^{p'}\zeta^{p'}\right)^{\frac{1}{p'}} \leq c_{17}\|\zeta\|_{W^{2,p'}}$$

and

$$\left(\int_\Omega |\nabla\zeta|^{2p'}dx\right)^{\frac{1}{p'}} \leq c_{18}\|\zeta\|_{W^{2,p'}}.$$

Finally, if $p > p_0 := \frac{N}{N-2}$, then $2p' < N$ which implies that there exists c_{19} independent of ζ (with value in $[0,1]$) such that

$$\left(\int_\Omega \frac{\zeta^{2p'}}{|x|^{2p'}}dx\right)^{\frac{1}{p'}} \le \left(\int_{B_1} \frac{dx}{|x|^{2p'}}\right)^{\frac{1}{p'}} := c_{19}.$$

Next we set

$$X = \left(\int_\Omega \zeta^{2p'} u^p dx\right)^{\frac{1}{p}},$$

and if $\mu \ge 0$, $p \ge p_0$, we have

$$X^p - (2p'(2p'-1)c_{18} - p'c_{18})\,\|\zeta\|_{W^{2,p'}} X \le 0; \tag{4.13}$$

and if $\mu < 0$, $p > p_0$, we have

$$X^p - ((2p'(2p'-1)c_{18} - p'c_{18})\,\|\zeta\|_{W^{2,p'}} - c_{19}\mu)\,X \le 0. \tag{4.14}$$

However, the condition $p > p_0$ is ensured when $\mu < 0$ since $p \ge p^*_\mu > p_0$. We consider a sequence $\{\eta_n\} \subset \mathcal{S}(\mathbb{R}^N)$ such that $0 \le \eta_n \le 1$, $\eta_n = 0$ on a neighborhood of K and such that $\|\eta_n\|_{W^{2,p'}} \to 0$ when $n \to \infty$. Such a sequence exists by the result in [24] since $c_{2,p'}(K) = 0$. Let $\xi \in C_0^\infty(\Omega)$ such that $0 \le \xi \le 1$ and with value 1 in a neighborhood of K. We take $\zeta := \zeta_n = (1-\eta_n)\xi$ in the above estimates. Letting $n \to \infty$, then $\zeta_n \to \xi$ in $W^{2,p'}$ and finally

$$X^{p-1} = \left(\int_\Omega \xi^{2p'} u^p dx\right)^{\frac{p-1}{p}} \le (2p'(2p'-1)c_{18} - p'c_{18})\,\|\xi\|_{W^{2,p'}} + c_{19}\mu_-, \tag{4.15}$$

under the condition that $p > p_0$ if $\mu < 0$, in which case there also holds

$$\int_\Omega \frac{u\zeta^{2p'}}{|x|^2}dx \le c_{19}X. \tag{4.16}$$

However the condition $p > p_0$ is not necessary in order the left-hand side of (4.16) is bounded, since we have

$$\mu\int_\Omega \frac{u\zeta^{2p'}}{|x|^2}dx + X^p \le (2p'(2p'-1)c_{18} - p'c_{18})\,\|\zeta\|_{W^{2,p'}} X, \tag{4.17}$$

and X is bounded.

Next we take $\zeta := \zeta_n = (1-\eta_n)\xi$ for test function in (1.37) and get

$$-\int_\Omega ((1-\eta_n)\Delta\xi - \xi\Delta\eta_n - 2\langle\nabla\eta_n, \nabla\xi\rangle)\,u dx + \mu\int_\Omega \frac{u\zeta_n}{|x|^2}dx + \int_\Omega \zeta_n u^p dx = 0.$$

Since

$$\int_\Omega u\xi\Delta\eta_n dx \le \left(\int_\Omega u^p \xi dx\right)^{\frac{1}{p}} \|\eta_n\|_{W^{2,p'}} \to 0 \quad \text{as } n \to \infty,$$

and

$$\left|\int_\Omega u\langle\nabla\eta_n, \nabla\xi\rangle dx\right| \le \left(\int_\Omega u^p|\nabla\xi|dx\right)^{\frac{1}{p}} \|\nabla\xi\|_{L^\infty}\|\eta_n\|_{W^{1,p'}} \quad \text{as } n \to \infty,$$

then we conclude that u satisfies

$$-\int_{\Omega} u\Delta\xi dx + \mu \int_{\Omega} \frac{u\xi}{|x|^2} dx + \int_{\Omega} \xi u^p dx = 0, \tag{4.18}$$

which proves that u satisfies the equation in the sense of distributions. By standard regularity u is C^2 in Ω^*, and by the maximum principle $u(x) \leq c_{20}\Gamma_\mu(x)$ in $B_{r_0} \subset \Omega$. Integrating by part as in the proof of Lemma 3.2 we obtain that u satisfies

$$\int_{\Omega} \left(u\mathcal{L}_\mu^* \xi + \xi u^p \right) d\gamma_\mu(x) = 0 \quad \text{for every } \xi \in \mathbb{X}_\mu(\Omega). \tag{4.19}$$

Finally, if u is a signed solution, then $|u|$ is a subsolution. For $\epsilon > 0$ we set $K_\epsilon = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) \leq \epsilon\}$. If ϵ is small enough $K_\epsilon \subset \Omega$. Let $v := v_\epsilon$ be the solution of

$$\begin{cases} -\Delta v + \dfrac{\mu}{|x|^2} v + v^p = 0 & \text{in } \Omega \setminus K_\epsilon, \\ v = |u|\lfloor_{\partial K_\epsilon} & \text{on } \partial K_\epsilon, \\ v = |u|\lfloor_{\partial\Omega} & \text{on } \partial\Omega. \end{cases} \tag{4.20}$$

Then $|u| \leq v_\epsilon$. Furthermore, by Keller-Osserman estimate as in [22, Lemma 1.1], there holds

$$v_\epsilon(x) \leq c_{21} \operatorname{dist}(x, K_\epsilon)^{-\frac{2}{p-1}} \quad \text{for all } x \in \Omega \setminus K_\epsilon, \tag{4.21}$$

where $c_{21} > 0$ depends on N, p and μ. Using local regularity theory and the Arzela-Ascoli Theorem, there exists a sequence $\{\epsilon_n\}$ converging to 0 and a function $v \in C^2(\Omega \setminus K) \cap C(\bar{\Omega} \setminus K)$ such that $\{v_{\epsilon_n}\}$ converges to v locally uniformly in $\bar{\Omega} \setminus K$ and in the $C^2_{loc}(\Omega \setminus K)$-topology. This implies that v is a positive solution of (1.37) in $\Omega \setminus K$. Hence it is a solution in Ω. This implies that $u \in L^p(\Omega)$ and $|u(x)| \leq v(x) \leq c_{20}\Gamma_\mu(x)$ in Ω^*. We conclude as in the nonnegative case that u is a weak solution in Ω. □

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Conflict of interest

The authors declare no conflict of interest.

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