Mathematical Biosciences and Engineering

https://www.aimspress.com/journal/mbe

MBE, 22(7): 1861–1874.
DOI: 10.3934/mbe.2025067
Received: 19 March 2025
Revised: 07 May 2025
Accepted: 19 May 2025
Published: 06 June 2025

Research article

Massera's theorem on arbitrary discrete time domains

Martin Bohner[1], **Jaqueline G. Mesquita**[2] **and Sabrina Streipert**[3,*]

[1] Department of Mathematics and Statistics, Missouri S&T, 400 W. 12th St., Rolla, MO 65409, USA
[2] Departamento de Matemática, Universidade de Brasília, 70910-900 Brasília, DF, Brazil
[3] Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA

* **Correspondence:** Email: streipert@pitt.edu; Tel: +14126248375; Fax: +14126248375.

Abstract: We present a general version of Massera's theorems for arbitrary discrete domains, based on a newly introduced definition for both linear and nonlinear equations. For scalar nonlinear equations, we identify sufficient conditions that ensure each μ-bounded solution approaches a periodic solution asymptotically. In the case of linear systems, we prove that the presence of a μ-bounded solution necessarily leads to a periodic solution. We also provide some examples to show the practical implications of our findings.

Keywords: isolated time scales; linear dynamic equations; nonlinear dynamic equations; boundedness; periodicity

1. Introduction

The pursuit of periodic solutions in differential equations is a foundational aspect of the qualitative theory of differential equations, and it has been explored by many researchers, due to its theoretical elegance and its extensive practical implications in modeling a variety of dynamical systems and types of differential equations. In 1950, J. L. Massera contributed a seminal paper to the field of differential equations, where he explored the existence of periodic solutions for ordinary differential equations characterized by periodic right-hand sides [1]. His analysis focused on n-dimensional systems that considered a smooth function f that is 1-periodic in the time variable. Massera's findings yielded several key insights: *i)* For scalar equations (where $n = 1$), the presence of a bounded solution invariably indicates the existence of a 1-periodic solution. *ii)* This correlation does not necessarily extend to systems with multiple equations ($n > 1$), where the scenario becomes more complex. *iii)* In the case of linear systems, similar to scalar equations, a bounded solution does suggest the existence of a 1-periodic solution.

Furthermore, Massera elucidated that, in the case of scalar equations, every bounded solution tends toward a 1-periodic solution, a principle that does not universally apply to linear systems. Massera's theorem provided deep insights into the behavior of differential systems, and it has inspired numerous adaptations to a range of equation types, such as functional differential equations [2–4], partial functional differential equations [5, 6], Liénard equations [7, 8], quantum time scales [9], and dynamical systems of higher dimensions [10].

It should be noted that delay differential equations [11] and externally forced systems [12] can be used to model nonlinear periodic dynamics in complex systems (see also, for example, [13–15]). These formulations allow for a more accurate representation of real-world nonlinear behaviors, especially when periodic solutions are involved. Although Massera's theorem was originally formulated for ordinary differential equations only, providing a relation between boundedness, stability, and periodic behavior in ODEs, its conceptual framework — that boundedness and asymptotic stability imply the existence of periodic solutions — has been attempted to generalize to these more complex systems such as functional equations with delay [16, 17] and partial differential equations [18, 19].

However, the application of Massera's theorem within the context of dynamic equations on time scales, particularly across all types of discrete time domains, e.g., the so-called isolated time scales, remains relatively unexplored. Although versions of Massera's theorem have been developed for time scales that possess additive properties, the restrictive nature of these properties significantly limits the scope of applicable time scales, even for discrete ones. More specifically, existing formulations of Massera's theorem for dynamic equations on time scales commonly rely on the restrictive assumption of a periodic time domain so that for all t in the time domain $\mathbb{T}$, $t \pm \omega \in \mathbb{T}$. This is too restrictive, as for example $\mathbb{T} = 2^{\mathbb{N}_0} \cup 3^{\mathbb{N}_0} = \{1, 2, 3, 4, 8, 9, 16, 27, 32, \ldots\}$ does not satisfy this condition for any ω, making a corresponding formulation inadequate for such time domains. In this work, we exploit a recently developed concept of periodicity [20] to overcome such limitations. Here, we remark that such arbitrary isolated time domains are specifically relevant for applications in physics, biology, and life sciences in general [21, 22], where processes are described at non-equidistant time points.

It should be noted that a specific iteration of Massera's theorem tailored to the particular time scale, the quantum scale $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$ was developed in [9]. This adaptation is especially significant, since quantum calculus plays an important role in describing phenomena in quantum physics, underscoring the importance of this result in both theoretical and applied contexts [23, 24].

Prompted by these gaps, this paper aims to expand Massera's theorem to all types of discrete time domains, introducing a comprehensive treatment and expanding to other more general extensions. Through this exploration, we seek to improve the theoretical landscape and enhance the applicability of this pivotal theorem to a broader spectrum of dynamic equations, which can also be used to deepen the understanding of some numerical simulations in the search for periodic solutions.

This paper is structured as follows: Section 2 outlines the essential concepts necessary for understanding the results presented in this article. Section 3 is dedicated to the nonhomogeneous linear dynamic equation on time scales, expressed as

$$x^{\Delta}(t) = a(t)x(t) + b(t). \tag{1.1}$$

To improve this analysis, we introduce a novel concept of boundedness within the framework of time scale theory, which is called μ-boundedness. We highlight that in the generalized version of Massera's theorem, it is not the traditional notion of boundedness that guarantees the existence of a periodic solution. Intriguingly, the concept of μ-boundedness is consistent with the established notion of boundedness for $\mathbb{T}_I = \mathbb{Z}$, as well as with the concept of q-boundedness for $\mathbb{T}_I = q^{\mathbb{N}_0}$ where $q > 1$.

This introduction of μ-boundedness is crucial for extending Massera's theorem to encompass any isolated time scales, applicable to both linear and nonlinear versions. This generalization not only broadens the applicability of the theorem but also deepens our understanding of dynamic equations across diverse discrete time scales. Besides the required new concept of μ-boundedness, the present work also established novel conditions for the functions a and b, as seen in (1.1), to be satisfied in the formulation of Massera's theorem on isolated time domains. We remark that these conditions replace the classical periodicity condition in the traditional formulation of Massera's theorem. Specifically, we require that the function $a\colon \mathbb{T}_I \to \mathbb{R}$ satisfies the condition

$$\left(a + \frac{1}{\mu}\right)\sigma^{\Delta} \in \mathcal{P}_{\omega}, \tag{1.2}$$

and the function $b\colon \mathbb{T}_I \to \mathbb{R}$ satisfies the condition

$$b\mu^{\sigma} \in \mathcal{P}_{\omega}. \tag{1.3}$$

These conditions, however, coincide with the classical conditions of periodicity in the special cases of a discrete time domain $\mathbb{Z}$ with equidistant time points and the quantum time domain $q^{\mathbb{N}_0}$ $(q > 1)$.

We note that in this work, we exploit the recently introduced concept of periodicity in [20] that differs from the definition in [25]. The interested reader is referred to the discussion in [20, appendix], specifically Example A.6. Furthermore, unlike the approach taken here, in the recent paper [26], the author proves Massera's theorem for both linear and nonlinear dynamic equations on time scales. However, the conclusion reached in that work defines the solution's periodicity in a different sense than what is used here. The author assumes Δ-periodicity for the functions as a hypothesis in the sense of [25], but the resulting periodicity for the solution aligns with the definition provided in [25]. On the other hand, another substantial difference is related to the different concept of boundedness employed in [26] to achieve the results, while here we work with μ-boundedness. These remarks make the results substantially different. For further details on this and the relationships between the different concepts of periodicity, we refer to the discussion in [20, appendix].

We present a generalized version of Massera's theorem for linear dynamic equations on all types of isolated time scales. This version asserts that the existence of a μ-bounded solution necessitates the existence of an ω-periodic solution, and conversely. Our approach involves applying the Brouwer fixed point theorem. We conclude the third section by demonstrating some examples that illustrate our main results.

In Section 4, we explore Massera's theorem for nonlinear dynamic equations on all types of isolated time scales, which are described by

$$x^{\Delta}(t) = f(t, \mu(t)x(t)). \tag{1.4}$$

We supplement this discussion with examples to clarify and highlight the practical implications of our findings.

The final section provides a conclusion that delves into our findings, with their limitations, and proposes potential future research directions related to this topic. This section aims to encourage further exploration into the dynamics of equations on isolated time scales.

We finish the introduction explaining some important notations that will be used in the entire paper (for details, see [21, 22]).

Notation.

- $\mathbb{T}$ is a time scale, that is, a closed nonempty subset of $\mathbb{R}$;
- $\mathbb{T}_I$ denotes isolated time scale, that is, all points in $\mathbb{T}_I$ are right-scattered and left-scattered at the same time;
- $\sigma\colon \mathbb{T} \to \mathbb{T}$, $\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$;
- $\mu\colon \mathbb{T} \to \mathbb{R}_0^+$, $\mu(t) := \sigma(t) - t$;
- $f^\sigma\colon \mathbb{T} \to \mathbb{R}$, $f^\sigma := f \circ \sigma$;
- $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has a left-scattered maximum M, otherwise $\mathbb{T}^\kappa = \mathbb{T}$;
- $e_p(t, t_0)$ is the exponential function for $p \in \mathcal{R}$ and $t_0, t \in \mathbb{T}$, that is, the unique solution of the IVP $y^\Delta = p(t)y$, $y(t_0) = 1$ (see [22, Theorem 2.33]);
- $\nu\colon \mathbb{T}_I \to \mathbb{T}_I$, $\nu(t) = \sigma^\omega(t)$;
- $\mathcal{R}$ denotes the set of all rd-continuous regressive functions, that is, all functions $p\colon \mathbb{T} \to \mathbb{R}$ such that $1 + \mu(t)p(t) \neq 0$;
- $\mathcal{R}^+$ denotes the set of all rd-continuous positively regressive functions, that is, all functions $p\colon \mathbb{T} \to \mathbb{R}$ such that $1 + \mu(t)p(t) > 0$;
- $\mathcal{P}_\omega = \mathcal{P} = \mathcal{P}(\mathbb{T}_I, \mathbb{R})$ denotes the set of all ω-periodic functions $f\colon \mathbb{T}_I \to \mathbb{R}$, that is, all functions that satisfy $\nu^\Delta p^\nu = p$.

For the definition of Δ-derivatives, Δ-integrals, and their properties, we refer to [21, 22].

2. Preliminaries

In this section, we present some properties that will be used to prove the main results. We assume that the reader has some familiarity with the theory of time scales. For a more in-depth exploration of this theory, we refer to [21, 22].

Let $t, t_0 \in \mathbb{T}_I$ and $p \in \mathcal{R}$. Then the exponential function satisfies $e_p(t, t_0) = \prod_{s\in[t_0,t)\cap\mathbb{T}_I}(1 + \mu(s)p(s))$, for $t > t_0$. In addition, if $p \in \mathcal{P}_\omega$, then $e_p(\nu(t), t)$ is independent of $t \in \mathbb{T}_I$ and $e_p(\nu(t), \nu(s)) = e_p(t, s)$ for all $s, t \in \mathbb{T}_I$. Let $p \in \mathcal{P}_\omega$. Then $\int_t^{\nu(t)} p(\tau)\Delta\tau$ is independent of $t \in \mathbb{T}$ and $\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau = \int_s^t p(\tau)\Delta\tau$ for all $s, t \in \mathbb{T}$.

Theorem 2.1. *Let $a \in \mathcal{R}$ and $b\colon \mathbb{T}_I \to \mathbb{R}$. Assume* (1.2). *If* (1.1) *has a nontrivial ω-periodic solution, then* (1.3) *holds.*

Theorem 2.2. *Let $a \in \mathcal{R}$ and assume* (1.2). *For $t_0 \in \mathbb{T}_I$, we have*

$$e_a(\nu(t), t) = e_a(\nu(t_0), t_0)\frac{\nu^\Delta(t_0)}{\nu^\Delta(t)} \quad \text{for all} \quad t \in \mathbb{T}_I. \tag{2.1}$$

Moreover, we have

$$e_a(\nu(t), \nu(s)) = e_a(t, s)\frac{\nu^\Delta(s)}{\nu^\Delta(t)} \quad \text{for all} \quad s, t \in \mathbb{T}_I. \tag{2.2}$$

Lemma 2.3. *If $b\colon \mathbb{T}_I \to \mathbb{R}$, then b satisfies* (1.3) *if and only if*

$$\nu^\Delta \nu^{\Delta\sigma} b^\nu = b. \tag{2.3}$$

Theorem 2.4. *Let $a \in \mathcal{R}$ and $b\colon \mathbb{T}_I \to \mathbb{R}$. Assume* (1.2) *and* (1.3). *If* (1.1) *has a solution x that satisfies $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$ for some $t_0 \in \mathbb{T}$, then $x \in \mathcal{P}_{\omega}$.*

Theorem 2.5. *If $a \in \mathcal{R}$, then a satisfies* (1.2) *if and only if*

$$\nu^{\Delta\Delta} + \nu^{\Delta}\nu^{\Delta\sigma}a^{\nu} = \nu^{\Delta}a. \tag{2.4}$$

3. Massera's theorem for linear dynamic equations

In this section, our goal is to prove Massera's theorem for linear dynamic equations of the form (1.1) on isolated time scales. We will first present some auxiliary results.

Definition 3.1. A function $x\colon \mathbb{T}_I \to \mathbb{R}$ is called *μ-bounded* if there exists $K > 0$ such that $\mu(t)|x(t)| \leq K$ for all $t \in \mathbb{T}_I$.

Remark 3.2. At first glance, the concept of boundedness might seem unconventional. However, in the case of $\mathbb{T}_I = \mathbb{Z}$, this concept aligns perfectly with the traditional notion of a bounded solution since $\mu(t) = 1$. Similarly, for $h\mathbb{Z}$, with $\mu(t) = h$. For the quantum case, where $\mu(t) = (q-1)t$, it aligns with a q-bounded function, introduced in [9].

This observation allows us to confidently affirm that our concept of boundedness is consistent with established definitions found in classical cases of time scales in the literature. We note that the concept of μ-boundedness, albeit crucial for the arbitrary isolated time domain, is not relevant as a new concept in the traditional formulations of Massera's theorem for differential or even difference equations. In these special cases, the concept of traditional boundedness suffices.

Definition 3.3. We say that a linear dynamic equation given by (1.1) is *ω-periodic* whenever $a\colon \mathbb{T}_I \to \mathbb{R}$ satisfies (1.2) and $b\colon \mathbb{T}_I \to \mathbb{R}$ satisfies (1.3).

Before proceeding, we clarify what it means for an equation on an isolated time scale to be ω-periodic in this context. In the case of $\mathbb{T}_I = \mathbb{Z}$, this condition simply means that both a and b are ω-periodic. Conversely, for $\mathbb{T}_I = q^{\mathbb{N}}$, the periodicity conditions are expressed as

$$\left(a + \frac{1}{(q-1)t}\right)q \in \mathcal{P}_{\omega} \quad \text{and} \quad b(q^2 - q)t \in \mathcal{P}_{\omega}.$$

This formulation provides a precise definition of ω-periodicity within these specific time scales.

In the sequel, we present our main result of this section. It is a version of Massera's theorem for linear dynamic equation (1.1). In its proof, we use the following well-known fixed point result.

Theorem 3.4 (Brouwer's Fixed Point Theorem)**.** *For any continuous function Γ mapping a compact convex set into itself, there exists an x_0 in that set satisfying $\Gamma(x_0) = x_0$.*

Theorem 3.5. *Assume* (1.1) *is ω-periodic and $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$ for some solution x of* (1.1). *Then x is ω-periodic.*

Proof. Let x be a solution of (1.1) satisfying $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$. Define $f(t) = \nu^{\Delta}(t)x(\nu(t)) - x(t)$ so that $f(t_0) = 0$ and

$$f^{\Delta}(t) \quad = \quad \nu^{\Delta\Delta}(t)x(\nu(t)) + \nu^{\Delta}(\sigma(t))x^{\Delta}(\nu(t))\nu^{\Delta}(t) - x^{\Delta}(t)$$

$$
\begin{aligned}
&= \nu^{\Delta\Delta}(t)x^{\nu}(t) + \nu^{\Delta\sigma}(t)\nu^{\Delta}a^{\nu}(t)x^{\nu}(t) \\
&\quad + \nu^{\Delta}(t)\nu^{\Delta\sigma}(t)b^{\nu}(t) - a(t)x(t) - b(t) \\
&\overset{(2.3)}{=} x^{\nu}(t)(\nu^{\Delta\Delta}(t) + \nu^{\Delta}(t)\nu^{\Delta\sigma}(t)a^{\nu}(t)) - a(t)x(t) \\
&\overset{(2.4)}{=} x^{\nu}(t)(\nu^{\Delta}(t)a(t)) - a(t)x(t) = a(t)f(t).
\end{aligned}
$$

Hence, $f(t) \equiv 0$, which implies that $\nu^{\Delta}(t)x(\nu(t)) - x(t) = 0$ for every $t \in \mathbb{T}_I$, proving the result. □

Theorem 3.6 (Massera's Theorem for Linear Dynamic Equation)**.** *The ω-periodic linear dynamic equation on time scales* (1.1) *has an ω-periodic solution if and only if it has a μ-bounded solution.*

Proof. First, let us assume (1.1) has an ω-periodic solution x. Define

$$K := \max_{0\le k\le \omega-1} \mu(\sigma^k(t_0))|x(\sigma^k(t_0))|.$$

Let $t \in \mathbb{T}_I$ such that $t \geq t_0$. Then there exist $n, k \in \mathbb{N}_0$ with $0 \leq k \leq \omega - 1$ such that $t = \nu^n(\sigma^k(t_0))$. Thus, $\mu(t)|x^{\nu}(t)| = \mu(\nu^n(\sigma^k(t_0))|x^{\nu}(\sigma^k(t_0))| = \mu(\sigma^k(t_0))|x(\sigma^k(t_0))| \leq K$. Hence, x is μ-bounded. Reciprocally, assume (1.1) has a μ-bounded solution $\tilde{x}$. Then there exists $K > 0$ such that $\mu(t)|\tilde{x}(t)| \leq K$ for all $t \in \mathbb{T}_I$. Define

$$\Omega := \{x_0 \in \mathbb{R} : |x_0| \leq K,\ \mu(t)|x(t, x_0)| \leq K \text{ for all } t \in \mathbb{T}_I\},$$

where $x(\cdot, x_0)$ is the unique solution of (1.1) with $x(t_0) = x_0$ (see [21, Theorem 2.77]), i.e.,

$$x(t; x_0) = e_a(t, t_0)x_0 + \int_{t_0}^{t} e_a(t, \sigma(s))b(s)\Delta s.$$

Since $\tilde{x}(t_0) \in \Omega$, we have $\Omega \neq \emptyset$. Since $\Omega \subset \mathbb{R}$ is closed and bounded, it is compact. Now, we will show that Ω is convex. Let $x_1, x_2 \in \Omega$ and $0 \leq \alpha \leq 1$. Then

$$|\alpha x_1 + (1-\alpha)x_2| \leq \alpha|x_1| + (1-\alpha)|x_2| \leq \alpha K + (1-\alpha)K = K$$

and

$$
\begin{aligned}
\mu(t)|x(t, \alpha x_1 + (1-\alpha)x_2)| &= t\left|e_a(t,1)(\alpha x_1 + (1-\alpha)x_2) + \int_1^t e_a(t,\sigma(s))b(s)\Delta s\right| \\
&\leq \mu(t)\left|\alpha\left[e_a(t,1)x_1 + \int_1^t e_a(t,\sigma(s))b(s)\Delta s\right] + \right. \\
&\quad \left. + \ (1-\alpha)\left[e_a(t,1)x_2 + \int_1^t e_a(t,\sigma(s))b(s)\Delta s\right]\right| \\
&= \mu(t)|\alpha x(t, x_1) + (1-\alpha)x(t, x_2)| \\
&= \alpha\mu(t)|x(t, x_1)| + (1-\alpha)\mu(t)|x(t, x_2)| \\
&\leq \alpha K + (1-\alpha)K = K
\end{aligned}
$$

for all $t \in \mathbb{T}_I$. So $\alpha x_1 + (1-\alpha)x_2 \in \Omega$ and hence, Ω is convex. Define now $P : \Omega \to \mathbb{R}$ by

$$P(x_0) := \nu^{\Delta}(t_0)x(\nu(t_0); x_0) = \nu^{\Delta}(t_0)e_a(\nu(t_0), t_0)x_0 + \nu^{\Delta}(t_0)\int_{t_0}^{\nu(t_0)} e_a(\nu(t_0), \sigma(s))b(s)\Delta s.$$

Since P is linear, it is continuous. Let $x_0 \in \Omega$. Since $\mu(t)|x(t; x_0)| \leq K$ for all $t \in \mathbb{T}_I$, we have $|P(x_0)| = |\nu^\Delta(t_0)x(\nu(t_0)); x_0)| \leq K$. Moreover, using [21, Theorem 2.77], we obtain

$$\begin{aligned}
x(t; P(x_0)) &= e_a(t, t_0)P(x_0) + \int_{t_0}^{t} e_a(t, \sigma(s))b(s)\Delta s \\
&= e_a(t, t_0)\left\{\nu^\Delta(t_0)e_a(\nu(t_0), t_0)x_0 + \nu^\Delta(t_0)\int_{t_0}^{\nu(t_0)} e_a(\nu(t_0), \sigma(s))b(s)\Delta s\right\} \\
&\quad + \int_{t_0}^{t} e_a(t, \sigma(s))b(s)\Delta s \\
&= e_a(t, t_0)\nu^\Delta(t)e_a(\nu(t), t)x_0 + e_a(\nu(t), \nu(t_0))\nu^\Delta(t)\int_{t_0}^{\nu(t_0)} e_a(\nu(t_0), \sigma(s))b(s)\Delta s \\
&\quad + \int_{\nu(t_0)}^{\nu(t)} \nu^\Delta(t)e_a(\nu(t), \sigma(s))b(s)\Delta s \\
&= \nu^\Delta(t)\left[e_a(\nu(t), t_0)x_0 + \int_{t_0}^{\nu(t)} e_a(\nu(t), \sigma(s))b(s)\Delta s\right] = \nu^\Delta(t)x(\nu(t); x_0)
\end{aligned}$$

which implies $\mu(t)|x(t, P(x_0))| = \mu(\nu(t))|x(\nu(t); x_0)| \leq K$ for all $t \in \mathbb{T}_I$. Hence $P(x_0) \in \Omega$. Thus $P : \Omega \to \Omega$. By Theorem 3.4, P has a fixed point in Ω, i.e., there exists $\tilde{x}_0 \in \Omega$ with $x(t_0; \tilde{x}_0) = \tilde{x}_0 = P(\tilde{x}_0) = \nu^\Delta(t_0)x(\nu(t_0); x_0)$. By Theorem 2.4, $\tilde{x} = x(\cdot, \tilde{x}_0)$ is an ω-periodic solution of (1.1). □

From a computational perspective, Theorem 3.6 is powerful as it allows us to check for bounded solutions, which are usually easier to identify in simulations than periodic solutions.

For the case $\mathbb{T} = \mathbb{Z}$, Theorem 3.6 can be read as follows.

Corollary 3.7. The ω-periodic linear difference equation

$$\Delta x(t) = a(t)x(t) + b(t)$$

has an ω-periodic solution if and only if it has a bounded solution.

Example 3.8. Consider, for example, (1.1) with

$$\mathbb{T} = \mathbb{Z}, \qquad a(t) = \sin\left(\frac{2t\pi}{\omega}\right), \qquad b(t) = \cos\left(\frac{t\pi}{\omega}\right). \tag{3.1}$$

Note that a is classically ω-periodic and b is classically 2ω-periodic. Then, Theorem 3.6 implies that there exists a bounded solution if and only if there exists a 2ω-periodic solution. However, if we change the time domain to $\mathbb{T}_I = 2^{\mathbb{N}_0} \cup 1.5^{\mathbb{N}_0}$, the coefficients no longer satisfy the conditions (1.2) and (1.3) to apply Massera's theorem, Theorem 3.6. A visualization of this distinction is provided in Figure 1, where we chose $\omega = 3$. The left panel illustrates solutions to three different initial conditions in the case of a classically discrete time domain $\mathbb{T}_I = \mathbb{Z}$. Since (1.2) and (1.3) are satisfied, we can conclude that since the solution is bounded, there must exist a 6-periodic. However, if we change the time domain to be some (here: randomly chosen) isolated time domain, the same coefficients are now not satisfying conditions (1.2) and (1.3) so that Theorem 3.6 cannot be applied. Thus, just because the solutions in the right panel seem bounded, we cannot conclude that there exists a 6-periodic solution. In fact, if there were to be a 6-periodic solution, then $x(t_0)\mu(t_0) = x(t_6)\mu(t_6)$. Iterating back $x(t_6)$ to $x(t_0)$ yields, for this particular time domain in the right panel, that $x(t_0) \approx -1.0709$. However, since it also must satisfy $x(t_1)\mu(t_1) = x(t_7)\mu(t_7)$, we require $x(t_1) \approx 1.3641 \neq x(t_0)(\mu(t_0)a(t_0) - 1) + \mu(t_0)b(t_0) = (a(1) - 1) + b(1) \approx 0.8758$. Thus, there does not exist a 6-periodic solution despite the existence of a seemingly bounded solution.

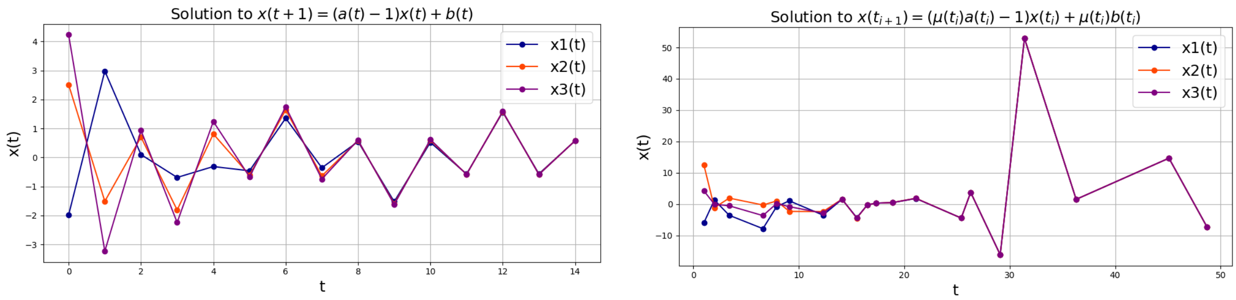

Figure 1. Color-coded solutions to three different initial conditions. Here, $\omega = 3$ so that the coefficients are 6 periodic. Although the dynamic equation is the same for both images, the time domain in the left panel is $\mathbb{T}_I = \mathbb{Z}$, in which case the coefficients satisfy (1.2) and (1.3) so that Theorem 3.6 can be applied. On the right panel, $\mathbb{T}_I$ is a randomly chosen isolated time scale with time points $\mathbb{T}_I = \{1, 2, 3.4, 6.6, 7.9, 9.1, 12.3, 14.1, 15.5, 16.5, 17.3, 18.9, 21.1, 25.4, 26.3, 29.1, \ldots\}$. In this case, (1.2) and (1.3) are not satisfied.

Example 3.9. Consider, for example, the classical Beverton–Holt model

$$x_{t+1} = \frac{K\rho x_t}{K + (\rho - 1)x_t}$$

that is used in the study of population models, describing the relation between the next generation x_{t+1} and the previous generation x_t [27]. A solution to this nonlinear recurrence can be obtained by the substitution $u_t = \frac{1}{x_t}$, which results in the linear recurrence $u_{t+1} = \frac{1}{\rho}u_t + \frac{\rho-1}{K\rho_t}$ [28], which allows the applicability of our formulation of Massera's theorem, Theorem 3.6, to gain insights into solutions of the original model. This example also highlights the relevance of studying isolated but arbitrary time domains. Based on its original derivation, $\rho = e^T > 1$, where T is the mean time it takes for newborns to become adults and therefore replace the previous generation [27]. While the classical formulation in discrete space, where time points are equidistant, assumes that it always takes exactly a time length of T to replace a generation (i.e., T is the generation time), external factors may speed up or slow down the process of generation time so that it truly should be T_i rather than T. In this case, not only would ρ be truly time-dependent, but also, x_{t+1} (which is the replacement for x_{t+T}) should be $x_{t_{i+1}}$.

For the case $\mathbb{T} = q^{\mathbb{N}_0}$, one can rewrite Theorem 3.6 as follows.

Corollary 3.10. The ω-periodic linear difference equation

$$x^{\Delta}(t) = a(t)x(t) + b(t)$$

has an ω-periodic solution if and only if it has a q-bounded solution.

Example 3.11. Consider the dynamic equation on an isolated time scale

$$x^{\Delta} = \frac{(5\sigma(t) - \sigma^2(t) - 4t)x + 2}{\mu(\sigma(t))\mu(t)}. \tag{3.2}$$

Notice that (3.2) is in the form (1.1) with

$$a(t) = \frac{4}{\mu(\sigma(t))} - \frac{1}{\mu(t)} \quad \text{and} \quad b(t) = \frac{2}{\mu(t)\mu(\sigma(t))}.$$

Clearly, $a \in \mathcal{R}$ since

$$1 + \mu(t)a(t) = \frac{4\mu(t)}{\mu(\sigma(t))} \neq 0$$

because of $\mu(t) > 0$ for all $t \in \mathbb{T}$. Also, clearly

$$\left(a + \frac{1}{\mu}\right)\sigma^{\Delta} = 4\frac{\sigma^{\Delta}}{\mu^{\sigma}} = \frac{4}{\mu} \in \mathcal{P}_{\omega} \quad \text{and} \quad b\mu^{\sigma} = \frac{2}{\mu} \in \mathcal{P}_{\omega},$$

so (1.2) and (1.3) are satisfied. Thus, by Theorem 3.5, the solution x of (3.2) satisfying $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$ is ω-periodic. Therefore, Theorem 3.6 implies the existence of a μ-bounded solution.

Example 3.12. Consider the linear dynamic equation on a time scale

$$x^{\Delta}(t) = -\frac{x(t)}{\mu(t)} + \frac{1}{(\mu(t))\mu(\sigma(t))}. \tag{3.3}$$

Here, $a(t) = -\frac{1}{\mu(t)}$ satisfies $(a + \frac{1}{\mu})\sigma^{\Delta} = 0 \in \mathcal{P}_1$ and $b(t)\mu(\sigma(t)) = \frac{1}{\mu(t)}$. Therefore,

$$\sigma^{\Delta}(t)b(\sigma(t))\mu(\sigma^2(t)) = \sigma^{\Delta}(t)\frac{1}{\mu(\sigma(t))} = \frac{1}{\mu(t)} = b(t)\mu(\sigma(t)).$$

Hence, $b\mu^{\sigma} \in \mathcal{P}_1$. The solution of (3.3) is given by $x(t) = \frac{1}{\mu(t)}$, which is clearly μ-bounded. Therefore, Theorem 3.6 implies the existence of a 1-periodic solution. On the other hand, clearly $x(t) = \frac{1}{\mu(t)}$ is 1-periodic.

4. Massera's theorem for nonlinear dynamic equations on time scales

In this section, we aim to prove a version of Massera's theorem for nonlinear dynamic equations on isolated time scales. We therefore consider (1.4) subject to the hypotheses

(H1) $f\colon \mathbb{T}_I \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to the second variable.
(H2) For each fixed $x \in \mathbb{R}$, assume $\nu^{\Delta}(t)\nu^{\Delta}(t)f(\nu(t), x) = f(t, x)$ for all $t \in \mathbb{T}_I$.

We note that if $\mathbb{T}_I = \mathbb{Z}$, then assumption (H2) is equivalent to requiring that f is ω-periodic (in the classical sense) in its first variable, that is, $f(t + \omega, x) = f(t, x)$ because in this case, $\nu(t) = t + \omega$ and $\nu^{\Delta} = 1$. However, for arbitrary isolated time domains $\mathbb{T}_I$, assumption (H2) is a new assumption not immediately obvious from its corresponding formulation of the discrete nor the continuous formulation of Massera's theorem.

The following auxiliary results will be useful.

Lemma 4.1. *Assume (H1)–(H2) and $\nu^{\Delta\Delta} \equiv 0$. Then the following assertions hold.*

(i) *If x is a solution of* (1.4)*, then so is y defined by $y(t) := \nu^{\Delta}(t)x(\nu(t))$.*
(ii) *Equation* (1.4) *has an ω-periodic solution if and only if there exists a solution x of* (1.4) *and $t_0 \in \mathbb{T}_I$ with $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$.*

Proof. Let us start by proving (i). Let x be a solution of (1.4) and define $y(t) := \nu^{\Delta}(t)x(\nu(t))$. Then, by the product rule,

$$\begin{aligned} y^{\Delta}(t) &= \nu^{\Delta\Delta}(t)x(\nu(\sigma(t))) + \nu^{\Delta}(t)\nu^{\Delta}(t)x^{\Delta}(\nu(t)) \\ &= \nu^{\Delta}(t)\frac{\sigma^{\Delta\nu}(t) - \sigma^{\Delta}(t)}{\mu^{\sigma}(t)}x(\nu(\sigma(t))) + \nu^{\Delta}(t)\nu^{\Delta}(t)x^{\Delta}(\nu(t)) \\ &= \nu^{\Delta}(t)\nu^{\Delta}(t)f(\nu(t), \mu(\nu(t))x(\nu(t))) \\ &= f(t, \mu(t)y(t)), \end{aligned}$$

proving our result.

Now, let us prove (ii). Suppose (1.4) has an ω-periodic solution x. Then, clearly for any $t_0 \in \mathbb{T}_I$, we get $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$.

Reciprocally, assume that x is a solution of (1.4) satisfying $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$ for some $t_0 \in \mathbb{T}_I$. Then, by (i), $y(t) = \nu^{\Delta}(t)x(\nu(t))$ is also a solution of (1.4) that satisfies $y(t_0) = \nu^{\Delta}(t_0)x(\nu(t_0))$. Therefore, by the uniqueness of solutions, it follows that $y(t) = x(t)$ for every $t \in \mathbb{T}_I$, so x is ω-periodic. □

In the proof of our main result below, we also use the hypothesis

$(H3)$ For all $t \in \mathbb{T}_I$, $x < y$ implies $x + \mu(t)f(t, \mu x) < y + \mu(t)f(t, \mu y)$.

Lemma 4.2. *Assume (H3). Then $x(1) < y(1)$ implies $x(t) < y(t)$ for all $t \in \mathbb{T}_I$.*

Proof. By induction, $x(1) < y(1)$ is given. Now suppose that $x(t) < y(t)$ holds. Then $x(\sigma(t)) = x(t) + \mu(t)f(t, \mu(t)x(t)) < y(t) + \mu(t)f(t, \mu(t)y(t)) = y(\sigma(t))$, obtaining the desired result. □

Lemma 4.3. *If $x_i(t) = (\nu^i)^{\Delta}(t)x(\nu^i(t))$ for all $t \in \mathbb{T}_I$ and $i \in \mathbb{N}$, then for all $t \in \mathbb{T}_I$ and $i \in \mathbb{N}$, we have $x_{i+1}(t) = \nu^{\Delta}(t)x_i(\nu(t))$.*

Proof. For $i = 1$, it follows directly. Notice that $(\nu^i)^{\Delta}(t) = \frac{\mu(\nu^i(t))}{\mu(t)}$. Therefore, we have

$$x_{i+1}(t) = \frac{\mu(\nu^{i+1}(t))}{\mu(t)}x(\nu^{i+1}(t)) = \frac{x_i(\nu(t))\mu(\nu(t))}{\mu(t)} = \nu^{\Delta}(t)x_i(\nu(t)),$$

proving the desired result. □

Theorem 4.4. *Assume (H1)–(H3). If* (1.4) *has a μ-bounded solution, then it has an ω-periodic solution.*

Proof. Let x be a μ-bounded solution of (1.4). Hence there exists $K > 0$ with $\mu(t)|x(t)| \leq K$ for all $t \in \mathbb{T}_I$. Define the sequence of functions $\{x_n\}$ by $x_n(t) = (\nu^n)^{\Delta}(t)x(\nu^n(t))$ on $\mathbb{T}_I$ for $n \in \mathbb{N}_0$.

Since $x_{n+1}(t) = (\nu^n)^{\Delta}(t)x_n(\nu(t))$ for all $n \in \mathbb{N}_0$, Lemma 4.1(i) shows that each $x_n, n \in \mathbb{N}$, is a solution of (1.4). Moreover, since x is μ-bounded and $|x_n(t)| = (\nu^n)^{\Delta}(t)|x(\nu^n(t))| \leqslant \frac{\mu(\nu(t))}{\mu_{\inf}}|x(\nu^n(t))| \leqslant \frac{K}{\mu_{\inf}}$, it follows that for $n \in \mathbb{N}_0$, each x_n is bounded.

First, assume $x(t_0) = x_1(t_0)$. Then $x(t_0) = \nu^\Delta(t_0)x(\nu(t_0))$ and by Lemma 4.1(ii), x is ω-periodic. Next, assume $x(t_0) < x_1(t_0)$. Then by Lemma 4.2, $x(t) < x_1(t)$ for all $t \in \mathbb{T}_I$. Hence $x(\nu^n(t)) < x_1(\nu^n(t))$ for all $t \in \mathbb{T}_I$, so that

$$\begin{aligned} x_n(t) &= (\nu^n)^\Delta(t)x(\nu^n(t)) < (\nu^n)^\Delta(t)x_1(\nu^n(t)) \\ &= (\nu^n)^\Delta(t)\nu^\Delta(\nu^n(t))x(\nu^{n+1}(t)) \\ &= \frac{\mu(\nu^n(t))}{\mu(t)}\frac{\mu(\nu^{n+1}(t))}{\mu(\nu^n(t))}x(\nu^{n+1}(t)) \\ &= \frac{\mu(\nu^{n+1}(t))}{\mu(t)}x(\nu^{n+1}(t)) = x_{n+1}(t) \end{aligned} \tag{4.1}$$

for all $t \in \mathbb{T}_I$. Thus, for each $t \in \mathbb{T}_I$, $\{x_n(t)\}_{n=1}^\infty$ is increasing and bounded, so we have $\lim_{n\to\infty} x_n(t) = \tilde{x}(t)$ pointwise for $t \in q^{\mathbb{N}_0}$, where $\tilde{x}$ is some function defined on $\mathbb{T}_I$. We have

$$\begin{aligned} \tilde{x}^\Delta(t) &= \frac{\tilde{x}(\sigma(t)) - \tilde{x}(t)}{\mu(t)} = \lim_{n\to\infty} \frac{x_n(\sigma(t)) - x_n(t)}{\mu(t)} = \lim_{n\to\infty} x_n^\Delta(t) \\ &= \lim_{n\to\infty} f(t, \mu(t)x_n(t)) = f(t, \mu(t)\tilde{x}(t)) \end{aligned}$$

for each $t \in \mathbb{T}_I$, as f is continuous in the second variable. So $\tilde{x}\colon \mathbb{T}_I \to \mathbb{R}$ solves (1.4). Moreover, $\nu^\Delta(t)\tilde{x}(\nu(t)) = \lim_{n\to\infty} \nu^\Delta(t)x_n(\nu(t)) = \lim_{n\to\infty} x_{n+1}(t) = \tilde{x}(t)$ for each $t \in \mathbb{T}_I$, so $\tilde{x}$ is ω-periodic. Finally, arguing similarly for the last case, $x(1) > x_1(1)$, we obtain an ω-periodic solution. □

Remark 4.5. If we replace the condition (H3) by

(H4) $f(t, \mu(t)x) > 0$ for all $t \in \mathbb{T}_I$ and $x \in \mathbb{R}$,

then it is not difficult to prove that the sequence $\{x_n\}$ obtained in the same way as in the proof of Theorem 4.4 is increasing and therefore satisfies the inequality (4.1). The rest of the proof follows in the same way as the proof of Theorem 4.4.

5. Conclusions

This paper extends some of the classical Massera theorems to dynamic equations on arbitrary isolated time scales by utilizing a newly introduced definition in [20] that adapts to the unique characteristics of these scales. We provided conditions for scalar nonlinear equations that guarantee that every bounded solution will asymptotically approach a periodic solution. This finding not only strengthens the theoretical underpinnings of dynamic systems theory but also provides practical insights into the behavior of such systems under non-standard conditions.

For linear systems, our results confirm that the existence of a bounded solution is a robust predictor for the emergence of periodic solutions. This relationship underscores the utility of the general version of Massera's theorem in predicting and analyzing the stability and long-term behavior of linear dynamic systems on isolated time scales.

We remark that several extensions of this work are possible. For example, in the case of nonlinear dynamic equations on isolated time scales, the formulation to bridge the existence of bounded solutions to the existence of periodic solutions relied on the assumption that the nonlinear function determining the dynamics satisfies an identity provided in (H2). It would be interesting to explore to what extent this assumption can be relaxed. More broadly, limitations of the presented work include the assumption of an arbitrary, yet discrete, time domain; the formulation for scalar systems; and the time scales analogue of ordinary differential equations. Future work could focus on extending this work to arbitrary, not necessarily discrete, time domains consisting, for example, of unions of continuous intervals and discrete points. Other extensions could focus on the formulation of the theorems to higher dimensions, which also requires the corresponding formulation of periodicity in higher dimensions [20, 29]. We remark that this would also allow the study of Massera's theorem for delay dynamic equations, as delay equations on arbitrary yet discrete time domains can be expressed as higher-order systems.

In general, our extension of Massera's theorems opens new avenues for exploring more complex dynamic systems where the interplay of nonlinearity and time-scale dynamics presents sophisticated challenges. Providing other formulations of Massera's theorem to arbitrary isolated time scales may be another future direction. The framework established here is meant to lay the groundwork for such potential future studies into the stability, control, and optimization of such systems, since periodicity and boundedness are key in investigating those properties.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Jaqueline Mesquita and Sabrina Streipert thank the Mathematisches Forschungsinstitut Oberwolfach for the Oberwolfach Research Fellows Program, where this work was initiated during their research stay in 2024.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. J. L. Massera, The existence of periodic solutions of systems of differential equations, *Duke Math. J.*, **17** (1950), 457–475. https://doi.org/10.1215/S0012-7094-50-01741-8
2. S. Murakami, T. Naito, N. V. Minh, Massera's theorem for almost periodic solutions of functional differential equations, *J. Math. Soc. Jpn.*, **56** (2004), 247–268. https://doi.org/10.2969/jmsj/1191418705
3. Q. Liu, N. Van Minh, G. Nguerekata, R. Yuan, Massera type theorems for abstract functional differential equations, *Funkcialaj Ekvacioj*, **51** (2008), 329–350. https://doi.org/10.1619/fesi.51.329
4. Y. Li, Z. Lin, Z. Li, A Massera type criterion for linear functional-differential equations with advance and delay, *J. Math. Anal. Appl.*, **200** (1996), 717–725. https://doi.org/10.1006/jmaa.1996.0235
5. R. Benkhalti, K. Ezzinbi, A Massera type criterion for some partial functional differential equations, *Dyn. Syst. Appl.*, **9** (2000), 221–228.

6. R. Benkhalti, H. Bouzahir, K. Ezzinbi, Existence of a periodic solution for some partial functional-differential equations with infinite delay, *J. Math. Anal. Appl.*, **256** (2001), 257–280. https://doi.org/10.1006/jmaa.2000.7321

7. M. Hayashi, On the improved Massera's theorem for the unique existence of the limit cycle for a Liénard equation, *Rend. Istit. Mat. Univ. Trieste*, **48** (2016), 487–493. https://doi.org/10.13137/2464-8728/13168

8. G. Villari, An improvement of Massera's theorem for the existence and uniqueness of a periodic solution for the Liénard equation, *Rend. Istit. Mat. Univ. Trieste*, **44** (2012), 187–195.

9. M. Bohner, J. G. Mesquita, Massera's theorem in quantum calculus, *Proc. Am. Math. Soc.*, **146** (2018), 4755–4766. https://doi.org/10.1090/proc/14116

10. J. Campos, Massera's theorem for monotone dynamical systems in three dimensions, *J. Math. Anal. Appl.*, **269** (2002), 607–615. https://doi.org/10.1016/S0022-247X(02)00039-2

11. C. Huang, B. Liu, H. Yang, J. Cao, Positive almost periodicity on SICNNs incorporatingmixed delays and D operator, *Nonlinear Anal. Mod. Control*, **27** (2022), 719–739. https://doi.org/10.15388/namc.2022.27.27417

12. Z. Yu, L. Li, W. Zhang, Dynamic behaviors of a symmetrically coupled period-doubling system, *J. Math. Anal. Appl.*, **512** (2022), 126189. https://doi.org/10.1016/j.jmaa.2022.126189

13. S. Gong, M. Han, An estimate of the number of limit cycles bifurcating from a planar integrable system, *Bull. Sci. Math.*, **176** (2022), 103118. https://doi.org/10.1016/j.bulsci.2022.103118

14. H. Liu, L. Zhao, Ground-state solution of a nonlinear fractional Schr?dinger-Poisson system, *Math. Methods Appl. Sci.*, **45** (2022), 1934–1958. https://doi.org/10.1002/mma.7899

15. L. Wang, H. Liu, Ground state solution of Kirchhoff problems with Hartree type nonlinearity, *Qual. Theory Dyn. Syst.*, **21** (2022), 136. https://doi.org/10.1007/s12346-022-00668-w

16. M. Es-saiydy, I. Oumadane, M. Zitane, Massera problem for some nonautonomous functional differential equations of neutral type with finite delay, *Russ. Math.*, **66** (2022), 49–59.

17. T. Naito, N. V. Minh, J. S. Shin, A Massera type theorem for functional differential equations with infinite delay, *Jpn. J. Math.*, **28** (2002), 31–49. https://doi.org/10.4099/math1924.28.31

18. E. Herńandez, A Massera type criterion for a partial neutral functional differential equation, *Cad. Mat.*, **3** (2002), 149–166.

19. K. Wang, M. Fan, M. Y. Li, A Massera theorem for quasi-linear partial differential equations of first order, *Rocky Mt. J. Math.*, **36** (2006), 1715–1727.

20. M. Bohner, J. G. Mesquita, S. Streipert, Periodicity on isolated time scales, *Math. Nachr.*, **295** (2022), 259–280. https://doi.org/10.1002/mana.201900360

21. M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Inc., Boston, MA, 2003. https://doi.org/10.1007/978-1-4612-0201-1

22. M. Bohner, A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser Boston, Inc., Boston, MA, 2001.

23. A. Lavagno, N. P. Swamy, q-deformed structures and nonextensive statistics: a comparative study, *Physica A*, **305** (2002), 310–315. https://doi.org/10.1016/S0378-4371(01)00680-X

24. A. Lavagno, A. M. Scarfone, N. P. Swamy, Basic-deformed thermostatistics, *J. Phys. A*, **40** (2007), 8635–8654. https://doi.org/10.1088/1751-8113/40/30/003
25. M. Adivar, A new periodicity concept for time scales, *Math. Slovaca*, **63** (2013), 817–828. https://doi.org/10.2478/s12175-013-0127-0
26. H. Koyuncuoğlu, Unified Massera type theorems for dynamic equations on time scales, *Filomat*, **37** (2023), 2405–2419. https://doi.org/10.2298/FIL2308405K
27. R. J. H. Beverton, S. J. Holt, *On the Dynamics of Exploited Fish Populations*, Chapman and Hall, London, 1957.
28. M. Bohner, S. Stević, H. Warth, *The Beverton–Holt Difference Equation*, World Sci. Publ. Hackensack, NJ, (2010), 189–193.
29. M. Bohner, T. Cuchta, S. Streipert, Delay dynamic equations on isolated time scales and the relevance of one-periodic coefficients, *Math. Methods Appl. Sci.*, **45** (2022), 5821–5838. https://doi.org/10.1002/mma.8141