Research article

Dynamic optimization of fishing tax and tourism fees for sustainable bioeconomic resource management

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Abstract: Balancing economic prosperity with environmental conservation is crucial in managing renewable bioeconomic resources. We explored a predator-prey fishery model that incorporates tourism, dynamic harvesting, and pricing strategies. Our analysis showed that increased fishing taxes reduce fishing efforts, enabling fish populations to recover. Furthermore, higher entrance fees for ecotourism support the predator population's growth. Bifurcation analysis revealed key dynamic transitions, including transcritical and Hopf bifurcations. A deeper look into coupled parameter bifurcation uncovered a transcritical bifurcation of the limit cycle, emphasizing the system's complexity. Using Pontryagin's maximum principle, we optimized fishing taxes and ecotourism entrance fees to achieve sustainable trade-offs between ecosystem health and societal revenue. The results highlighted that societal revenue peaked at an intermediate level of entrance fees, suggesting diminishing returns beyond a certain point. Revenue landscape analysis further showed that centralized, two-parameter optimization strategies outperform decentralized, single-parameter approaches. These insights provide policymakers with effective tools to design regulations that promote ecological resilience and economic viability through balanced fishing and tourism practices.

Keywords: ecotourism; tourist entry fee; fishing tax; bifurcations; transcritical limit cycle; two-parameter optimization

1. Introduction

The sustainable management of marine ecosystems is paramount for ensuring the well-being of coastal communities and the preservation of biodiversity [1]. It is crucial to examine how the global community can address the gap in marine economy and marine tourism development by leveraging their expertise, innovation, capabilities, and financial resources to prevent underdeveloped nations from

falling behind in the realm of sustainable marine economy [2]. Coastal and marine tourism constitutes a crucial aspect of a sustainable blue economy, generating employment for millions of individuals [3]. Tourism is considered a sustainable and environmentally friendly option, forming a mutually beneficial relationship with the ecosystem over time [4]. Furthermore, it is characterized as a "smokeless industry", leveraging social, historical, physical, and cultural resources. Projections suggest a worldwide expansion of 3.5% in the ocean economy and marine tourism sector [5]. By 2030, marine and coastal tourism are anticipated to emerge as the primary value-added segment of the marine economy, comprising 26% of total production [6]. However, increasing evidence suggests that high human visitation rates in natural areas can lead to significant environmental impacts, mainly due to the scale and concentration of tourism activities [7, 8]. For instance, globally, terrestrial protected areas receive billions of visits annually, generating substantial revenue and posing threats to native wildlife due to infrastructure development and access facilitation for large crowds [9, 10]. One notable example is the Great Smoky Mountains National Park, which attracts over ten million visitors per year in the USA [11]. Several researchers [12, 13] have raised concerns about this matter, with Hall and McArthur [14] characterizing it as a paradoxical situation.

Fishing also serves as a significant and sustainable means of livelihood, as highlighted in a recent report by the Food and Agriculture Organization (FAO) [15]. The report indicates that nearly 38.98 million individuals are engaged in fishing, contributing to approximately 7% of the global protein supply sourced from seafood. However, despite its importance, coastal communities reliant on fisheries face many challenges stemming from social, economic, political, and environmental factors [16, 17]. The exacerbation of these issues by anthropogenic climate change and the overexploitation of marine resources intensifies the pressure on these communities and their fishing sectors [18–20]. Acknowledging the inherent intricacy and unpredictability, fisheries management strategies should adopt intricate marine socio-ecological-economic systems (SEESs) [21, 22]. However, traditional linear decision-making processes often fail to address the complexity of SEESs, leading to recurring management problems [23]. The Ecosystem approach to fisheries management (EAFM), widely adopted in governing these SEESs, seeks to holistically manage capture fisheries to ensure healthy marine ecosystems and sustainable livelihoods derived from fishing activities [24, 25]. Such an approach actively promotes sustainable thinking and stakeholder participation to address these challenges [26, 27]. Effective implementation of the EAFM requires balancing multiple objectives and considering priorities and trade-offs in a multi-stakeholder context [28, 29], emphasizing the integration of various criteria to support decision-making processes.

Wetlands, termed the "kidneys of the earth", are crucial ecological components, now recognized for their role in environmental diversity and supported by the rise of fishing and ecotourism [30]. However, their development has suffered from exploitation, resulting in environmental harm [31, 32]. Preserving ecosystem balance is essential for sustainable development to prevent reduced species diversity, resource depletion, and environmental degradation [33, 34]. Fortunately, growing awareness among the public and government has spurred concerted actions towards wetland conservation and sustainable fishing and ecotourism [35]. Enforcing regulatory policies concerning fishing activities and ecotourism is crucial for striking a delicate equilibrium between economic progress and ecological sustainability [36]. Fishing tax and tourist entrance fee policies are vital for regulating fishing activities and promoting sustainable tourism practices in wetland ecosystems [37, 38]. These policies influence the behavior of fishers, tourism operators, and consumers, shaping the dynamics of resource utilization and revenue

generation [39]. By strategically adjusting fishing taxes and tourist entrance fees, governing bodies can incentivize sustainable fishing practices, reduce overfishing, and mitigate the negative impacts of tourism on marine habitats [40]. The implementation of such regulating policies supports sustainable development goals (SDGs) 8 and 14, promoting decent work, economic growth, and sustainable marine ecosystems, thereby aiding poverty reduction and economic development in coastal communities, particularly for fishers and tourism stakeholders [41]. Furthermore, these policies foster a sustainable blue economy and contribute to other SDGs such as SDG 1, 2, 13, and 15 by enhancing food security, poverty alleviation, climate resilience, and biodiversity conservation in coastal regions, promoting holistic and sustainable development [42].

Numerous studies have been undertaken in this direction to explore research and its implications. Cohen-Shacham et al. [43] performed a stakeholder analysis at the Galilee Sea Hula wetland, revealing that insufficient coordination among stakeholders could result in competition for cultural services, particularly tourism, jeopardizing habitat services like biodiversity conservation. Boncoeur et al. [44] examined the bioeconomic effects of both ecotourism and fishing and found that the introduction of a marine reserve alongside a fishing zone brings about more intricate economic outcomes than expected by studies concentrating solely on one stock and/or commercial fisheries. Kar and Ghosh [45] investigated the financial implications and sustainability of reserve areas involving multiple species and activities, suggesting that the combined revenue from ecotourism and fisheries could surpass that from fisheries alone. Gallagher and Hammerschlag [46] examined a South African shark tourism operation, analyzing 12 years of demographic and economic data, uncovering rising trends in customer numbers and trip costs. They highlighted the economic potential of shark-based tourism, benefiting national economies. Lee and Iwasa [47] optimized tourist entrance fees in a shared fishing and ecotourism area, demonstrating that resource availability significantly impacts benefit distribution among stakeholders. In contrast, in a comparable fishing area, Sarkar et al. [48] primarily concentrated on optimizing fishing taxes while keeping tourist entrance fees constant. Many other studies have been conducted in this direction, focusing solely on either fishing management or ecotourism governance [49–53]. Despite recent progress, the impacts of ecotourism and fishing on ecosystems with multiple species and activities remain largely unexplored. Researchers mainly focus on decentralized decision-making schemes (DDMS), characterized by challenges such as lack of coordination, suboptimal resource allocation, and inefficiencies due to independent optimization of control parameters [54]. This underscores the importance of a centralized decision-making scheme (CDMS) enabling coordinated governance strategies for optimizing resource allocation and promoting ecosystem sustainability [55]. Therefore, we seek to investigate a two-parameter dynamic optimization approach within a CDMS to comprehensively understand system dynamics and societal benefits.

We build upon the classical predator-prey framework, extending it to incorporate harvesting dynamics and fishery-based tourism. Predator-prey models have been extensively studied in ecological modeling, with recent advancements focusing on nonlocal interactions, functional responses, and behavioral effects [56, 57]. For instance, bifurcation phenomena in modified Leslie-Gower systems with nonlocal competition and Beddington-DeAngelis functional response have been explored in Ma and Yang [58], while the influence of fear effects in such models has been analyzed in Zhu and Yang [59]. Chatterjee et al. [60] examined the dynamic behavior of a predator-prey model in the presence of migratory prey and predator preference. Ghorai et al. [61] further examined the effect of preferential selection of predators and the spatiotemporal patterns formation in the predator-prey model. A predator-prey model with

prey refuge was studied under a stochastic and deterministic environment in [62]. Here, we investigate the socio-ecological-economic implications of regulatory policies to preserve fisheries and ecotourism activities within a predator-prey ecological framework. Specifically, we explore implementing a taxation fee system for managing a fishing zone alongside a no-take zone within a region governed by a tourist entry fee system inhabited by interdependent prey (fish) and predator (whale, seal, and dolphin) populations. While the former is targeted by commercial fishing, the latter is a potential basis for non-extractive commercial activity such as watching whales, seals, and dolphins. To augment the realism of commercial species dynamics, we integrate dynamic harvesting efforts and market price strategies into the system. Through single-parameter bifurcation analysis, we discern multiple instances of transcritical and Hopf bifurcations, highlighting the intricate dynamics within the system. However, our exploration into coupled-parameter bifurcation uncovers a distinctive regime shift where a stable limit cycle encompassing the harvested equilibrium state transits to a stable limit cycle centred around the harvesting-free equilibrium state. This transition is facilitated by a transcritical bifurcation of the limit cycle (TBLC), showcasing a unique dynamical behavior. Such regime shift phenomenon is rare within bioeconomic fishery models. Here, we delve into optimizing fishing tax and tourist entrance fee policies to maximize societal revenue while promoting the sustainability of marine ecosystems. Through rigorous analysis, we identify optimal policy configurations that balance ecological conservation and economic prosperity, fostering a win-win scenario for marine biodiversity and coastal communities.

The organization of this paper is as follows: In Section 2, we introduce an integrated fishery-ecotourism model featuring dynamic harvesting and market price, followed by an analysis of the positivity and boundedness conditions. Subsequently, the model is analyzed to determine equilibrium points and their local asymptotic stability conditions in Section 3. Global stability analysis of the equilibrium point is provided in Section 4. The existence conditions of numerous bifurcation points are evaluated in Section 5. In Section 6, we present an analysis corresponding to the DDMS and CDMS using Pontryagin's maximum principle. Numerical simulations corresponding to the analytical results are presented in Section 7. Finally, we conclude with a discussion in Section 8.

2. Model construction

Suppose the rate equations of two fish species sharing a common habitat with a typical predator-prey relationship are defined as

$$
\begin{aligned}
\frac{dP}{dt} &= rP\left(1 - \frac{P}{K}\right) - \frac{\rho_1 PQ}{P + D'} = f(P, Q), \\
\frac{dQ}{dt} &= \frac{\rho_2 PQ}{P + D'} - \mu Q = g(P, Q),
\end{aligned}
$$

where P and Q are the prey and predatory fish biomasses at time t. The predator-prey interaction follows Rosenzweig–MacArthur type reaction kinetics [63]. Prey grows with the biotic potential r and environmental carrying capacity K. The parameters ρ_1, D', and ρ_2 involved in the response function are the maximum prey attack rate, half-saturation constant of predation and predator's food conversion rate, respectively. The mortality rate of predatory fish, denoted as μ, must adhere to the condition $\rho_2 > \mu$.

In the context of a prey harvesting fishery model, the extraction of prey fish is subject to a harvesting rate denoted as $\Xi(t)$. Traditionally, this harvesting policy has been shaped by the catch-per-unit effort (CPUE) hypothesis, as introduced by Clark [64]. The CPUE hypothesis is mathematically expressed

as $\Xi(P, E) = \xi EP$, where ξ represents the catchability coefficient, and E stands for the harvesting effort. Since the harvesting effort is specifically targeted towards the prey species, the harvesting process is rendered selective in nature. However, this classical CPUE-based approach exhibits two shortcomings: First, it exhibits an unbounded linear increase in harvesting rate (Ξ) as a function of the prey population (P), given a fixed level of effort (E). Second, it also results in an unbounded linear increase in harvesting rate (Ξ) with respect to the effort (E), keeping the prey population (P) constant. It is worth highlighting that the harvesting efforts, encompassing elements such as boats, fishermen, and various fishing instruments, essentially act as predators in the ecological context [65]. In line with this idea, numerous researchers have put forth an alternative harvesting rate, expressed as $\Xi(P, E) = \frac{\xi PE}{P+D''}$ [66, 67]. This formulation aims to rectify the impractical traits observed in the conventional harvesting rate. Here, D'' represents the critical half-saturation constant that governs the harvesting rate. However, it is worth noting that this alternative formulation fails to rectify the second flaw previously identified. To simultaneously overcome both of these limitations, we advocate for the adoption of a saturated nonlinear harvesting rate, expressed as $\Xi(P, E) = \frac{\xi PE}{D_0 P + D_1 E}$ [68]. In this context, the parameters D_0 and D_1 assume distinct roles: D_0 quantifies the extent of competition among various elements involved in fishing, including boats, fishing nets, fishermen and other fishing technologies [69], while D_1 represents the product of the capture rate and the handling time for the prey fish [70]. It is worth noting two essential characteristics of this modified harvesting rate function:

(i) As the prey population (P) approaches infinity while keeping the effort (E) constant, the harvesting rate (Ξ) converges to a finite value, specifically, $\Xi \to \frac{\xi E}{D_0}$.

(ii) Conversely, when the harvesting effort (E) tends to infinity while maintaining a fixed prey population (P), the harvesting rate (Ξ) converges to a finite value as well, given by $\Xi \to \frac{\xi P}{D_1}$.

This behavior underscores the importance of incorporating handling time to achieve a more realistic and bounded functional form for the harvesting function. Consequently, this modified catch rate function effectively exhibits saturation effects concerning prey stock and harvesting effort levels. Then, the interactive dynamics of the system in the presence of harvesting becomes

$$\begin{aligned}
\frac{dP}{dt} &= f(P, Q) - \frac{\xi PE}{D_0 P + D_1 E}, \\
\frac{dQ}{dt} &= g(P, Q).
\end{aligned}$$

Next, we turn our attention to the utilization of predatory fish species (such as whales, seals, dolphins, etc.) not for harvesting but for ecotourism endeavors. These species hold economic significance as a resource for non-extractive recreational activities. Therefore, the predatory fish are utilized for recreational purposes among visitors, who are charged an entry fee, denoted as b. The number of tourists, denoted as $\mathscr{T}$, is influenced by two key factors: The population density of predatory fish (Q) and the entry fee for visitors (b). It is commonly assumed that the number of tourists decreases exponentially as the entry fee increases in this recreational setting, as suggested by Boncoeur et al. [44]. Consequently, the tourist population $\mathscr{T}$ at any given time t can be mathematically expressed in accordance with the work of Lee and Iwasa [47]:

$$\mathscr{T}(Q) = T_0 Q^a e^{-bg},$$

where T_0 represents the intrinsic number of tourists, and g is a tourism sensitivity parameter, quantifying how sensitive visitors are to a change in the entry fee. The negative sign in the exponential signifies the

diminishing interest of visitors in the ecotourism site as the entrance fee (b) increases. Parameter a plays a crucial role in determining the relationship between the tourist population and the predatory fish's level utilized for recreational purposes. Different values of a convey distinct dependencies: When $a = 0$, it implies no dependence on the predatory fish level. A value of $a = 1$ indicates a linear dependency on the predatory fish level Q. Values of a greater than 1 suggest a dependency that increases faster than the predatory fish level. Values of a less than 1 imply a dependency that increases slower than the predatory fish level Q. Subsequently, the growth equation for each species, accounting for both harvesting and ecotourism, can be formulated as

$$
\begin{aligned}
\frac{dP}{dt} &= f(P, Q) - \Xi(P, E), \\
\frac{dQ}{dt} &= g(P, Q) - \beta T_0 Q^{(a+1)} e^{-bg},
\end{aligned}
$$

where β is a proportionality constant.

We introduce two additional assumptions in the line of some studies [48, 67, 71]. First, we consider that the level of effort (E) deployed for catching prey fish at any given time t is contingent upon the profit margin. This margin is calculated by subtracting the costs of employing an effort level of E from the revenue generated from selling harvested fish. Second, designating M as the market price per unit of fish and τ as the fixed fishing tax applied to each unit of landed fish, the net selling price becomes $(M - \tau)$. Subsequently, it is assumed that the per capita fish price at any time t is determined by the equilibrium between fish supply and demand within an open market environment. Under the presumption that D represents the maximum fish demand and that α_1 and α_2 are the rates at which demand decreases quadratically with increasing prices, we can mathematically express the demand (Γ) as [72]:

$$
\Gamma(M) = D - \alpha_1 M - \alpha_2 M^2.
$$

With these assumptions in mind, the following predator-prey fishery model is formulated, incorporating ecotourism, prey fish harvesting, and the implementation of two regulating policies:

$$
\begin{aligned}
\frac{dP}{dt} &= rP\left(1 - \frac{P}{K}\right) - \frac{\rho_1 PQ}{P + D'} - \frac{\xi PE}{D_0 P + D_1 E} \equiv \mathscr{F}_1(P, Q, E, M), \\
\frac{dQ}{dt} &= \frac{\rho_2 PQ}{P + D'} - \mu Q - \beta T_0 Q^{a+1} e^{-bg} \equiv \mathscr{F}_2(P, Q, E, M), \\
\frac{dE}{dt} &= \eta_1\left(\frac{\xi PE}{D_0 P + D_1 E}(M - \tau) - cE\right) \equiv \mathscr{F}_3(P, Q, E, M), \\
\frac{dM}{dt} &= \eta_2 M\left(D - \alpha_1 M - \alpha_2 M^2 - \frac{\xi PE}{D_0 P + D_1 E}\right) \equiv \mathscr{F}_4(P, Q, E, M),
\end{aligned}
\tag{2.1}
$$

with initial conditions

$$
P(0) = P_0 > 0, Q(0) = Q_0 > 0, E(0) = E_0 > 0, M(0) = M_0 > 0,
$$

where η_1 is a stiffness parameter that acts as a proportionality constant, regulating the rate of change of fishing effort in response to net profit. Similarly, η_2 is a price-sensitive parameter that governs the rate of market price change based on open market demand-supply dynamics. All parameters in system (2.1) are positive, and their significance is detailed in Table 1.

The system serves as a generalization of much existing research in this field. Sarkar et al. [48] considered a similar model with generalist predator, CPUE harvesting rate and linear demand function to optimize the fishing tax. In their work, Paul et al. [51] considered the first two equations of system (2.1), focusing on the impacts of entrance fees on total benefits, prey, predators, and tourists with CPUE harvesting, $\xi(P, E) = \xi P E$. Das and Kar [73] extended this study by incorporating logistic growth in both species. Conversely, Lee and Iwasa [47] simplified the model by considering only fishing and ecotourism on the same species, focusing solely on the first equation with the CPUE harvesting effort. Additionally, several researchers [66, 71, 72, 74] have examined bioeconomic models of fisheries with dynamic catch and variable prices without considering the impact of ecotourism on socioeconomic development. Here, we amalgamate two separate regulatory policies, fishing taxation and tourist entrance fees, within the context of dynamic fishing efforts and market-based price fluctuations of harvested fish. Our aim is to identify an optimal equilibrium decision-making scenario and evaluate their collective ecological and economic impacts.

2.1. Preliminary results

The right-hand side of the system (2.1) is a smooth function of the variables (P, Q, E, M) and the parameters. As long as these quantities are non-negative, local existence and uniqueness properties hold in $\mathbb{R}+^{4,0}$ [75].

LEMMA 2.1 [48, 67] *For any initial values* $(P_0, Q_0, E_0, M_0) \in \mathbb{R}+^{4,0}$, *all solutions* $(P(t), Q(t), E(t), M(t))$ *of the system* (2.1) *remain positive and uniformly bounded within* $\mathscr{B}_{\mathscr{P}}$, *where*

$$\mathscr{B}_{\mathscr{P}} = \left\{ (P, Q, E, M) \ : \ 0 < P < \zeta + \hat{\epsilon}, \ 0 < M < \hat{\zeta} + \tilde{\epsilon}, \ 0 < S(P, Q, E, M) < \frac{s_2}{s_1} + \epsilon, \text{ for any positive } \hat{\epsilon}, \tilde{\epsilon}, \epsilon \right\}, \text{ where } s_1 = \min\{\frac{r}{K}, \frac{\rho_1}{K+D'}, \frac{\xi}{D_0}, \eta_2\alpha_1\} \text{ and } s_2 = r + \rho_2 + \eta_2 D + \frac{\eta_1 \xi \hat{\zeta}}{D_0}.$$

3. Equilibria and stability analysis

System (2.1) admits eight equilibrium points. The conditions for their existence are summarized in the following theorem.

Theorem 1. *(i) The trivial equilibrium* $\Sigma_1 = (0, 0, 0, 0)$, *the only prey equilibrium* $\Sigma_2 = (K, 0, 0, 0)$, *the only price equilibrium* $\Sigma_3 = (0, 0, 0, M_3)$, *and the prey fish and price equilibrium* $\Sigma_4 = (K, 0, 0, M_3)$ *always exist, where* $M_3 = \frac{-\alpha_1 + \sqrt{\alpha_1^2 + 4D\alpha_2}}{2\alpha_2}$.

(ii) The predator-prey equilibrium $\Sigma_5 = (P_5, Q_5, 0, 0)$, *where* $Q_5 = \frac{r(K-P_5)(D'+P_5)}{K\rho_1}$, $P_5 = \frac{D'(\mu e^{bg} + T_0 \beta \hat{\sigma}_1)}{(\rho_2 - \mu)e^{bg} - T_0 \beta \hat{\sigma}_1}$ *and* $\hat{\sigma}_1 = \left(\frac{(P_5+D')(K-P_5)r}{K\rho_1} \right)^a > 0$, *exists uniquely if the inequality* $(\rho_2 - \mu)e^{bg} > T_0 \beta \hat{\sigma}_1$ *is satisfied.*

(iii) The harvesting effort-free equilibrium is given by $\Sigma_6 = (P_6, Q_6, 0, M_3)$, *where* $Q_6 = \frac{r(K-P_6)(D'+P_6)}{K\rho_1}$ *and* P_6 *shares the same expression as* P_5. *This equilibrium exists if* $(\rho_2 - \mu)e^{bg} > T_0 \beta \hat{\sigma}_1$.

(iv) The predator fish-free equilibrium can be written as $\Sigma_7 = (P_7, 0, E_7, M_7)$, *where* $E_7 = \frac{1}{c}(M_7 - \tau)(D - \alpha_1 M_7 - \alpha_2 M_7^2)$, $P_7 = \frac{D_1(M_7-\tau)(\alpha_2 M_7^2 + \alpha_1 M_7 - D)}{D_0 c - M_7 \xi + \tau \xi}$, *and* M_7 *is the positive root of the equation*

$$\zeta_7 M_7^7 + \zeta_6 M_7^6 + \zeta_5 M_7^5 + \zeta_4 M_7^4 + \zeta_3 M_7^3 + \zeta_2 M_7^2 + \zeta_1 M_7 + \zeta_0 = 0,$$

Table 1. Variables and parameters with their descriptions.

State Variable	Description	Unit	
$P(t)$	Prey fish biomass at time t	MT*	
$Q(t)$	Predator fish biomass at time t	MT	
$E(t)$	Fishing effort of prey species at time t	SFU$^\odot$	
$M(t)$	Market price per unit biomass of prey species at time t	M\$$^\otimes$/MT	
Dependent Variable	**Description**	**Unit**	
$\Xi(P, E)$	Harvested prey fish biomass at time t	MT/year	
$\mathscr{T}(Q)$	Number of visiting tourist at time t	million/year	
$\Gamma(M)$	Demand rate, a function of price at time t	MT/year	
Parameter	**Description**	**Default Value**	**Reference**
r	Intrinsic growth rate of prey species	4/year	[48]
K	Environmental carrying capacity of prey fish	variables MT	Estimated
ρ_1	Maximum attack rate of predatory fish	0.7 /year	[48]
ρ_2	Food conversion rate of predatory fish	0.35 /year	[76]
ξ	Catchability coefficient of prey fish	0.34 MT/SFU/year	[77]
D'	Half saturation constant of predation	10 MT	This study
μ	Mortality rate of predatory fish	0.15 /year	[51]
D_0	Competition coefficient among various elements involved in fishing	1.5 /MT	This study
D_1	Product of the capture rate and the handling time for the fishing	2 /SFU	This study
β	Predator mortality rate due to tourism activities	0.2 /million/year	[48]
T_0	Intrinsic number of tourist	2.5 million/year/MT^a	[48]
a	Tourist sensitivity parameter	0.2 unitless	[48]
b	Tourist entrance fee	variable \$/tourist	Estimated
g	Tourist fee sensitivity parameter	0.4 tourist/\$	[51]
c	Cost per unit of fishing effort	0.14 M\$/SFU/year	[48]
D	Maximal demand	5 MT/year	This study
α_1	The linear demand decreasing rate	0.6 (MT)^2/M\$/year	[71]
α_2	The nonlinear demand decreasing rate	0.1 (MT)^3/M\$^{2}/year	This study
τ	Regulating tax corresponding to the fishing	variable M\$/MT	Estimated
η_1	Stiffness parameter	0.6 SFU/M\$	[48]
η_2	Proportionality constant	0.5 /MT	[48]

*MT represents metric tons, $^\odot$ SFU stands for Standardized Fishing Unit [78, 79], and $^\otimes$ M\$ indicates million USD.

where $c\xi^2 M_5^2 + \left(-2D_0 c^2 \xi - 2\tau c\xi^2\right) M_5 + D_0^2 c^3 + 2D_0 c^2 \tau \xi + c\tau^2 \xi^2 \neq 0$ and the coefficients are given in Appendix A. This equilibrium exists uniquely if M_7 exists uniquely and $\tau < \frac{M_7 \xi - D_0 c}{\xi}$.

(v) *The coexisting equilibrium point $\Sigma^* = (P^*, Q^*, E^*, M^*)$, where*

$$P^* = \frac{D'\left(\mu + Q^{*a} T_0 \beta e^{-bg}\right)}{\rho_2 - \mu - Q^{*a} T_0 \beta e^{-bg}},$$

$$E^* = \frac{D_0 D' \sigma_4 (\sigma_1 K Q^* \rho_1 + D' r \rho_2 e^{bg} \sigma_2)}{\sigma_3 (DK Q^* \rho_1 \sigma_1 + D' \rho_2 e^{bg} (Dr\sigma_2 - K\xi))}, \text{ and}$$

$$M^* = \tau + \frac{D_0 c}{\xi} - \frac{\sigma_3 D_1 c E^*}{\xi D' \sigma_4},$$

with $\sigma_1 = \mu e^{bg} - \rho_2 e^{bg} + Q^{*a} T_0 \beta$, $\sigma_2 = K + \frac{D' \sigma_4}{\sigma_3}$, $\sigma_3 = \mu - \rho_2 + Q^{*a} T_0 \beta e^{-bg}$, $\sigma_4 = \mu +$

$Q^{*a} T_0 \beta e^{-bg}$, $\sigma_5 = K + \frac{D'(\mu+\sigma_6)}{\mu-\rho_2+\sigma_6}$, $\sigma_6 = Q^{*a} T_0 \beta e^{-bg}$, and Q^* is the positive root of the equation

$$C_1 Q^{*(3a)} - C_2 Q^{*(2a+1)} - C_3 Q^{*(2a)} + C_4 Q^{*(a+2)} - C_5 Q^{*(a+1)} + C_6 Q^{*(a)} + C_7 Q^{*3} - C_8 Q^{*2}$$
$$-C_9 Q^* - C_{10} = 0, \tag{3.1}$$

where the coefficients are given in Appendix B. Then Σ^* exists uniquely under the conditions $\rho_2 > \mu + Q^{*a} T_0 \beta e^{-bg}$, $K > \frac{D' \rho_2 Dr\sigma_2 e^{bg}}{DQ^* \rho_2 \sigma_1 + D' \rho_2 \xi e^{bg}}$, *and* $E^* < \frac{\xi D' \sigma_4}{\sigma_3 D_1 c}\left(\tau + \frac{D_0 c}{\xi}\right)$.

Proof. (i) Since all components of these equilibrium points are strictly positive and uniquely determined, the equilibrium points Σ_1, Σ_2, Σ_3, and Σ_4 exist and are unique.

(ii) For the predator-prey equilibrium $\Sigma_5 = (P_5, Q_5, 0, 0)$, observe that $Q_5 > 0$ since $K > P_5$ and P_5 remains positive when the inequality $(\rho_2 - \mu)e^{bg} > T_0\beta\hat{\sigma}_1$ holds. Thus, the predator-prey equilibrium exists and is unique when the inequality $(\rho_2 - \mu)e^{bg} > T_0\beta\hat{\sigma}_1$ is satisfied.

(iii) Following arguments similar to those for the existence of Σ_5, the harvesting effort-free equilibrium $\Sigma_6 = (P_6, Q_6, 0, M_3)$ exists uniquely under the same condition: $(\rho_2 - \mu)e^{bg} > T_0\beta\hat{\sigma}_1$.

(iv) Since $E_7 > 0$ hold unconditionally, the equilibrium Σ_7 exists uniquely provided M_7 exists uniquely, and $P_7 > 0$. Observe that $P_7 > 0$ yields $\tau < \frac{M_7\xi - D_0 c}{\xi}$. Hence, the required conditions are: M_7 should exist uniquely, and the fishing tax (τ) is less than the threshold $\frac{M_7\xi - D_0 c}{\xi}$.

(v) For the coexisting equilibrium point $\Sigma^* = (P^*, Q^*, E^*, M^*)$, observe that $P^* > 0$ if $\rho_2 > \mu + Q^{*a} T_0 \beta e^{-bg}$, $E^* > 0$ if $K > \frac{D' \rho_2 Dr\sigma_2 e^{bg}}{DQ^* \rho_2 \sigma_1 + D' \rho_2 \xi e^{bg}}$, $M^* > 0$ if $E^* < \frac{\xi D' \sigma_4}{\sigma_3 D_1 c}\left(\tau + \frac{D_0 c}{\xi}\right)$, and Q^* is the unique positive root of Eq (3.1). Therefore, whenever Q^* exists uniquely and the conditions $\rho_2 > \mu + Q^{*a} T_0 \beta e^{-bg}$, $K > \frac{D' \rho_2 Dr\sigma_2 e^{bg}}{DQ^* \rho_2 \sigma_1 + D' \rho_2 \xi e^{bg}}$, and $E^* < \frac{\xi D' \sigma_4}{\sigma_3 D_1 c}\left(\tau + \frac{D_0 c}{\xi}\right)$ are satisfied, Σ^* exists uniquely.

Hence, the proof. $\qquad\square$

Remark 1 (Ecological interpretation of equilibria). *System (2.1) exhibits eight biologically meaningful equilibrium points, each representing distinct ecological-economic scenarios:*

1) $\Sigma_1 = (0, 0, 0, 0)$: *Trivial equilibrium representing complete ecological collapse ($P = Q = 0$) and economic inactivity ($E = M = 0$).*

2) $\Sigma_2 = (K, 0, 0, 0)$: *Prey population at carrying capacity ($P = K$) with absent predators ($Q = 0$) and no economic activity.*

3) $\Sigma_3 = (0, 0, 0, M_3)$: *Market equilibrium ($M = M_3$) with collapsed ecosystem ($P = Q = E = 0$), nonviable in practice.*

4) $\Sigma_4 = (K, 0, 0, M_3)$: *Sustainable prey population ($P = K$) with market activity but depleted predators ($Q = 0$) and no harvesting ($E = 0$).*

5) $\Sigma_5 = (P_5, Q_5, 0, 0)$: *Natural predator-prey coexistence ($P_5, Q_5 > 0$) without anthropogenic influence ($E = M = 0$).*

6) $\Sigma_6 = (P_6, Q_6, 0, M_3)$: *Ecotourism-dominated system with intact predator-prey dynamics ($P_6, Q_6 > 0$) but no direct harvesting ($E = 0$).*

7) $\Sigma_7 = (P_7, 0, E_7, M_7)$: *Fishery-optimized equilibrium showing predator extinction ($Q = 0$) due to excessive harvesting ($E_7 > 0$).*

8) $\Sigma^* = (P^*, Q^*, E^*, M^*)$: *Optimal co-existence state balancing ecological preservation ($P^*, Q^* > 0$) with sustainable economic activity ($E^*, M^* > 0$).*

Understanding the stability of a system is fundamental in predicting its behavior over time. A widely used method for assessing stability is the linearization technique, which approximates the behavior of nonlinear systems near an equilibrium point. This technique, often leveraging tools such as the Jacobian matrix, aids in identifying the parametric conditions that promote stable equilibrium states. The Jacobian matrix associated with the system (2.1) at any arbitrary equilibrium point $\hat{\Sigma}(\hat{P}, \hat{Q}, \hat{E}, \hat{M})$ reads

$$J(\hat{P}, \hat{Q}, \hat{E}, \hat{M}) = \begin{pmatrix} e_{11} & e_{12} & e_{13} & 0 \\ e_{21} & e_{22} & 0 & 0 \\ e_{31} & 0 & e_{33} & e_{34} \\ e_{41} & 0 & e_{43} & e_{44} \end{pmatrix}, \tag{3.2}$$

where

$e_{11} = \frac{\hat{P}\hat{Q}\rho_1}{(D'+\hat{P})^2} - \frac{\hat{Q}\rho_1}{D'+\hat{P}} - \frac{\hat{E}\xi}{D_1\hat{E}+D_0\hat{P}} - \frac{\hat{P}r}{K} - r\left(\frac{\hat{P}}{K} - 1\right) + \frac{D_0\hat{E}\hat{P}\xi}{(D_1\hat{E}+D_0\hat{P})^2}$, $e_{12} = -\frac{\hat{P}\rho_1}{D'+\hat{P}}$, $e_{13} = -\frac{D_0\hat{P}^2\xi}{(D_1\hat{E}+D_0\hat{P})^2}$, $e_{21} = \frac{D'\hat{Q}\rho_2}{(D'+\hat{P})^2}$, $e_{22} = \frac{\hat{P}\rho_2}{D'+\hat{P}} - \mu - \hat{Q}^a T_0 \beta e^{-bg}(a+1)$, $e_{31} = \frac{D_1\hat{E}^2\eta_1\xi(\hat{M}-\tau)}{(D_1\hat{E}+D_0\hat{P})^2}$, $e_{33} = -\frac{\eta_1}{(D_1\hat{E}+D_0\hat{P})^2}(D_1{}^2\hat{E}^2 c + D_0{}^2\hat{P}^2 c - D_0\hat{M}\hat{P}^2\xi + D_0\hat{P}^2\tau\xi + 2D_0 D_1\hat{E}\hat{P}c)$, $e_{34} = \frac{\hat{E}\hat{P}\eta_1\xi}{D_1\hat{E}+D_0\hat{P}}$, $e_{41} = -\frac{D_1\hat{E}^2\hat{M}\eta_2\xi}{(D_1\hat{E}+D_0\hat{P})^2}$, $e_{43} = -\frac{D_0\hat{M}\hat{P}^2\eta_2\xi}{(D_1\hat{E}+D_0\hat{P})^2}$, $e_{44} = D\eta_2 - 2\hat{M}\alpha_1\eta_2 - 3\hat{M}^2\alpha_2\eta_2 - \frac{\hat{E}\hat{P}\eta_2\xi}{D_1\hat{E}+D_0\hat{P}}$.

Theorem 2. *(i) Whenever the equilibrium points $\Sigma_1(0,0,0,0)$, $\Sigma_2(K,0,0,0)$, $\Sigma_3 = (0,0,0,M_3)$, and $\Sigma_5(P_5, Q_5, 0, 0)$ exist, they are always unstable.*
(ii) The equilibrium point $\Sigma_4(K, 0, 0, M_3)$ is locally asymptotically stable if the enrichment level and the fishing tax follow the inequalities $K < \frac{\mu D'}{\rho_2 - \mu}$, $\rho_2 > \mu$ and $\tau > M_3 - \frac{cD_0}{\xi}$.
(iii) Whenever the tourist entrance fee (b) and fishing tax (τ) satisfy the relationships $\tau > M_3 - \frac{cD_0}{\xi}$, and $b < \min\left\{\frac{1}{g}\log\left(\frac{a\beta T_0 K Q_6^a(D'+P_6)^2}{P_6(KQ_6\rho_1 - r(D'+P_6)^2)}\right), \frac{1}{g}\log\left(\frac{a\beta T_0 Q_6^{a-1}(D'+P_6)(\rho_1 K Q_6 - r(D'+P_6)^2)}{KD'\rho_1\rho_2}\right)\right\}$, the harvesting effort-free equilibrium point $\Sigma_6 = (P_6, Q_6, 0, M_3)$ (if exists) becomes locally asymptotically stable.
(iv) The equilibrium point $\Sigma_7(P_7, 0, E_7, M_7)$, whenever exists, is locally asymptotically stable under the conditions $\mu > \frac{P_7\rho_2}{D'+P_7}$, $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$.
(v) Whenever the interior equilibrium point $\Sigma^*(P^*, Q^*, E^*, M^*)$ exists, it is locally asymptotically stable if $\mathrm{Tr}(J(P^*, Q^*, E^*, M^*)) < 0$, $a_6 > 0$, $a_7 > 0$ and $a_4 a_5 a_6 > a_6^2 + a_4^2 a_7$.

Proof. Please see Appendix C for its proof. □

4. Global stability analysis

In this section, the high-dimensional Bendixson criterion [80] is employed to observe whether an equilibrium point is globally asymptotically stable (GAS). Given the challenges posed by applying the conventional Lyapunov function method [81] to complex systems like (2.1), the high-dimensional

Bendixson criterion emerges as an alternative approach, which is suitable for intricate systems. This method thoroughly examines the global stability properties of an arbitrary equilibrium point $\hat{\Sigma}$ in the socio-ecological-economic model (2.1). The following theorem guarantees the globally asymptotically stability of the equilibrium $\hat{\Sigma}$.

Theorem 3. *Whenever an equilibrium point $\hat{\Sigma}$ exists and the condition*

$$
\Upsilon = \max\left\{\theta_{11} + \frac{\theta_{14}\vartheta_1}{\vartheta_4}, \ \theta_{22} + \theta_{23}\vartheta_2 + \frac{\theta_{24}\vartheta_2}{\vartheta_4}, \ \frac{\theta_{32}}{\vartheta_2} + \theta_{33} + \frac{\theta_{35}}{\vartheta_4} + \frac{\theta_{36}}{\vartheta_5}, \ \frac{\theta_{41}\vartheta_3}{\vartheta_1} + \frac{\theta_{42}\vartheta_3}{\vartheta_2} + \frac{\theta_{44}\vartheta_2}{\vartheta_3} \right.
$$
$$
\left. + \frac{\theta_{45}\vartheta_3}{\vartheta_4}, \ \frac{\theta_{51}\vartheta_4}{\vartheta_1} + \theta_{53}\vartheta_4 + \frac{\theta_{54}\vartheta_4}{\vartheta_3} + \theta_{55}, \ \frac{\theta_{62}\vartheta_5}{\vartheta_2} + \theta_{63}\vartheta_5 + \theta_{66}\right\} < 0 \tag{4.1}
$$

is satisfied for a set of positive real number $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5)$, the system (2.1) cannot demonstrate any non-trivial periodic solutions. Additionally, the equilibrium point $\hat{\Sigma}$ becomes GAS in $\mathbb{R}_+^{4,0}$.

Proof. Let us assume O be an open subset of $\mathbb{R}_+^{4,0}$ and consider the function $\mathcal{G} : \mathcal{Y} \to \mathcal{G}(\mathcal{Y}) \in \mathbb{R}_+^{4,0}$, which is differentiable and has a continuous gradient on $\mathcal{Y} \in O$. Presume system (2.1) as

$$
\frac{d\mathcal{Y}}{dt} = \mathcal{G}(\mathcal{Y}), \tag{4.2}
$$

where $\mathcal{Y} = (P, Q, E, M)^T$ and $\mathcal{G}(\mathcal{Y}) = \left[\mathscr{F}_1(P, Q, E, M), \ \mathscr{F}_2(P, Q, E, M), \ \mathscr{F}_3(P, Q, E, M), \ \mathscr{F}_4(P, Q, E, M)\right]^T$. One can utilize the high-dimensional Bendixson criterion by showing that the second additive compound matrix

$$
\frac{d\mathcal{Z}}{dt} = \frac{\partial \mathcal{G}}{\partial \mathcal{Y}}^{[2]} (\mathcal{Y}(t, \mathcal{Y}_0))(\mathcal{Z}(t)) \tag{4.3}
$$

at a solution $\mathcal{Y}(t, \mathcal{Y}_0) \in O$ of Eq (4.2) is equi-uniformly asymptotically stable [82,83], where $\mathcal{Z}$ is a 6×1 matrix denoted by $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6)^T$. The second additive compound matrix $(\frac{\partial \mathcal{G}}{\partial \mathcal{Y}}^{[2]})$ corresponding to the variational matrix (3.2), i.e., $J(\hat{P}, \hat{Q}, \hat{E}, \hat{M})$, or

$$
\frac{\partial \mathcal{G}}{\partial \mathcal{Y}} = \left(e_{ij}\right)_{4\times 4} \tag{4.4}
$$

will be a $\left({}^4C_2 \times {}^4C_2\right) = (6 \times 6)$ matrix and can be represented as [71]

$$
\frac{\partial \mathcal{G}}{\partial \mathcal{Y}}^{[2]} = \begin{pmatrix} e_{11} + e_{22} & 0 & 0 & -e_{13} & 0 & 0 \\ 0 & e_{11} + e_{33} & e_{34} & e_{12} & 0 & 0 \\ 0 & e_{43} & e_{11} + e_{44} & 0 & e_{12} & e_{13} \\ -e_{31} & e_{21} & 0 & e_{22} + e_{33} & e_{34} & 0 \\ -e_{41} & 0 & e_{21} & e_{43} & e_{22} + e_{44} & 0 \\ 0 & -e_{41} & e_{31} & 0 & 0 & e_{33} + e_{44} \end{pmatrix}.
$$

The region contained in any compact two-dimensional space in O falls exponentially if the system (4.2) exhibits equi-uniform asymptotic stability nature. However, there cannot exist any invariant simple closed rectifiable curve within O, including periodic orbits, if O is simply connected. Thus, one can obtain the following proposition.

PROPOSITION 4.1 *Consider a simply connected region $O \subset \mathbb{R}_+^{4,0}$. Assume that the family of nonlinear system (4.2) is equi-uniformly asymptotically stable. Then, if O is positively invariant and contains an unique arbitrary equilibrium point $\hat{\Sigma}(\hat{P}, \hat{Q}, \hat{E}, \hat{M})$, then $\hat{\Sigma}$ is globally asymptotically stable in O.*

One can write the second compound system (4.3) as

$$(\dot{\delta}_1, \dot{\delta}_2, \dot{\delta}_3, \dot{\delta}_4, \dot{\delta}_5, \dot{\delta}_6)^T = \frac{\partial \mathcal{G}^{[2]}}{\partial \mathcal{Y}} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6)^T,$$

where

$$\begin{aligned}
\dot{\delta}_1 &= \delta_1 \theta_{11} + \delta_4 \theta_{14}, \quad \dot{\delta}_2 = \delta_2 \theta_{22} + \delta_3 \theta_{23} + \delta_4 \theta_{24}, \\
\dot{\delta}_3 &= \delta_2 \theta_{32} + \delta_3 \theta_{33} + \delta_5 \theta_{35} + \delta_6 \theta_{36}, \quad \dot{\delta}_4 = \delta_1 \theta_{41} + \delta_2 \theta_{42} + \delta_4 \theta_{44} + \delta_5 \theta_{45}, \\
\dot{\delta}_5 &= \delta_1 \theta_{51} + \delta_3 \theta_{53} + \delta_4 \theta_{54} + \delta_5 \theta_{55}, \quad \dot{\delta}_6 = \delta_2 \theta_{62} + \delta_3 \theta_{63} + \delta_6 \theta_{66},
\end{aligned} \tag{4.5}$$

and

$\theta_{11} = -\left(\mu + r\left(\frac{2P}{K} - 1\right) + \frac{Q\rho_1 D' - (D'+P)P\rho_2}{(D'+P)^2} + \frac{D_1 E^2 \xi}{(D_1 E + D_0 P)^2} + Q^a T_0 \beta e^{-bg}(a+1)\right)$, $\theta_{14} = \frac{D_0 P^2 \xi}{(D_1 E + D_0 P)^2}$, $\theta_{22} = r\left(1 - \frac{2P}{K}\right) - \frac{\eta_1 \left(c(D_1 E + D_0 P)^2 - D_0 P^2 \xi(M-\tau)\right) + D_1 E^2 \xi}{(D_1 E + D_0 P)^2} - \frac{QD'\rho_1}{(D'+P)^2}$, $\theta_{23} = \frac{E P \eta_1 \xi}{(D_1 E + D_0 P)}$, $\theta_{24} = -\frac{P\rho_1}{D'+P}$, $\theta_{32} = -\frac{D_0 M P^2 \eta_2 \xi}{(D_1 E + D_0 P)^2}$, $\theta_{33} = r\left(\frac{2P}{K}\right) - D\eta_2 + 2M\alpha_1 \eta_2 + \frac{Q\rho_1 D'}{(D'+P)^2} + \frac{D_1 E^2 \xi}{(D_1 E + D_0 P)^2} + 3M^2 \alpha_2 \eta_2 + \frac{E P \eta_2 \xi}{(D_1 E + D_0 P)}$, $\theta_{35} = -\frac{P\rho_1}{D'+P}$, $\theta_{36} = -\frac{D_0 P^2 \xi}{(D_1 E + D_0 P)^2}$, $\theta_{41} = -\frac{D_1 E^2 \eta_1 \xi(M-\tau)}{(D_1 E + D_0 P)^2}$, $\theta_{42} = \frac{D' Q\rho_2}{(D'+P)^2}$, $\theta_{44} = -\left(\mu + \frac{\eta_1 \left(c(D_1 E + D_0 P)^2 - D_0 P^2 \xi(M-\tau)\right)}{(D_1 E + D_0 P)^2} - \frac{P\rho_2}{D'+P} + Q^a T_0 \beta e^{-bg}(a+1)\right)$, $\theta_{45} = \frac{E P \eta_1 \xi}{(D_1 E + D_0 P)}$, $\theta_{51} = \frac{D_1 E^2 M \eta_2 \xi}{(D_1 E + D_0 P)^2}$, $\theta_{53} = \frac{D' Q\rho_2}{(D'+P)^2}$, $\theta_{54} = -\frac{D_0 M P^2 \eta_2 \xi}{(D_1 E + D_0 P)^2}$, $\theta_{55} = -\left(\mu - D\eta_2 + 2M\alpha_1 \eta_2 - \frac{P\rho_2}{D'+P} + 3M^2 \alpha_2 \eta_2 + \frac{E P \eta_2 \xi}{(D_1 E + D_0 P)} + Q^a T_0 \beta e^{-bg}(a+1)\right)$, $\theta_{62} = \frac{D_1 E^2 M \eta_2 \xi}{(D_1 E + D_0 P)^2}$, $\theta_{63} = \frac{D_1 E^2 \eta_1 \xi(M-\tau)}{(D_1 E + D_0 P)^2}$, $\theta_{66} = -\left(\frac{\eta_1 \left(c(D_1 E + D_0 P)^2 - D_0 P^2 \xi(M-\tau)\right)}{(D_1 E + D_0 P)^2} - D\eta_2 + 2M\alpha_1 \eta_2 + 3M^2 \alpha_2 \eta_2 + \frac{E P \eta_2 \xi}{(D_1 E + D_0 P)}\right)$.

Consider the following set:

$$\mathcal{U}(\mathcal{Z}) = \max\{\vartheta_1 |\delta_1|, \vartheta_2 |\delta_2|, |\delta_3|, \vartheta_3 |\delta_4|, \vartheta_4 |\delta_5|, \vartheta_5 |\delta_6|\},$$

where $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5)$ is a set of positive real numbers. At an arbitrary equilibrium point $\hat{\Sigma}(\hat{P}, \hat{Q}, \hat{E}, \hat{M})$, the sequence of system (4.5) can be represented by the following inequalities:

$$\begin{aligned}
\frac{d^+}{dt} \vartheta_1 |\delta_1| &\leq \bar{\theta}_{11} \vartheta_1 |\delta_1| + \frac{\bar{\theta}_{14} \vartheta_1}{\vartheta_4} \vartheta_4 |\delta_4|, \\
\frac{d^+}{dt} \vartheta_2 |\delta_2| &\leq \bar{\theta}_{22} \vartheta_2 |\delta_2| + \bar{\theta}_{23} \vartheta_2 |\delta_3| + \frac{\bar{\theta}_{24} \vartheta_2}{\vartheta_4} \vartheta_4 |\delta_4|, \\
\frac{d^+}{dt} |\delta_3| &\leq \frac{\bar{\theta}_{32}}{\vartheta_2} \vartheta_2 |\delta_2| + \bar{\theta}_{33} |\delta_3| + \frac{\bar{\theta}_{35}}{\vartheta_4} \vartheta_4 |\delta_5| + \frac{\bar{\theta}_{36}}{\vartheta_5} \vartheta_5 |\delta_6|, \\
\frac{d^+}{dt} \vartheta_3 |\delta_4| &\leq \frac{\bar{\theta}_{41} \vartheta_3}{\vartheta_1} \vartheta_1 |\delta_1| + \frac{\bar{\theta}_{42} \vartheta_3}{\vartheta_2} \vartheta_2 |\delta_2| + \frac{\bar{\theta}_{44} \vartheta_2}{\vartheta_3} \vartheta_3 |\delta_4| + \frac{\bar{\theta}_{45} \vartheta_3}{\vartheta_4} \vartheta_4 |\delta_5|, \\
\frac{d^+}{dt} \vartheta_4 |\delta_5| &\leq \frac{\bar{\theta}_{51} \vartheta_4}{\vartheta_1} \vartheta_1 |\delta_1| + \bar{\theta}_{53} \vartheta_4 |\delta_3| + \frac{\bar{\theta}_{54} \vartheta_4}{\vartheta_3} \vartheta_3 |\delta_4| + \bar{\theta}_{55} \vartheta_4 |\delta_5|, \\
\frac{d^+}{dt} \vartheta_5 |\delta_6| &\leq \frac{\bar{\theta}_{62} \vartheta_5}{\vartheta_2} \vartheta_2 |\delta_2| + \bar{\theta}_{63} \vartheta_5 |\delta_3| + \bar{\theta}_{66} \vartheta_5 |\delta_6|,
\end{aligned} \tag{4.6}$$

where $\frac{d^+}{dt}$ denotes the right-hand derivative, and $\bar{\theta}_{ij}$ represents the value of θ_{ij} at the equilibrium point $\hat{\Sigma}$ for $1 \le i, j \le 6$.

This results in

$$\frac{d^+}{dt}\mathcal{U}(\mathcal{Z}(t)) \le \Upsilon \mathcal{U}(\mathcal{Z}(t)),$$

where

$$\Upsilon = \max\left\{\bar{\theta}_{11} + \frac{\bar{\theta}_{14}\vartheta_1}{\vartheta_4}, \ \bar{\theta}_{22} + \bar{\theta}_{23}\vartheta_2 + \frac{\bar{\theta}_{24}\vartheta_2}{\vartheta_4}, \ \frac{\bar{\theta}_{32}}{\vartheta_2} + \bar{\theta}_{33} + \frac{\bar{\theta}_{35}}{\vartheta_4} + \frac{\bar{\theta}_{36}}{\vartheta_5}, \ \frac{\bar{\theta}_{41}\vartheta_3}{\vartheta_1} + \frac{\bar{\theta}_{42}\vartheta_3}{\vartheta_2} + \frac{\bar{\theta}_{44}\vartheta_2}{\vartheta_3} \right.$$
$$\left. + \frac{\bar{\theta}_{45}\vartheta_3}{\vartheta_4}, \ \frac{\bar{\theta}_{51}\vartheta_4}{\vartheta_1} + \bar{\theta}_{53}\vartheta_4 + \frac{\bar{\theta}_{54}\vartheta_4}{\vartheta_3} + \bar{\theta}_{55}, \ \frac{\bar{\theta}_{62}\vartheta_5}{\vartheta_2} + \bar{\theta}_{63}\vartheta_5 + \bar{\theta}_{66} \right\}.$$

Having established the boundedness of the system (2.1), when the parametric condition $\Upsilon < 0$ holds, one can obtain a positive constant $\hat{\Upsilon}$ such that $\Upsilon \le -\hat{\Upsilon} < 0$. Consequently, we have

$$\mathcal{U}(\mathcal{Z}(t)) \le \mathcal{U}(\mathcal{Z}((\hat{t})) \exp(-\hat{\Upsilon}(t - \hat{t})), t \ge \hat{t} > 0.$$

This ensures the equi-uniform asymptotic stability of the second compound system (4.3). Consequently, by Proposition (4.1), whenever an equilibrium point $\hat{\Sigma}$ of the system (2.1) satisfies the condition $\Upsilon < 0$, it becomes GAS. This concludes the proof. □

5. Local bifurcation analysis

Several local bifurcations may be observed in the system (2.1) with the variation of the parameters τ and b. We show that the system may experience transcritical and Hopf bifurcations at some critical values of these parameters.

Theorem 4. *(i) The system (2.1) undergoes a transcritical bifurcation at the harvesting-free equilibrium point $\Sigma_6(P_6, Q_6, 0, M_3)$ if τ arrives the threshold level $\tau = M_3 - \frac{cD_0}{\xi}$, and the associate transversality condition $(2\eta_1 \xi \bar{\zeta}_3 (D_0 P_6 \bar{\zeta}_4 - D_1 M_3 \bar{\zeta}_3 + D_1 \tau \bar{\zeta}_3))/D_0^2 P_6 \ne 0$ is hold.*

(ii) The system (2.1) undergoes a transcritical bifurcation at the predatory-free equilibrium point $\Sigma_7(P_7, 0, E_7, M_7)$ if b attains the critical value $b = b^{TC}$, where b^{TC} is the positive root of the equation

$$c_{22} (c_{11}(c_{33}c_{44} - c_{43}c_{34}) + c_{13}(c_{34}c_{41} - c_{31}c_{44})) = 0,$$

where c_{ij} are the coefficients of the Jacobian matrix (A.3) (See Appendix C), and the associate transversality condition $\frac{D' \rho_2 \iota_1 \iota_2}{(D' + P_7)^2} \ne 0$ is hold.

Proof. (i) From (A.2), one can observe that the Jacobian matrix leaves a zero eigenvalue if

$$\tau = M_3 - \frac{cD_0}{\xi} = \tau^{TC}.$$

At $\tau = \tau^{TC}$, the eigenvector corresponding to the zero eigenvalue of $J(P_6, Q_6, 0, M_3)$ and $J(P_6, Q_6, 0, M_3)^T$ are given, respectively, by $\bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3, \bar{\zeta}_4)$, and $\bar{\eta} = (0, 0, 1, 0)^T$, where

$J(P_6, Q_6, 0, M_3)^T$ is the transpose of $J(P_6, Q_6, 0, M_3)$, and $\bar{\zeta}_1 = -\frac{b_{22}}{b_{21}}$, $\bar{\zeta}_2 = 1$, $\bar{\zeta}_3 = \frac{b_{11}b_{22}-b_{21}b_{12}}{b_{21}b_{13}}$, $\bar{\zeta}_4 = \frac{b_{43}(b_{21}b_{12}-b_{11}b_{22})}{b_{44}b_{21}b_{13}}$. Now, the three conditions of Sotomayor's theorem [84] for the existence of a degenerate transcritical bifurcation at $\tau = \tau^{TC}$ can be given as

$$\bar{\eta}^T R_\tau \Big(\Sigma_6(P_6, Q_6, 0, M_3); \tau = \tau^{TC}\Big) = 0,$$

$$\bar{\eta}^T DR_\tau \Big(\Sigma_6(P_6, Q_6, 0, M_3); \tau = \tau^{TC}\Big)\bar{\zeta} = 0, \tag{5.1}$$

$$\bar{\eta}^T D^2 R\Big(\Sigma_6(P_6, Q_6, 0, M_3); \tau = \tau^{TC}\Big)(\bar{\zeta}, \bar{\zeta}) \neq 0.$$

Here, $R_\tau = (\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \mathscr{F}_4)^T$ and $DR_\tau \Big(J(P_6, Q_6, 0, M_3); \tau = \tau^{TC}\Big)\bar{\zeta}$ is the linear transformation formed by the matrix of partial derivatives of the components of R_τ with respect to the state variables (P, Q, E, M). Similarly, one can define the other linear transformation $D^2 R\Big(J(P_6, Q_6, 0, M_3); \tau = \tau^{TC}\Big)(\bar{\zeta}, \bar{\zeta})$. It is to be noted that for the appearance of the non-degenerate transcritical bifurcation, the second condition of (5.1) needs to be non-zero [84].

Now,

$$\bar{\eta}^T R_\tau \Big(\Sigma_6(P_6, Q_6, 0, M_3); \tau = \tau^{TC}\Big) = (0\ 0\ 1\ 0)\begin{pmatrix} \mathscr{F}_1(P_6, Q_6, 0, M_3) \\ \mathscr{F}_2(P_6, Q_6, 0, M_3) \\ 0 \\ \mathscr{F}_4(P_6, Q_6, 0, M_3) \end{pmatrix}_{\tau=\tau^{TC}}$$

$$= 0,$$

$$\bar{\eta}^T DR_\tau \Big(\Sigma_6(P_6, Q_6, 0, M_3); \tau = \tau^{TC}\Big)\bar{\zeta} = (0\ 0\ 1\ 0)\begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} \end{pmatrix}_{\tau=\tau^{TC}}\bar{\zeta}$$

$$= 0,$$

$$\bar{\eta}^T D^2 R\Big(\Sigma_6(P_6, Q_6, 0, M_3); \tau = \tau^{TC}\Big)(\bar{\zeta}, \bar{\zeta}) = (0\ 0\ 1\ 0)\begin{pmatrix} f_{11} & f_{12} & f_{13} & 0 \\ f_{21} & f_{22} & 0 & 0 \\ 0 & 0 & f_{33} & f_{34} \\ 0 & 0 & f_{43} & f_{44} \end{pmatrix}_{\tau=\tau^{TC}}\bar{\zeta}$$

$$= \frac{2\,\eta_1\,\xi\,\bar{\zeta}_3\Big(D_0\,P_6\,\bar{\zeta}_4 - D_1\,M_3\,\bar{\zeta}_3 + D_1\,\tau\,\bar{\zeta}_3\Big)}{D_0{}^2\,P_6},$$

where $f_{11} = \frac{P_6\rho_1}{(D'+P_6)^2} - \bar{\zeta}_1\left(\frac{2r}{K} - \frac{2Q_6\rho_1}{(D'+P_6)^2} + \frac{2P_6Q_6\rho_1}{(D'+P_6)^3}\right) - \frac{\rho_1}{D'+P_6}$, $f_{12} = -\frac{D'\rho_1\bar{\zeta}_1}{(D'+P_6)^2}$, $f_{13} = \frac{2D_1\xi\bar{\zeta}_3}{D_0{}^2 P_6}$, $f_{21} = \frac{D'\rho_2\left(D'+P_6-2Q_6\bar{\zeta}_1\right)}{(D'+P_6)^3}$, $f_{22} = \frac{D'\rho_2\bar{\zeta}_1}{(D'+P_6)^2} - Q_6{}^{a-1}T_0\,a\beta\,e^{-bg}(a+1)$, $f_{33} = \frac{\eta_1\xi\bar{\zeta}_4}{D_0} - \frac{2D_1\eta_1\xi\bar{\zeta}_3(M_3-\tau)}{D_0{}^2 P_6}$, $f_{34} = \frac{\eta_1\xi\bar{\zeta}_3}{D_0}$, $f_{43} = \frac{2D_1 M_3\eta_2\xi\bar{\zeta}_3}{D_0{}^2 P_6} - \frac{\eta_2\xi\bar{\zeta}_4}{D_0}$, $f_{44} = -\bar{\zeta}_4\left(2\eta_2\left(\alpha_1 + 2M_3\alpha_2\right) + 2M_3\alpha_2\eta_2\right) - \frac{\eta_2\xi\bar{\zeta}_3}{D_0}$.

Thus, following Sotomayor's theorem [84], whenever the control parameter τ reaches the critical value $\tau = \tau^{TC}$, a degenerate transcritical bifurcation point occurs under the condition $(2\,\eta_1\,\xi\,\bar{\zeta}_3(D_0\,P_6\,\bar{\zeta}_4 - D_1\,\bar{\zeta}_3 + D_1\,\tau\,\bar{\zeta}_3))/D_0{}^2\,P_6 \neq 0$.

(ii) Once more, a zero eigenvalue of the Jacobian matrix (A.3), for the equilibrium state $\Sigma_7(P_7, 0, E_7, M_7)$, occurs if and only if $\det(J_{\Sigma_7}(P_7, 0, E_7, M_7)) = 0$, leading to the condition $c_{22}(c_{11}(c_{33}c_{44} - c_{43}c_{34}) + c_{13}(c_{34}c_{41} - c_{31}c_{44})) = 0$. Let $b = b^{TC}$ denote the positive solution of this equation. Subsequently, the eigenvectors of $J_{\Sigma_7}(P_7, 0, E_7, M_7)$ and $J_{\Sigma_7}(P_7, 0, E_7, M_7)^T$, associated with the zero eigenvalue at b^{TC}, manifest as

$$\bar{\zeta} = (\iota_1, \iota_2, \iota_3, 1)^T \text{ and } \bar{\eta} = (0, 1, 0, 0)^T,$$

where $\iota_1 = -\frac{c_{33}\iota_3 + c_{34}}{c_{31}}$, $\iota_2 = -\frac{c_{11}\iota_1 + c_{13}\iota_3}{c_{12}}$, and $\iota_3 = \frac{c_{31}c_{44} - c_{34}c_{41}}{c_{41}c_{33} - c_{31}c_{44}}$. Similar computations indicate the presence of a degenerate transcritical bifurcation point at $b = b^{TC}$ whenever the condition $\frac{D' \rho_2 \iota_1 \iota_2}{(D' + P_7)^2} \neq 0$ holds.

$\square$

Theorem 5. *The interior equilibrium $\Sigma^*(P^*, Q^*, E^*, M^*)$ of system (2.1) experiences a supercritical Hopf bifurcation at $b = b^{HF}$ provided*

$$\text{Det}(J(\Sigma^*; b = b^{HF})) > 0 \text{ and } \frac{d}{db}\left(\text{Tr}(J(\Sigma^*; b = b^{HF}))\right) \neq 0,$$

where $b^{HF} = \frac{1}{g} \log \frac{a\beta T_0 Q^{*a}}{d_{11} + d_{33} + d_{44}}$.

Proof. In Theorem 2(v), it is evident that the sign of $\text{Tr}(J(P^*, Q^*, E^*, M^*))$ determines the stability nature of the interior equilibrium point Σ^*. The equilibrium point Σ^* loses its stability as $\text{Tr}(J(P^*, Q^*, E^*, M^*))$ changes its sign from negative to positive due to the smooth variation of the control parameter b. By solving $\text{Tr}(J(P^*, Q^*, E^*, M^*)) = 0$ for b, one can derive

$$b = \frac{1}{g} \log \frac{a\beta T_0 Q^{*a}}{d_{11} + d_{33} + d_{44}} = b^{HF}.$$

Hence, at $b = b^{HF}$, one pair of eigenvalues will become purely imaginary, triggering a Hopf bifurcation, provided the transversality condition $\frac{d}{db}\left(\text{Tr}(J(\Sigma^*; b = b^{HF}))\right) \neq 0$ holds [84].

$\square$

Remark 2. *One can similarly prove that the harvesting effort-free equilibrium $\Sigma_6(P_6, Q_6, E_6, M_6)$ experiences a supercritical Hopf bifurcation at $b = \hat{b}^{HF}$ provided*

$$\text{Det}(J(\Sigma_6; b = \hat{b}^{HF})) > 0 \text{ and } \frac{d}{db}\left(\text{Tr}(J(\Sigma_6; b = \hat{b}^{HF}))\right) \neq 0,$$

where $\hat{b}^{HF} = \frac{1}{g} \log \frac{a\beta T_0 Q_6 a}{b_{11} + b_{33} + b_{44}}$.

6. Optimal economic policies

The societal revenue linked to the system (2.1) is derived by combining the fisher's income (Π_1) from fish sales, earning (Π_2) from ecotourism, and revenue (Π_3) from fishing tax. These components of earnings are represented as

$$\Pi_1(P, Q, E, M, \mathscr{C}_{\mathscr{P}}) = (M - \tau)\left(\frac{\xi PE}{D_0 P + D_1 E}\right), \tag{6.1}$$

$$\Pi_2(P, Q, E, M, \mathscr{C}_{\mathscr{P}}) = bT_0 Q^a e^{-bg}, \tag{6.2}$$

$$\Pi_3(P, Q, E, M, \mathscr{C}_{\mathscr{P}}) = \frac{\tau \xi PE}{D_0 P + D_1 E}, \tag{6.3}$$

where $\mathscr{C}_{\mathscr{P}}$ indicates the control parameter/parameters associated with the optimal problem. The societal revenue, denoted as $\Pi(P, Q, E, M, \mathscr{C}_{\mathscr{P}})$, is the aggregate of the three aforementioned components and is expressed as

$$\Pi(P, Q, E, M, \mathscr{C}_{\mathscr{P}}) = \frac{\xi PEM}{D_0 P + D_1 E} + bT_0 Q^a e^{-bg}. \tag{6.4}$$

To identify the optimal fishing tax that maximizes societal revenue, the objective function is formulated as

$$\chi = \int_0^\infty e^{-\gamma t} \Pi(P, Q, E, M, \mathscr{C}_{\mathscr{P}}) dt, \tag{6.5}$$

subject to the constraints (2.1). Here, γ represents the discount rate, encompassing factors such as inflation and the time value of money. The control variable is constrained within its lower to upper limit range.

Applying Pontryagin's maximum principle, the Hamiltonian of the system is obtained as

$$\begin{aligned}
\Omega(P, Q, E, M, \mathscr{C}_{\mathscr{P}}) = {} & e^{-\gamma t} \left(\frac{\xi PEM}{D_0 P + D_1 E} + bT_0 Q^a e^{-bg} \right) + \epsilon_1 \left(rP \left(1 - \frac{P}{K} \right) - \frac{\rho_1 PQ}{P + D'} - \frac{\xi PE}{D_0 P + D_1 E} \right) \\
& + \epsilon_2 \left(\frac{\rho_2 PQ}{P + D'} - \mu Q - \beta T_0 Q^{a+1} e^{-bg} \right) + \epsilon_3 \left(\eta_1 \left(\frac{\xi PE}{D_0 P + D_1 E} (M - \tau) - cE \right) \right) \\
& + \epsilon_4 \left(\eta_2 M \left(D - \alpha_1 M - \alpha_2 M^2 - \frac{\xi PE}{D_0 P + D_1 E} \right) \right). \tag{6.6}
\end{aligned}$$

Here, $\epsilon_1, \epsilon_2, \epsilon_3$, and ϵ_4 denote the adjoint variables, which are always bounded.

6.1. Optimal entrance fee policy

Tourist entrance fees provide crucial financial support for marine ecosystem conservation and sustainable resource management. The revenue aids in promoting ecotourism, preserving culture, and fostering climate resilience, contributing to the overall robustness of the blue economy. However, high tourist entrance fees may discourage tourism participation. Therefore, adopting an optimal entrance fee policy is imperative, striking a balance that ensures both tourist engagement and societal benefits. Consequently, the control parameter of interest becomes $\mathscr{C}_{\mathscr{P}} = b$. To enhance the societal revenue Π, it is necessary to fulfil the specified conditions related to the Hamiltonian:

$$\frac{\partial \Omega}{\partial b} = 0, \; \frac{d\epsilon_1}{dt} = -\frac{\partial \Omega}{\partial P}, \; \frac{d\epsilon_2}{dt} = -\frac{\partial \Omega}{\partial Q}, \; \frac{d\epsilon_3}{dt} = -\frac{\partial \Omega}{\partial E}, \; \frac{d\epsilon_4}{dt} = -\frac{\partial \Omega}{\partial M}. \tag{6.7}$$

Observe that tourist visits and fish harvesting, contingent upon the coexistence of both species, are feasible exclusively in the coexisting state. Thus, the focus is on scrutinizing optimality at this equilibrium. The first condition of (6.7) evaluated at any arbitrary equilibrium point $\hat{\Sigma}(\hat{P}, \hat{Q}, \hat{E}, \hat{M})$ gives

$$\epsilon_2 = \frac{(bg - 1)}{g\beta \hat{Q}} e^{-\gamma t}.$$

Similarly, $\frac{d\epsilon_2}{dt} = -[\frac{\partial\Omega}{\partial Q}]_{(\hat{P},\hat{Q},\hat{E},\hat{M})}$ gives

$$\epsilon_1 = \phi_1 e^{-\gamma t},$$

where $\phi_1 = \frac{\hat{P}+D'}{\rho_1 \hat{P}}\left(abT_0\hat{Q}^{(a+1)}e^{-bg} - \frac{(bg-1)}{g\beta\hat{Q}}(\beta T_0 a\hat{Q}^a e^{-bg} + \gamma)\right)$.

By inserting the value of ϵ_1 into the equation $\frac{d\epsilon_1}{dt} = -[\frac{\partial\Omega}{\partial P}]_{(\hat{P},\hat{Q},\hat{E},\hat{M})}$, the following expression can be derived:

$$\epsilon_3 = \frac{\eta_2\hat{M}}{\eta_1(\hat{M}-\tau)}\epsilon_4 + \phi_2 e^{-\gamma t}, \tag{6.8}$$

where $\phi_2 = \frac{(D_0\hat{P}+D_1\hat{E})^2}{\eta_1\xi\hat{E}^2 D_1(\hat{M}-\tau)}\left(\phi_1(\gamma + \frac{r\hat{P}}{K} - \frac{\xi\hat{E}(D_1\hat{E}\hat{M}+\phi_1 D_0\hat{P})}{(D_0\hat{P}+D_1\hat{E})^2} - \frac{((bg-1)\rho_2 D' + \phi_1\rho_1 g\beta\hat{Q}\hat{P})}{g\beta(\hat{P}+D')^2})\right)$.

Subsequently, employing the relation (6.8) in the equation $\frac{d\epsilon_3}{dt} = -[\frac{\partial\Omega}{\partial E}]_{(\hat{P},\hat{Q},\hat{E},\hat{M})}$ leads to

$$\frac{d\epsilon_4}{dt} = \hat{\phi}\epsilon_4 + \bar{\phi}e^{-\gamma t}, \tag{6.9}$$

where $\hat{\phi} = c - \frac{2\eta_1\xi\hat{P}^2 D_0(\hat{M}-\tau)}{(D_0\hat{P}+D_1\hat{E})^2}$ and $\bar{\phi} = \frac{\xi\hat{P}^2 D_0(\phi_1-\hat{M}+\phi_2\eta_1(\hat{M}-\tau))}{(D_0\hat{P}+D_1\hat{E})^2} + \phi_2 c$.

Upon solving the ordinary differential equation (6.9), one gets

$$\epsilon_4 = -\frac{\bar{\phi}}{\hat{\phi}+\gamma}e^{-\gamma t}.$$

Substitution of ϵ_4 into Eq (6.8) gives

$$\epsilon_3 = -\frac{\eta_2\hat{M}\bar{\phi}}{(\hat{\phi}+\gamma)(\hat{M}-\tau)\eta_1 + \phi_2}e^{-\gamma t}.$$

Note that all the adjoint variables (ϵ_i, $i = 1, 2, 3, 4$) are bounded. Eventually, by inserting the values of ϵ_3 and ϵ_4 into $\frac{d\epsilon_4}{dt} = -[\frac{\partial\Omega}{\partial M}]_{(\hat{P},\hat{Q},\hat{E},\hat{M})}$, the resulting optimal entrance fee equation for a fixed choice of annual discount rate γ at the coexisting equilibrium point is

$$\hat{\Theta}(b) = \frac{\bar{\phi}}{\hat{\phi}+\gamma}(\eta_2\hat{M}(\alpha_1 + 2\alpha_2\hat{M}) - \gamma) + \frac{\xi\hat{P}\hat{E}}{(D_0\hat{P}+D_1\hat{E})}\left(\frac{\eta_1\eta_2\hat{M}\bar{\phi}}{((\hat{\phi}+\gamma)(\hat{M}-\tau)\eta_1 + \phi_2)} - 1\right) = 0. \tag{6.10}$$

Observe that $(\hat{P}, \hat{Q}, \hat{E}, \hat{M})$ contains control parameter b. The positive solution of Eq (6.10) gives the optimal entrance fee b^*. Subsequently, the corresponding optimal societal revenue can be obtained from (6.4) at the coexisting equilibrium point for the optimum entrance fee $b = b^*$.

Remark 3. *To optimize the individual income components, namely those from fishing, ecotourism, or fishing tax, function $\Pi(P, Q, E, M, \mathscr{C}_{\mathscr{P}})$ in Eq (6.5) needs to be substituted with $\Pi_1(P, Q, E, M, \mathscr{C}_{\mathscr{P}})$, $\Pi_2(P, Q, E, M, \mathscr{C}_{\mathscr{P}})$, or $\Pi_3(P, Q, E, M, \mathscr{C}_{\mathscr{P}})$, respectively. Following this substitution, further analysis is conducted to derive an equation akin to Eq (6.10).*

6.2. Optimum fishing tax policy

Fishing tax is essential for regulating harvest levels, preserving renewable resources, and ensuring long-term availability. These measures sustain abundant prey fish populations, support predatory fish

growth, attract recreational tourists, and benefit society economically. However, excessive taxation may discourage fishers, risking a decline in fishing activity. Thus, establishing an optimal fishing tax policy is crucial for balancing taxation and societal benefits. In this section, we introduce such a policy, where the societal revenue and the Hamiltonian function remain consistent with Eq (6.6), and the control parameter becomes $\mathscr{C}_{\mathscr{P}} = \tau$. Following a comparable analysis in this case, one can obtain bounded adjoint variables as

$$\epsilon_1 = (\eta_2\phi_3 + 1)\hat{M}e^{-\gamma_1 t}, \ \epsilon_2 = \phi_4 e^{-\gamma_1 t}, \ \epsilon_3 = 0, \ \epsilon_4 = \phi_3 e^{-\gamma_1 t},$$

where $\phi_3 = \xi\hat{P}\hat{E}/\{\eta_2\hat{M}(\alpha_1 + 2\alpha_2\hat{M}) + \gamma_1\}(D_0\hat{P} + D_1\hat{E})$, $\phi_4 = \frac{(\hat{P}+D')^2}{\rho_2 D'\hat{Q}}\Big(\gamma_1(\eta_2\phi_3 + 1)\hat{M} - \frac{\xi\hat{E}^2\hat{M}D_1}{(D_0\hat{P}+D_1\hat{E})^2} - (\eta_2\phi_3 + 1)\hat{M}(-\frac{r\hat{P}}{K} + \frac{\rho_1\hat{P}\hat{Q}}{(\hat{P}+D')^2} + \frac{\xi\hat{P}\hat{E}}{(D_0\hat{P}+D_1\hat{E})^2}) + \phi_3\frac{\eta_2\hat{M}\xi D_1\hat{E}^2}{(D_0\hat{P}+D_1\hat{E})^2}\Big)$ and γ_1 is the annual discount rate.
Eventually, the equation for optimal taxation can be derived as

$$\bar{\Theta}(\tau) = abT_0\hat{Q}^{(a-1)}e^{-bg} - \phi_4\beta aT_0\hat{Q}^a e^{-bg} + \frac{(\eta_2\phi_3 + 1)\rho_1\hat{M}\hat{P}}{\hat{P} + D'} + \gamma_1\phi_4 = 0. \tag{6.11}$$

6.3. Combined optimum policy

In the earlier analyses, we focused on a DDMS, optimizing individual control parameters independently. However, collaborative efforts among regulating bodies are essential for effective socio-ecological-economic management, necessitating a CDMS. To achieve comprehensive resource management, we extend our analysis beyond single-parameter optimization to consider the interaction between fishing tax (τ) and tourist entrance fee (b). This expands the scope of control parameters to a pair, denoted as $\mathscr{C}_{\mathscr{P}} = (b, \tau)$. By optimizing both parameters simultaneously, we aim to identify centralized strategies that maximize societal revenue while ensuring sustainability in both the fishing and ecotourism sectors. In this scenario, the Hamiltonian function remains consistent with Eq (6.6), but parameters b and τ vary simultaneously.
The conditions prescribed by Pontryagin's maximum principle, associated with this Hamiltonian involving two control parameters, are

$$\frac{\partial\Omega}{\partial b} = 0, \ \frac{\partial\Omega}{\partial\tau} = 0, \ \frac{d\epsilon_1}{dt} = -\frac{\partial\Omega}{\partial P}, \ \frac{d\epsilon_2}{dt} = -\frac{\partial\Omega}{\partial Q}, \ \frac{d\epsilon_3}{dt} = -\frac{\partial\Omega}{\partial E}, \ \frac{d\epsilon_4}{dt} = -\frac{\partial\Omega}{\partial M}. \tag{6.12}$$

By solving this set of equations, one can similarly obtain bounded adjoint variables for this case as

$$\epsilon_1 = \phi_5 e^{-\gamma_2 t}, \ \epsilon_2 = \frac{(bg-1)}{g\beta\hat{Q}}e^{-\gamma_2 t}, \ \epsilon_3 = 0, \ \epsilon_4 = \frac{(M-\phi_5)}{\eta_2 M}e^{-\gamma_2 t},$$

where $\phi_5 = \left[\frac{bg-1}{\beta g\hat{Q}}\left\{\frac{\rho_2\hat{P}}{\hat{P}+D'} - \mu - \beta T_0(a+1)\hat{Q}^a e^{-bg} - \gamma_2\right\}\right]\frac{\hat{P}+D'}{\rho_1\hat{P}}$, and γ_2 is the annual discount rate.
Ultimately, the determination of the combined optimal solution for fishing tax (τ) and tourist entrance fee (b) at the coexisting equilibrium hinges upon uncovering solution pair to the subsequent set of

equations:

$$\Theta_1(b,\tau) = \frac{\xi\phi_5\hat{P}\hat{E}}{\hat{M}(D_0\hat{P}+D_1\hat{E})} + \frac{bg-1}{\beta g\hat{Q}}\left\{\frac{\rho_2\hat{P}}{\hat{P}+D'} - \mu - \beta T_0(a+1)\hat{Q}^a e^{-bg}\right\} + \frac{\hat{M}-\phi_5}{\hat{M}}\Big(D-$$
$$2\alpha_1\hat{M} - 3\alpha_2\hat{M}^2 - \frac{1}{\eta_2}\Big) = 0,$$

$$\Theta_2(b,\tau) = \phi_5\left\{r\left(1-\frac{2\hat{P}}{K}\right) - \frac{\rho_1\hat{Q}D'}{\hat{P}+D'} - \frac{\xi D_1\hat{E}^2}{(D_0\hat{P}+D_1\hat{E})^2} - \gamma_2\right\} + \frac{\xi D_1\hat{E}^2}{(D_0\hat{P}+D_1\hat{E})^2}\Big(\hat{M}-1$$

$$+ \frac{\phi_5}{\hat{M}}\Big) + \frac{\rho_2 D'(bg-1)}{\beta g(\hat{P}+D')^2} = 0.$$

(6.13)

7. Numerical simulations

The investigation commences with a detailed examination of the parameters specified in Table 1. Following this, the values of b, τ, and K are systematically varied to explore the diverse equilibrium regions of stability and instability. The rationale behind selecting b and τ as variables lies in their significance from a sustainability perspective, while K is chosen for its ecological implications. All numerical simulations and graphical illustrations presented in this section are performed using MATLAB. With $\tau = 4.5$, $b = 5.6$, and $K = 7$, the local stability conditions of the equilibrium $\Sigma_4(K,0,0,M_3)$, where predator and fishing are absent, as outlined in Theorem 2(ii), are satisfied, ensuring $K < \frac{\mu D'}{\rho_2 - \mu} = 7.5$ and $\tau > M_3 - \frac{cD_0}{\xi} = 4.0635$. Additionally, with the parameter set $\vartheta = (0.2, 0.5, 0.04, 0.03, 0.01, 0.6)$, the global stability conditions specified in Theorem 3 are met, demonstrated by $\max\{-3.6262, -10.3889, -25.3794, -0.6916, -3.9647 - 3.6342\} = -0.6916 < 0$. From an ecological standpoint, the equilibrium is characterized by a lower carrying capacity of the prey species, which hampers predatory fish from acquiring sufficient sustenance, resulting in their decline to death. Economically, a higher fishing tax restricts harvesting activities, guiding the system towards global asymptotic stability at the prey fish and price equilibrium $\Sigma_4 = (7, 0, 0, 4.68)$. The corresponding time series evaluation is shown in Figure 1(a). Subsequently, the value of K is increased from 7 to 30, satisfying the local stability condition of the equilibrium $\Sigma_6(P_6, Q_6, 0, M_3)$ given in Theorem 2(iii) as $b_{11} + b_{22} = -3.12 < 0$, $b_{11}b_{22} - b_{12}b_{21} = 0.1 > 0$, and $\tau - \left(M_3 - \frac{cD_0}{\xi}\right) = 4.0624 > 0$. Furthermore, with the set $\vartheta = (0.001, 0.1, 0.05, 0.1, 0.01, 0.6)$, the global stability conditions specified in Theorem 3 are met, demonstrated by $\max\{-3.113, -3.66, -26.341, -0.123, -4.67, -3.65\} = -0.12 < 0$. As anticipated, the predatory species can persist under such higher enrichment levels, as illustrated in Figure 1(b). However, harvesting remains unsustainable in this scenario. To enable harvesting, the fishing tax is decreased from $\tau = 4.5$ to 0.9. Then, the local stability condition given in Theorem 2(v) for the equilibrium $\Sigma^*(P^*, Q^*, E^*, M^*)$ is satisfied as $\text{Tr}(J(P^*, Q^*, E^*, M^*)) = -4.5333 < 0$, $a_6 = 0.5247 > 0$, $a_7 = 0.0090 > 0$ and $a_4a_5a_6 - a_6^2 + a_4^2a_7 = 11.2613 > 0$. Moreover, with the set $\vartheta = (0.1, 0.9, 0.05, 0.1, 0.01, 0.6)$, the global stability conditions for Σ^* specified in Theorem 3 are met, demonstrated by $\max\{-2.99, -6.09, -3.59, -0.53, -1.46, -1.51\} = -0.53 < 0$. Consequently, the coexisting equilibrium achieves both local and global asymptotic stability, as depicted in Figure 1(c). Similarly, manipulation of the tourist entrance fee is conducted to observe the system's response towards the predatory fish-free equilibrium. Decreasing b from 5.6 to 1 leads to the global asymptotic stability of the predatory fish-free equilibrium $\Sigma_7 = (29.24, 0, 32.17, 2.43)$ by satisfying Theorem 3 (See Figure
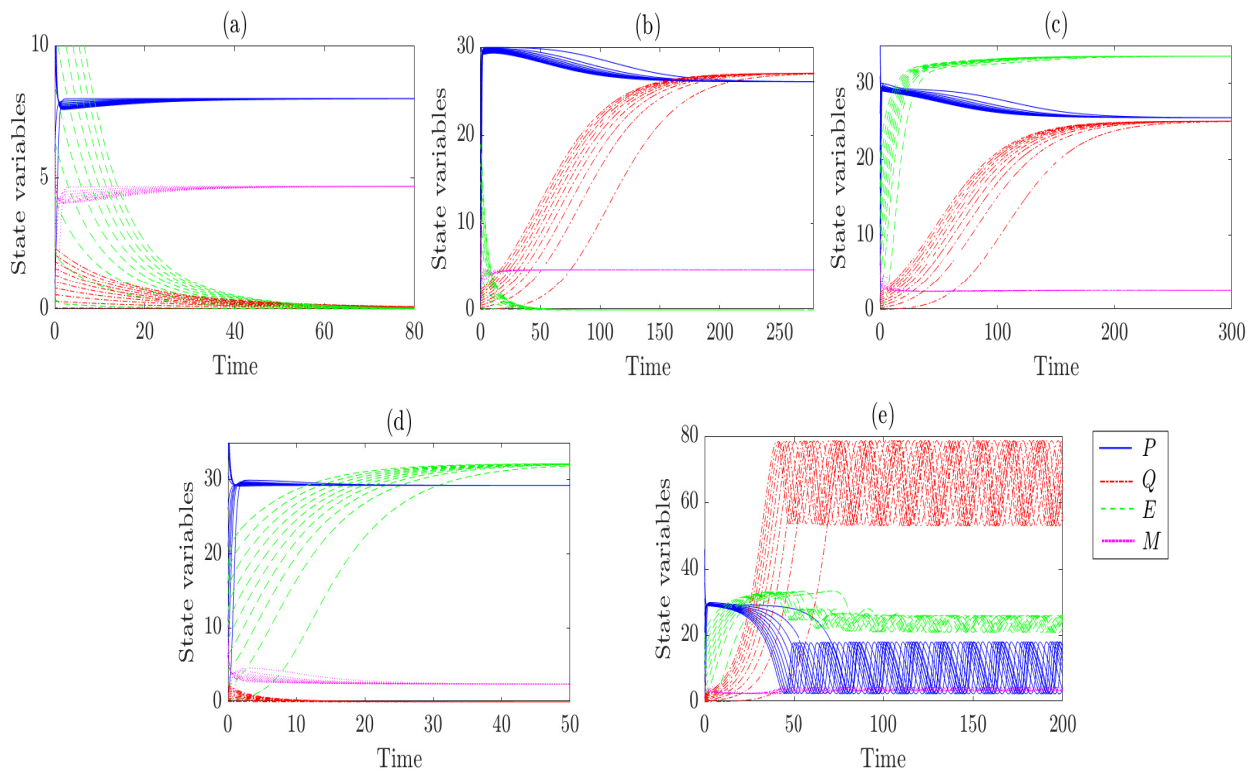
Figure 1. Global stability of the (a) prey fish and price equilibrium $\Sigma_4 = (7, 0, 0, 4.68)$, (b) harvesting effort-free equilibrium $\Sigma_6 = (26.06, 27.04, 0, 4.68)$, (c) coexisting equilibrium $\Sigma^* = (25.48, 25.03, 33.58, 2.60)$, (d) predatory fish-free equilibrium $\Sigma_7 = (29.24, 0, 32.17, 2.43)$, and (e) instability of coexisting equilibrium. In each case, the system started from ten distinct random initial points within the interval $(0, 40]$ and converged to the respective equilibrium points. The blue, red, green, and magenta curves respectively indicate P, Q, E and M. For figures (a) $\tau = 4.5$, $b = 5.6$, $K = 8$, (b) $\tau = 4.5$, $b = 5.6$, $K = 30$, (c) $\tau = 0.9$, $b = 5.6$, $K = 30$, (d) $\tau = 0.9$, $b = 1$, $K = 30$, and (e) $\tau = 0.9$, $b = 12$, $K = 30$. Other parameters are as in Table 1.

1(d)). Conversely, increasing b from 5.6 to 12 results in oscillations within the coexisting state of the system (see Figure 1(e)).

In Figure 2, the switching phenomena of the system (2.1) is demonstrated under the variation of parameters b and τ. The tourist entrance fee parameter, b, was varied within a specified range, $0 \leq b \leq 18$, revealing three distinct dynamic regions, namely R_1, R_2, and R_3, as illustrated in Figure 2. A transition from the stable predator-free equilibrium at lower values of b to a stable coexistence state occurs via a transcritical bifurcation at $b = 2.3$, satisfying Theorem 4(ii) as $c_{22}(c_{11}(c_{33}c_{44} - c_{43}c_{34}) + c_{13}(c_{34}c_{41} - c_{31}c_{44})) = 0$. Subsequently, a Hopf bifurcation emerges at $b = b^{HF} = \frac{a\beta T_0 Q^{*a}}{d_{11}+d_{33}+d_{44}} = 9.91$ in the forward direction, satisfying the transversality condition of Theorem 5 as $\text{Det}(J(\Sigma^*; b = b^{HF})) = 0.039 > 0$ and $\frac{d}{db}\left(\text{Tr}(J(\Sigma^*; b = b^{HF}))\right) = 0.0148 > 0$. Consequently, the system populations transit from stable to unstable states as b crosses $b = 9.91$ from lower to higher values. Thus, a stable limit cycle emerges around the unstable spiral interior equilibrium in the region R_3. Notably, within the lower range of b, the predatory population cannot persist even at higher prey (food)
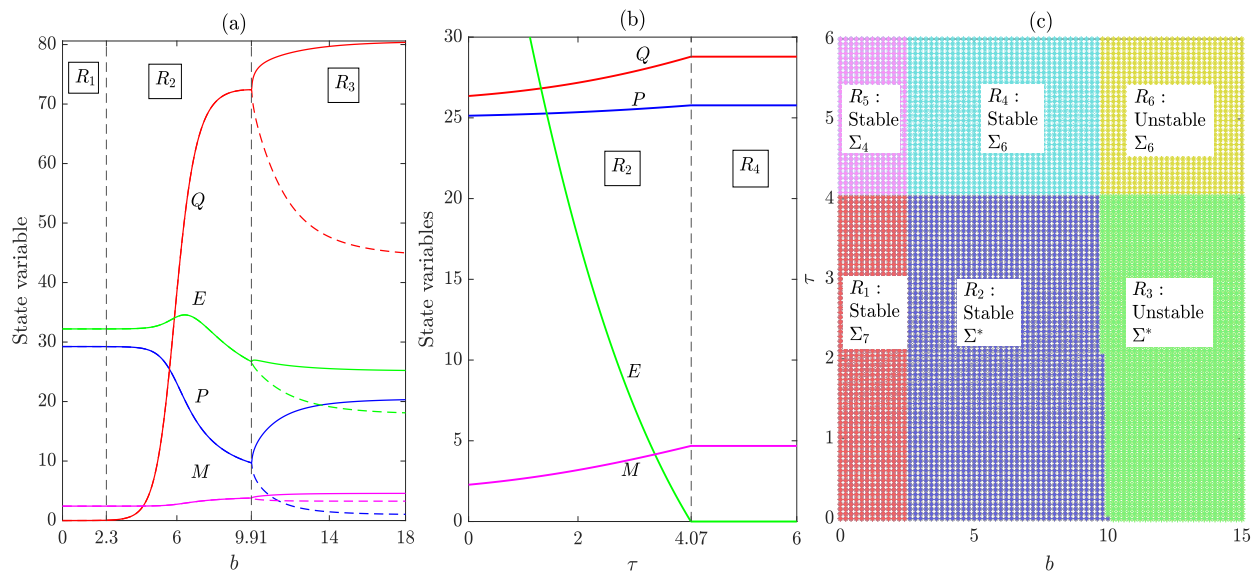
Figure 2. One-parameter bifurcation diagrams illustrate the state variables as they vary with respect to the parameters: (a) Tourist entrance fee and (b) fishing tax. In this context, the blue, red, green, and magenta curves represent the evaluations of state variables P, Q, E, and M, respectively. A two-parameter bifurcation diagram in the $b - \tau$ parametric plane is presented in panel (c). The parameters remain the same as in Figure 1(c) unless and until it varies.

levels. This behavior highlights the impact of imposing an entrance fee. Predatory fish populations are threatened without a tourist entrance fee or if the fee is too low. Drastic tourist visits may be the reason for this situation. In the intermediate range of b, $2.3 < b < 9.91$, tourist visit declines, resulting in the existence of predatory species and, hence, the existence and stability of the coexisting equilibrium. In this region, a steady increase in predatory fish biomass and a consequent decline in prey population and market price of prey fish may be observed. Fishing effort increases initially, reaching a threshold before declining, indicating a tradeoff between fishing effort and tourist entrance fee. The system exhibits oscillatory behavior as b surpasses the higher threshold value of 9.91. A time series solution in this region corresponding to $b = 12$ is depicted in Figure 1(e). Therefore, sustainable control of the entrance fee parameter is necessary, and the challenge for regulatory agencies lies in optimizing this parameter for sustainable socio-economic benefits. In Figure 2(b), a similar bifurcation diagram is presented under the variation of fishing tax (τ). Two distinct dynamic regimes are observed: For lower tax levels, the interior equilibrium stabilizes (region R_2), while for higher tax levels, the harvesting effort-free equilibrium becomes stable (region R_4), following a transcritical bifurcation at $\tau = M_3 - \frac{cD_0}{\xi} = 4.07$. At this point, the transversality condition of Theorem (4)(i) is satisfied as $(2\eta_1 \xi \bar{\zeta}_3 (D_0 P_6 \bar{\zeta}_4 - D_1 M_3 \bar{\zeta}_3 + D_1 \tau \bar{\zeta}_3))/D_0{}^2 P_6 = -1.8 \neq 0$. This variation shows that the harvesting effort significantly increases if the tax level is low. Conversely, harvesting becomes unfeasible if the tax level exceeds the transcritical threshold value of 4.07. Thus, regulatory bodies also need to regulate fishing taxes for sustainable resource management.

The two-parameter bifurcation analysis, depicted in Figure 2(c) reveals the intricate dynamic behavior of the system across the $b - \tau$ parametric plane. This visualization distinguishes four stable regions, corresponding to equilibrium points Σ_7, Σ^*, Σ_6, and Σ_4, represented in red, blue, cyan, and magenta

colors. Additionally, two unstable oscillatory regions associated with Σ^* and Σ_6 are depicted in green and yellow, respectively. Notably, the domain of attraction of the coexisting equilibrium Σ^* exhibits the largest stability domain, while the stable domain of attraction of Σ_4 occupies the smallest area. Transcritical bifurcation curves separate each stable regime, while Hopf bifurcation curves separate a stable and unstable equilibrium regime. An interesting situation occurs when the parameter τ varies, and b is fixed at higher values. In this case, two stable limit cycles (around the equilibrium points Σ^* and the other is Σ_6) intersect and interchange the oscillation through a transcritical bifurcation of limit cycle (TBLC). For a more profound comprehension of this transition, Figure 3 is presented, where
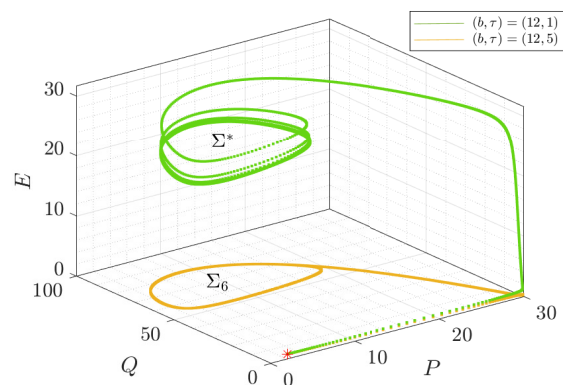

Figure 3. The $3D$ trajectory in the $P - Q - E$ space vividly illustrates the transcritical bifurcation of a limit cycle. The trajectory depicted in yellow corresponds to the stable limit cycle around Σ_6 equilibrium, while the trajectory in green demonstrates the stable limit cycle around Σ^* equilibrium, signifying an exchange of stability of the limit cycles. Each of these trajectories initiates from the same initial point $(2, 0.2, 0.8, 0.5)$. The parameters are consistent with those defined in Figure 2.

the TBLC is prominently displayed. This diagram plots a three-dimensional ($3D$) trajectory in the $P - Q - E$ parametric space. One can generate a similar three-dimensional diagram by selecting E along with any two state variables among P, Q, and M to observe identical dynamics. The yellow curve corresponds to the trajectory on the $E = 0$ plane, representing the stable limit cycle around the equilibrium Σ_6 for the parameter pair $(b, \tau) = (12, 1)$. Conversely, the green curve depicts another trajectory, representing the coexisting equilibrium for the parameter pair $(b, \tau) = (12, 5)$, indicating the stable limit cycle around the equilibrium Σ^*. Notably, this trajectory pops up from the $E = 0$ plane through the TBLC at $(b, \tau) = (12, 4.07)$.

For the numerical optimization of the optimal entrance fee (b) under DDMS, we consider the parameter values from Figure 2(a) with an annual discount rate of $\delta = 0.75$ /year, and solve Eq (6.10) for b. The green curve in Figure 4(a) represents the curve $\hat{\Theta}(b) = 0$ at an arbitrary equilibrium point $\hat{\Sigma}$, with corresponding solutions denoted by red dots. Four solutions are identified, viz., $b_1 = 2.5$ \$/tourist, $b_2 = 5.5$ \$/tourist, $b_3 = 5.65$ \$/tourist, and $b_4 = 13$ \$/tourist. Among these solutions, the first three belong to the stable coexisting equilibrium region, while the last one ($b_4 = 13$) corresponds to the unstable oscillatory region (compare Figure 2(a)). Subsequently, the societal revenue is computed for the three stable solutions using Eq (6.4), yielding $\Pi(b_1) = 8.56$ M\$/year, $\Pi(b_2) = 10.01$ M\$/year, and $\Pi(b_3) = 10.02$ M\$/year. It is evident that among these entrance fees, Π achieves its maximum at b_3. The
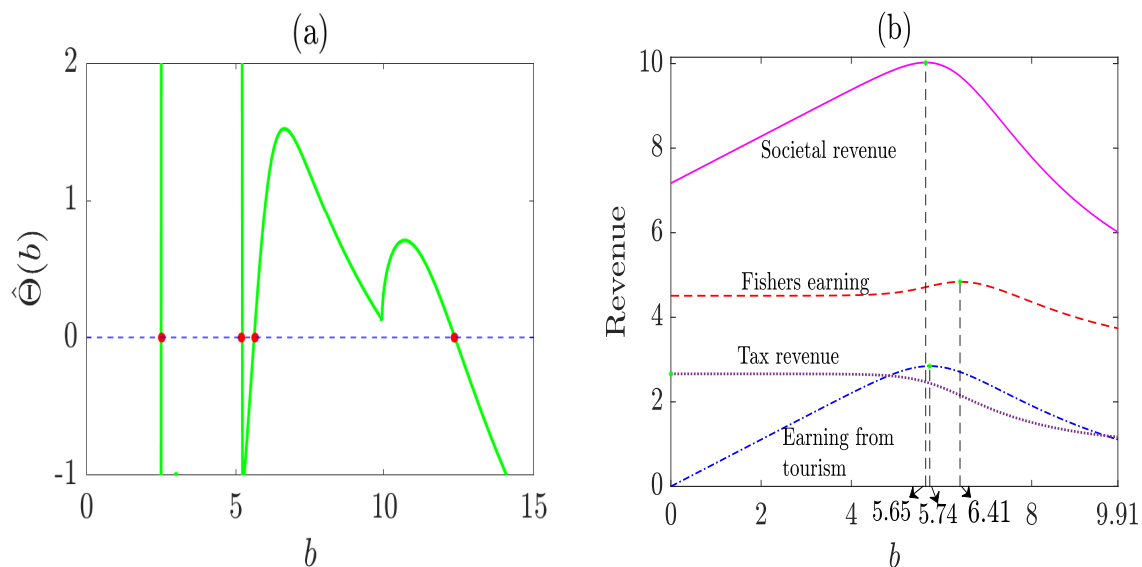
Figure 4. (a) Plot of $\hat{\theta}(b)$ under the variation of tourist entry fee, b. The plot reveals four solutions of Eq (6.10), represented by red dots, indicating multiple optimal values of b, where $\hat{\theta}(b) = 0$. (b) The magenta, red, blue, and black lines represent societal revenue (Π), income from fishing after tax (Π_1), earnings from ecotourism (Π_2), and income from fishing tax (Π_3), respectively. The optimal societal revenue ($\Pi(b_3) = 10.02$ M\$/year) is achieved at $b_3 = 5.65$ \$/tourist. Individual optimal revenue levels are obtained from fishers after paying tax ($\Pi_1 = 4.84$ M\$/year) at $b = 6.41$ \$/tourist, from the imposed entrance fee ($\Pi_2 = 2.84$ M\$/year) at $b = 5.74$ \$/tourist, and from fishing tax ($\Pi_3 = 2.66$ M\$/MT) at $b = 0$ \$/tourist. The parameters correspond to those in Figure 2(a) with an annual discount rate of $\delta = 0.75$ /year.

top magenta curve in Figure 4(b) represents the societal revenue function $\Pi(b)$ at the equilibrium level, confirming the maximum societal revenue at $b_3 = 5.65 = b^*$ \$/tourist. Following Remark 3, one can similarly optimize the individual income components in this parametric plane. Figure 4(b) illustrates that the income from fishing after tax, Π_1, is maximum (4.84 M\$/year) at the corresponding optimum entrance fee level $b = 6.41$ \$/tourist. Similarly, the income from ecotourism, Π_2, achieves its maximum (2.85 M\$/year) at $b = 5.74$ \$/tourist. Whereas the income from fishing tax, Π_3, is a decreasing function of b, reaching its maximum (2.66 M\$/year) at $b = 0$ \$/tourist. Notably, each income component attains its maximum value at different entrance fee levels, yet the maximum societal revenue is earned at the intermediate optimal entrance fee $b = 5.65$ \$/tourist. The trajectories of the adjoint variables ϵ_1, ϵ_2, ϵ_3, and ϵ_4 are presented in Figure 5 to illustrate their bounded nature. The figure demonstrates that each adjoint variable remains within finite limits over the considered time interval.

The subsequent optimization focuses on the fishing tax (τ) under DDMS, as illustrated in Figure 6. We maintain parameter values identical to those in Table 1 with $K = 30$ and $b = 3$ (an arbitrary value following a DDMS), and solve Eq (6.11) for τ. The green curve in Figure 6(a) represents the curve $\bar{\Theta}(\tau) = 0$ at an arbitrary equilibrium point $\hat{\Sigma}$, with the corresponding solution marked by red dots. A unique solution is identified at $\tau_1 = 1.13$ M\$/MT, and the corresponding societal revenue can be computed from Eq (6.5), resulting in $\Pi(\tau_1) = 8.85$ M\$/year. The top magenta curve depicted in Figure 6(b) illustrates the societal revenue function $\Pi(\tau)$ at the coexisting equilibrium state, affirming
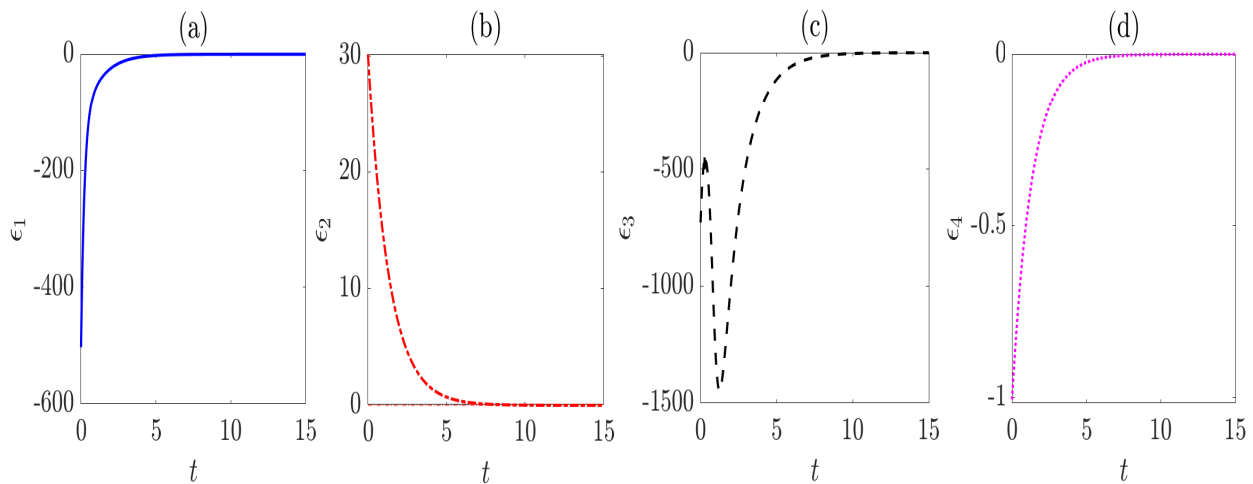
Figure 5. The plots of ϵ_1, ϵ_2, ϵ_3, and ϵ_4 are shown in Figure (a)–(d), respectively. The simulations use the same parameter values as in Figure 4 with $b = 5.65$.
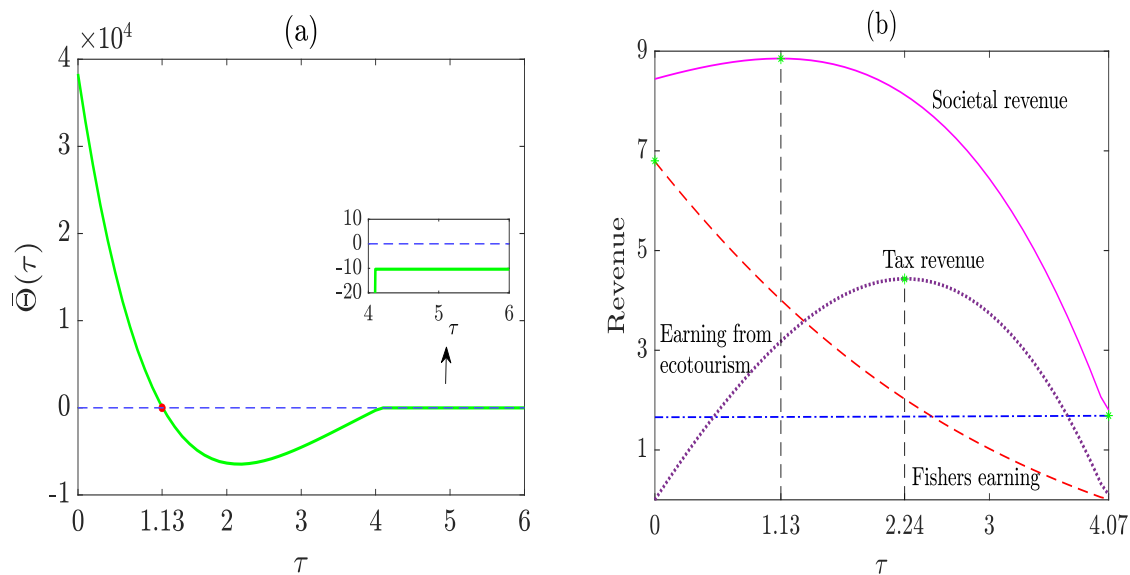

Figure 6. (a) Plot of $\bar{\theta}(\tau)$ under the variation of the harvesting control parameter τ. It shows that the Eq (6.11) has a unique solution, that is, there is a unique optimal value of τ, namely $\tau = 1.13$ M\$/MT for which $\bar{\theta}(\tau) = 0$. (b) Magenta, red, blue, and black lines indicate the societal revenue, income from fishing after tax, earnings from ecotourism and income from fishing tax. The optimal societal revenue ($\Pi(\tau_1) = 8.85$ M\$/year) is attained at $\tau_1 = 1.13$, M\$/MT and the optimal revenue earned from the imposed taxation ($\Pi_3 = 4.43$ M\$/year) is obtained at the optimal tax $\tau = 2.24$ M\$/MT. The parameters are as in Figure 2(b) with $b = 3$ and an annual discount rate of $\gamma = 0.022$ /year.

the presence of maximum societal revenue at $\tau_1 = 1.13 = \tau^*$ M\$/MT. Furthermore, the income derived from fishing tax, denoted as Π_3, reaches its peak value (4.43 M\$/year) at the corresponding optimal taxation level $\tau = 2.24$ M\$/MT. Conversely, the income from fishing after tax, denoted as Π_1, exhibits a decreasing trend with respect to τ, achieving its maximum (6.79 M\$/year) at $\tau = 0$. In contrast, the

income from ecotourism (Π_2) displays a slow pattern relative to τ, reaching its peak (1.69 M\$/year) at the corresponding optimal tax level $\tau = 4.07$ M\$/MT. Notably, each income component attains its maximum value at distinct tax levels. Intriguingly, the maximum societal revenue is garnered at the lower non-zero optimal tax level $\tau = 1.13$ M\$/MT. At this stage, there is ambiguity in determining the optimal solution pair. Different pairs, such as $(b, \tau) = (5.65, 0.9)$ or $(3, 1.13)$, emerge from multiple DDMS analyses, leading to inconclusiveness. Therefore, a two-parameter optimization following CDMS is required for a clearer understanding.
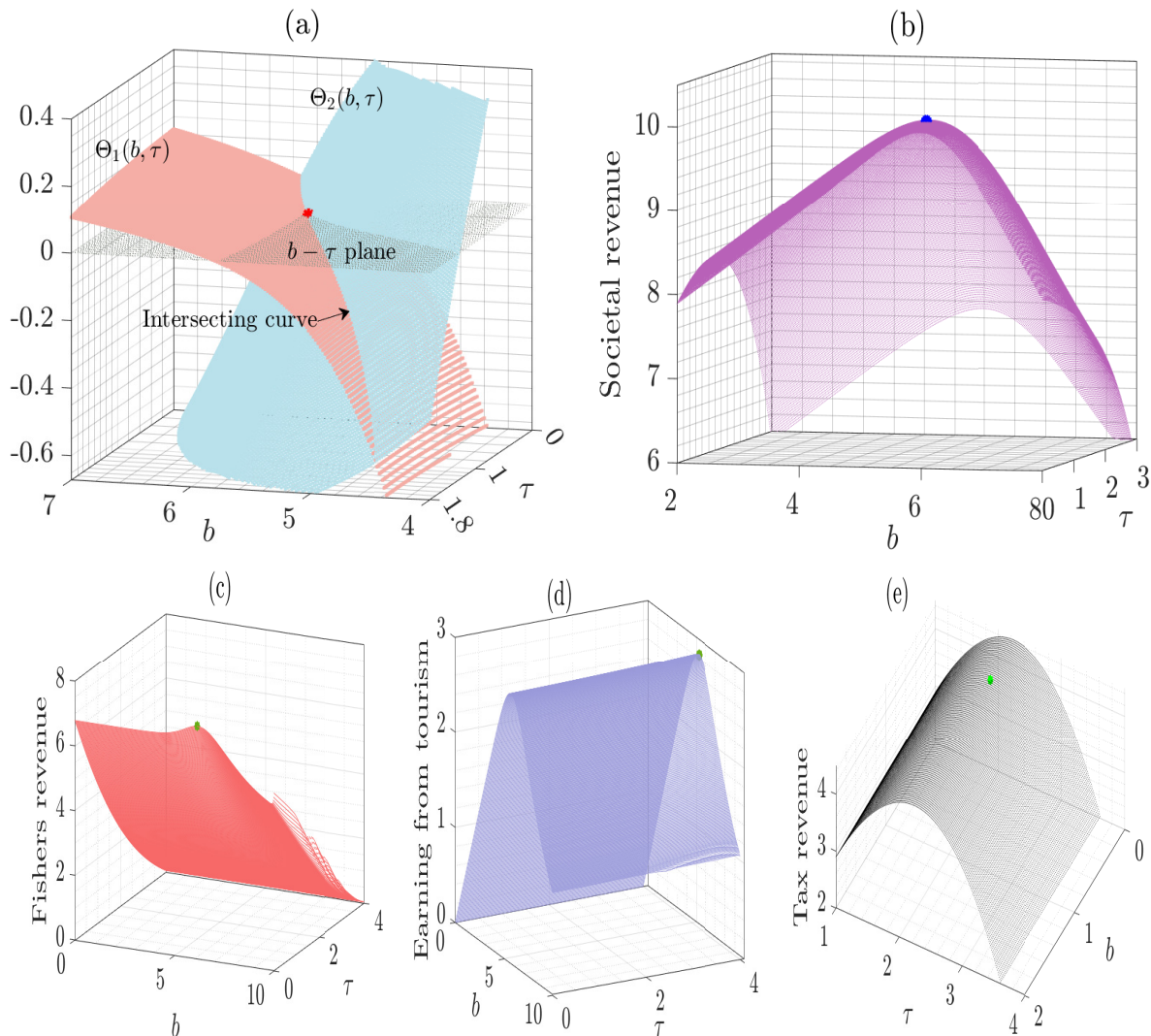

Figure 7. Upper panel: (a) Plot of $\Theta_1(b, \tau)$ and $\Theta_2(b, \tau)$ under the variation of the control parameter pair (b, τ). The plot reveals that Eq (6.13) has a unique solution, denoted as $\tau^* = 0.67$ and $b^* = 5.74$, where $\Theta_1(b^*, \tau^*) = 0$ and $\Theta_2(b^*, \tau^*) = 0$. (b) The purple surface represents societal revenue in the $\tau - b$ equilibrium space. The optimal societal revenue $(\Pi(b^*, \tau^*) = 10.035$ M\$/year) is attained at $\tau^* = 0.67$ and $b^* = 5.74$ M\$/MT. Lower panel: The surface plot of individual earnings components: (c) Fishing activities, (d) Ecotourism. (e) Fishing tax, showcasing their dependence on the fishing tax (τ) and entrance fee (b). The parameters are as in Table 1 with $K = 30$ and an annual discount rate of $\gamma = 0.008$ /year.

Table 2. This table presents the societal revenue (Π), income from fishing after tax (Π_1), income from ecotourism (Π_2), and tax from fishing (Π_3) at specific fixed values of the tourist entrance fee (b), and fishing tax (τ). The optimum earning components and the corresponding optimal pairs are highlighted in boldface. The parameters are consistent with those in Figure 7.

b^* M\$/million tourist	τ^* (M\$/MT)	$\hat{P}$ (MT)	$\hat{Q}$ (MT)	$\hat{E}$ (SFU)	$\hat{M}$ (M\$/MT)	Π_1 (M\$/year)	Π_2 (M\$/year)	Π_3 (M\$/year)	$\Pi = \Pi_1 + \Pi_2 + \Pi_3$ (M\$/year)
0 ($\mathcal{V}$)	0.9 ($\mathcal{F}$)	29.24	0	32.2	2.42	4.51	0	2.66	7.17
0.48 ($\mathcal{V}$)	**2.25** ($\mathcal{V}$)	29.50	0.001	14.33	3.27	2.01	0.27	**4.44**	6.71
3 ($\mathcal{F}$)	0 ($\mathcal{V}$)	29.09	0.21	48.47	1.99	6.79	1.66	0	8.44
3 ($\mathcal{F}$)	1.13 ($\mathcal{V}$)	29.25	0.22	28.63	2.55	4.01	1.66	3.18	8.85
3 ($\mathcal{F}$)	2.24 ($\mathcal{V}$)	29.47	0.22	14.44	3.26	2.02	1.67	4.43	8.13
3 ($\mathcal{F}$)	4.07 ($\mathcal{V}$)	29.95	0.23	0.37	4.63	0.05	1.69	0.33	2.07
5.65 ($\mathcal{V}$)	0.9 ($\mathcal{F}$)	25.15	26.97	33.68	2.62	4.74	2.81	2.47	10.02
5.74 ($\mathcal{V}$)	**0.67** ($\mathcal{V}$)	24.67	29.53	37.87	2.55	5.31	2.83	1.89	**10.035**
5.74 ($\mathcal{V}$)	0.9 ($\mathcal{F}$)	24.69	29.62	33.82	2.65	4.72	2.85	2.43	10.00
5.74 ($\mathcal{V}$)	**4.07** ($\mathcal{V}$)	25.22	31.93	0.36	4.63	0.05	**2.89**	0.32	3.26
6.14 ($\mathcal{V}$)	**0** ($\mathcal{V}$)	22.06	42.60	51.36	2.54	**7.19**	2.79	0	9.98
6.41 ($\mathcal{V}$)	0.9 ($\mathcal{F}$)	20.23	51.15	34.56	2.92	4.84	2.70	2.15	9.69

Note: In this table, the symbol $\mathcal{V}$ is used to denote the parameter value obtained through its variation, while $\mathcal{F}$ is employed to represent the assumed constant value of the parameter.

The next focus is visualizing the two-parameter optimization as analyzed in Section 6.3. In Figure 7(a), the surfaces $\Theta_1(b, \tau)$ and $\Theta_2(b, \tau)$ with respect to the two control parameters τ and b are plotted for an arbitrary equilibrium point $\hat{\Sigma}$, depicted in red and blue, respectively. From this diagram, it is evident that the intersection of these two surfaces forms a decreasing curve. Consequently, its intersection with the $b - \tau$ plane must correspond to a unique point marked with a red dot, indicating a unique solution at the pair $(b^*, \tau^*) = (5.74, 0.67)$. The corresponding societal revenue can be computed from Eq (6.5) as $\Pi(b^*, \tau^*) = 10.035$ M\$/year. Figure 7(b) illustrates the corresponding societal revenue surface $\Pi(b, \tau)$ at the equilibrium level, highlighting the occurrence of maximum societal revenue at $(b^*, \tau^*) = (5.74, 0.67)$ marked by a blue dot. Following a similar analysis, optimal pairs for (b, τ) that maximize incomes from fishing, fishing tax, or ecotourism can be determined. Figure 7(c) shows the income from fishing excluding tax ($\Pi_1(b, \tau)$) in the $b - \tau$ parametric space. Π_1 attains its maximum (7.19 M\$/year) at the corresponding optimum pair $(6.14, 0)$. The income from ecotourism, Π_2, is plotted in Figure 7(d), achieves its maximum (2.89 M\$/year) at $(5.74, 4.07)$. The income from fishing tax (Π_3), depicted in Figure 7(e), reaches its maximum (4.44 M\$/year) at the corresponding optimum level $(0.48, 2.25)$. All single-parameter optimization results can be derived from this two-parametric optimization. For instance, fixing the value of $\tau = 0.9$ in Figure 7(b)–(d) enables observation of the effects of various earning components with respect to b, analogous to the optimization results provided in Figure 4(b). Table 2 is presented for better comprehension and comparison of these optimal policies. This table showcases all the optimal equilibrium solutions obtained from the aforementioned single and

coupled parameter analyses, along with the corresponding values of the state variables. For example, one can observe that the societal revenue reaches its maximum (Π = 10.035 M\$/year) when (b, τ) = (5.74, 0.67), corresponding to state variable values (P, Q, E, M) = (24.67, 29.53, 37.87, 2.55), with other income components noted as (Π_1, Π_2, Π_3) = (5.31, 2.83, 1.89). The comprehensive optimal solutions for each earning component are highlighted in boldface. Notably, employing CDMS (two-parameter optimization) yields a more favorable optimal pair, indicating that the optimal societal revenue is achieved at a lower fishing tax and a higher entrance fee than any DDMS (single-parameter optimization strategy). Additionally, optimal earnings for each individual also occur within this CDMS framework.

8. Discussion

Despite traditional management methods, commercial aquatic species continue to face risks stemming from the tension between economic expansion and environmental stability. This presents significant challenges and uncertainties for coastal communities and fisheries management. However, non-consumptive activities such as marine ecotourism and recreational fishing offer viable alternatives, contributing to local economies and global socio-economic dynamics. Notably, marine ecotourism sheds light on the evolving roles of predator species beyond their traditional value as fisheries commodities, particularly in regions experiencing declining fisheries. As societal awareness grows, there is increasing recognition of the importance of ecotourism in fostering the coexistence of humans and the environment, catering to both material and spiritual needs. Sustainable management efforts for popular destinations have gained traction, with an emphasis on the involvement of local communities in environmentally friendly practices. Fishing activities and fishery-based ecotourism play pivotal roles in the socio-economic fabric of coastal regions, providing livelihoods, economic opportunities, and cultural heritage.

In this study, we employ a bioeconomic model that integrates features of the marine reserve and multispecies modelling. The model merges two traditionally distinct domains, fishing and ecotourism, into a simplified modelling framework of predator-prey dynamics. Fish species are subject to fishing pressure and non-extractive ecotourism activities involving predators such as whales, seals, and dolphins. This innovative approach recognizes the economic value of predator species in non-consumptive recreational use. The system comprises eight equilibrium points, of which four are consistently unstable, and four are locally and globally asymptotically stable under certain parametric conditions. Among these equilibria, the points Σ_6, Σ_7, and Σ^* represent bioeconomic equilibrium. Single parameter bifurcation analysis reveals that higher tourist entrance fees destabilize the system's interior equilibrium point, leading to an oscillatory state. Conversely, higher fishing taxes shift the stable coexisting state towards a stable harvesting-free region. Moreover, we found another critical dynamic in fisheries by identifying a unique transition where one stable oscillatory harvesting-free equilibrium state (Σ_6) transitions to a stable oscillatory coexistence state (Σ^*) following a transcritical bifurcation of the limit cycle. This finding highlights the dynamic nature of fishery systems and underscores the importance of adaptive management strategies to address the inherent variability in marine ecosystems.

The societal revenue is an ensemble income, including the revenue from fish sales after tax deductions, ecotourism entrance fees, and fishing taxes. Each earning may be optimized differently based on the control parameter levels, leading to varying societal revenues. Consequently, an optimal policy is necessary to determine the optimal pair of control parameters. To address this challenge, we first optimize parameter b while keeping τ fixed at 0.9, following a decentralized approach and obtain the

optimal value of $b = 5.65$. Subsequently, we conduct a similar optimization with respect to τ for a random value of $b\ (= 3)$. However, this approach yields a higher taxation level ($\tau = 1.13$) compared to the previously assumed fixed tax value ($\tau = 0.9$). Thus, these obtained pairs are all different from the actual optimal pair of CDMS as it is derived from two separate decentralized decision-making schemes. The inconclusive results arising from multiple DDMS analyses yield different societal revenue outcomes ($\Pi(5.65,\ 0.9) = 10.02$) versus ($\Pi(3,\ 1.13) = 8.85$). We perform a two-parameter optimization using a centralized decision-making scheme to find the optimal pair (b, τ). This approach provides an optimal solution pair as $(5.74, 0.67)$ that offers policymakers and governing bodies actionable insights for crafting policies conducive to the harmonious coexistence envisioned within the blue economy paradigm. Ultimately, the choice between DDMS and CDMS frameworks depends on the specific goals and constraints of the regulatory agency, with DDMS being suitable for maximizing revenue for one component and CDMS being necessary for achieving a balance between fishing and ecotourism revenue. Moreover, the investigation of revenue landscapes underscores the superior performance of CDMS compared to decentralized counterparts.

We now delve deeper into the intricate interplay between regulatory parameters and societal revenue, offering valuable insights into the complexities of decision-making within fisheries management. In exploring the optimization analysis results of our study, we uncover a notable discrepancy between the optimal regulatory control parameters and the corresponding societal revenue across various decision-making schemes. Within the DDMS analysis, a methodological approach involves assigning a random value to one regulatory parameter, followed by optimization of the other. This procedure prompts a pivotal question: What if we adopt the optimal value of one parameter from a preceding DDMS analysis and integrate it into the subsequent analysis? To address this inquiry, we focus on the value of b delineated in Eq (6.11), regarded as the optimum value derived from Figure 6. Thus, we set $b = 5.65$ and then variate τ at the equilibrium level, effectively transforming Figure 6(b) into Figure 8. This strategic shift enables a deeper investigation into the dynamic interplay between regulatory parameters and societal revenue, shedding light on the nuanced intricacies of decision-making in fisheries management.

In Figure 8, the top magenta curve represents the societal revenue function $\Pi(\tau)$ at the coexisting equilibrium state, highlighting the maximum societal revenue ($\Pi(\tau^*) = 10.025$) at $\tau_1 = 0.79$ M$/MT ($\tau^*$). The income from fishing tax, Π_3, peaks at 4.22 M$/year with an optimal taxation level of $\tau = 2.28$ M$/MT. Conversely, fishing income after tax, Π_1, decreases with increasing τ, giving its maximum (7.09 M$/year) at $\tau = 0$. Ecotourism income (Π_2) exhibits a slower trend, peaking at 2.89 M$/year with $\tau = 4.07$ M$/MT. Notably, each income component peaks at different tax levels. The optimal pair of solutions is found to be $(b, \tau) = (5.65,\ 0.79)$, differing significantly from the original optimal pair $(5.74, 0.67)$. This emphasizes the importance of coupled parameter optimization for socioeconomic sustainability, as each optimal pair differs significantly from those obtained from coupled parameter optimization.

The equilibria and bifurcations observed in this predator-prey-fishery system reflect well-documented ecological processes. The extinction of predators at low prey carrying capacity (Σ_4; Figure 1a) exemplifies resource limitation, where insufficient prey biomass ($K = 7$) cannot sustain predator populations, consistent with threshold densities required for predator persistence in natural systems (e.g., wolf-moose dynamics in Isle Royale) [85]. Conversely, higher $K = 30$ (Figure 1b) enables predator survival (Σ_6), but excessive fishing taxes ($\tau = 4.5$) suppress harvesting, mirroring top-down human control overriding bottom-up productivity. It is to be mentioned that the harvesting effort-free equilibrium (Σ_6) represents
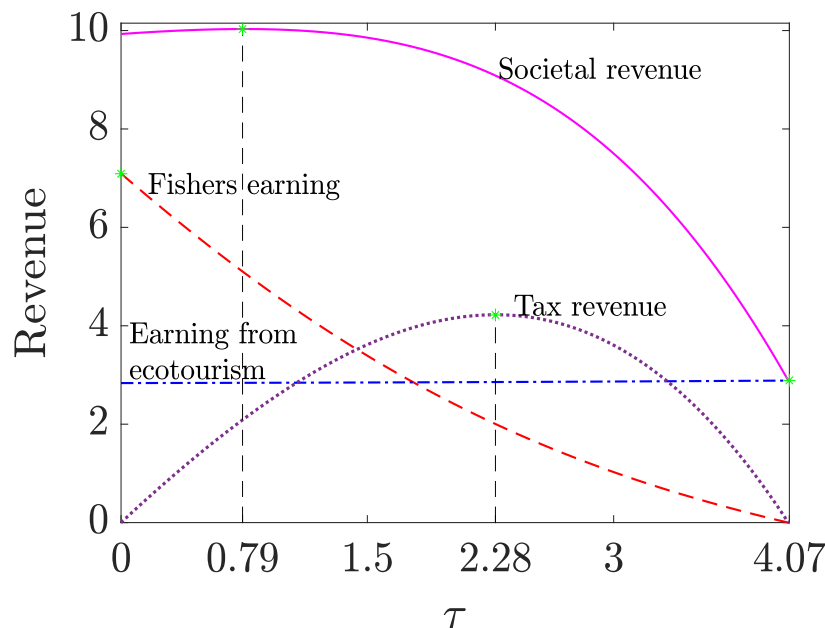
Figure 8. Magenta, red, blue, and black lines indicate the societal revenue, income from fishing after tax, earnings from ecotourism and income from fishing tax. The optimal societal revenue ($\Pi(\tau_1) = 10.025$ M\$/year) is attained at $\tau_1 = 0.79$, M\$/MT and the optimal revenue earned from the imposed taxation ($\Pi_3 = 4.22$ M\$/year) is obtained at the optimal tax $\tau = 2.28$ M\$/MT. The parameters are as in Figure (2)(b) with $b = 5.6$.

the ecological balance between predator and prey populations in the absence of fishing. It reflects a baseline scenario where population dynamics are governed solely by biological interactions, ecotourism, and environmental constraints. This equilibrium is reference point to assess the impact of harvesting and indicates the maximum sustainable population levels under undisturbed conditions. The coexistence equilibrium Σ^* (Figure 1c) emerges when moderate fishing tax ($\tau = 0.9$) and ecotourism fees ($b = 5.6$) balance exploitation and predator-mediated *trophic cascades* akin to kelp-urchin-otter systems where regulated harvesting stabilizes communities [86]. The predator-free state Σ_7 (Figure 1d) reveals an anthropogenic Allee effect: Low tourist fees ($b = 1$) increase visitation, potentially disrupting predator behavior (e.g., dolphin avoidance) or habitat, analogous to ecotourism impacts on coral reefs or nesting birds [87]. Oscillations at high $b = 12$ (Figure 1e) arise from overcompensation, where delayed density-dependent responses (e.g., predator satiation and time-lagged fishing effort) destabilize the system, as seen in hare-lynx cycles. The transcritical bifurcation at $b = 2.3$ (Figure 2a) marks a critical threshold for predator invasion, while the Hopf bifurcation at $b = 9.91$ signifies nonlinear feedback between tourism and predation. These dynamics underscore that sustainable management must account for energy flow constraints (carrying capacity K limits predator survival), human behavioral feedbacks (taxes/fees alter exploitation rates), and non-consumptive stress (tourism can indirectly extirpate predators). The optimal control pair ($b^* = 5.74$, $\tau^* = 0.67$) thus represents a biodiversity-economic trade-off, balancing harvest yields, ecotourism revenue, and ecosystem stability, a principle applicable to shark-dive tourism or whale-watching fisheries.

Our study contributes to the broader discourse on sustainable development by demonstrating how

coupled-parameter optimization can reconcile economic profitability with ecological conservation in fishery-tourism systems. The proposed framework aligns with the United Nations' Sustainable Development Goals (SDGs), particularly SDG 14 (Life Below Water) and SDG 8 (Decent Work and Economic Growth), by offering actionable levers, such as adaptive taxation and tourism fees, to mitigate overexploitation while supporting coastal livelihoods. While we uncover promising findings regarding the dynamics of socio-ecological-economic interaction, it has limitations. First, the impact of infectious diseases on fish stocks is a critical consideration. Contagious diseases caused by viruses, bacteria, protists, and metazoans can significantly reduce biological productivity and commercial value. Therefore, integrating infection dynamics into the study would represent a valuable extension, providing insights into managing fisheries in the face of biological threats. Second, while our model is well-suited for large-scale commercial fisheries, its applicability to small-scale fisheries may be limited. Small-scale fisheries often operate under different socio-economic and ecological conditions, which our model may not adequately capture. Future research could explore the development of more tailored models for small-scale fisheries. Additionally, our study relies on a deterministic system, overlooking marine environments' inherent stochastic and noisy nature. In the future, researchers could enhance the study's applicability by incorporating uncertainty into fish harvesting and stock recruitment through stochastic optimization techniques, thus yielding more realistic outcomes. Such extensions would provide policymakers with robust tools to balance short-term economic pressures against long-term ecosystem resilience, a cornerstone of sustainable resource management.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgment

Conflict of interest

The authors declare there is no conflict of interest.

References

1. S. J. Pittman, L. D. Rodwell, R. J. Shellock, M. Williams, M. J. Attrill, J. Bedford, Marine parks for coastal cities: A concept for enhanced community well-being, prosperity and sustainable city living, *Mar. Policy*, **103** (2019), 160–171. https://doi.org/10.1016/j.marpol.2019.02.012

2. H. Österblom, C. C. Wabnitz, D. Tladi, E. H. Allison, S. Arnaud-Haond, J. Bebbington, et al., *Towards Ocean Equity*, The Blue Compendium: From knowledge to action for a sustainable ocean economy. Cham: Springer International Publishing, (2023), 485–521.

3. P. Karani, P. Failler, Comparative coastal and marine tourism, climate change, and the blue economy in African Large Marine Ecosystems, *Environ. Dev.*, **36** (2020), 100572. https://doi.org/10.1016/j.envdev.2020.100572

4.   E. K. O. Yuwono, R. Illa Maulany, R. A Barkey,   Site suitability evaluation for ecotourism development: A case study in Bulue village, Soppeng district, Indonesia, *J. Sustainability Sci. Manage.*, **16** (2021), 129–140. https://doi.org/10.46754/jssm.2021.01.012

5.   L. Li, B. Wu, A. K. Patwary, How marine tourism promote financial development in sustainable economy: New evidences from South Asia and implications to future tourism students, *Environ. Sci. Pollut. Res.*, **29** (2022), 1155–1172. https://doi.org/10.1007/s11356-024-32868-z

6.   M. Khokhar, Y. Hou, M. A. Rafique, W. Iqbal, Evaluating the social sustainability criteria of supply chain management in manufacturing industries: A role of BWM in MCDM, *Probl. Ekoroz.*, **15** (2020), 185–194. https://doi.org/10.35784/pe.2020.2.18

7.   A. Balmford, J. M. H. Green, M. Anderson, J. Beresford, C. Huang, R. Naidoo, et al., Walk on the wild side: estimating the global magnitude of visits to protected areas, *PLoS Biol.*, **13** (2015), e1002074. https://doi.org/10.1371/journal.pbio.1002074

8.   I. D. Wolf, D. B. Croft, R. J. Green, Nature conservation and nature-based tourism: a paradox?, *Environments*, **6** (2019), 104. https://doi.org/10.3390/environments6090104

9.   Y. F. Leung, J. L. Marion,   Characterizing backcountry camping impacts in Great Smoky Mountains national park, USA,   *J. Environ. Manage.*, **57** (1999), 193–203. https://doi.org/10.1006/jema.1999.0303

10.   G. Shannon, C. L. Larson, S. E. Reed, K. R. Crooks, L. M. Angeloni, Ecological consequences of ecotourism for wildlife populations and communities, *Ecotourism's Promise Peril: Biol. Eval.*, (2017), 29–46. https://doi.org/10.1007/978-3-319-58331-0

11.   A. S. Dye, S. L. Shaw,   A GIS-based spatial decision support system for tourists of Great Smoky Mountains National Park,   *J. Retail. Consum. Serv.*, **14** (2007), 269–278. https://doi.org/10.1016/j.jretconser.2006.07.005

12.   P. Deaden, S. Harron, Alternative tourism and adaptive change, *Ann. Tour. Res.*, **21** (1994), 81–102. https://doi.org/10.1016/0160-7383(94)90006-x

13.   A. Barros, C. Monz, C. Pickering, Is tourism damaging ecosystems in the Andes? Current knowledge and an agenda for future research, *Ambio*, **44** (2015), 82–98. https://doi.org/10.1007/s13280-014-0550-7

14.   C. M. Hall, S. McArthur, *Heritage Management in Australia and New Zealand: The Human Dimension*, Oxford University Press, 1996.

15.   A. Stankus, State of world aquaculture 2020 and regional reviews: FAO webinar series, *FAO Aquacult. Newsl.*, **63** (2021), 17–18. http://www.fao.org/3/cb4850en/cb4850en.pdf

16.   O. Hoegh-Guldberg, J. F. Bruno, The impact of climate change on the world's marine ecosystems, *Science*, **328** (2010), 1523–1528. https://doi.org/10.1126/science.1189930

17.   A. S. Brierley, M. J. Kingsford, Impacts of climate change on marine organisms and ecosystems, *Curr. Biol.*, **19** (2009), R602–R614. https://doi.org/10.1016/j.cub.2009.05.046

18.   M. J. Tegner, P. K. Dayton, Ecosystem effects of fishing in kelp forest communities, *ICES J. Mar. Sci.*, **57** (2000), 579–589. https://doi.org/10.1006/jmsc.2000.0715

19. E. S. Poloczanska, C. J. Brown, W. J. Sydeman, W. Kiessling, D. S. Schoeman, P. J. Moore, Global imprint of climate change on marine life, *Nat. Clim. Change*, **3** (2013), 919–925. https://doi.org/10.1038/nclimate1958

20. M. Dixon, G. Grilli, B. D. Stewart, R. H. Bark, S. Ferrini, The importance of rebuilding trust in fisheries governance in post-Brexit England, *Mar. Policy*, **161** (2024), 106034. https://doi.org/10.1016/j.marpol.2024.106034

21. R. E. Ommer, *Coasts Under Stress: Restructuring and Social-Ecological Health*, McGill-Queen's Press-MQUP, 2007.

22. E. Ostrom, A general framework for analyzing sustainability of social-ecological systems, *Science*, **325** (2009), 419–422. https://doi.org/10.1126/science.1172133

23. L. C. Gammage, A. Jarre, Using structured decision-making tools with marginalised fishers to promote system-based fisheries management approaches in South Africa, *Front. Mar. Sci.*, **7** (2020), 477. https://doi.org/10.3389/fmars.2020.00477

24. S. M. Garcia, *The Ecosystem Approach to Fisheries: Issues, Terminology, Principles, Institutional Foundations, Implementation and Outlook*, Food & Agriculture Organization, **443** (2003).

25. B. Sarkar, N. Bairagi, S. Bhattacharya, An integrated dynamic biological supply chain management with three layers under logistic strategies, *Comput. Ind. Eng.*, **194** (2024), 110387. https://doi.org/10.1016/j.cie.2024.110387

26. *FAO, The State of the World Fisheries and Aquaculture 2020*, Food and Agriculture Organization of the United Nations, Rome, 2020. https://doi.org/10.4060/ca9229en

27. M. Sarkar, A. Majumder, S. Bhattacharya, B. Sarkar, Optimization of energy cycle under a sustainable supply chain management, *RAIRO-Oper. Res.*, **57** (2023), 2177–2196. https://doi.org/10.1051/ro/2023061

28. S. M. Garcia, D. J. Staples, J. Chesson, The FAO guidelines for the development and use of indicators for sustainable development of marine capture fisheries and an australian example of their application, *Ocean Coast. Manage.*, **43** (2000), 537–556. doi:https://doi.org/10.1016/s0964-5691(00)00045-4

29. P. Degnbol, A. Jarre, Review of indicators in fisheries management–a development perspective, *Afr. J. Mar. Sci.*, **26** (2004), 303–326. https://doi.org/10.2989/18142320409504063

30. W. K. Balwan, S. Kour, Wetland-an ecological boon for the environment, *East Afr. Sch. J. Agric. Life Sci.*, **4** (2021), 38–48. https://doi.org/10.36349/easjals.2021.v04i03.001

31. T. Xu, B. Weng, D. Yan, K. Wang, X. Li, W. Bi, et al., Wetlands of international importance: status, threats, and future protection, *Int. J. Environ. Res. Public Health*, **16** (2019), 1818. https://doi.org/10.3390/ijerph16101818

32. E. R. Herbert, P. Boon, A. J. Burgin, S. C. Neubauer, R. B. Franklin, M. Ardón, et al., A global perspective on wetland salinization: ecological consequences of a growing threat to freshwater wetlands, *Ecosphere*, **6** (2015), 1–43. https://doi.org/10.1890/es14-00534.1

33. E. Aceves-Bueno, A. J. Read, M. A. Cisneros-Mata, Illegal fisheries, environmental crime, and the conservation of marine resources, *Conserv. Biol.*, **35** (2021), 1120–1129. https://doi.org/10.1111/cobi.13674

34. L. Cui, G. Li, H. Liao, N. Ouyang, Y. Zhang, Integrated approach based on a regional habitat succession model to assess wetland landscape ecological degradation, *Wetlands*, **35** (2015), 281–289. https://doi.org/10.1007/s13157-014-0617-z

35. E. B. Barbier, Valuing ecosystem services for coastal wetland protection and restoration: progress and challenges, *Resources*, **2** (2013), 213–230. https://doi.org/10.3390/resources2030213

36. R. K. Dowling, D. A. Fennell, The context of ecotourism policy and planning, *Ecotourism Policy Plan.*, (2003), 21–38. https://doi.org/10.1079/9780851996097.0001

37. I. Logar, Sustainable tourism management in Crikvenica, Croatia: An assessment of policy instruments, *Tour. Manage.*, **31** (2010), 125–135. https://doi.org/10.1016/j.tourman.2009.02.005

38. P. W. Wang, J. B. Jia, Tourists' willingness to pay for biodiversity conservation and environment protection, Dalai Lake protected area: Implications for entrance fee and sustainable management, *Ocean Coast. Manage.*, **62** (2012), 24–33. https://doi.org/10.1016/j.ocecoaman.2012.03.001

39. N. Gooroochurn, M. T. Sinclair, Economics of tourism taxation: evidence from Mauritius, *Ann. Tour. Res.*, **32** (2015), 478–498. https://doi.org/10.1007/s13280-014-0550-7

40. R. B. Ditton, S. M. Holland, D. K. Anderson, Recreational fishing as tourism, *Fisheries*, **27** (2002), 17–24. https://doi.org/10.1577/1548-8446(2002)027¡0017:RFAT¿2.0.CO;2

41. S. Morton, D. Pencheon, N. Squires, Sustainable development goals (SDGs), and their implementation: a national global framework for health, development and equity needs a systems approach at every level, *Br. Med. Bull.*, **124** (2017), 81–90. https://doi.org/10.1093/bmb/ldx031

42. E. Y. Mohammed, D. Steinbach, P. Steele, Fiscal reforms for sustainable marine fisheries governance: delivering the SDGs and ensuring no one is left behind, *Mar. Policy*, **93** (2018), 262–270. https://doi.org/10.1016/j.marpol.2017.05.017

43. E. Cohen-Shacham, T. Dayan, R. de Groot, C. Beltrame, F. Guillet, E. Feitelson, Using the ecosystem services concept to analyse stakeholder involvement in wetland management, *Wetlands Ecol. Manage.*, **23** (2015), 241–256. https://doi.org/10.1007/s11273-014-9375-1

44. J. Boncoeur, F. Alban, O. Guyader, O. Thébaud, Fish, fishers, seals and tourists: economic consequences of creating a marine reserve in a multi-species, multi-activity context, *Nat. Resour. Model.*, **15** (2002), 387–411. https://doi.org/10.1111/j.1939-7445.2002.tb00095.x

45. T. K. Kar, B. Ghosh, Sustainability and economic consequences of creating marine protected areas in multispecies multiactivity context, *J. Theor. Biol.*, **318** (2013), 81–90. https://doi.org/10.1016/j.jtbi.2012.11.004

46. A. J. Gallagher, N. Hammerschlag, Global shark currency: the distribution, frequency, and economic value of shark ecotourism, *Curr. Issues Tour.*, **14** (2011), 797–812. https://doi.org/10.1080/13683500.2011.585227

47. J. H. Lee, Y. Iwasa, Tourists and traditional divers in a common fishing ground, *Ecol. Econ.*, **70** (2011), 2350–2360. https://doi.org/10.1016/j.ecolecon.2011.07.013

48. B. Sarkar, S. Bhattacharya, N. Bairagi, An ecological-economic fishery model: Maximizing the societal benefit through an integrated approach of fishing and ecotourism, *Math. Methods Appl. Sci.*, **46** (2023), 14962–14982. https://doi.org/10.1002/mma.9356

49. R. Mchich, N. Charouki, P. Auger, N. Raïssi, O. Ettahiri, Optimal spatial distribution of the fishing effort in a multi fishing zone model, *Ecol. Model.*, **197** (2006), 274–280. https://doi.org/10.1016/j.ecolmodel.2006.03.026

50. J. H. Lee, Y. Iwasa, Ecotourism development and the heterogeneity of tourists, *Theor. Ecol.*, **13** (2020), 371–383. https://doi.org/10.1007/s12080-020-00458-7

51. P. Paul, T. K. Kar, A. Ghorai, Ecotourism and fishing in a common ground of two interacting species, *Ecol. Model.*, **328** (2016), 1–13. https://doi.org/10.1016/j.ecolmodel.2016.02.010

52. S. Marick, S. Bhattacharya, N. Bairagi, Dynamic properties of a reaction–diffusion predator–prey model with nonlinear harvesting: a linear and weakly nonlinear analysis, *Chaos Solit. Fractals*, **175** (2023), 113996. https://doi.org/10.1016/j.chaos.2023.113996

53. S. Bhattacharya, S. Ghorai, N. Bairagi, Dynamic patterns in herding predator–prey system: analyzing the impact of inertial delays and harvesting, *Chaos: Interdiscip. J. Nonlinear Sci.*, **34** (2024). https://doi.org/10.1063/5.0239612

54. J. N. Tsitsiklis, *Problems in Decentralized Decision Making and Computation*, PhD Thesis, Massachusetts Institute of Technology, 1984.

55. J. Zabojnik, Centralized and decentralized decision making in organizations, *J. Labor Econ.*, **20** (2002), 1–22. https://doi.org/10.2139/ssrn.244581

56. S. X. Wu, X. Y. Meng, Hopf bifurcation analysis of a multiple delays stage-structure predator-prey model with refuge and cooperation, *Electron. Res. Arch.*, **33** (2025). https://doi.org/10.3934/era.2025045

57. X. Y. Meng, Y. Q. Wu, Bifurcation and control in a singular phytoplankton-zooplankton-fish model with nonlinear fish harvesting and taxation, *Int. J. Bifurcation Chaos*, **28** (2018), 1850042. https://doi.org/10.1142/s0218127418500426

58. Y. Ma, R. Yang, Bifurcation analysis in a modified Leslie-Gower with nonlocal competition and Beddington-Deangelis functional response, *J. Appl. Anal. Comput.*, **15** (2025), 2152–2184. https://doi.org/10.11948/20240415

59. F. Zhu, R. Yang, Bifurcation in a modified Leslie-Gower model with nonlocal competition and fear effect, *Discrete Contin. Dyn. Syst.-B*, (2025), 2865–2893. https://doi.org/10.3934/dcdsb.2024195

60. S. Chatterjee, D. Kesh, N. Bairagi, How population dynamics change in presence of migratory prey and predator's preference, *Ecol. Complex.*, **11** (2012), 53–66. https://doi.org/10.1016/j.ecocom.2012.03.001

61. S. Ghorai, B. Chakraborty, N. Bairagi Preferential selection of zooplankton and emergence of spatiotemporal patterns in plankton population, *Chaos Soliton Fract.*, **153** (2021), 111471. https://doi.org/10.1016/j.chaos.2021.111471

62. A. Chatterjee, M. A. Abbasi, E. Venturino, J. Zhen, M. Haque, A predator–prey model with prey refuge: under a stochastic and deterministic environment, *Nonlinear Dyn.*, **112** (2024), 13667–13693. https://doi.org/10.1007/s11071-024-09756-9

63. M. L. Rosenzweig, R. H. MacArthur, Graphical representation and stability conditions of predator-prey interactions, *Am. Nat.*, **97** (1963), 209–223. https://doi.org/10.1086/282272

64. C. W. Clark, The optimal management of renewable resources, *Math. Bioeconomics*, **2** (1990), 1–10. https://doi.org/10.2307/2532591

65. S. V. Krishna, P. D. N. Srinivasu, B. Kaymakcalan, Conservation of an ecosystem through optimal taxation, *Bull. Math. Biol.*, **60** (1998), 569–584. https://doi.org/10.1006/bulm.1997.0023

66. A. Moussaoui, P. Auger, A bioeconomic model of a fishery with saturated catch and variable price: Stabilizing effect of marine reserves on fishery dynamics, *Ecol. Complex.*, **45** (2021), 100906. https://doi.org/10.1016/j.ecocom.2020.100906

67. N. Bairagi, S. Bhattacharya, B. Sarkar, Demand-induced regime shift in fishery: a mathematical perspective, *Math. Biosci.*, **361** (2023), 109008. https://doi.org/10.1016/j.mbs.2023.109008

68. B. Ghosh, T. K. Kar, Sustainable use of prey species in a prey–predator system: jointly determined ecological thresholds and economic trade-offs, *Ecol. Model.*, **272** (2014), 49–58. https://doi.org/10.1016/j.ecolmodel.2013.09.013

69. T. T. Agnew, Optimal exploitation of a fishery employing a non-linear harvesting function, *Ecol. Model.*, **6** (1979), 47–57. https://doi.org/10.1016/0304-3800(79)90057-7

70. P. A. Abrams, L. R. Ginzburg, The nature of predation: prey dependent, ratio dependent or neither?, *Trends Ecol. Evol.*, **15** (2000), 337–341. https://doi.org/10.1016/s0169-5347(00)01908-x

71. N. Bairagi, S. Bhattacharya, P. Auger, B. Sarkar, Bioeconomics fishery model in presence of infection: Sustainability and demand-price perspectives, *Appl. Math. Comput.*, **405** (2021), 126225. https://doi.org/10.1016/j.amc.2021.126225

72. B. Sarkar, S. Bhattacharya, N. Bairagi, Dynamic behaviour of a single-species nonlinear fishery model with infection: the role of fishing tax and time-dependent market price, *J. Nonlinear Sci. Appl. (JNSA)*, **16** (2023). https://doi.org/10.22436/jnsa.016.03.02

73. D. Das, T. K. Kar, Marine reserve and its consequences in a predator-prey system for ecotourism and fishing, *Int. J. Math. Model. Numer. Optim.*, **11** (2021), 20–36. https://doi.org/10.1504/ijmmno.2021.111720

74. T. Brochier, P. Auger, D. Thiao, A. Bah, S. Ly, T. Nguyen-Huu, Can overexploited fisheries recover by self-organization? Reallocation of fishing effort as an emergent form of governance, *Mar. Policy*, **95** (2018), 46–56. https://doi.org/10.1016/j.marpol.2018.06.009

75. G. Birkhoff, G. C. Rota, *Ordinary Differential Equations*, John Wiley & Sons, Inc, 1978.

76. B. Petrie, K. T. Frank, N. L. Shackell, W. C. Leggett, Structure and stability in exploited marine fish communities: quantifying critical transitions, *Fish. Oceanogr.*, **18** (2009), 83–101. https://doi.org/10.1111/j.1365-2419.2009.00500.x

77. P. B. Bayley, D. J. Austen, Capture efficiency of a boat electrofisher, *Trans. Am. Fish. Soc.*, **131** (2002), 435–451. https://doi.org/10.1577/1548-8659(2002)131¡0435:ceoabe¿2.0.co;2

78. F. B. Hanson, D. Ryan, Optimal harvesting with both population and price dynamics, *Math. Biosci.*, **148** (1998), 129–146. https://doi.org/10.1016/s0025-5564(97)10011-6

79. N. M. Brites, C. A. Braumann, Fisheries management in random environments: comparison of harvesting policies for the logistic model, *Fish. Res.*, **195** (2017), 238–246. https://doi.org/10.1016/j.fishres.2017.07.016

80. Y. Li, J. S. Muldowney, On Bendixson's criterion, *J. Differ. Equations*, **106** (1993), 27–39. https://doi.org/10.1006/jdeq.1993.1097

81. J. K. Hale, *Ordinary Differential Equations*, Courier Corporation, 2009.

82. M. Fiedler, Additive compound matrices and an inequality for eigenvalues of symmetric stochastic matrices, *Czechoslovak Math. J.*, **24** (1974), 392–402. http://dml.cz/dmlcz/101253

83. J. S. Muldowney, Compound matrices and ordinary differential equations, *Rocky Mt. J. Math.*, **20** (1990). https://doi.org/10.1216/rmjm/1181073047

84. L. Perko, *Differential Equations and Dynamical Systems*, Springer Science & Business Media, 2013.

85. N. R. De Jager, J. J. Rohweder, M. J. Duveneck, Climate change is likely to alter future wolf–moose–forest interactions at Isle Royale National Park, United States, *Front. Ecol. Evol.*, **8** (2020), 543915. https://doi.org/10.3389/fevo.2020.543915

86. S. Kiel, J. L. Goedert, T. L. Huynh, M. Krings, D. Parkinson, R. Romero, et al., Early Oligocene kelp holdfasts and stepwise evolution of the kelp ecosystem in the North Pacific, *Proc. Natl. Acad. Sci.*, **121** (2024), e2317054121. https://doi.org/10.1073/pnas.2317054121

87. S. Barteneva-Vitry, C. Appadoo, S. Plön, Sperm whales—Island specialists, are they on the way to extinction? Systematic literature review in a global context, *Mammal Rev.*, (2025), e70000. https://doi.org/10.1111/mam.70000

88. A. Hurwitz, On the conditions under which an equation has only roots with negative real parts, *Sel. Papers Math. Trends Control Theory*, **65** (1964), 273–284.

Appendix A.

$$\zeta_7 = -D_1{}^2 \alpha_2{}^2 r \xi,$$

$$\zeta_6 = 3 D_1{}^2 \alpha_2{}^2 r \tau \xi - 2 D_1{}^2 \alpha_1 \alpha_2 r \xi,$$

$$\zeta_5 = -r D_1{}^2 \alpha_1{}^2 \xi + 6 r D_1{}^2 \alpha_1 \alpha_2 \tau \xi - 3 r D_1{}^2 \alpha_2{}^2 \tau^2 \xi + 2 D r D_1{}^2 \alpha_2 \xi - K r D_1 \alpha_2 \xi^2 + K \alpha_2 \xi^3,$$

$$\begin{aligned}\zeta_4 = {} & 3 r D_1{}^2 \alpha_1{}^2 \tau \xi - 6 r D_1{}^2 \alpha_1 \alpha_2 \tau^2 \xi + 2 D r D_1{}^2 \alpha_1 \xi + r D_1{}^2 \alpha_2{}^2 \tau^3 \xi - 6 D r D_1{}^2 \alpha_2 \tau \xi - K r D_1 \alpha_1 \xi^2 \\ & + 3 K r D_1 \alpha_2 \tau \xi^2 + D_0 K c r D_1 \alpha_2 \xi + K \alpha_1 \xi^3 - 3 K \alpha_2 \tau \xi^3 - 2 D_0 K c \alpha_2 \xi^2,\end{aligned}$$

$$\begin{aligned}\zeta_3 = {} & K \alpha_2 D_0{}^2 c^2 \xi + K r D_0 D_1 \alpha_1 c \xi - 2 K \alpha_2 r D_0 D_1 c \tau \xi - 2 K D_0 \alpha_1 c \xi^2 + 4 K \alpha_2 D_0 c \tau \xi^2 - r D_1{}^2 D^2 \xi \\ & - 6 r D_1{}^2 D \alpha_1 \tau \xi + 6 \alpha_2 r D_1{}^2 D \tau^2 \xi - 3 r D_1{}^2 \alpha_1{}^2 \tau^2 \xi + 2 \alpha_2 r D_1{}^2 \alpha_1 \tau^3 \xi + K r D_1 D \xi^2 - K D \xi^3 \\ & + 3 K r D_1 \alpha_1 \tau \xi^2 - 3 K \alpha_2 r D_1 \tau^2 \xi^2 - 3 K \alpha_1 \tau \xi^3 + 3 K \alpha_2 \tau^2 \xi^3,\end{aligned}$$

$$\begin{aligned}\zeta_2 = {} & K D_0{}^2 \alpha_1 c^2 \xi - K \alpha_2 D_0{}^2 c^2 \tau \xi - K r D_0 D_1 D c \xi - 2 K r D_0 D_1 \alpha_1 c \tau \xi + K \alpha_2 r D_0 D_1 c \tau^2 \xi + 2 K D_0 D c \xi^2 \\ & + 4 K D_0 \alpha_1 c \tau \xi^2 - 2 K \alpha_2 D_0 c \tau^2 \xi^2 + 3 r D_1{}^2 D^2 \tau \xi + 6 r D_1{}^2 D \alpha_1 \tau^2 \xi - 2 \alpha_2 r D_1{}^2 D \tau^3 \xi + r D_1{}^2 \alpha_1{}^2 \tau^3 \xi \\ & - 3 K r D_1 D \tau \xi^2 - 3 K r D_1 \alpha_1 \tau^2 \xi^2 + K \alpha_2 r D_1 \tau^3 \xi^2 + 3 K D \tau \xi^3 + 3 K \alpha_1 \tau^2 \xi^3 - K \alpha_2 \tau^3 \xi^3,\end{aligned}$$

$$\zeta_1 = 2 K r D_0 D_1 D c \tau \xi - K D_0{}^2 D c^2 \xi - K \alpha_1 D_0{}^2 c^2 \tau \xi + K \alpha_1 r D_0 D_1 c \tau^2 \xi - 4 K D_0 D c \tau \xi^2 - K \alpha_1 \tau^3 \xi^3$$

$$-2\,K\,\alpha_1\,D_0\,c\,\tau^2\,\xi^2 - 3\,r\,D_1^{\,2}\,D^2\,\tau^2\,\xi - 2\,\alpha_1\,r\,D_1^{\,2}\,D\,\tau^3\,\xi + 3\,K\,r\,D_1\,D\,\tau^2\,\xi^2 + K\,\alpha_1\,r\,D_1\,\tau^3\,\xi^2 - 3\,K\,D\,\tau^2\,\xi^3,$$

$$\zeta_0 \;=\; K\,D_0^{\,2}\,D\,c^2\,\tau\,\xi - K\,r\,D_0\,D_1\,D\,c\,\tau^2\,\xi + 2\,K\,D_0\,D\,c\,\tau^2\,\xi^2 + r\,D_1^{\,2}\,D^2\,\tau^3\,\xi - K\,r\,D_1\,D\,\tau^3\,\xi^2 + K\,D\,\tau^3\,\xi^3.$$

Appendix B.

$$C_1 \;=\; K_2^{\,3}N_0^{\,3}\Sigma^3\rho_1\xi,\; C_2 = K_2^{\,2}\,N_0^{\,2}\,\Sigma^2\,\xi\,e^{bg}(3\,\rho_1\,\rho_2 + K_1\,K_2\,\eta_1\,\eta_2),\; C_3 = \rho_1\Sigma^2 N_0^2\xi K_2^3 e^{bg}(K_1\eta_2 + 3\rho_2),$$

$$C_4 \;=\; K_2 N_0\beta\rho_2\xi\,e^{2bg}(3\rho_1\rho_2 + 2\,K_1 K_2\,\eta_1\,\eta_2), C_5 = 2\,K_2^{\,2}\,N_0\,\beta\rho_2\,\xi\,e^{2bg}(3\,\rho_1\rho_2 + K_1 K_2\,\eta_1\,\rho_2 + K_1\,\eta_2\,\rho_1),$$

$$C_6 \;=\; K_2^{\,3}\,N_0\,\beta e^{2bg}(3\,\rho_1\,\rho_2^{\,2}\,\xi + K_1\,\Phi\,\eta_2^{\,2}\,\xi - K_1\,\eta_2^{\,2}\,\kappa\,r + 2\,K_1\,\eta_2\,\rho_1\,\rho_2\,\xi - K_1\,\eta_2^{\,2}\,r\,\tau\,\xi),$$

$$C_7 \;=\; \xi\,e^{3bg}\,\rho_2^{\,2}(\rho_1\rho_2 K_1 K_2\,\eta_1\,\eta_2),\; C_8 = K_2\rho_2^{2}\xi e^{3bg}(3\rho_1\rho_2 + K_1\,\eta_2\,\rho_1 + 2\,K_1\,\eta_1\,\eta_2),$$

$$C_9 \;=\; K_2^{2}\rho_2 e^{3bg} + K_1\,\eta_2^{\,2}\,\kappa\,r - 2\,K_1\,\eta_2\,\rho_1\rho_2\,\xi - K_1\,\Phi\,\eta_2^{\,2}\,\xi - 3\rho_1\rho_2^{\,2}\,\xi - K_1\,K_2\,\eta_1\,\eta_2\,\rho_2\,\xi + K_1\,\eta_2^{\,2}\,r\,\tau\,\xi,$$

$$C_{10} \;=\; K_2^{\,3}\,e^{3bg} + \rho_1\rho_2^{\,3}\,\xi + K_1\,\eta_2\,\rho_1\rho_2^{\,2}\,\xi + D_1 K_1\,\eta_2^{\,3}\,\kappa\,r + K_1\,\Phi\,\eta_2^{\,2}\,\rho_2\,\xi - K_1\,\eta_2^{\,2}\,\kappa\,r\rho_2 - K_1\,\eta_2^{\,2}\,r\,\rho_2\,\tau\,\xi.$$

Appendix C.

Proof of Theorem 2

Proof. (i) Upon analyzing the equilibrium points Σ_1, Σ_2, and Σ_5, it becomes evident that a consistent pattern emerges regarding the eigenvalues of the Jacobian matrix (3.2). Specifically, it is noted that at each of these equilibrium points, one eigenvalue of the Jacobian matrix consistently manifests as $\eta_2 D$, while for the equilibrium point Σ_3, one eigenvalue presents as r. This consistent observation implies the presence of at least one positive eigenvalue across all equilibrium points, leading to their inherent instability.

 (ii) The Jacobian matrix at $\Sigma_4(K, 0, 0, M_3)$ is

$$J(K, 0, 0, M_3) = \left(a_{ij}\right)_{4\times 4},\; 1 \le i, j \le 4, \tag{A.1}$$

where $a_{11} = -r$, $a_{12} = -\frac{K\rho_1}{D'+K}$, $a_{13} = -\frac{\xi}{D_0}$, $a_{22} = \frac{K\rho_2}{D'+K} - \mu$, $a_{33} = -\eta_1\left(c - \frac{\xi(M_3-\tau)}{D_0}\right)$, $a_{43} = -\frac{M_3\,\eta_2\,\xi}{D_0}$, $a_{44} = -M_3\,\eta_2\,(\alpha_1 + 2\,M_3\,\alpha_2)$, and $a_{ij} = 0$ for other values of i, j.

The eigenvalues are $-r(< 0)$, $\frac{K\rho_2}{D'+K} - \mu$, $-\eta_1\left(c - \frac{\xi(M_3-\tau)}{D_0}\right)$ and $-M_3\,\eta_2\,(\alpha_1 + 2\,M_3\,\alpha_2)(< 0)$. So, whenever $K < \frac{\mu D'}{\rho_2-\mu}$ and $\tau > M_3 - \frac{cD_0}{\xi}$ holds simultaneously, the equilibrium point $\Sigma_4(K, 0, 0, M_3)$ becomes locally asymptotically stable.

 (iii) The Jacobian matrix at $\Sigma_6 = (P_6, Q_6, 0, M_3)$ reads

$$J(P_6, Q_6, 0, M_3) = \left(b_{ij}\right)_{4\times 4},\; 1 \le i, j \le 4, \tag{A.2}$$

where $b_{11} = \frac{P_6 Q_6\rho_1}{(D'+P_6)^2} - \frac{P_6 r}{K}$, $b_{12} = -\frac{P_6\rho_1}{D'+P_6}$, $b_{13} = -\frac{\xi}{D_0}$, $b_{21} = \frac{D'\,Q_6\rho_2}{(D'+P_6)^2}$, $b_{22} = -Q_6^{\,a}\,T_0\,\beta\,e^{-bg}a$, $b_{33} = -\eta_1\left(c - \frac{\xi(M_3-\tau)}{D_0}\right)$, $b_{43} = -\frac{M_3\,\eta_2\,\xi}{D_0}$, $b_{44} = -M_3\eta_2(\alpha_1 + 2\,M_3\,\alpha_2)$, and $b_{ij} = 0$ for other values of i, j. One can obtain the eigenvalues as: $\lambda_1 = -M_3\eta_2(\alpha_1 + 2\,M_3\,\alpha_2)$, $\lambda_{2,3} = \{\pm\sqrt{(b_{11} + b_{22})^2 - 4(b_{11}b_{22} - b_{12}b_{21})} + (b_{11} + b_{22})\}/2$ and $\lambda_4 = -\eta_1\left(c - \frac{\xi(M_3-\tau)}{D_0}\right)$. Observe that λ_1 is always negative, $\lambda_{2,3} < 0$ if $b_{11} + b_{22} < 0$ and $b_{11}b_{22} - b_{12}b_{21} > 0$ holds simultaneously,

which implies $b < \min\left\{\frac{1}{g}\log\left(\frac{a\beta T_0 KQ_6^a(D'+P_6)^2}{P_6(KQ_6\rho_1-r(D'+P_6)^2)}\right), \frac{1}{g}\log\left(\frac{a\beta T_0 Q_6^{a-1}(D'+P_6)(\rho_1 KQ_6-r(D'+P_6)^2)}{KD'\rho_1\rho_2}\right)\right\}$, and $\lambda_4 < 0$ implies $\tau > M_3 - \frac{cD_0}{\xi}$.

(iv) The Jacobian matrix at $\Sigma_7(P_7, 0, E_7, M_7)$ becomes

$$J(P_7, 0, E_7, M_7) = \left(c_{ij}\right)_{4\times4}, \ 1 \leq i, j \leq 4, \tag{A.3}$$

where $c_{11} = \frac{D_0 E_7 P_7 \xi}{(D_1 E_7+D_0 P_7)^2} - \frac{P_7 r}{K}$, $c_{12} = -\frac{P_7 \rho_1}{D'+P_7}$, $c_{13} = -\frac{D_0 P_7^2 \xi}{(D_1 E_7+D_0 P_7)^2}$, $c_{22} = \frac{P_7 \rho_2}{D'+P_7} - \mu$, $c_{31} = \frac{D_1 E_7^2 \eta_1 \xi (M_7-\tau)}{(D_1 E_7+D_0 P_7)^2}$, $c_{33} = -\frac{\eta_1\left(c(D_1 E_7+D_0 P_7)^2-D_0 P_7^2\xi(M_7-\tau)\right)}{(D_1 E_7+D_0 P_7)^2}$, $c_{34} = \frac{E_7 P_7 \eta_1 \xi}{D_1 E_7+D_0 P_7}$, $c_{41} = -\frac{D_1 E_7^2 M_7 \eta_2 \xi}{(D_1 E_7+D_0 P_7)^2}$, $c_{43} = -\frac{D_0 M_7 P_7^2 \eta_2 \xi}{(D_1 E_7+D_0 P_7)^2}$, $c_{44} = -M_7 \alpha_1 \eta_2 - 2M_7^2 \alpha_2 \eta_2$, and $c_{ij} = 0$ for other values of i, j.

One eigenvalue is $\frac{P_7 \rho_2}{D'+P_7} - \mu$, and other three are the roots of the equation

$$\bar{\lambda}^3 + a_1\bar{\lambda}^2 + a_2\bar{\lambda} + a_3 = 0, \tag{A.4}$$

where

$$
\begin{aligned}
a_1 &= -c_{11} - c_{33} - c_{44}, \\
a_2 &= c_{11} c_{33} - c_{13} c_{31} + c_{11} c_{44} + c_{33} c_{44} - c_{34} c_{43}, \\
a_3 &= -c_{11} c_{33} c_{44} + c_{11} c_{34} c_{43} + c_{13} c_{31} c_{44} - c_{13} c_{34} c_{41}.
\end{aligned} \tag{A.5}
$$

All roots of Eq (A.4) will be negative or have negative real parts if $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$. Hence, whenever the conditions $\mu > \frac{P_7 \rho_2}{D'+P_7}$, $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$ hold simultaneously, predator free equilibrium $\Sigma_7(P_7, 0, E_7, M_7)$ will become locally asymptotically stable.

(v) The Jacobian matrix at $\Sigma^*(P^*, Q^*, E^*, M^*)$ is

$$J(P^*, Q^*, E^*, M^*) = \left(d_{ij}\right)_{4\times4}, \ 1 \leq i, j \leq 4, \tag{A.6}$$

where $d_{11} = \frac{P^\star Q^\star \rho_1}{(D'+P^\star)^2} - \frac{P^\star r}{K} + \frac{D_0 P^\star E^\star \xi}{(D_1 E^\star+D_0 P^\star)^2}$, $d_{12} = -\frac{P^\star \rho_1}{D'+P^\star}$, $d_{13} = -\frac{D_0 P^{*2}\xi}{(D_1 E^\star+D_0 P^\star)^2}$, $d_{21} = \frac{D' Q^* \rho_2}{(D'+P^\star)^2}$, $d_{22} = -Q^{*a}T_0\beta e^{-bg}a$, $d_{31} = \frac{D_1 E^{*2}\eta_1\xi(M^*-\tau)}{(D_1 E^*+D_0 P^*)^2}$, $d_{33} = -\eta_1\left(c - \frac{\xi D_0 P^{*2}(M^*-\tau)}{(D_1 E^*+D_0 P^*)^2}\right)$, $d_{34} = \frac{E^* P^* \eta_1 \xi}{D_1 E^*+D_0 P^*}$, $d_{41} = -\frac{D_1 E^{*2}M^*\eta_2\xi}{(D_1 E^*+D_0 P^*)^2}$, $d_{43} = -\frac{D_0 M^* P^{*2}\eta_2\xi}{(D_1 E^*+D_0 P^*)^2}$, $d_{44} = -M^*\alpha_1\eta_2 - 2M^{*2}\alpha_2\eta_2$, and $d_{ij} = 0$ for other values of i, j.

The characteristic equation becomes

$$\lambda^4 + a_4\lambda^3 + a_5\lambda^2 + a_6\lambda + a_7 = 0,$$

where

$$
\begin{aligned}
a_4 &= -\text{Tr}(J(P^*, Q^*, E^*, M^*)) = -d_{11} - d_{21} - d_{33} - d_{44}, \\
a_5 &= -d_{12}^2 + d_{11} d_{21} + d_{11} d_{33} - d_{13} d_{31} + d_{21} d_{33} + d_{11} d_{44} + d_{21} d_{44} + d_{33} d_{44} - d_{34} d_{43}, \\
a_6 &= d_{12}^2 d_{33} + d_{12}^2 d_{44} - d_{11} d_{21} d_{33} + d_{13} d_{21} d_{31} - d_{11} d_{21} d_{44} - d_{11} d_{33} d_{44} + d_{11} d_{34} d_{43} \\
&\quad + d_{13} d_{31} d_{44} - d_{13} d_{34} d_{41} - d_{21} d_{33} d_{44} + d_{21} d_{34} d_{43}, \\
a_7 &= d_{12}^2 d_{34} d_{43} - d_{12}^2 d_{33} d_{44} + d_{11} d_{21} d_{33} d_{44} - d_{11} d_{21} d_{34} d_{43} - d_{13} d_{21} d_{31} d_{44} + d_{13} d_{21} d_{34} d_{41}.
\end{aligned}
$$

All roots of (A.4) have negative real parts [88], and the locally asymptotically stability of Σ^* is assured if $a_4 > 0$, $a_6 > 0$, $a_7 > 0$ and $a_4 a_5 a_6 > a_6^2 + a_4^2 a_7$ or $\text{Tr}(J(P^*, Q^*, E^*, M^*)) < 0$, $a_6 > 0$, $a_7 > 0$ and $a_4 a_5 a_6 > a_6^2 + a_4^2 a_7$.

$\square$