Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 21(1): 1650–1671.
DOI: 10.3934/mbe.2024072
Received: 11 September 2023
Revised: 19 November 2023
Accepted: 03 December 2023
Published: 02 January 2024

Research article

Stationary distribution and extinction of a stochastic HIV/AIDS model with nonlinear incidence rate

Helong Liu[1,*] **and Xinyu Song**[2]

[1] School of Mathematics and Statistics, Xinyang College, Xinyang 464000,China
[2] School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China

* **Correspondence:** Email: liuhelong2013@126.com.

Abstract: This paper studies a stochastic HIV/AIDS model with nonlinear incidence rate. In the model, the infection rate coefficient and the natural death rates are affected by white noise, and infected people are affected by an intervention strategy. We derive the conditions of extinction and permanence for the stochastic HIV/AIDS model, that is, if $R_0^s < 1$, HIV/AIDS will die out with probability one and the distribution of the susceptible converges weakly to a boundary distribution; if $R_0^s > 1$, HIV/AIDS will be persistent almost surely and there exists a unique stationary distribution. The conclusions are verified by numerical simulation.

Keywords: stochastic HIV/AIDS model; nonlinear incidence rate; ergodicity; extinction; stationary distribution

1. Introduction

HIV/AIDS is not only a medical problem, but also a serious social problem. Once there is a large-scale epidemic of HIV/AIDS, it seriously affects economic development and social stability. Therefore, it is of theoretical and practical significance to establish a reasonable mathematical model to analyze the epidemic trend of HIV/AIDS. Research has shown that epidemic systems are always affected by environmental variations, which brings some randomness to the birth rates, death rates, and transmission coefficient. Therefore, epidemic systems with stochastic perturbations can provide an additional degree of realism compared to their corresponding deterministic systems [1–4]. From a biological and mathematical perspective, there are different methods for adding white noise disturbances to epidemic models [5–10]. In addition, intervention strategies including the impact of media coverage has a critical influence on the spread of HIV/AIDS, such as the public health department taking some necessary disease prevention measures which can change people's behavior and reduce the effective contact rate between the susceptible and the infected, so as to more effectively prevent and control the spread of diseases. In recent years, a number of mathematical models have been formulated to describe the impact of intervention strategies on the dynamics of infectious diseases [11–20].

Many scholars have already studied HIV/AIDS infection models with nonlinear stochastic perturbations [21–25]. In [21], Zhou et al. considered the complexity of environmental variations in the real world, and also studied a stochastic staged progression HIV/AIDS infection model with third-order perturbations. [22] took second-order perturbation into consideration for realism, and obtained some sufficient conditions about the extinction of stochastic systems. In this paper, we consider that the death rates and the incidence rate are perturbed by white noise [2,6,9], and the incidence rate is a nonlinear functional response, which describes the intervention strategy. The rest of the paper is arranged as follows: Section 2 provides some preliminary results about the system. In Section 3, we derive the condition for the extinction of the system (2.1), which is equivalent to the case $R_0^s < 1$. Section 4 focuses on the condition for permanence, corresponding to the case $R_0^s > 1$. The last section is devoted to some numerical examples as well as discussing the obtained results in this paper.

The HIV/AIDS model is designed by dividing the population into three compartments containing susceptible, infected, and symptomatic individuals. Let $S(t), I(t)$, and $A(t)$ be the number of susceptible, infected, and symptomatic individuals at time t, respectively. During the spread of HIV/AIDS, we assume that the symptomatic individuals (AIDS) have received antiretroviral treatment, and they are less likely to transmit the virus to others. This assumption is based on the following facts: people living with HIV are less likely to transmit the virus to others if they know they have been infected; antiretroviral therapy lowers infectivity, and treatment may be coupled to safer sex education. Additionally, assuming that HIV/AIDS is intervened in the process of transmission, we obtain the following HIV/AIDS model:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - \frac{\beta S(t)I(t)}{a+I^{\alpha}(t)}, \\ \frac{dI(t)}{dt} = \frac{\beta S(t)I(t)}{a+I^{\alpha}(t)} - (\mu+\nu+\gamma_1)I(t), \\ \frac{dA(t)}{dt} = \nu I(t) - (\mu+\gamma_2)A(t) \end{cases} \tag{1.1}$$

where Λ is the recruitment rate of susceptible individuals, μ the natural death rate of the population, β the effective contact rate, ν the rate of transition from infective class to AIDS class, γ_1 the extra-mortality due to infection for infected individuals, γ_2 the extra-mortality due to infection for symptomatic individuals, $\frac{\beta S(t)I(t)}{a+I^{\alpha}(t)}$ the nonlinear incidence rate, $\alpha > 1,\ a > 0$ are positive constant, and similar incidence rates can be found in [9,10,14].

In the following, we analyze the characteristics of the infectious rate $\frac{\beta I(t)}{a+I^{\alpha}(t)}$. The model (1.1) includes an intervention strategy in the transmission of HIV/AIDS. In the early stages of infection, due to the small scale of infection and insufficient understanding of the disease infection, the infectious rate is increasing. As the scale of infection increases, it attracts people's attention, and some intervention strategies are implemented. People might reduce their number of contacts per unit of time, and the infectious rate decreases. For the infectious rate $\frac{\beta I(t)}{a+I^{\alpha}(t)}$, since $(\frac{I}{a+I^{\alpha}})' = \frac{a-(\alpha-1)I^{\alpha}}{(a+I^{\alpha})^2}$, then $(\frac{I}{a+I^{\alpha}})' > 0$ for $I \leq \sqrt[\alpha]{\frac{a}{\alpha-1}}$, and $(\frac{I}{a+I^{\alpha}})' < 0$ for $I \geq \sqrt[\alpha]{\frac{a}{\alpha-1}}$, and we can find that the above intervention characteristics are consistent with the properties of the function $\frac{\beta I(t)}{a+I^{\alpha}(t)}$, where $\frac{\beta I(t)}{a+I^{\alpha}(t)}$ is monotone increasing if $I(t)$ is less than $\sqrt[\alpha]{\frac{a}{\alpha-1}}$, and $\frac{\beta I(t)}{a+I^{\alpha}(t)}$ monotonically decreases if $I(t)$ is greater than $\sqrt[\alpha]{\frac{a}{\alpha-1}}$. The parameter a characterizes the size of the infected population at which intervention strategies are implemented.

The basic reproduction number of model (1.1) is

$$R_0 = \frac{\Lambda\beta}{a\mu(\mu+\nu+\gamma_1)}.$$

There are two equilibrium points in model (1.1): the disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ and the endemic equilibrium $E_0^* = (S_0^*, I_0^*, A_0^*)$, where

$$S_0^* = \frac{(\mu+\nu+\gamma_1)\left(a+(I_0^*)^\alpha\right)}{\beta}, \quad A_0^* = \frac{\nu I_0^*}{\mu+\gamma_2}$$

and I_0^* satisfies

$$\Lambda - \mu\frac{(\mu+\nu+\gamma_1)\left(a+(I_0^*)^\alpha\right)}{\beta} - (\nu+\mu+\gamma_1)I_0^* = 0.$$

Let

$$\rho(I^*) = \Lambda - \mu\frac{(\mu+\nu+\gamma_1)\left(a+(I_0^*)^\alpha\right)}{\beta} - (\nu+\mu+\gamma_1)I_0^*.$$

We have $\rho'(I^*) \le -(\mu+\nu+\gamma_1)$, where $\rho(I^*)$ is monotonic decreasing. If $a < \frac{\beta\Lambda}{\mu(\mu+\nu+\gamma_1)}$, then

$$\rho(0) = \Lambda - a\mu\frac{(\mu+\nu+\gamma_1)}{\beta} = \frac{a\mu(\mu+\nu+\gamma_1)}{\beta}(R_0-1) > 0.$$

So, for $\rho(I^*)$ there exists a unique positive solution I_0^* if $a < \frac{\beta\Lambda}{\mu(\mu+\nu+\gamma_1)}$, i.e., $R_0 > 1$.

For model (1.1), if $R_0 \le 1$, the disease-free equilibrium E_0 is globally asymptotically stable, while if $R_0 > 1$, there is a unique endemic equilibrium E_0^* which is globally asymptotically stable.

The deterministic model (1.1) describes that HIV/AIDS is intervened in the process of transmission. However, environmental variations bring some randomness to the spread and development process of HIV/AIDS. Based on this situation, we consider that the natural death rates are affected by white noises, i.e.,

$$\begin{aligned}
&\mu \hookrightarrow \mu - \sigma_1\dot{B}_1(t),\\
&\mu+\nu+\gamma_1 \hookrightarrow \mu+\nu+\gamma_1 - \sigma_3\dot{B}_3(t),\\
&\mu+\gamma_2 \hookrightarrow \mu+\gamma_2 - \sigma_4\dot{B}_4(t)
\end{aligned}$$

and the deterministic model (1.1) by perturbing dimensionless valid contact coefficient β by $\beta+\sigma_2\dot{B}_2(t)$ to obtain the following stochastic differential equations.

$$\begin{cases}
dS(t) &= \left[\Lambda - \mu S(t) - \dfrac{\beta S(t)I(t)}{a+I^\alpha(t)}\right]dt + \sigma_1 S(t)dB_1(t) - \sigma_2\dfrac{S(t)I(t)}{a+I^\alpha(t)}dB_2(t),\\
dI(t) &= \left[\dfrac{\beta S(t)I(t)}{a+I^\alpha(t)} - (\mu+\nu+\gamma_1)I(t)\right]dt + \sigma_3 I(t)dB_3(t) + \sigma_2\dfrac{S(t)I(t)}{a+I^\alpha(t)}dB_2(t),\\
dA(t) &= \left[\nu I(t) - (\mu+\gamma_2)A(t)\right]dt + \sigma_4 A(t)dB_4(t).
\end{cases} \tag{1.2}$$

Model (1.2) considers both media influence and random factors, which can more accurately describe the development and transmission process of HIV/AIDS.

Throughout this paper, let $\left(\Omega, \tilde{F}, P\right)$ be a complete probability space with a filtration $\{\tilde{F}_t\}_{t\ge0}$ satisfying the usual conditions, $B_i(t)(i = 1,2,3,4)$ represent independent standard Brownian motions defined on this probability space, and $\sigma_i(i = 1,2,3,4)$ represent the intensities of the white noises.

2. Preliminary analyses and results

As the third equation of system (1.2) can be represented by the first two equations, system (1.2) can be written as

$$\begin{cases} dS(t) &= \left[\Lambda - \mu S(t) - \dfrac{\beta S(t)I(t)}{a + I^{\alpha}(t)}\right]dt + \sigma_1 S(t)dB_1(t) - \sigma_2 \dfrac{S(t)I(t)}{a + I^{\alpha}(t)}dB_2(t), \\ dI(t) &= \left[\dfrac{\beta S(t)I(t)}{a + I^{\alpha}(t)} - (\mu + \nu + \gamma_1)I(t)\right]dt + \sigma_3 I(t)dB_3(t) + \sigma_2 \dfrac{S(t)I(t)}{a + I^{\alpha}(t)}dB_2(t). \end{cases} \tag{2.1}$$

$S(t)$ and $I(t)$ in system (2.1) represent the sizes of the susceptible individuals and the infected individuals at time t, respectively. So we define the state space of system (2.1) as

$$\mathbb{R}^2_+ = \{(s, i) : s \geq 0, i \geq 0\},$$

and denote

$$\mathbb{R}^{2,\circ}_+ = \{(s, i) : s > 0, i > 0\}, \mathbb{R}^{2,i}_+ = \{(s, i) : s \geq 0, i > 0\}.$$

Similar to the method in [26,27], one can obtain the global existence and uniqueness of the positive solution of model (2.1), so we omit it here.

Let $(S_u(t), I_u(t))$ be the solution of model (2.1) with initial value $u = (s, i) = (S(0), I(0)) \in \mathbb{R}^2_+$. For further investigation, we give some assertions on $(S_u(t), I_u(t))$.

Lemma 2.1. Assume $\mu > 4\max\{\sigma_1^2, \sigma_3^2\}$, then the following assertions hold:

(i) For any $h > 0$, there exists $C = C(h)$ such that

$$\mathbb{E}(S_u(t) + I_u(t))^8 \leq C, t \geq 0, u \in [0, h] \times [0, h].$$

(ii) For any $\varepsilon > 0, h > 0$, we have $H = H(\varepsilon, h) > 1$ such that

$$\mathbb{P}(0 \leq S_u(t) + I_u(t)) \leq H) \geq 1 - \varepsilon, t \geq 0, u \in [0, h] \times [0, h].$$

Proof. Let $V(t) = N^8(t), N(t) = S_u(t) + I_u(t)$. By Itô's formula, we have

$$\begin{aligned} LV(t)) &= 8N^7(t)(\Lambda - \mu N(t) - (\nu + \gamma_1)I_u(t)) + 28N^6(t)(\sigma_1^2 S_u^2(t) + \sigma_3^2 I_u^2(t)) \\ &\leq 8\Lambda N^7(t) - 8\mu V(t) + 28\max\{\sigma_1^2, \sigma_3^2\}V(t). \end{aligned}$$

Applying the generalized Itô's formula to $e^{\mu t}V(t)$, we obtain

$$\begin{aligned} d(e^{\mu t}V(t)) &= \mu e^{\mu t}V(t)dt + e^{\mu t}(LV(t)dt + \sigma_1 S_u(t)dB_1(t) + \sigma_3 I_u(t)dB_3(t)) \\ &\leq e^{\mu t}(8\Lambda N^7(t) - 7\mu V(t) + 28\max\{\sigma_1^2, \sigma_3^2\}V(t))dt + \sigma_1 S_u(t)dB_1(t) + \sigma_3 I_u(t)dB_3(t) \\ &= e^{\mu t}[8\Lambda N^7(t) - 7(\mu - 4\max\{\sigma_1^2, \sigma_3^2\})V(t)]dt + \sigma_1 S_u(t)dB_1(t) + \sigma_3 I_u(t)dB_3(t). \end{aligned}$$

By $\mu - 4\max\{\sigma_1^2, \sigma_3^2\} > 0$, we have

$$M := \sup_{x \geq 0}\{8\Lambda x^7 - 7(\mu - 4\max\{\sigma_1^2, \sigma_3^2\})x^8\} < \infty.$$

For each $n \in N^+$, define the stopping time

$$\tau_n = \inf\{t \geq 0, N(t) \geq n\}.$$

From the above inequality, we have

$$\begin{aligned}\mathbb{E}(e^{\mu(t\wedge\tau_n)}V(t\wedge\tau_n)) &\leq V(0)+\mathbb{E}(\int_0^{t\wedge\tau_n} Me^{\mu\tau}d\tau)\\ &\leq V(0)+\tfrac{M}{\mu}(e^{\mu t}-1).\end{aligned}$$

By letting $n\to+\infty$, we have

$$e^{\mu t}\mathbb{E}(V(t))\leq V(0)+\frac{M}{\mu}(e^{\mu t}-1).$$

This implies that

$$\begin{aligned}\mathbb{E}(V(t)) &\leq (V(0)-\tfrac{M}{\mu})e^{-\mu t}+\tfrac{M}{\mu}\\ &< V(0)e^{-\mu t}+\tfrac{M}{\mu}=C(h).\end{aligned}$$

Then we complete the proof of item (i).

For any $u\in[0,h]\times[0,h]$, applying Chebyshev's inequality, we have

$$\mathbb{P}((N(t)\geq H)\leq\frac{\mathbb{E}(V(t))}{H^8}\leq\frac{C}{H^8}.$$

Setting $H>(\frac{C}{\varepsilon})^{\frac{1}{8}}$, we obtain the result of item (ii). We complete the proof of the lemma.

Remark 2.1. In Lemma 2.1, $C=C(h)$ depends on h, which depicts the initial value of susceptible $S_u(t)$ and infectious $I_u(t)$. From a biological perspective, the initial value of the population is bounded. So, we may assume that $C(h)$ is bounded, that is to say, $C(h)$ is a constant. It is also based on this viewpoint that, throughout this paper, we restrict that initial value $u=(s,i)$ of the system (2.1) to the region $[0,h^*]\times[0,h^*]$, h^* to be a positive constant.

3. Extinction

In this section, we establish a condition for the extinction of HIV/ADS. If $I_u(t)\equiv 0$, from system (2.1) , we have the following one-dimensional homogeneous Markov process

$$d\varphi(t)=(\Lambda-\mu\varphi(t))dt+\sigma_1\varphi(t)dB_1(t),\varphi(0)=s\geq 0,t\geq 0. \tag{3.1}$$

Applying [28], we can obtain that the process (3.1) has ergodic properties with the invariant density given by

$$f(\varphi)=\left(\frac{2\Lambda}{\sigma_1^2}\right)^{\frac{2\mu}{\sigma_1^2}+1}\Gamma^{-1}\left(\frac{2\mu}{\sigma_1^2}+1\right)\varphi^{-(\frac{2\mu}{\sigma_1^2}+2)}e^{-\frac{2\Lambda}{\sigma_1^2}\frac{1}{\varphi}}, \tag{3.2}$$

and

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t\varphi(\tau)d\tau=\int_0^{+\infty}\varphi f(\varphi)d\varphi$$

where $\Gamma(\tau)$ is the Gamma function.

In the following, we apply the Lyapunov exponent $\limsup_{t\to+\infty}\frac{\ln I_u(t)}{t}$ to show that $I_u(t)$ converges to 0. By the definition of the almost surely exponential stability [29, 30], if $\limsup_{t\to+\infty}\frac{\ln I_u(t)}{t}<c_1(c_1<0)$ holds, then $I_u(t)$ is almost surely exponentially stability, which implies that $I_u(t)$ trends to zero exponentially fast, i.e., $\lim_{t\to+\infty}I_u(t)=0$.

From the second equation of system (2.1), using Itô's formula, we have

$$\begin{aligned}\frac{\ln I_u(t)}{t} &= \frac{\ln I_u(0)}{t} - (\mu+\nu+\gamma_1+\tfrac{1}{2}\sigma_3^2) + \frac{\sigma_3 B_3(t)}{t} + \frac{1}{t}\int_0^t \sigma_2 \frac{S_u(\tau)}{a+I_u^\alpha(\tau)} dB_2(\tau) \\ &+\frac{1}{t}\int_0^t \left(\beta \frac{S_u(\tau)}{a+I_u^\alpha(\tau)} - \frac{1}{2}\sigma_2^2 \frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2}\right) d\tau.\end{aligned} \tag{3.3}$$

If t is sufficiently large, $I_u(t)$ is sufficiently small, and $S_u(t)$ is close to $\varphi(t)$, we have

$$\frac{1}{t}\int_0^t \left(\beta \frac{S_u(\tau)}{a+I_u^\alpha(\tau)} - \frac{1}{2}\sigma_2^2 \frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2}\right) d\tau \approx \frac{1}{t}\int_0^t (\frac{\beta}{a}\varphi(\tau) - \frac{1}{2}\frac{\sigma_2^2}{a^2}\varphi^2(\tau))d\tau. \tag{3.4}$$

From (3.3) and (3.4), by the ergodicity property of φ we have

$$\begin{aligned}\limsup_{t\to+\infty}\frac{\ln I_u(t)}{t} &\le \limsup_{t\to+\infty}\frac{1}{t}\int_0^t \left(\beta \frac{S_u(\tau)}{a+I_u^\alpha(\tau)} - \frac{1}{2}\sigma_2^2 \frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2}\right) d\tau - (\mu+\nu+\gamma_1+\tfrac{1}{2}\sigma_3^2) \\ &\approx \frac{1}{t}\int_0^t \left(\frac{\beta}{a}\varphi(\tau) - \frac{1}{2}\frac{\sigma_2^2}{a^2}\varphi^2(\tau)\right)d\tau - (\mu+\nu+\gamma_1+\tfrac{1}{2}\sigma_3^2) \\ &= \frac{\beta}{a}\int_0^\infty \varphi f(\varphi)d\varphi - \frac{1}{2}\frac{\sigma_2^2}{a^2}\int_0^\infty \varphi^2 f(\varphi)d\varphi - (\mu+\nu+\gamma_1+\tfrac{1}{2}\sigma_3^2) \\ &= (c_2 + \frac{1}{2}\frac{\sigma_2^2}{a^2}\int_0^\infty \varphi^2 f(\varphi)d\varphi)(R_0^s - 1)\end{aligned} \tag{3.5}$$

where $R_0^s = \frac{\frac{\beta}{a}\int_0^\infty \varphi f(\varphi)d\varphi}{c_2+\frac{1}{2}\frac{\sigma_2^2}{a^2}\int_0^\infty \varphi^2 f(\varphi)d\varphi}, c_2 = \mu+\nu+\gamma_1+\frac{1}{2}\sigma_3^2$.

If $R_0^s < 1$, then $\limsup_{t\to+\infty}\frac{\ln I_u(t)}{t} < c_1 < 0$, and it follows that $\lim_{t\to+\infty} I_u(t) = 0$.

From the above arguments, we need to show that, when $I_u(t)$ is small, $S_u(t)$ is close to $\varphi(t)$, and (3.5) holds. To prove this assertion, we first use the methods mentioned in [2,19,29] to give the following lemmas.

Lemma 3.1. Assume that $\mu > 4\max\{\sigma_1^2, \sigma_3^2\}$ holds. For any $T, h > 1, \varepsilon, \sigma > 0$, there is a $\delta = \delta(T, h, \varepsilon, \sigma) > 0$, such that

$$P(\tau^\sigma \ge T) \ge 1-\varepsilon, u \in [0,h]\times(0,\delta],$$

where τ^σ is the stopping time

$$\tau^\sigma = \inf\{t\ge 0, I_u(t)\ge\sigma\}.$$

Proof. By the exponential martingale inequality [29], we have $P(\Omega_1) \ge 1-\frac{\varepsilon}{2}$, where

$$\Omega_1 = \{\sigma_3 B_3(t) + \int_0^t \sigma_2 \frac{S_u(\tau)}{a+I_u^\alpha(\tau)}dB_2(\tau) \le \frac{1}{2}\sigma_3^2 t + \frac{1}{2}\int_0^t \sigma_2^2 \frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2}d\tau + 2\ln\frac{4}{\varepsilon}, t\ge 0\}.$$

In view of Lemma 2.1, there exists $H = H(h,\varepsilon)$ such that $P(\Omega_2) \ge 1-\frac{\varepsilon}{2}$, where

$$\Omega_2 = \{S_u(t) \le H, t\in[0,T], u\in[0,h]\times[0,h]\}.$$

For $\omega \in \Omega_1 \bigcap \Omega_2$, we obtain

$$\begin{aligned}\ln I_u(t) &= \ln i - c_2 t + \int_0^t \sigma_2\frac{S_u(\tau)}{a+I_u^\alpha(\tau)}dB_2(\tau) + \int_0^t \sigma_3 dB_3(\tau) + \int_0^t (\beta\frac{S_u(\tau)}{a+I_u^\alpha(\tau)} \\ &-\frac{1}{2}\sigma_2^2\frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2})d\tau \\ &\le \ln i + \beta\frac{H}{a}t + 2\ln\frac{4}{\varepsilon} \\ &= \ln\frac{16ie^{\tilde{M}t}}{\varepsilon^2}\end{aligned}$$

where $\tilde{M} = \beta\frac{H}{a}$.
Let $\delta = \frac{\varepsilon^2\sigma}{16}e^{-\tilde{M}T} < h$. For all $u \in [0, h] \times (0, \delta]$, then $I_u(t) < \sigma, t \in [0, T]$. The proof is complete.
Lemma 3.2. Assume that $\mu > 4\max\{\sigma_1^2, \sigma_3^2\}$ holds. For any $T, h > 1, \varepsilon, \eta > 0$, there is $\sigma > 0$ such that, for $u \in [0, h] \times (0, \sigma]$,

$$P(|\varphi(t) - S_u(t)| < \eta, t \in [0, T \wedge \tau^\sigma]) \geq 1 - \varepsilon.$$

Proof. Let $\varphi(t)$ be the solution of Eq (3.1). An argument similar to Lemma 2.1, for any $\varepsilon, \bar{h} > 1$, shows that there is $H > 1$, such that

$$P(0 \leq \varphi(t) \leq H) \geq 1 - \frac{\varepsilon}{4}, s \in [0, \bar{h}]. \tag{3.6}$$

From Lemma 2.1 and (3.6), for any ε we have that there exist H and $1 < h \leq \bar{h}$, such that

$$P(\varphi(t) \vee S_u(t) \leq H) \geq 1 - \frac{\varepsilon}{4}, t \in [0, T], u \in [0, h] \times (0, h]. \tag{3.7}$$

By Itô·s formula, from systems (2.1) and (3.1), we obtain

$$\begin{aligned} |\varphi(t) - S_u(t)| &\leq \mu\int_0^t |\varphi(\tau) - S_u(\tau)|d\tau + \beta\int_0^t \frac{S_u(\tau)I_u(\tau)}{a+I_u^\alpha(\tau)}d\tau + \sigma_1|\int_0^t(\varphi(\tau) - S_u(\tau))dB_1(\tau)| \\ &+\sigma_2|\int_0^t \frac{S_u(\tau)I_u(\tau)}{a+I_u^\alpha(\tau)})dB_2(\tau)|. \end{aligned}$$

Noticing that $(\sum_{i=1}^4 a_i)^2 \leq 16\sum_{i=1}^4 a_i^2$ holds, then

$$\begin{aligned} (\varphi(t) - S_u(t))^2 &\leq 16[\mu^2(\int_0^t |\varphi(\tau) - S_u(\tau)|d\tau)^2 + \beta^2(\int_0^t \frac{S_u(\tau)I_u(\tau)}{a+I_u^\alpha(\tau)}d\tau)^2 \\ &+\sigma_1^2|\int_0^t(\varphi(\tau) - S_u(\tau))dB_1(\tau)|^2 + \sigma_2^2|\int_0^t \frac{S_u(\tau)I_u(\tau)}{a+I_u^\alpha(\tau)})dB_2(\tau)|^2]. \end{aligned} \tag{3.8}$$

Let $\tau_u^H = \{t \geq 0, \varphi(t) \vee S_u(t) \geq H\}, \rho = \tau^\sigma \wedge \tau_u^H$. From (3.8), we obtain

$$\begin{aligned} E[\sup_{\tau\leq t}(\varphi(\tau\wedge\rho) - S_u(\tau\wedge\rho))^2] &\leq 16[\mu^2 E(\int_0^{t\wedge\rho} |\varphi(\tau) - S_u(\tau)|d\tau)^2 + \beta^2 E(\int_0^{t\wedge\rho} \frac{S_u(\tau)I_u(\tau)}{a+I_u^\alpha(\tau)}d\tau)^2 \\ &+\sigma_1^2 E\sup_{\bar{\tau}\leq t}|\int_0^{\bar{\tau}\wedge\rho}(\varphi(\tau) - S_u(\tau))dB_1(\tau)|^2 \\ &+\sigma_2^2 E\sup_{\bar{\tau}\leq t}|\int_0^{\bar{\tau}\wedge\rho} \frac{S_u(\tau)I_u(\tau)}{a+I_u^\alpha(\tau)}dB_2(\tau)|^2]. \end{aligned} \tag{3.9}$$

For $t \in [0, T]$, applying Hölder's inequality we have

$$\beta^2 E(\int_0^{t\wedge\rho} \frac{S_u(\tau)I_u(\tau)}{a + I_u^\alpha(\tau)}d\tau)^2 \leq \frac{T\beta^2H^2}{a^2}\sigma^2.$$

Using the Burkholder-Davis-Gundy inequality [29], we obtain

$$\sigma_1^2 E[\sup_{\bar{\tau}\leq t}|\int_0^{\bar{\tau}\wedge\rho}(\varphi(\tau) - S_u(\tau))dB_1(\tau)|^2] \leq 4\sigma_1^2 E[\int_0^{t\wedge\rho}(\varphi(\tau) - S_u(\tau))^2 d\tau],$$

$$\sigma_2^2 E[\sup_{\bar{\tau}\leq t}|\int_0^{\bar{\tau}\wedge\rho} \frac{S_u(\tau)I_u(\tau)}{a + I_u^\alpha(\tau)}dB_2(\tau)|^2] \leq 4\sigma_2^2\frac{TH^2}{a^2}\sigma^2.$$

From (3.9) and the above estimates, we have

$$\begin{aligned} E[\sup_{\tau\leq t}(\varphi(\tau\wedge\rho) - S_u(\tau\wedge\rho))^2] &\leq c_3\sigma^2 + c_4 E[\int_0^{t\wedge\rho}(\varphi(\tau) - S_u(\tau))^2 d\tau] \\ &\leq c_3\sigma^2 + c_4\int_0^t E[\sup_{\tau\leq t}(\varphi(\tau\wedge\rho) - S_u(\tau\wedge\rho))^2]d\tau \end{aligned} \tag{3.10}$$

where $c_3 = 16(\beta^2 + 4\sigma_2^2)\frac{TH^2}{a^2}, c_4 = 16(\mu^2 + 4\sigma_1^2)$.
Applying Gronwall's inequality, we have

$$\mathrm{E}[\sup_{\tau\le T}(\varphi(\tau\wedge\rho) - S_u(\tau\wedge\rho))^2] \le c_3 e^{c_4 T}\sigma^2.$$

By Chebyshev's inequality, we have

$$\mathrm{P}[\sup_{\tau\le T}(\varphi(\tau\wedge\rho) - S_u(\tau\wedge\rho))^2 \ge \eta^2] \le \frac{c_3 e^{c_4 T}}{\eta^2}\sigma^2 < \frac{\varepsilon}{2}. \tag{3.11}$$

In addition, $\{\sup_{t\le T}(\varphi(t)\vee S_u(t)) \le H, t\in[0,T]\} \subset \{t\wedge\rho = t\wedge\tau^\sigma, t\in[0,T]\}$. As a result,

$$\mathrm{P}\{t\wedge\rho = t\wedge\tau^\sigma, t\in[0,T]\} \ge \mathrm{P}\{\sup_{t\le T}(\varphi(t)\vee S_u(t)) \le H, t\in[0,T]\} \ge 1 - \frac{\varepsilon}{2}. \tag{3.12}$$

Combining (3.11) and (3.12), we complete the proof.
Lemma 3.3. Assume that $\mu > 4\max\{\sigma_1^2, \sigma_3^2\}$ and $R_0^s < 1$ hold. For any $h > 1, \varepsilon > 0$, there is a $\bar\delta = \bar\delta(h,\varepsilon) > 0$ such that

$$\mathrm{P}(\limsup_{t\to+\infty}|\frac{\ln I_u(t)}{t} - \lambda| \le \varepsilon) \ge 1 - 7\varepsilon, u\in[0,h]\times(0,\bar\delta],$$

where $\lambda = (\frac{1}{2}\frac{\sigma_2^2}{a^2}\int_0^\infty \varphi^2 f(\varphi)d\varphi + c_2)(R_0^s - 1)$.
Proof. From (3.5) and the ergodicity of φ, we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t(\frac{\beta}{a}\varphi(\tau) - \frac{1}{2}\frac{\sigma_2^2}{a^2}\varphi^2(\tau))d\tau - c_2 = \lambda,$$

and then there exists $T_1 = T_1(\varepsilon)$ and $\mathrm{P}(\Omega_3) \ge 1 - \varepsilon$, where

$$\Omega_3 = \{\frac{1}{t}\int_0^t(\frac{\beta}{a}\varphi(\tau) - \frac{1}{2}\frac{\sigma_2^2}{a^2}\varphi^2(\tau))d\tau - c_2 \le \lambda + \varepsilon, t \ge T_1\}.$$

Since $\lim_{t\to+\infty}\frac{B_3(t)}{t} = 0$ a.s., there is a $T_2 = T_2(\varepsilon) > 0$, such that $\mathrm{P}(\Omega_4) \ge 1 - \varepsilon$, where

$$\Omega_4 = \{\frac{\sigma_3 B_3(t)}{t} \le \varepsilon, t \ge T_2\}.$$

From Lemma 2.1 and (3.7), there is an $H, h > 1$, such that $\mathrm{P}(\Omega_5) \ge 1 - \varepsilon$,

$$\Omega_5 = \{0 \le \varphi(t), S_u(t), I_u(t) \le H, u\in[0,h]\times(0,h], t\in[0,T]\} \tag{3.13}$$

where $T = T_1 \vee T_2$.
Supposing $|S_u(t) - \varphi(t)| < \eta$ and $|I_u(t)| < \eta$, from (3.13) we obtain

$$\beta|\frac{S_u(t)}{a + I_u^\alpha(t)} - \frac{1}{a}\varphi(t)| \le b_1\eta \tag{3.14}$$

where $b_1 = \frac{\beta(H+a)}{a^2}$.

$$\frac{\sigma_2^2}{2}|\frac{S_u^2(t)}{(a + I_u^\alpha(t))^2} - \frac{1}{a^2}\varphi^2(t)| \le b_2\eta \tag{3.15}$$

where $b_2 = \frac{\sigma_2^2 H}{2a^2}(2 + \frac{1+2a}{a^2}H)$. From (3.14) and (3.15), there is an $\eta < \min\{1, \frac{\varepsilon}{2(b_1+b_2)}\}$ such that

$$\beta\Big|\frac{S_u(t)}{a + I_u^\alpha(t)} - \frac{1}{a}\varphi(t)\Big| + \frac{\sigma_2^2}{2}\Big|\frac{S_u^2(t)}{(a + I_u^\alpha(t))^2} - \frac{1}{a^2}\varphi^2(t)\Big| < \frac{\varepsilon}{2}. \tag{3.16}$$

On the other hand, by the exponential martingale inequality we have $P(\Omega_6) \geq 1 - \varepsilon$, where

$$\Omega_6 = \{\int_0^t \sigma_2 \frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau) \leq \frac{k^2}{\varepsilon} \ln\frac{1}{\varepsilon} + \frac{\varepsilon}{2k^2}\int_0^t \sigma_2^2 \frac{S_u^2(\tau)}{(a + I_u^\alpha(\tau))^2} d\tau, t \geq 0\},$$

$k = \frac{\sigma_2 H}{a}$.

In view of [2], there exists a $c_\varepsilon > 0$, such that $P(\Omega_7) \geq 1 - \varepsilon$, where

$$\begin{aligned}\Omega_7 = &\{|\sigma_1 B_1(t)| \leq q_\varepsilon(t), t \geq 0\} \cap \\ &\{\Big|\int_T^t e^{c_5\tau - \sigma_1 B_1(\tau)} \frac{I_u(\tau)S_u(\tau)}{a + I_u^\alpha(\tau)} dB_3(\tau)\Big| \leq c_\varepsilon \sqrt{n(t)(|\ln n(t)| + 1)}, t \geq T\},\end{aligned}$$

$q_\varepsilon(t) = c_\varepsilon\sqrt{t(|\ln t| + 1)}, n(t) = \int_T^t e^{2c_5\tau + 2c_\varepsilon\sqrt{\tau(|\ln\tau|+1)}} \frac{I_u^2(\tau)S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2} d\tau, c_5 = \mu + \frac{\sigma_1^2}{2}$. It is clear that

$$\Phi_1(\varepsilon) := \sup_{t\geq 0} e^{-c_5 t + q_\varepsilon(t) + c_5 T + q_\varepsilon(T)}\Big(\frac{(\beta + \sigma_2)H}{a} + \frac{k^2}{\varepsilon}\ln\frac{1}{\varepsilon} + \frac{\varepsilon}{2}\Big)T < \infty. \tag{3.17}$$

From Lemma 3.2, there exists $\eta_1 > 0$ satisfying

$$\eta_1 < \min\{\eta, \frac{\eta}{2\Phi_1(\varepsilon)}\}$$

such that $u \in [0, h] \times (0, \eta_1]$ and $P(\Omega_8) \geq 1 - \varepsilon$, where $\Omega_8 = \{|S_u(t) - \varphi(t)| < \eta, t \leq T \wedge \tau^{\eta_1}\}$, $\tau^{\eta_1} = \inf\{t \geq 0, I_u(t) > \eta_1\}$.

Let $\tau_{\varphi,S}^\eta = \inf\{t : |S_u(t) - \varphi(t)| > \eta\}$ and $\bar\rho = \tau^{\eta_1} \wedge \tau_{\varphi,S}^\eta$. From Lemma 3.1, there is a $0 < \delta < \eta_1$ such that $u \in [0, h] \times (0, \delta], P(\Omega_9) \geq 1 - \varepsilon$, where

$$\Omega_9 = \{\tau^{\eta_1} \geq T\}.$$

Therefore, for all $u \in [0, h] \times (0, \delta], \omega \in \cap_{i=3}^9 \Omega_i$, we have $\bar\rho \geq T$.

For $u \in [0, h] \times (0, \delta], t \in [T, \bar\rho], \omega \in \cap_{i=3}^9 \Omega_i$, from the system (2.1), we have

$$\begin{aligned}\ln I_u(t) &= \ln i + \int_0^t \Big(\beta\frac{S_u(\tau)}{a+I_u^\alpha(\tau)} - \frac{1}{2}\sigma_2^2\frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2} - c_2\Big)d\tau \\ &\quad+\sigma_3 B_3(t) + \int_0^t \sigma_2\frac{S_u(\tau)}{a+I_u^\alpha(\tau)}dB_2(\tau) \\ &\leq \ln i + \int_0^t \Big(\frac{\beta}{a}\varphi(\tau) - \frac{\sigma_2^2}{2a^2}\varphi^2(\tau) - c_2\Big)d\tau \\ &\quad+\beta\int_0^t \Big|\frac{S_u(\tau)}{a+I_u^\alpha(\tau)} - \frac{1}{a}\varphi(\tau)\Big|d\tau + \frac{\sigma_2^2}{2}\int_0^t \Big|\frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2} - \frac{1}{a^2}\varphi^2(\tau)\Big|d\tau \\ &\quad+\sigma_3 B_3(t) + \int_0^t \sigma_2\frac{S_u(\tau)}{a+I_u^\alpha(\tau)}dB_2(\tau) \\ &\leq \ln i + \frac{k^2}{\varepsilon}\ln\frac{1}{\varepsilon} + (\lambda + 3\varepsilon)t.\end{aligned} \tag{3.18}$$

From (3.18), we have

$$I_u(t) \leq i\varepsilon^{-\frac{k^2}{\varepsilon}} e^{(\lambda+3\varepsilon)t}, t \in [T, \bar\rho]. \tag{3.19}$$

In the following, we estimate $|\varphi(t) - S_u(t)|$. From systems (2.1) and (3.1), we obtain

$$\begin{aligned} d(\varphi(t) - S_u(t)) &= \mu(\varphi(t) - S_u(t))dt + \beta\tfrac{S_u(t)I_u(t)}{a+I_u^\alpha(t)}dt + \sigma_1(\varphi(t) - S_u(t))dB_1(t) \\ &+\sigma_2\tfrac{S_u(t)I_u(t)}{a+I_u^\alpha(t)}dB_2(t). \end{aligned}$$

It follows that

$$\begin{aligned} \varphi(t) - S_u(t) &= e^{-c_5 t+\sigma_1 B_1(t)}\int_0^t e^{c_5\tau-\sigma_1 B_1(\tau)}(\beta\tfrac{S_u(\tau)I_u(\tau)}{a+I_u^\alpha(\tau)}d\tau + \sigma_2\tfrac{S_u(\tau)I_u(\tau)}{a+I_u^\alpha(\tau)}dB_2(\tau)) \\ &= A_1(t) + A_2(t) \end{aligned} \tag{3.20}$$

where

$$A_1(t) = \beta e^{-c_5 t+\sigma_1 B_1(t)}\int_0^t e^{c_5\tau-\sigma_1 B_1(\tau)}\frac{S_u(\tau)I_u(\tau)}{a + I_u^\alpha(\tau)}d\tau,$$
$$A_2(t) = \sigma_2 e^{-c_5 t+\sigma_1 B_1(t)}\int_0^t e^{c_5\tau-\sigma_1 B_1(\tau)}\frac{S_u(\tau)I_u(\tau)}{a + I_u^\alpha(\tau)}dB_2(\tau).$$

For all $u \in [0, h] \times (0, \delta], \omega \in \cap_{i=3}^{9}\Omega_i, t \geq T$, from (3.19) we have

$$\begin{aligned} A_1(t\wedge\bar\rho) &= \beta e^{-c_5(t\wedge\bar\rho)+\sigma_1 B_1(t\wedge\bar\rho)}(\int_0^T e^{c_5\tau-\sigma_1 B_1(\tau)}\frac{S_u(\tau)I_u(\tau)}{a + I_u^\alpha(\tau)}d\tau + \int_T^{t\wedge\bar\rho} e^{c_5\tau-\sigma_1 B_1(\tau)}\frac{S_u(\tau)I_u(\tau)}{a + I_u^\alpha(\tau)}d\tau) \\ &\leq \frac{\beta H}{a}e^{-c_5(t\wedge\bar\rho)+q_\varepsilon(t\wedge\bar\rho)}(\eta_1\int_0^T e^{c_5\tau+q_\varepsilon(\tau)}d\tau + i\varepsilon^{-\frac{k^2}{\varepsilon}}\int_T^{t\wedge\bar\rho} e^{c_5\tau+q_\varepsilon(\tau)}e^{(\lambda+3\varepsilon)\tau}d\tau) \\ &\leq \Phi_1(\varepsilon)\eta_1 + \Phi_2(\varepsilon)i \end{aligned} \tag{3.21}$$

where

$$\Phi_2(\varepsilon) := \frac{\beta H}{a}\varepsilon^{-\frac{k^2}{\varepsilon}}\sup_{t\geq0} e^{-c_5 t+q_\varepsilon(t)}\int_T^t e^{c_5\tau+q_\varepsilon(\tau)}e^{(\lambda+3\varepsilon)\tau}d\tau < +\infty.$$

Similarly, for $u \in [0, h] \times (0, \delta], \omega \in \cap_{i=3}^{9}\Omega_i, t \geq T$, we have

$$\begin{aligned} A_2(t\wedge\bar\rho) &\leq \eta_1 e^{-c_5(t\wedge\bar\rho)+\sigma_1 B_1(t\wedge\bar\rho)}e^{c_5 T+q_\varepsilon(T)}|\int_0^T \sigma_2\frac{S_u(\tau)I_u(\tau)}{a + I_u^\alpha(\tau)}dB_2(\tau)| \\ &+ e^{-c_5(t\wedge\bar\rho)+\sigma_1 B_1(t\wedge\bar\rho)}|\int_T^{t\wedge\bar\rho} \sigma_2 e^{c_5\tau-\sigma_1 B_1(\tau)}\frac{S_u(\tau)I_u(\tau)}{a + I_u^\alpha(\tau)}dB_2(\tau)| \\ &\leq \Phi_1(\varepsilon)\eta_1 + e^{-c_5(t\wedge\bar\rho)+q_\varepsilon(t\wedge\bar\rho)}\int_T^{t\wedge\bar\rho} c_\varepsilon\sqrt{n(\tau)(|\ln n(\tau)| + 1)}d\tau \end{aligned} \tag{3.22}$$

where

$$n(t) = \int_T^t \sigma_2^2 e^{2c_5\tau+2q_\varepsilon(\tau)}\frac{I_u^2(\tau)S_u^2(\tau)}{(a + I_u^\alpha(\tau))^2}d\tau.$$

From (3.19), we have $n(t) \leq i^2 m(t)$, $m(t) = k^2\varepsilon^{-\frac{2k^2}{\varepsilon}}\int_T^t e^{2c_5\tau+2q_\varepsilon(\tau)}e^{2(\lambda+3\varepsilon)\tau}d\tau$. Then,

$$\Phi_3(\varepsilon) := \sup_{t\geq0} e^{-c_5 t+q_\varepsilon(t)}\int_T^t c_\varepsilon\sqrt{m(\tau)(|\ln m(\tau)| + 1)}d\tau < \infty$$

and

$$A_2(t \wedge \bar{\rho}) \le \eta_1 \Phi_1(\varepsilon) + i\Phi_3(\varepsilon). \tag{3.23}$$

Let $\bar{\delta} \in (0, \delta)$, satisfying

$$2\bar{\delta}(\Phi_2(\varepsilon) + \Phi_3(\varepsilon)) < \frac{\eta}{2}.$$

From (3.20)–(3.23), for $u \in [0, h] \times (0, \bar{\delta}], \omega \in \cap_{i=3}^{9} \Omega_i, t \ge T$, we have

$$|\varphi(t \wedge \bar{\rho}) - S_u(t \wedge \bar{\rho})| \le 2\eta_1 \Phi_1(\varepsilon) + (\Phi_2(\varepsilon) + \Phi_3(\varepsilon))i < \frac{\eta}{2} + \frac{\eta}{2} = \eta. \tag{3.24}$$

It follows that $t \wedge \bar{\rho} \le \tau_{\varphi,s}^{\eta}, t \ge T$ holds. Therefore, for all $\omega \in \cap_{i=3}^{9} \Omega_i, \bar{\rho} \le \tau_{\varphi,s}^{\eta}$, the equality only occurs when $\bar{\rho} = \tau_{\varphi,s}^{\eta} = \infty$. As a consequence, $\omega \in \cap_{i=3}^{9} \Omega_i \subset \{\tau^{\eta_1} \le \tau_{\varphi,s}^{\eta}\}$. From (3.19) we obtain that, for $u \in [0, h] \times (0, \bar{\delta}], \omega \in \cap_{i=3}^{9} \Omega_i, t \ge T$ and

$$I_u(t \wedge \tau^{\eta_1}) \le \bar{\delta} \varepsilon^{-\frac{k^2}{\varepsilon}} e^{(\lambda + 5\varepsilon)T} < \eta_1.$$

This implies that $t \wedge \tau^{\eta_1} < \tau^{\eta_1}, t \ge T$ or $\tau^{\eta_1} = \infty$ for $u \in [0, h] \times (0, \bar{\delta}], \omega \in \cap_{i=3}^{9} \Omega_i$.

$$\begin{aligned}
\limsup_{t \to \infty} \left|\tfrac{\ln I_u(t)}{t} - \lambda\right| &\le \limsup_{t \to \infty} \tfrac{\ln I_u(0)}{t} + \limsup_{t \to \infty} \left|\tfrac{\sigma_3 B_3(t)}{t}\right| \\
&+ \limsup_{t \to \infty} \left|\tfrac{1}{t} \int_0^t \sigma_2 \tfrac{S_u(\tau)}{a + I_u^{\alpha}(\tau)} dB_2(\tau)\right| \\
&+ \limsup_{t \to \infty} \tfrac{\beta}{t} \int_0^t \left|\tfrac{S_u(\tau)}{a + I_u^{\alpha}(\tau)} - \tfrac{\varphi(\tau)}{a}\right| d\tau \\
&+ \tfrac{1}{2}\sigma_2^2 \limsup_{t \to \infty} \int_0^t \left|\tfrac{S_u^2(\tau)}{(a + I_u^{\alpha}(\tau))^2} - \tfrac{\varphi^2(\tau)}{a^2}\right| d\tau \\
&\le \tfrac{\varepsilon}{2} + (b_1 + b_2)\eta < \varepsilon.
\end{aligned}$$

The proof is completed by noting that $P(\cap_{i=3}^{9} \Omega_i) \ge 1 - 7\varepsilon$.

In the following, we give the property of the stochastic extinction of HIV/AIDS.

Theorem 3.1. Assume that $\mu > 4 \max\{\sigma_1^2, \sigma_3^2\}$ holds. If $R_0^s < 1$, then the solution $(S_u(t), I_u(t))$ of system (2.1) satisfies

$$P\left\{\lim_{t \to +\infty} \frac{\ln I_u(t)}{t} = (\frac{1}{2}\frac{\sigma_2^2}{a^2} \int_0^{\infty} \varphi^2 f(\varphi) d\varphi + c_2)(R_0^s - 1)\right\} = 1, u \in [0, h^*] \times (0, h^*] \subset R_+^{2,i}$$

and the distribution of $S_u(t)$ converges weakly to φ, which is the unique stationary distribution of (3.1) with density (3.2).

Proof. Let $u_0 \in [0, h^*] \times (0, h^*]$ and $\varepsilon > 0$. From Lemma 1.1, there is a constant $H > 1$ such that

$$\lim_{t \to +\infty} P\{(S_{u_0}(t), I_{u_0}(t)) \in A_0\} \ge 1 - \varepsilon \tag{3.25}$$

where $A_0 = \{[0, H] \times (0, H]\}$.

From Lemma 3.3, we have that the process $(S_{u_0}(t), I_{u_0}(t))$ is not recurrent in the invariant set $R_+^{2,i}$. Meanwhile, the diffusion Eq (2.1) is non-degenerate, its solution process must be transient [26]. For ε above, let $\bar{\delta} > 0$ satisfy Lemma 3.3, and by the transience of $(S_{u_0}(t), I_{u_0}(t))$, we have

$$\lim_{t \to +\infty} P\{(S_{u_0}(t), I_{u_0}(t)) \in A_1\} = 0 \tag{3.26}$$

where $A_1 = \{[0, H] \times [\bar{\delta}, H]\}$.
Since $A_0 - A_1 \subset [0, H] \times (0, \bar{\delta}] = A_2$, from (3.25) and (3.26) we have

$$\lim_{t\to+\infty} \mathrm{P}\{(S_{u_0}(t), I_{u_0}(t)) \in A_2\} \geq 1 - \varepsilon.$$

Therefore, there exists a $T_0 > 0$ such that

$$\mathrm{P}\{(S_{u_0}(T_0), I_{u_0}(T_0)) \in A_2\} \geq 1 - 2\varepsilon. \tag{3.27}$$

By the Markov property of stochastic process $(S_{u_0}(t), I_{u_0}(t))$ and the results in Lemma 3.3, from (3.27) we obtain

$$\mathrm{P}(\limsup_{t\to+\infty} |\frac{\ln I_{u_0}(t)}{t} - \lambda| \leq \varepsilon) \geq (1 - 7\varepsilon)(1 - 2\varepsilon) \geq 1 - 9\varepsilon.$$

From the arbitrariness of ε, we have

$$\mathrm{P}(\limsup_{t\to+\infty} \frac{\ln I_{u_0}(t)}{t} = \lambda) = 1, u_0 \in [0, h^*] \times (0, h^*] \tag{3.28}$$

that is, $I_{u_0}(t)$ converges almost surely to 0 at an exponential rate.

Next, we will prove that the distribution of $S_{u_0}(t)$ converges weakly to the measure φ. By Portmanteau's theorem [14], it is sufficient to prove that for any $g(x) : \mathrm{R} \to \mathrm{R}$ satisfying

(i) $|g(x)| \leq M, x \in \mathrm{R}$, M is a positive constant,

(ii) $|g(x_1) - g(x_2)| \leq M|x_1 - x_2|, x_1, x_2 \in \mathrm{R}$, then

$$\lim_{t\to+\infty} \mathrm{E}(g(S_{u_0}(t)) = \int_0^{+\infty} g(\phi)f(\phi)d\phi£ = g^*, u_0 \in [0, h^*] \times (0, h^*].$$

Since the diffusion Eq (3.1) is non-degenerate, it is well known that the distribution of $\varphi(t)$ weakly converges to f. So, we have

$$\lim_{t\to+\infty} \mathrm{E}[g(\varphi(t))] = g^*. \tag{3.29}$$

From (3.28), it follows for $\varepsilon > 0$ that there exists $T > 0$ such that $\mathrm{P}(\Omega_{10}) > 1 - \varepsilon$, where

$$\Omega_{10} = \{I_{u_0}(t) \leq \exp\{\frac{\lambda t}{2}\}, t \geq T\}.$$

For $\sigma > 0$ in Lemma 3.2, we choose $\bar{T} > T$ satisfying $I_{u_0}(t) \leq \exp\{\frac{\lambda\bar{T}}{2}\} < \sigma$. From Lemma 3.2 and Lemma 3.3, we have that

$$\mathrm{P}\{|S_{u_0}(t) - \varphi(t)| < \eta, t > \bar{T}\} > 1 - \varepsilon. \tag{3.30}$$

In addition,

$$\begin{aligned} |\mathrm{E}(g(S_{u_0}(t)) - g^*| &\leq |\mathrm{E}g(S_{u_0}(t)) - \mathrm{E}g(\varphi(t))| + |\mathrm{E}g(\varphi(t)) - g^*| \\ &\leq M\mathrm{P}(|S_{u_0}(t) - \varphi(t)| < \eta)\eta + 2M\mathrm{P}(|S_{u_0}(t) - \varphi(t)| \geq \eta) + |\mathrm{E}g(\varphi(t)) - g^*|. \end{aligned} \tag{3.31}$$

From (3.29)–(3.31), we obtain

$$\limsup_{t\to+\infty} \mathrm{E}[|g(\varphi(t)) - g^*|] < M(\eta + 2\varepsilon) + \varepsilon.$$

From the arbitrariness of ε, η, we get that $\mathrm{E}(g(S_{u_0}(t))$ converges to g^*. The proof is complete.

It is clear that $R_0^s = R_0$ holds if $\sigma_i = 0 (i = 1, 2, 3, 4)$.

4. Existence of stationary distribution

In this section, we will focus on the stationary distribution of the solution of model (2.1). Since $\lambda = (\frac{1}{2}\frac{\sigma_2^2}{a^2}\int_0^\infty \varphi^2 f(\varphi)d\varphi + c_2)(R_0^s - 1)$, $R_0^s > 1$ is equivalent to $\lambda > 0$, we deal with the case $R_0^s > 1$. In the following, we present some useful lemmas.

Lemma 4.1. Assume that $\mu > 4\max\{\sigma_1^2, \sigma_3^2\}$ holds. There are constants $K_1 > 0$ and $K_2 > 0$ such that, for any $u = (s, i) \in [0, h^*] \times (0, h^*]$, $t \geq 1$, and $A \in \tilde{F}$,

$$\mathrm{E}(\ln^- I_u(t))^2 I_A \leq \mathrm{P}(A)(\ln^- i)^2 + K_1\sqrt{\mathrm{P}(A)}(\ln^- i)t + K_2\sqrt{\mathrm{P}(A)}t^2 \tag{4.1}$$

where $\ln^-\xi = \max\{0, -\ln\xi\}$ and I_A denotes the indicator function of A.

Proof. For any $u = (s, i) \in [0, h^*] \times (0, h^*]$, in view of Lemma 2.1, we have $\mathrm{E}S_u^8(t) \leq C$, which implies that

$$\mathrm{E}S_u^2(t) \leq \bar{C}, \mathrm{E}S_u^4(t) \leq \bar{C}, t \geq 0$$

where $\bar{C} > 0$ is a constant. From system (2.1), we have

$$\begin{aligned} \ln I_u(t) &= \ln i - c_2 t + \int_0^t \sigma_2 \frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau) + \int_0^t \sigma_3 dB_3(\tau) + \int_0^t (\beta\frac{S_u(\tau)}{a + I_u^\alpha(\tau)} - \frac{1}{2}\sigma_2^2 \frac{S_u^2(\tau)}{(a + I_u^\alpha(\tau))^2})d\tau \\ &\geq \ln i - c_2 t + \int_0^t \sigma_2 \frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau) + \int_0^t \sigma_3 dB_3(\tau) - \frac{1}{2}\sigma_2^2 \int_0^t \frac{S_u^2(\tau)}{(a + I_u^\alpha(\tau))^2} d\tau. \end{aligned}$$

Hence,

$$\ln^- I_u(t) \leq \ln^- i + c_2 t + \sigma_2 \left|\int_0^t \frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau)\right| + \sigma_3|B_3(t)| + \frac{1}{2a^2}\sigma_2^2 \int_0^t S_u^2(\tau)d\tau.$$

Using the inequality $(\sum_{m=1}^5 a_m)^2 \leq a_1^2 + 2a_2^2 + 4(a_3^2 + a_4^2 + a_5^2) + 2a_1(a_2 + a_3 + a_4 + a_5)$, we obtain

$$\begin{aligned} (\ln^- I_u(t))^2 I_A &\leq (\ln^- i)^2 I_A + 2c_2^2 t^2 I_A + 4\sigma_2^2 \left|\int_0^t \frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau)\right|^2 I_A + 4\sigma_3^2 |B_3(t)|^2 I_A \\ &+ \frac{1}{a^4}\sigma_2^4 (\int_0^t S_u^2(\tau)d\tau)^2 I_A + 2(\ln^- i)c_2 t I_A + 2(\ln^- i)\sigma_2 \left|\int_0^t \frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau)\right| I_A \\ &+ 2(\ln^- i)\sigma_3 |B_3(t)| I_A + 2(\ln^- i)\frac{1}{2a^2}\sigma_2^2 \int_0^t S_u^2(\tau)d\tau I_A. \end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned} &\mathrm{E}\int_0^t S_u^2(\tau)d\tau I_A \leq \sqrt{\bar{C}}\sqrt{P(A)}t, \\ &\mathrm{E}(\int_0^t S_u^2(\tau)d\tau)^2 I_A \leq \mathrm{E}(t\int_0^t S_u^4(\tau)d\tau)I_A \leq \sqrt{C}\sqrt{P(A)}t^2. \end{aligned}$$

Applying Hölder's inequality and the Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned} &\mathrm{E}|B_3(t)|I_A \leq \sqrt{\mathrm{E}B_3^2(t)}\sqrt{P(A)} \leq \sqrt{t}\sqrt{P(A)}, \mathrm{E}B_3^2(t)I_A \leq \sqrt{\mathrm{E}B_3^4(t)}\sqrt{P(A)} \leq 3t\sqrt{P(A)}, \\ &\mathrm{E}\left|\int_0^t \frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau)\right| I_A \leq \sqrt{P(A)}(\mathrm{E}\int_0^t \frac{S_u^2(\tau)}{(a + I_u^\alpha(\tau))^2} d\tau)^{\frac{1}{2}} \leq \sqrt{\bar{C}}\sqrt{P(A)}\sqrt{t}, \\ &\mathrm{E}\left|\int_0^t \frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau)\right|^2 I_A \leq \sqrt{P(A)}(\mathrm{E}\int_0^t \left|\frac{S_u(\tau)}{a + I_u^\alpha(\tau)} dB_2(\tau)\right|^4)^{\frac{1}{2}} \leq \sqrt{P(A)}(3\mathrm{E}(\int_0^t \frac{S_u^2(\tau)}{(a + I_u^\alpha(\tau))^2} d\tau)^2)^{\frac{1}{2}} \\ &\leq \sqrt{P(A)}(3t^2\mathrm{E}\int_0^t S_u^4(\tau)d\tau)^{\frac{1}{2}} \leq \sqrt{3\bar{C}}\sqrt{P(A)}t. \end{aligned}$$

Therefore, there exist two constants K_1, K_2 such that

$$\mathrm{E}(\ln^- I_u(t))^2 I_A \leq \mathrm{P}(A)(\ln^- i)^2 + K_1\sqrt{\mathrm{P}(A)}(\ln^- i)t + K_2\sqrt{\mathrm{P}(A)}t^2.$$

The proof of Lemma 4.1 is complete.
Let $\varepsilon \in (0,1)$ satisfy

$$-\frac{5\lambda}{4}(1-\varepsilon) + K_1\sqrt{\varepsilon} < -\lambda. \tag{4.2}$$

Lemma 4.2. Assume that $\mu > 4\max\{\sigma_1^2, \sigma_3^2\}$ and $R_0^s > 1$ hold. For ε chosen in (4.2), there are $T^* = T^*(\varepsilon) > 1$ and $\delta^* \in (0,1)$ such that

$$\mathrm{P}\{\ln i + \frac{5\lambda}{8}t \leq \ln I_u(t) < 0, t \in [T^*, 2T^*]\} > 1-\varepsilon$$

for $u = (s,i) \in [0,h^*] \times (0,\delta^*]$.
Proof. By the ergodicity of φ, there exists $T_1^* > 1$ such that $\mathrm{P}(\Omega_1^*) \geq 1 - \frac{\varepsilon}{4}$, where

$$\Omega_1^* = \{\frac{1}{t}(\int_0^t \frac{\beta}{a}\varphi(\tau)d\tau - \frac{1}{2}\int_0^t \frac{\sigma_2^2}{a^2}\varphi^2(\tau)d\tau) - c_2 \geq \frac{3}{4}\lambda, t \geq T_1^*\}.$$

Noting

$$\begin{array}{l} \mathrm{E}|\int_0^t \sigma_3 dB_3(\tau)|^2 = \mathrm{E}\int_0^t \sigma_3^2 d\tau = \sigma_3^2 t, \\ \mathrm{E}|\int_0^t \sigma_2 \frac{S_u(\tau)}{a+I_u^\alpha(\tau)}dB_2(\tau)| \leq \sigma_2\left(\mathrm{E}\int_0^t \frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2}d\tau\right)^{\frac{1}{2}} \leq \sigma_2\sqrt{\bar{C}t} \end{array}$$

and (ii) of Lemma 2.1, by Chebyshev's inequality we obtain $\mathrm{P}(\Omega_2^*) \geq 1 - \frac{\varepsilon}{4}$, where

$$\Omega_2^* = \{|\sigma_3 B_3(t)| + |\int_0^t \sigma_2 \frac{S_u(\tau)}{a + I_u^\alpha(\tau)}dB_2(\tau)| \leq M(\varepsilon)\sqrt{t}, t \geq 1\}$$

and $M(\varepsilon) > 0$.
Using Itô's formula, we obtain

$$\begin{array}{rl} \ln I_u(t) & \geq \ln i + \int_0^t (\frac{\beta}{a}\varphi(\tau) - \frac{\sigma_2^2}{2a^2}\varphi^2(\tau) - c_2)d\tau \\ & +\beta\int_0^t (\frac{S_u(\tau)}{a+I_u^\alpha(\tau)} - \frac{1}{a}\varphi(\tau))d\tau + \frac{\sigma_2^2}{2}\int_0^t (\frac{1}{a^2}\varphi^2(\tau) - \frac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2})d\tau \\ & -|\sigma_3 B_3(t)| - |\int_0^t \sigma_2 \frac{S_u(\tau)}{a+I_u^\alpha(\tau)}dB_2(\tau)|. \end{array} \tag{4.3}$$

Let $T^* > \max\{1, T_1^*, \frac{16^2 M^2(\varepsilon)}{\lambda^2}\}$. From Lemma 3.2 we can choose $0 < \eta^* < 1$ and $0 < \eta_1^* < \eta^*$ such that $\mathrm{P}(\Omega_3^*) \geq 1 - \frac{\varepsilon}{4}$ and, for $\omega \in \Omega_3^*$,

$$\beta|\frac{S_u(t)}{a + I_u^\alpha(t)} - \frac{1}{a}\varphi(t)| + \frac{\sigma_2^2}{2}|\frac{1}{a^2}\varphi^2(t) - \frac{S_u^2(t)}{(a + I_u^\alpha(t))^2}| < \frac{\lambda}{16}$$

where $\Omega_3^* = \{|\varphi(t) - S_u(t)| < \eta^*, t \in [0, 2T^* \wedge \tau^{\eta_1^*}]\}, \tau^{\eta_1^*} = \inf\{t \geq 0, I_u(t) > \eta_1^*\}$.
By Lemma 3.1, there exists $\delta^* \in (0, \eta_1^*)$ such that $u \in [0,h^*] \times (0,\delta^*], \mathrm{P}(\Omega_4^*) \geq 1 - \frac{\varepsilon}{4}$, where

$$\Omega_4^* = \{\tau^{\eta_1^*} \geq 2T^*\}.$$

From (4.3), for $u \in [0, h^*] \times (0, \delta^*], t \in [T^*, 2T^*]$ and $\omega \in \bigcap_{i=1}^4 \Omega_i^*$, we obtain

$$\begin{aligned} 0 > \ln \eta_1^* > \ln I_u(t) \;&\geq \ln i + \int_0^t (\tfrac{\beta}{a}\varphi(\tau) - \tfrac{\sigma_2^2}{2a^2}\varphi^2(\tau) - c_2)d\tau \\ &-\beta \int_0^t |\tfrac{S_u(\tau)}{a+I_u^\alpha(\tau)} - \tfrac{1}{a}\varphi(\tau)|d\tau - \tfrac{\sigma_2^2}{2}\int_0^t |\tfrac{1}{a^2}\varphi^2(\tau) - \tfrac{S_u^2(\tau)}{(a+I_u^\alpha(\tau))^2}|d\tau \\ &-|\sigma_3 B_3(t)| - |\int_0^t \sigma_2 \tfrac{S_u(\tau)}{a+I_u^\alpha(\tau)} dB_2(\tau)| \\ &> \ln i + \tfrac{3\lambda}{4}t - \tfrac{\lambda}{16}t - \tfrac{\lambda}{16}t \geq \ln i + \tfrac{5\lambda}{8}t. \end{aligned}$$

Therefore, we complete the proof by $\mathrm{P}(\bigcap_{i=1}^4 \Omega_i^*) > 1 - \varepsilon$.

Lemma 4.3. Assume that $\mu > 4\max\{\sigma_1^2, \sigma_3^2\}$ and $R_0^s > 1$ hold. For ε, T^* chosen as in (4.2) and Lemma 4.2, respectively, there is a $K_3 > 0$ such that

$$\mathrm{E}(\ln^- I_u(t))^2 \leq (\ln^- i)^2 - \lambda t \ln^- i + K_3 T^* \tag{4.4}$$

for $u = (s, i) \in [0, h^*] \times (0, h^*], t \in [T^*, 2T^*]$.

Proof. First, consider $u = (s, i) \in [0, h^*] \times (0, \delta^*]$, where δ^* is as in Lemma 4.2. By Lemma 4.2, we have $\mathrm{P}(\Omega_5^*) \geq 1 - \varepsilon$, where

$$\Omega_5^* = \{\ln^- I_u(t) \leq \ln^- i - \frac{5\lambda}{8}t, t \in [T^*, 2T^*]\}.$$

From the above inequality, we obtain

$$(\ln^- I_u(t))^2 \leq (\ln^- i)^2 - \frac{5\lambda}{4}(\ln^- i)t + \frac{25\lambda^2}{64}t^2, t \in [T^*, 2T^*]$$

which implies that

$$\mathrm{E}(\ln^- I_u(t))^2 I_{\Omega_5^*} \leq (\ln^- i)^2 \mathrm{P}(\Omega_5^*) - \frac{5\lambda}{4}\mathrm{P}(\Omega_5^*)(\ln^- i)t + \frac{25\lambda^2}{64}\mathrm{P}(\Omega_5^*)t^2. \tag{4.5}$$

Let $(\Omega_5^*)^c = \Omega - \Omega_5^*$, then $\mathrm{P}((\Omega_5^*)^c) < \varepsilon$. From Lemma 4.1, we have

$$\mathrm{E}(\ln^- I_u(t))^2 I_{(\Omega_5^*)^c} \leq (\ln^- i)^2 \mathrm{P}((\Omega_5^*)^c) + K_1\sqrt{\mathrm{P}((\Omega_5^*)^c)}(\ln^- i)t + K_2\sqrt{\mathrm{P}((\Omega_5^*)^c)}t^2. \tag{4.6}$$

Combining (4.5) and (4.6), we have

$$\mathrm{E}(\ln^- I_u(t))^2 \leq (\ln^- i)^2 - (\frac{5\lambda}{4}(1-\varepsilon) - K_1\sqrt{\varepsilon})(\ln^- i)t + (K_2 + \frac{25\lambda^2}{64})t^2. \tag{4.7}$$

From (4.2), we obtain

$$\mathrm{E}(\ln^- I_u(t))^2 \leq (\ln^- i)^2 - \lambda(\ln^- i)t + (K_2 + \frac{25\lambda^2}{64})t^2. \tag{4.8}$$

Now, for $u = (s, i) \in [0, h^*] \times [\delta^*, h^*]$, it follows from Lemma 4.1 that

$$\begin{aligned} \mathrm{E}(\ln^- I_u(t))^2 &\leq (\ln^- i)^2 + K_1(\ln^- i)t + K_2 t^2 \\ &\leq |\ln^- \delta^*|^2 + K_1|\ln^- \delta^*|t + K_2 t^2. \end{aligned} \tag{4.9}$$

Since $t \in [T^*, 2T^*]$, from (4.8) and (4.9) we take K_3 sufficiently large such that $K_3 > K_2 + \frac{25\lambda^2}{64}$ and

$$|\ln^- \delta^*|^2 + 2K_1|\ln^- \delta^*|T^* + 4K_2T^* \leq K_3T^*$$

completing the proof of Lemma 4.3.

Theorem 4.1. If $\mu > 4\max\{\sigma_1^2, \sigma_3^2\}$ and $R_0^s > 1$, system (2.1) with initial condition $u = (s, i) \in [0, h^*] \times (0, h^*]$ is permanent, i.e., the solution $(S_u(t), I_u(t))$ of system (2.1) has a unique invariant probability $\pi^*(\cdot)$ concentrated on $u \in \mathrm{R}_+^{2,\circ}$. Moreover,

(i) For any $u = (s, i) \in [0, h^*] \times (0, h^*]$,

$$\lim_{t\to\infty} t^{q^*} \|\mathrm{P}(t, u, \cdot) - \pi^*(\cdot)\| = 0$$

where $\|\cdot\|$ is the total variation norm, q^* is any positive number, and $\mathrm{P}(t, u, \cdot)$ is the transition probability of the solution $(S_u(t), I_u(t))$.

(ii) The law of large numbers holds, i.e., for any π^*-integrable $l :\in [0, h^*] \times (0, h^*] \to \mathrm{R}$, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t l(S_u(\tau), I_u(\tau))d\tau = \int_{\mathrm{R}_+^{2,\circ}} l(\tau_1, \tau_2)\pi^*(d\tau_1, d\tau_2) \ \ a.s. \ \ u = (s, i) \in [0, h^*] \times (0, h^*]. \tag{4.10}$$

Proof. Let $N(t) = S_u(t) + I_u(t)$. From Lemma 2.1, there are $p_1, p_2 > 0$ such that

$$\mathrm{E}(N(2T^*)) \leq (1 - p_1)N(0) + p_2, u \in [0, h^*] \times (0, h^*]. \tag{4.11}$$

Let $\bar{V}(t) = N(t) + (\ln^- I_u(t))^2$. Then, $\bar{V}(2T^*) = N(2T^*) + (\ln^- I_u(2T^*))^2$, $\bar{V}(0) = N(0) + (\ln^- i)^2$. From Lemma 4.3 and (4.10), there is a compact set $\mathrm{K} \subset [0, h^*] \times (0, h^*]$, $p_1^*, p_2^* > 0$ such that

$$\mathrm{E}(\bar{V}(2T^*)) \leq \bar{V}(0) - p_1^* \sqrt{\bar{V}(0)} + p_2^* I_{u\in\mathrm{K}}, u \in [0, h^*] \times (0, h^*]. \tag{4.12}$$

Since system (2.1) is a non-degeneracy of the diffusion, from (4.12) and [32], as $n \to \infty$, we have

$$n\|\mathrm{P}(2nT^*, u, \cdot) - \pi^*\| \to 0 \tag{4.13}$$

for some invariant probability measure π^* of the Markov chain $(S_u(2nT^*), I_u(2nT^*))$. Let

$$\tau_{\mathrm{K}} = \inf\{n \in \mathrm{N} : (S_u(2nT^*), I_u(2nT^*)) \in \mathrm{K}\}.$$

From the proof of [32] and (4.12), we obtain $\mathrm{E}\tau_{\mathrm{K}} < \infty$. In view of [26], the Markov process $(S_u(t), I_u(t))$ has an invariant probability measure π_*. Therefore, π^* is also an invariant probability measure of the Markov chain $(S_u(2nT^*), I_u(2nT^*))$. In light of (4.13), $\pi^* = \pi_*$, i.e., π^* is an invariant measure of the Markov process $(S_u(t), I_u(t))$. The proof is complete.

5. Numerical simulations

In this section, we prepare some numerical simulations to demonstrate the impact of environmental noise on the HIV/AIDS model and our analytical results. The numerical simulations are given by the following Milstein scheme [33].

Consider the following discretization of system (1.2) for $t = 0, \Delta t, 2\Delta t, ..., n\Delta t$:

$$\begin{aligned}
S_{k+1} &= S_k + \left(\Lambda - \mu S_k - \tfrac{\beta S_k I_k}{a+I_k^{\alpha}}\right)\Delta t + \sigma_1 S_k \sqrt{\Delta t}\xi_k + \tfrac{\sigma_1^2}{2} S_k(\xi_k^2 - 1)\Delta t \\
&\quad - \tfrac{\sigma_2 S_k I_k}{a+I_k^{\alpha}} \sqrt{\Delta t}\xi_k - \tfrac{\sigma_2^2 S_k I_k}{2(a+I_k^{\alpha})}(\xi_k^2 - 1)\Delta t, \\
I_{k+1} &= I_k + \left(\tfrac{\beta S_k I_k}{a+I_k^{\alpha}} - (\nu + \mu + \gamma_1)I_k\right)\Delta t + \sigma_3 I_k \sqrt{\Delta t}\eta_k \\
&\quad + \tfrac{\sigma_3^2}{2} I_k(\eta_k^2 - 1)\Delta t + \tfrac{\sigma_2 S_k I_k}{a+I_k^{\alpha}} \sqrt{\Delta t}\xi_k + \tfrac{\sigma_2^2 S_k I_k}{2(a+I_k^{\alpha})}(\xi_k^2 - 1)\Delta t, \\
A_{k+1} &= A_k + (\nu I_k - (\mu + \gamma_2)A_k)\Delta t + \sigma_4 A_k \sqrt{\Delta t}\zeta_k + \tfrac{\sigma_4^2}{2} A_k(\zeta_k^2 - 1)\Delta t,
\end{aligned} \tag{5.1}$$

where $\Delta t > 0$ is time increment, $\xi_k, \eta_k, \zeta_k (k = 1, 2, \cdots, n)$ is the Gaussian random variable which follows $N(0, 1)$. For simplicity, we adopt $\sigma = \sigma_i (i = 1, 2, 3, 4)$.
We use the parameters given by [14,22]. The detailed values of the parameters are taken as follows:

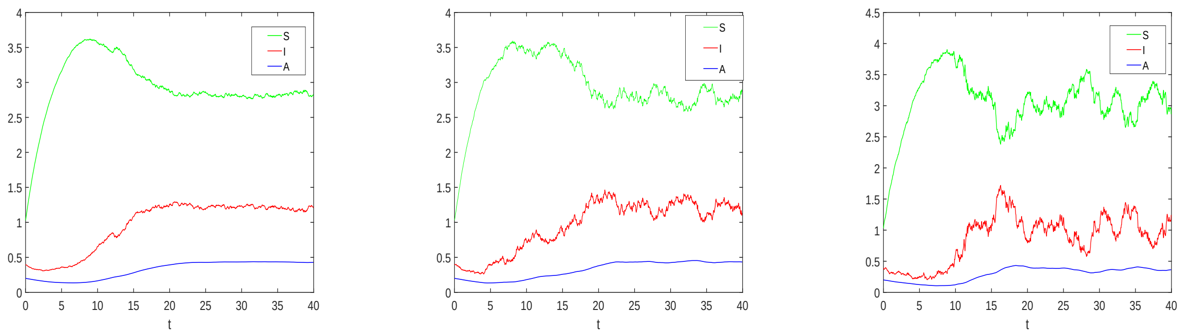
$$\Lambda = 1, \mu = 0.2, \beta = 0.1, a = 0.7, \alpha = 6, \nu = 0.2, \gamma_1 = 0.02, \gamma_2 = 0.05. \tag{5.2}$$

5.1. The stochastic persistence dynamics of HIV/AIDS model (1.2)

First, we take $\sigma = 0.01$ with the parameters in (5.2). Direct calculations show that

$$\begin{aligned}
R_0^s &= \frac{\frac{\beta}{a}\int_0^{\infty} \varphi f(\varphi) d\varphi}{\mu+\nu+\gamma_1+\frac{1}{2}\sigma^2+\frac{1}{2}\frac{\sigma^2}{a^2}\int_0^{\infty} \varphi^2 f(\varphi) d\varphi} \\
&= \frac{\frac{\beta}{a}\int_0^{\infty} \varphi[(\int_0^{\infty} t^{\frac{2\mu}{\sigma^2}} e^{-t} dt)^{-1} (\frac{2\Lambda}{\sigma^2})^{\frac{2\mu}{\sigma^2}+1} \varphi^{-(\frac{2\mu}{\sigma^2}+2)} e^{-\frac{2\Lambda}{\sigma^2}\frac{1}{\varphi}}] d\varphi}{\mu+\nu+\gamma_1+\frac{1}{2}\sigma^2+\frac{1}{2}\frac{\sigma^2}{a^2}\int_0^{\infty} \varphi^2[(\int_0^{\infty} t^{\frac{2\mu}{\sigma^2}} e^{-t} dt)^{-1} (\frac{2\Lambda}{\sigma^2})^{\frac{2\mu}{\sigma^2}+1} \varphi^{-(\frac{2\mu}{\sigma^2}+2)} e^{-\frac{2\Lambda}{\sigma^2}\frac{1}{\varphi}}] d\varphi} = 1.9051 > 1
\end{aligned}$$

which satisfies the conditions in Theorem 4.1. Then, HIV/AIDS is almost surely persistent (see Figure 1(a)). When increasing the intensity of white noise, $\sigma = 0.03, 0.05$, and then $R_0^s = 1.8321, 1.7969 > 1$. We can easily observe that HIV/AIDS persists. However, the amplitude of fluctuations increases around the epidemic equilibrium $E_0^* = (2.8532, 1.1782, 0.3135)$ of the corresponding deterministic system (1.1) of the stochastic model (1.2) (see Figure 1(b),(c)). Figure 1 shows that the intensity of white noise affects the stability of endemic disease of the deterministic system (1.1).



(a) (b) (c)

Figure 1. The path of $(S(t), I(t), A(t))$ for the stochastic model (1.2) with $\sigma =$ 0.01,0.03, 0.05 if $R_0^s > 1$, respectively.

We adopt $\sigma = 0.01, 0.03, 0.05$, for $R_0^s > 1$, which satisfy the conditions in Theorem 4.1. When t is sufficiently large, numerical simulations show that the amplitude of fluctuations is slight and the oscillations are more symmetrically distributed around the equilibrium $E_0^* = (2.8532, 1.1782, 0.3135)$ of the corresponding deterministic model of the stochastic model (1.2). Figures 2–4 are the histograms of the probability density function of $S(t), I(t)$, and $A(t)$ for model (1.2) with $\sigma = 0.01, 0.03, 0.05$ at $t = 100$.

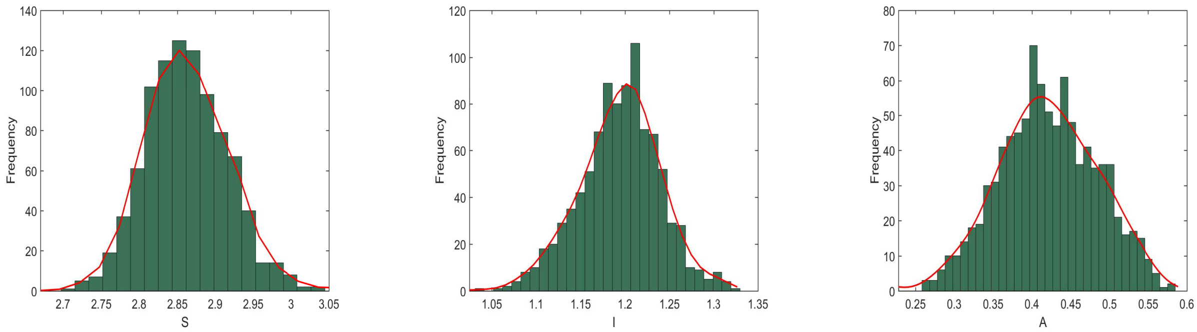


Figure 2. Histogram of the probability density functions of $S(t), I(t)$, and $A(t)$ for model (1.2) with $\sigma = 0.01$.

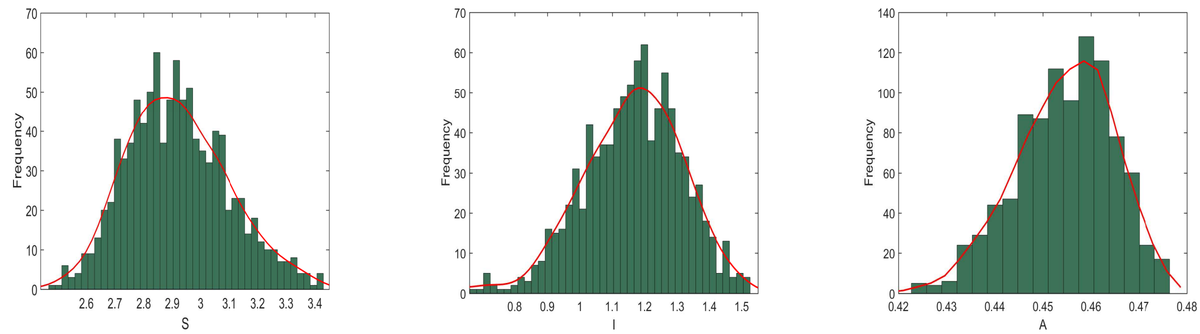


Figure 3. Histogram of the probability density functions of $S(t), I(t)$, and $A(t)$ for model (1.2) with $\sigma = 0.03$.

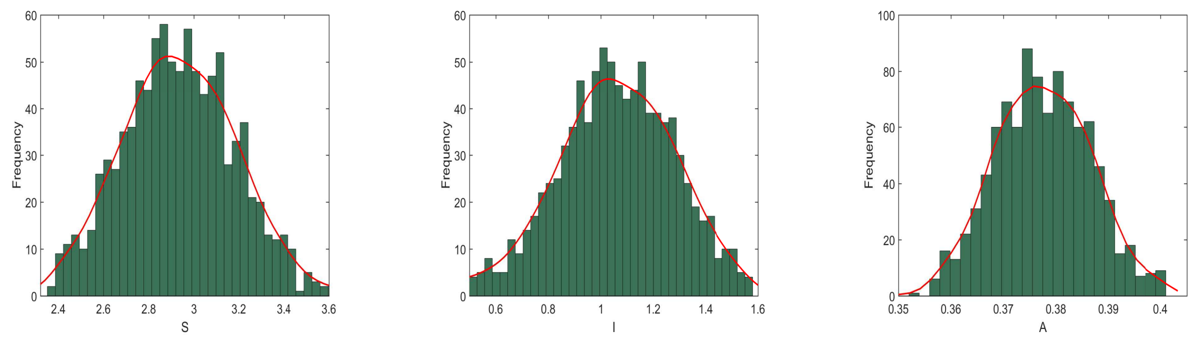


Figure 4. Histogram of the probability density functions of $S(t), I(t)$, and $A(t)$ for model (1.2) with $\sigma = 0.05$.

5.2. *The stochastic extinction of HIV/ADS model (1.2)*

When increasing the intensity of white noise, we choose $\sigma = 0.118$, $R_0^s = 0.9765 < 1$, and HIV/AIDS is almost surely extinct (see Figure 5).

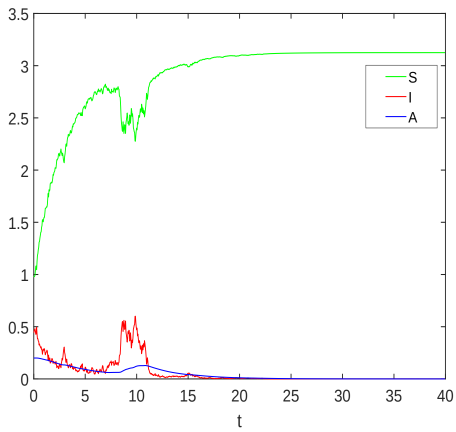


Figure 5. The path of $(S(t), I(t), A(t))$ for the stochastic model (2) with initial value $(S_0, I_0, A_0) = (1, 0.4, 0.2)$.

6. Discussion

In model (1.2), we adopt the saturated incidence rate $\frac{\beta S(t)I(t)}{a+I^{\alpha}(t)}$, replace the parameter β by $\beta+\sigma_1\dot{B}_1(t)$, and obtain the threshold R_0^s which governs the stochastic dynamics of the stochastic HIV/AIDS model. For the stochastic HIV/AIDS model with intervention strategy, in view of the following incidence

$$(\beta_1 - \frac{\beta_2 I(t)}{a + I^{\alpha}(t)})S(t)I(t) + \sigma\frac{S(t)I(t)}{a + I^{\alpha}(t)}\dot{B}(t)$$

where β_1 is the usual contact rate without considering the infective individuals, the term $\frac{\beta_2 I(t)}{a+I^{\alpha}(t)}$ represents the diminished value of the transmission rate when infectious individuals are taken into account; investigating such problem is important and meaningful. In addition, the HIV/AIDS model with higher order perturbation such as

$$\begin{aligned}
&\mu \hookrightarrow \mu - (\sigma_{11} + \sigma_{12}S^m(t))\dot{B}_1(t),\\
&\mu + \nu + \gamma_1 \hookrightarrow \mu + \nu + \gamma_1 - (\sigma_{31} + \sigma_{32}I^m(t))\dot{B}_3(t),\\
&\mu + \gamma_2 \hookrightarrow \mu + \gamma_2 - (\sigma_{41} + \sigma_{41}A^m(t))\dot{B}_4(t)
\end{aligned}$$

where $m \geq 2$ is also interesting, and we look forward to solving it in the near future.

Theorem 3.1 shows that, when increasing the intensity of white noise that $R_0^s < 1$ holds, HIV/AIDS is almost surely extinct. However, HIV transmission is affected by many factors, such as economics, education, and policies, so it is difficult to eradicate HIV/AIDS in real life.

In addition, for the HIV/AIDS epidemic, the relevant interventions are PrEP and HAART, which would help to control viral level. However, for simplicity, we did not bring those interventions into the consideration.

Hepatitis B and C are similar to HIV/AIDS in transmission. Hence, the results of this theoretical study also are instructive to the control of hepatitis B and C. In addition, like HIV/AIDS, media reports affect the transmission of COVID-19. So, model (1.2) is also of theoretical significance to the study of COVID-19.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Science Foundation of China (No. 12071407) .

Conflict of interest

The authors declare there is no conflict of interest.

References

1. N. Dalal, D. Greenhalgh, X. Mao, A stochastic model for internal HIV dynamics, *J. Math. Anal. Appl.*, **341** (2008), 1084–1101. https://doi.org/10.1016/j.jmaa.2007.11.005
2. A. Nhd, B. Nnn, Permanence and extinction for the stochastic SIR epidemic model, *J. Differ. Equation*, **269** (2020), 9619–9652. https://doi.org/10.1016/j.jde.2020.06.049
3. A. Gral, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A stochastic differential equation SIS epidemic model, *SIAM J. Appl. Math.*, **71** (2011), 876–902. https://doi.org/10.1137/10081856X
4. Z. Liu, Dynamics of positive solutions to SIR and SEIR epidemic models with saturated incidence rates, *Nonlinear Anal. RWA*, **14** (2013), 1286–1299. https://doi.org/10.1016/j.nonrwa.2012.09.016
5. L. Imhof, S. Walcher, Exclusion and persistence in deterministic and stochastic chemostat models, *J. Differ. Equation*, **217** (2005), 26–53. https://doi.org/10.1016/j.jde.2005.06.017
6. X. Mao, G. Marion, E. Renshaw, Environmental brownian noise suppresses explosions in population dynamics, *Stoch. Process. Appl.*, **97** (2002), 95–110. https://doi.org/10.1016/S0304-4149(01)00126-0
7. X. Y. Zhou, X. Gao, X. Y. Shi, Analysis of an SQEIAR stochastic epidemic model with media coverage and asymptomatic infection, *Int. J. Biomath.*, **15** (2022), 2250083. https://doi.org/10.1142/S1793524522500838
8. Q. Liu, D. Q. Jiang, N. Shi, B. Ahmad, Stationary distribution and extinction of a stochastic SEIR epidemic model with standard incidence, *Phys. A*, **469** (2017), 510–517. https://doi.org/10.1016/j.physa.2017.02.028
9. W. D. Wang, Epidemic models with nonlinear infection forces, *Math. Biosci. Eng.*, **3** (2006), 267–279. https://doi.org/10.3934/mbe.2006.3.267

10. D. Xiao, S. Ruan, Global analysis of an epidemic model with nonmonotone incidence rate, *Math. Biosci.*, **208** (2007), 419–429. https://doi.org/10.1016/j.mbs.2006.09.025

11. J. Cui, X. Tao, H. Zhu, An SIS infection model incorporating media coverage, *Rocky Mountain J. Math.*, **38** (2008), 1323–1334. https://doi.org/10.1216/RMJ-2008-38-5-1323

12. J. Cui, Y. Sun, H. Zhu, The impact of media on the control of infectious diseases, *J. Dynam. Differ. Equations*, **20** (2008), 31–53. https://doi.org/10.1007/s10884-007-9075-0

13. C. T. Bauch, Dynamics of an infectious disease where media coverage influences transmission, *ISRN Biomath.*, (2012), 581274. https://doi.org/10.5402/2012/581274

14. Y. Cai, Y. Kang, M. Banerjee, W. Wang, A stochastic SIRS epidemic model with infectious force under intervention strategies, *J. Differ. Equation*, **259** (2015), 7463–7502. https://doi.org/10.1016/j.jde.2015.08.024

15. W. Liu, A SIRS epidemic model incorporating media coverage with random perturbation, *Abst. Appl. Anal.*, (2013), 792308. https://doi.org/10.1155/2013/792308

16. Y. Zhang, K. Fan, S. Gao, Y. Liu, S. Chen, Ergodic stationary distribution of a stochastic SIRS epidemic model incorporating media coverage and saturated incidence rate, *Phys. A*, **514** (2019), 671–685. https://doi.org/10.1016/j.physa.2018.09.124

17. W. Guo, Q. Zhang, X. Li, W. Wang, Dynamic behavior of a stochastic SIRS epidemic model with media coverage, *Math. Meth. Appl. Sci.*, **41** (2018), 5506–5525. https://doi.org/10.1002/mma.5094

18. W. Liu, Q. Zheng, A stochastic SIS epidemic model incorporating media coverage in a two patch setting, *Appl. Math. Comput.*, **62** (2015), 160–168. https://doi.org/10.1016/j.amc.2015.04.025

19. Y. P. Tan, Y. L. Cai, Z. Peng, K. Wang, R. Yao, et al., Stochastic dynamics of an SIS epidemiological model with media coverage, *Math. Comput. Simulat.*, **204** (2–23), 1–27. https://doi.org/10.1016/j.matcom.2022.08.001

20. B. Q. Zhou, D. Q. Jiang, B. Han, T. Hayat, Threshold dynamics and density function of a stochastic epidemic model with media coverage and mean-reverting OrnsteinCUhlenbeck process, *Math. Comput. Simulat.*, **196** (2022), 15–44. https://doi.org/10.1016/j.matcom.2022.01.014

21. B. Q. Zhou, B. T. Han, D. Q. Jiang, T. Hayat, A. Alsaedi, Ergodic stationary distribution and extinction of a staged progression HIV/AIDS infection model with nonlinear stochastic perturbations, *Nonlinear Dyn.*, **104** (2022), 3863–3886. https://doi.org/10.1007/s11071-021-07116-5

22. B. T. Han, D. Q. Jiang, T. Hayat, A. Alsaedi, B. Ahmad, Stationary distribution and extinction of a stochastic staged progression AIDS model with staged treatment and second-order perturbation, *Chaos Soliton Fract.*, **140** (2020), 110238. https://doi.org/10.1016/j.chaos.2020.110238

23. Q. Liu, D. Q. Jiang, T. Hayat, B. Ahmad, Asymptotic behavior of a stochastic delayed HIV-1 infection model with nonlinear incidence, *Phys. A*, **486** (2017), 867–882. https://doi.org/10.1016/j.physa.2017.05.069

24. M. M. Gao, D. Q. Jiang, T. Hayat, Qualitative analysis of an HIV/AIDS model with treatment and nonlinear perturbation, *Qual. Theor. Dyn. Syst.*, **21** (2022), 12346-022-00615-9. https://doi.org/10.1007/s12346-022-00615-9

25. Q. Liu, D. Q. Jiang, Dynamics of a stochastic multigroup S-DI-A model for the transmission of HIV, *Appl. Anal.*, **99** (2020), 1–26. https://doi.org/10.1080/00036811.2020.1758310

26. S. Ruan, W. Wang, Dynamical behavior of an epidemic model with a nonlinear incidence rate, *J. Differ. Equ.*,**188** (2003), 135–163. https://doi.org/10.1016/S0022-0396(02)00089-X

27. Q. S. Yang, D. Q. Jiang, N. Z. Shi, C. Y. Ji, The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence, *J. Math. Anal. Appl.*, **388** (2012), 248–271. https://doi.org/10.1016/j.jmaa.2011.11.072

28. E. Nummelin, *General Irreducible Markov Chains and Non-Negative Operations*, Cambridge: Cambridge University Press, 1984.

29. X. Mao,*Stochastic Differential Equations and Applications*, Chichester: Elsevier, 2007.

30. L. Allen, *An introduction to stochastic epidemic models*, Berlin Heidelberg: Springer, 2008.

31. D. Nguyen, G. Yin, Z. Chu, Certain properties related to well posedness of switching diffusions, *Stoch. Process. Appl.*,**127** (2017), 3135–3158. https://doi.org/10.1016/j.spa.2017.02.004

32. N. Nguyen, G. Yin, Stochastic partial differential equation SIS epidemic models: modeling and analysis, *Commun. Stoch. Anal.*, **13** (2019), 8. https://doi.org/10.31390/cosa.13.3.08

33. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. https://doi.org/10.1137/S0036144500378302