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Research article

Stationary distribution and probability density function analysis of a stochastic Microcystins degradation model with distributed delay

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Abstract: A stochastic Microcystins degradation model with distributed delay is studied in this paper. We first demonstrate the existence and uniqueness of a global positive solution to the stochastic system. Second, we derive a stochastic critical value R_0^s related to the basic reproduction number R_0. By constructing suitable Lyapunov function types, we obtain the existence of an ergodic stationary distribution of the stochastic system if $R_0^s > 1$. Next, by means of the method developed to solve the general four-dimensional Fokker-Planck equation, the exact expression of the probability density function of the stochastic model around the quasi-endemic equilibrium is derived, which is the key aim of the present paper. In the analysis of statistical significance, the explicit density function can reflect all dynamical properties of a chemostat model. To validate our theoretical conclusions, we present examples and numerical simulations.

Keywords: Microcystins degradation model; distributed delay; stationary distribution; probability density function; extinction

1. Introduction

Microcystins (MCs) represent a highly prevalent form of cyanotoxins derived from harmful cyanobacterial blooms. These toxins can introduce toxicity into vital organs including liver, kidney, heart etc., posing major threats to human health [1]. Biodegradation has emerged as an economically and environmentally favorable approach for MCs degradation, garnering considerable research attention [2,3]. Recently, Song et al. [4] proposed a degradation model as follows:

$$\begin{cases} x_1'(t) = Da_{10} - a_{12}x_1(t)x_2(t) - a_{13}x_1(t)x_3(t) - (D + d_1)x_1(t), \\ x_2'(t) = a_{21}x_2(t)x_1(t) - a_{20}x_2(t) - (D + d_2)x_2(t), \\ x_3'(t) = a_{30}x_2(t) - a_{31}x_1(t)x_3(t) - (D + d_3)x_3(t), \end{cases} \tag{1.1}$$

where $x_1(t), x_2(t)$ and $x_3(t)$ denote the concentrations of MCs, MC-degrading bacteria and degrading enzymes produced by MC-degrading bacteria at time t, respectively. $a_{10} > 0$ is the input concentration of MCs; D is the washout rate; $a_{12} \geq 0$ is the consumption rate of MCs; $a_{21} \geq 0$ is the maximal growth rate of MC-degrading bacteria; $a_{20} \geq 0$ is the consumption rate of MC-degrading bacteria; $a_{30} \geq 0$ is the maximal growth rate of degrading enzymes; $a_{13} \geq 0$ and $a_{31} \geq 0$ are the degradation rate of MCs and the consumption rate of degrading enzymes, respectively; $d_i > 0\,(i = 1, 2, 3)$ represents the death rates of MCs, MC-degrading bacteria and degrading enzymes, as distinguished by, i = 1, 2, 3, respectively. Re-scaling model (1.1) by

$$x_1 = a_{10}x,\ x_2 = y,\ x_3 = z, t = \frac{T}{D},\ D_1 = \frac{D + d_1}{D}, D_2 = \frac{a_{20} + D + d_2}{D},\ D_3 = \frac{D + d_3}{D},$$
$$a_1 = \frac{a_{12}}{D},\ a_2 = \frac{a_{13}}{D},\ b_1 = \frac{a_{21}a_{10}}{D}, c_1 = \frac{a_{30}}{D},\ c_2 = \frac{a_{31}a_{10}}{D},$$

the authors of [5] obtained the following dimensionless model:

$$\begin{cases} x'(t) = 1 - a_1x(t)y(t) - a_2x(t)z(t) - D_1x(t), \\ y'(t) = b_1x(t)y(t) - D_2y(t), \\ z'(t) = c_1y(t) - c_2x(t)z(t) - D_3z_3(t). \end{cases} \tag{1.2}$$

Time delays cannot be ignored in models for MCs degradation, especially the delay between MC-degrading bacteria and the production of new degrading enzymes particles. There has been extensive discussion of the presence of such delays and their impact on microbial population dynamics (see [6–8]). Song et al. [9] considered the following model with time delay:

$$\begin{cases} x'(t) = 1 - a_1x(t)y(t) - a_2x(t)z(t) - D_1x(t), \\ y'(t) = b_1e^{-\delta\tau}x(t - \tau)y(t - \tau) - D_2y(t), \\ z'(t) = c_1y(t) - c_2x(t)z(t) - D_3z_3(t). \end{cases}$$

Here the constant $\tau \geq 0$ represents the time consumed by MC-degrading bacteria to convert MCs, once consumed, into viable biomass and $e^{-\delta\tau}$ represents the survival probability of degrading bacteria during the conversion process. However, a more realistic delayed MC-degrading model should be an equation with distributed delay, which describes the cumulative effect of the past history of a system. In this paper, we will mainly consider the following degradation model with distributed delay

$$\mathrm{d}z(t) = [c_1 \int_{-\infty}^{t} K(t - s)y(s)ds - c_2x(t)z(t) - D_3z_3(t)]\mathrm{d}t.$$

According to MacDonald [10], the distributed delay function $f(t)$ can be described by the Gamma distribution

$$K(t) = \frac{\alpha^{n+1}t^ne^{-\alpha t}}{n!},$$

as a kernel, where $\alpha > 0$ and n is a nonnegative integer. In this article, we mainly consider the weak kernel, that is $K(t) = \alpha e^{-\alpha t}$ $(n = 0)$ with $\alpha > 0$.

On the other hand, in the real world, population systems are inevitably influenced by the environmental noise (see [11–15]). According to May [16], the birth rate, carrying capacity, competition coefficient, and other parameters should exhibit random fluctuations as a result of environmental noise. In order to investigate the impact of environmental noise on population dynamics, various scholars have added white noise into deterministic systems (see [17–21]).

Therefore, based on the model (1.2), we further consider the following stochastic MCs degradation model with distribution delay

$$\begin{cases} \mathrm{d}x(t) = [1 - a_1x(t)y(t) - a_2x(t)z(t) - D_1x(t)]\mathrm{d}t + \sigma_1x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = [b_1x(t)y(t) - D_2y(t)]\mathrm{d}t + \sigma_2y(t)\mathrm{d}B_2(t), \\ \mathrm{d}z(t) = [c_1\int_{-\infty}^{t} K(t-s)y(s)ds - c_2x(t)z(t) - D_3z_3(t)]\mathrm{d}t + \sigma_3z(t)\mathrm{d}B_3(t), \end{cases} \tag{1.3}$$

where $B_i(t)$ denotes mutually independent standard Brownian motions and $\sigma_i^2 > 0$ denotes the intensity of white noises $i = 1, 2, 3$. Set

$$w(t) = \int_{-\infty}^{t} K(t-s)y(s)ds = \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}y(s)ds.$$

Then, by using the linear chain technique, system (1.3) is transformed into the following equivalent system:

$$\begin{cases} \mathrm{d}x(t) = [1 - a_1x(t)y(t) - a_2x(t)z(t) - D_1x(t)]\mathrm{d}t + \sigma_1x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = [b_1x(t)y(t) - D_2y(t)]\mathrm{d}t + \sigma_2y(t)\mathrm{d}B_2(t), \\ \mathrm{d}z(t) = [c_1w(t) - c_2x(t)z(t) - D_3z(t)]\mathrm{d}t + \sigma_3z(t)\mathrm{d}B_3(t), \\ \mathrm{d}w(t) = [\alpha y(t) - \alpha w(t)]\mathrm{d}t. \end{cases} \tag{1.4}$$

In this paper, we propose a Microcystins degradation model with distributed delay and stochastic perturbation. As is known to us, there is no work concerning the existence of stationary distribution in system (1.3). The main difficulty is to find appropriate Lyapunov functions. It should be emphasized that there are relatively few studies focusing on the explicit expression of probability density functions due to the difficulty of solving the corresponding Fokker-Planck equation.

The paper is organized as follows. In Section 2, we prove that the stochastic Microcystins degradation system (1.4) with distributed delay has a unique global positive solution. In Section 3, a unique global solution of system (1.4) is proved to be stationary by constructing several suitable Lyapunov functions. Numerical simulations are presented to illustrate our analytical results in Section 4.

2. The positivity of the solution

To study the dynamical behavior of Microcystins degradation model, we first need to consider whether the solution is global and positive. In this section, we will prove the existence and uniqueness of a global solution to system (1.4) with any positive initial value.

Theorem 2.1: For any initial value $(x(0), y(0), z(0), w(0)) \in \mathbb{R}_+^4$, there is a unique solution of the system (1.4) on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^4$ with probability one.

Proof. The coefficients of system (1.3) satisfy the local Lipschitz condition, so there is a unique local solution $(x(t), y(t), z(t), w(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. Now we shall show that this solution is global, i.e., prove $\tau_e = \infty$ a.s. Let $k_0 \geq 1$ be sufficiently large so that $x(0), y(0), z(0)$ and $w(0)$ all lie within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time:

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \min\{x(t), y(t), z(t), w(t)\} \leq \frac{1}{k} \text{ or } \max\{x(t), y(t), z(t), w(t)\} \geq k\right\},$$

where throughout this paper we set $\inf \emptyset = \infty$. Evidently, τ_k is increasing as $k \to \infty$. Denote $\tau_\infty = \lim_{k\to\infty} \tau_k$, where $\tau_\infty \leq \tau_e$ a.s. If we can prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(x(t), y(t), z(t), w(t)) \in \mathbb{R}^4_+$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to prove is that $\tau_\infty = \infty$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} > \varepsilon \text{ for all } k \geq k_1. \tag{2.1}$$

Define the nonnegative C^2- Lyapunov function V_0 as follows

$$V_0(x, y, z, w) = x - 1 - \ln x + ay - \ln y - (1 + \ln a) + cz - d\ln z - (d + \ln\frac{c}{d}) + bw - \ln w - (1 + \ln b),$$

where a, b, c, d are positive constants which will be determined later.

Based on the basic inequality $u - 1 - \ln u \geq 0$, we have

$$cz - d\ln z - (d + \ln\frac{c}{d}) = d(\frac{cz}{d} - 1 - \ln\frac{cz}{d}) \geq 0.$$

Applying Itô's formula, we have

$$\mathrm{d}V_0 = LV_0dt + \sigma_1(x - 1)dB_1(t) + \sigma_2(ay - 1)dB_2(t) + \sigma_3(cz - d)dB_3(t), \tag{2.2}$$

$$\begin{aligned}
LV_0 = {} & 1 - a_1xy - a_2xz - D_1x + a(b_1xy - D_2y) + c(c_1w - c_2xz - D_3z) + b(\alpha y - \alpha w) - \frac{1}{x} + a_1y + a_2z \\
& + D_1 + \frac{1}{2}\sigma_1^2 - b_1x + D_2 + \frac{1}{2}\sigma_2^2 + d(-\frac{c_1w}{z} + c_2x + D_3 + \frac{1}{2}\sigma_3^2) - \frac{\alpha y}{w} + \alpha \\
& \leq 1 + D_1 + D_2 + dD_3 + \alpha + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + d\sigma_3^2) + (ab_1 - a_1)xy + (a_2 - cD_3)z + (-D_1 - b_1 + dc_2)x \\
& + (-aD_2 + a_1 + b\alpha)y + (cc_1 - b\alpha)w.
\end{aligned}$$

Choose $b = \frac{a_2c_1}{\alpha D_3}, c = \frac{a_2}{D_3}, d = \frac{b+D_1}{c}$ and $a = \min\{\frac{a_1}{b_1}, \frac{a_1D_3+c_1a_2}{D_2D_3}\}$ such that $ab_1 - a_1 \leq 0$ and $a_1 - aD_2 + \alpha b \leq 0$, then, we have

$$LV_0 \leq 1 + D_1 + D_2 + dD_3 + \alpha + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + d\sigma_3^2) := K_1,$$

where K_1 is a positive constant. Integrating equation (2.2) from 0 to $\tau_k \wedge T$ and then taking the expectation on both sides, we have

$$\mathbb{E}V_0\Big(x(\tau_k\wedge T),y(\tau_k\wedge T),z(\tau_k\wedge T),w(\tau_k\wedge T))\Big)\le V_0(x(0),y(0),z(0),w(0))+K_1\mathbb{E}(\tau_k\wedge T)$$
$$\le V_0(x(0),y(0),z(0),w(0))+K_1T \tag{2.3}$$

Set $\Omega_k=\{\tau_k\le T\}$ for $k\ge k_1$ and in view of Eq (2.1), we have $\mathbb{P}(\Omega_k)\ge\epsilon$. Note that for every $\omega\in\Omega_k$, there is $x(\tau_k,\omega)$ or $y(\tau_k,\omega)$ or $z(\tau_k,\omega)$ or $w(\tau_k,\omega)$ that is equal to either k or $\frac{1}{k}$. Thus $V_0(x(\tau_k,\omega),y(\tau_k,\omega),z(\tau_k,\omega),w(\tau_k,\omega))$ is no less than either

$$\begin{aligned}
&\Big(k-1-\ln k\Big)\wedge\Big(ak-\ln k-(1+\ln a)\Big)\wedge d\Big(\frac{c}{d}k-1-\ln\frac{ck}{d}\Big)\wedge\Big(bk-\ln k-(1+\ln b)\Big)\\
\text{or }&\Big(\frac{1}{k}-1-\ln\frac{1}{k}\Big)\wedge\Big(a\frac{1}{k}-\ln\frac{1}{k}-(1+\ln a)\Big)\wedge d\Big(\frac{c}{d}\frac{1}{k}-1-\ln\frac{c}{dk}\Big)\wedge\Big(b\frac{1}{k}-\ln\frac{1}{k}-(1+\ln b)\Big)\\
=&\Big(k-1-\ln k\Big)\wedge\Big(ak-\ln k-(1+\ln a)\Big)\wedge d\Big(\frac{c}{d}k-1-\ln(ck)+\ln d\Big)\wedge\Big(bk-\ln k-(1+\ln b)\Big)\\
\text{or }&\Big(\frac{1}{k}-1+\ln k\Big)\wedge\Big(a\frac{1}{k}+\ln k-(1+\ln a)\Big)\wedge d\Big(\frac{c}{d}\frac{1}{k}-1-\ln c+\ln(dk)\Big)\wedge\Big(b\frac{1}{k}+\ln k-(1+\ln b)\Big)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&V_0\Big(x(\tau_k,\omega),y(\tau_k,\omega),z(\tau_k,\omega),w(\tau_k,\omega))\Big)\\
&\ge\Big(k-1-\ln k\Big)\wedge\Big(ak-\ln k-(1+\ln a)\Big)\wedge d\Big(\frac{c}{d}k-1-\ln(ck)+\ln d\Big)\wedge\Big(bk-\ln k-(1+\ln b)\Big)\\
&\wedge\Big(\frac{1}{k}-1+\ln k\Big)\wedge\Big(a\frac{1}{k}+\ln k-(1+\ln a)\Big)\wedge d\Big(\frac{c}{d}\frac{1}{k}-1-\ln c+\ln(dk)\Big)\wedge\Big(b\frac{1}{k}+\ln k-(1+\ln b)\Big)
\end{aligned}$$

It follows from Eq (2.3) that

$$\begin{aligned}
V_0(x(0),y(0),z(0),w(0))+K_1T\ \ge&\mathbb{E}\Big[I_{\Omega_k(\omega)}V_0(x(\tau_k,\omega),y(\tau_k,\omega),z(\tau_k,\omega),w(\tau_k,\omega)))\Big]\\
\ge&\epsilon\Big(k-1-\ln k\Big)\wedge\Big(ak-\ln k-(1+\ln a)\Big)\wedge d\Big(\frac{c}{d}k-1-\ln(ck)+\ln d\Big)\\
&\wedge\Big(bk-\ln k-(1+\ln b)\Big)\wedge\Big(\frac{1}{k}-1+\ln k\Big)\wedge\Big(a\frac{1}{k}+\ln k-(1+\ln a)\Big)\\
&\wedge d\Big(\frac{c}{d}\frac{1}{k}-1-\ln c+\ln(dk)\Big)\wedge\Big(b\frac{1}{k}+\ln k-(1+\ln b)\Big)
\end{aligned}$$

where $I_{\Omega_k(\omega)}$ is the indicator function of Ω_k. Let $k\to\infty$, we obtain

$$\infty>V_0(x(0),y(0),z(0),w(0))+K_1T=\infty$$

which is a contradiction and so we must have that $\tau_\infty=\infty$. This completes the proof.

3. Stationary distribution

For the stochastic model, we mainly focus on the existence of a stationary distribution. Define a critical value

$$R_0^s=\frac{b_1}{(D_1+\frac{1}{2}\sigma_1^2)(D_2+\frac{1}{2}\sigma_2^2)}.$$

Consider the integral equation

$$X(t) = X(t_0) + \int_{t_0}^{t} b(s, X(s))\mathrm{d}s + \sum_{r=1}^{k} \int_{t_0}^{t} \sigma_r(s, X(s))\mathrm{d}B(s). \tag{3.1}$$

Lemma 3.1. ([22]) Suppose that the coefficient of (3.1) is independent of t and satisfies the following condition for some constant B:

$$\begin{aligned} &|b(s,x) - b(s,y)| + \sum_{r=1}^{k} |\sigma_r(s,x) - \sigma_r(s,y)| \le B\,|x-y|\,, \\ &|b(s,x)| + \sum_{r=1}^{k} |\sigma_r(s,x)| \le B(1+|x|), \end{aligned} \tag{3.2}$$

in $U_R \subset \mathbb{R}_+^d$ for every R>0, also suppose that there exists a nonnegative C^2-function $V(x) \in \mathbb{R}_+^d$ such that

$$LV(x) \le -1 \ \text{ outside some compact set.}$$

Then system (3.1) has a solution, which is a stationary distribution.

Remark 3.1. According to the proof of Lemma 2.1 in [23], we know that condition (3.2) in Lemma 3.1 can be replaced by the global existence of the solutions. Hence, Theorem 2.1 indicates that (3.2) in Lemma 3.1 is satisfied.

Theorem 3.1. For any initial value $(x(0), y(0), z(0), w(0)) \in \mathbb{R}_+^4$, if R_0^s>1, then system (1.4) has at least one stationary solution $(x(t), y(t), z(t), w(t)) \in \mathbb{R}_+^4$.

Proof. By Lemma 3.1, we only need to construct a nonnegative C^2-function $\widetilde{V}(x,y,z,w)$ and a closed set $U \subset \mathbb{R}_+^4$ such that

$$L\widetilde{V}(x,y,z,w) \le -1, \quad \text{for any } (x,y,t,w) \in \mathbb{R}_+^4 \setminus U.$$

Define

$$V_1(x,y,z,w) = -\ln x - e_1 \ln y + \frac{a_2}{D_3} z + \frac{c_1 a_2}{\alpha D_3} w,$$

where e_1 is a positive constant to be chosen later. Then applying Itô's formula to V_1, we have

$$\begin{aligned} LV_1 \le& -\frac{1}{x} - e_1 b_1 x + e_1(D_2 + \frac{1}{2}\sigma_2^2) + D_1 + \frac{1}{2}\sigma_1^2 + a_1 y + a_2 z + \frac{a_2}{D_3}(c_1 w - c_2 xz - D_3 z) \\ &+ \frac{c_1 a_2}{D_3 \alpha}(\alpha y - \alpha w) \\ \le& -2\sqrt{b_1 e_1} + e_1(D_2 + \frac{1}{2}\sigma_2^2) + D_1 + \frac{1}{2}\sigma_1^2 + (a_1 + \frac{c_1 a_2}{D_3})y \end{aligned} \tag{3.3}$$

Taking $e_1 = \frac{b_1}{(D_2 + \frac{1}{2}\sigma_2^2)^2}$, then in view of (3.3) one can see that

$$\begin{aligned} LV_1 &\le -\frac{b_1}{D_2 + \frac{1}{2}\sigma_2^2} + D_1 + \frac{1}{2}\sigma_1^2 + (a_1 + \frac{c_1 a_2}{D_3})y \\ &= -(D_1 + \frac{1}{2}\sigma_1^2)(R_0^s - 1) + (a_1 + \frac{c_1 a_2}{D_3})y \\ &=: -\lambda + (a_1 + \frac{c_1 a_2}{D_3})y, \end{aligned} \tag{3.4}$$

where

$$R_0^s = \frac{b_1}{(D_1 + \frac{1}{2}\sigma_1^2)(D_2 + \frac{1}{2}\sigma_2^2)},\ \lambda = (D_1 + \frac{1}{2}\sigma_1^2)(R_0^s - 1).$$

Define

$$V_2 = -\ln x - \ln w - \ln z.$$

Then applying Itô's formula to V_2, one gets

$$LV_2 \leq -\frac{1}{x} - \frac{\alpha y}{w} - \frac{c_1 w}{z} + a_1 y + a_2 z + D_1 + \frac{1}{2}\sigma_1^2 + c_2 x + D_3 + \frac{1}{2}\sigma_3^2 + \alpha. \tag{3.5}$$

Select

$$V_3(x, y, z, w) = \frac{1}{\theta + 2}(x + \frac{a_1}{b_1}y + \frac{a_1 D_2}{4b_1 c_1}z + \frac{a_1 D_2}{2\alpha b_1}w)^{\theta+2},$$

where $0 < \theta < \min\{\frac{D_1 - \frac{\sigma_1^2}{2}}{D_1 + \frac{\sigma_1^2}{2}}, \frac{D_2 - \sigma_2^2}{D_2 + \sigma_2^2}, \frac{D_3 - \frac{\sigma_3^2}{2}}{D_3 + \frac{\sigma_3^2}{2}}\}$. Applying the same method to V_3, we have

$$\begin{aligned}
LV_3 \leq{}& (x + \frac{a_1}{b_1}y + \frac{a_1 D_2}{4b_1 c_1}z + \frac{a_1 D_2}{2\alpha b_1}w)^{\theta+1}\left\{1 - D_1 x - \frac{a_1 D_2}{2b_1}y - \frac{a_1 D_2 D_3}{4b_1 c_1}z - \frac{a_1 D_2}{4b_1}w\right\} \\
&+ \frac{\theta+1}{2}(x + \frac{a_1}{b_1}y + \frac{a_1 D_2}{4b_1 c_1}z + \frac{a_1 D_2}{2\alpha b_1}w)^{\theta}[\sigma_1^2 x^2 + \sigma_2^2(\frac{a_1}{b_1})^2 y^2 + \sigma_3^2(\frac{a_1 D_2}{4c_1 b_1})^2 z^2]\Big\} \\
\leq{}& (x + \frac{a_1}{b_1}y + \frac{a_1 D_2}{4b_1 c_1}z + \frac{a_1 D_2}{2\alpha b_1}w)^{\theta+1} - D_1 x^{\theta+2} - \frac{D_2}{2}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2} - D_3(\frac{a_1 D_2}{4b_1 c_1})^{\theta+2}z^{\theta+2} \\
&- \frac{\alpha}{2}(\frac{a_1 D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2} + \frac{\theta+1}{2}(x + \frac{a_1}{b_1}y + \frac{a_1 D_2}{4b_1 c_1}z + \frac{a_1 D_2}{2\alpha b_1}w)^{\theta}[\sigma_1^2 x^2 + \sigma_2^2(\frac{a_1}{b_1})^2 y^2 + \sigma_3^2(\frac{a_1 D_2}{4c_1 b_1})^2 z^2] \\
\leq{}& -D_1\theta x^{\theta+2} - \frac{D_2\theta}{2}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2} - D_3\theta(\frac{a_1 D_2}{4b_1 c_1})^{\theta+2}z^{\theta+2} - \frac{\alpha}{4}(\frac{a_1 D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2} + B_0,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
B_0 ={}& \sup_{(x,y,z,w)\in\mathbb{R}_+^4}\Big\{(x + \frac{a_1}{b_1}y + \frac{a_1 D_2}{4b_1 c_1}z + \frac{a_1 D_2}{2\alpha b_1}w)^{\theta+1} - D_1(1-\theta)x^{\theta+2} - \frac{D_2(1-\theta)}{2}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2} \\
&- D_3(1-\theta)(\frac{a_1 D_2}{4b_1 c_1})^{\theta+2}z^{\theta+2} - \frac{\alpha}{4}(\frac{a_1 D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2} + \frac{\theta+1}{2}(x + \frac{a_1}{b_1}y + \frac{a_1 D_2}{4b_1 c_1}z + \frac{a_1 D_2}{2\alpha b_1}w)^{\theta} \\
&\cdot [\sigma_1^2 x^2 + \sigma_2^2(\frac{a_1}{b_1})^2 y^2 + \sigma_3^2(\frac{a_1 D_2}{4c_1 b_1})^2 z^2]\Big\} < \infty.
\end{aligned}$$

Consequently, we have

$$V = M_0 V_1 + V_2 + V_3, \tag{3.7}$$

where $M_0 > 0$ satisfies that $-M_0 R_0^s + c_0 \leq -2$. Also,

$$\begin{aligned}
c_0 = \sup_{(x,y,z,w)\in\mathbb{R}_+^4}\Big\{&- D_1\theta x^{\theta+2} - \frac{D_2\theta}{2}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2} - D_3\theta(\frac{a_1 D_2}{4b_1 c_1})^{\theta+2}z^{\theta+2} - \frac{\alpha}{4}(\frac{a_1 D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2} \\
&+ a_2 z + c_2 x + D_1 + D_3 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^3 + \alpha + B_0\Big\}.
\end{aligned}$$

It is obvious that

$$\liminf_{n\to\infty,\,(x,y,z,w)\in\mathbb{R}_+^4\setminus K_n} V(x,y,z,w)=+\infty,$$

and $K_n=(\frac{1}{n},n)\times(\frac{1}{n},n)\times(\frac{1}{n},n)\times(\frac{1}{n},n)$. It is evident that $V(x,y,z,w)$ is a continuous functions, so there exists a minimum point $(x_0,y_0,z_0,w_0)\in\mathbb{R}_+^4$. Thereafter, we formulate the nonnegative C^2 function $\widetilde{V}$ as follows:

$$\widetilde{V}(x,y,z,w)=V(x,y,z,w)-V(x_0,y_0,z_0,w_0).$$

Combining with (3.4), (3.5), (3.6) and (3.7), we obtain that

$$\begin{aligned}L\widetilde{V}\le& -M_0\lambda+\left[M_0(a_1+\frac{a_2c_1}{D_3})+a_1\right]y+a_2z+c_2x+D_1+\frac{1}{2}\sigma_1^2+D_3+\frac{1}{2}\sigma_3^2+\alpha-D_1\theta x^{\theta+2}-\frac{D_2\theta}{2}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2}\\&-D_3\theta(\frac{a_1D_2}{4b_1c_1})^{\theta+2}z^{\theta+2}-\frac{\alpha}{4}(\frac{a_1D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2}+B_0-\frac{1}{x}-\frac{\alpha y}{w}-\frac{c_1w}{z}.\end{aligned}$$

Next, we define the following bounded closed set

$$U_\varepsilon=\left\{\varepsilon\le x\le\frac{1}{\varepsilon},\ \varepsilon\le y\le\frac{1}{\varepsilon},\ \varepsilon^2\le w\le\frac{1}{\varepsilon^3},\ \varepsilon^3\le z\le\frac{1}{\varepsilon^3}\right\}.$$

For convenience, we can divide $\mathbb{R}_+^4\setminus U_\varepsilon$ into eight regions:

$$\begin{aligned}&U_1=\left\{(x,y,z,w)\in\mathbb{R}_+^4,\ 0<y<\varepsilon\right\},U_2=\left\{(x,y,z,w)\in\mathbb{R}_+^4,0<x<\varepsilon\right\},\\&U_3=\left\{(x,y,z,w)\in\mathbb{R}_+^4,0<w<\varepsilon^2,\ y>\varepsilon\right\},U_4=\left\{(x,y,z,w)\in\mathbb{R}_+^4,\ 0<z<\varepsilon^3,\ w\ge\varepsilon^2\right\},\\&U_5=\left\{(x,y,z,w)\in\mathbb{R}_+^4,\ y>\frac{1}{\varepsilon}\right\},U_6=\left\{(x,y,z,w)\in\mathbb{R}_+^4,\ x>\frac{1}{\varepsilon}\right\},\\&U_7=\left\{(x,y,z,w)\in\mathbb{R}_+^4,\ z>\frac{1}{\varepsilon^3}\right\},U_8=\left\{(x,y,z,w)\in\mathbb{R}_+^4,w>\frac{1}{\varepsilon^2}\right\}.\end{aligned}$$

Case 1. If $(x,y,z,w)\in U_1$, in view of (3.8) we get

$$\begin{aligned}L\widetilde{V}\le& -M_0\lambda+\left[M_0(a_1+\frac{a_2c_1}{D_3})+a_1\right]y+a_2z+c_2x-D_1\theta x^{\theta+2}-\frac{D_2\theta}{2}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2}\\&-D_3\theta(\frac{a_1D_2}{4b_1c_1})^{\theta+2}z^{\theta+2}-\frac{\alpha}{4}(\frac{a_1D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2}+B_0+D_1+\frac{1}{2}\sigma_1^2+D_3+\frac{1}{2}\sigma_3^2+\alpha\\\le& -M_0\lambda+\left\{M_0(a_1+\frac{a_2c_1}{D_3})+a_1\right]\varepsilon+c_0\le-1;\end{aligned}$$

here, ε is a positive constant small enough to satisfy the following conditions:

$$-M_0\lambda+\left[M_0(a_1+\frac{a_2c_1}{D_3})+a_1\right]\varepsilon+c_0\le-1, \tag{3.8}$$

$$-\frac{1}{\varepsilon}+E\le-1, \tag{3.9}$$

$$-\frac{\alpha}{\varepsilon}+E\leq-1, \tag{3.10}$$

$$-\frac{c_1}{\varepsilon}+E\leq-1, \tag{3.11}$$

$$-\frac{D_2\theta}{4}(\frac{a_1}{b_1\varepsilon})^{\theta+2}+E\leq-1, \tag{3.12}$$

$$-\frac{D_1\theta}{2}(\frac{1}{\varepsilon})^{\theta+2}+E\leq-1, \tag{3.13}$$

$$-\frac{D_3\theta}{2}(\frac{a_1D_2}{4b_1c_1\varepsilon^3})^{\theta+2}+E\leq-1, \tag{3.14}$$

$$-\frac{\alpha}{8}(\frac{a_1D_2}{2\alpha b_1\varepsilon^2})^{\theta+2}+E\leq-1, \tag{3.15}$$

where

$$E=\sup_{(x,y,z,w)\in\mathbb{R}^4_+}\Big\{-\frac{D_1\theta}{2}x^{\theta+2}-\frac{D_2\theta}{4}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2}-\frac{D_3\theta}{2}(\frac{a_1D_2}{4b_1c_1})^{\theta+2}z^{\theta+2}-\frac{\alpha}{8}(\frac{a_1D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2}+a_2z+c_2x$$
$$+\Big[M(a_1+\frac{a_2c_1}{D_3})+a_1\Big]y+D_1+\frac{1}{2}\sigma_1^2+D_3+\frac{1}{2}\sigma_3^2+\alpha+B_0\Big\}.$$

Case 2 : If $(x,y,z,w)\in U_2$ according to (3.9), it implies that

$$\begin{aligned}L\widetilde{V}\leq&-\frac{1}{x}+\Big[M_0(a_1+\frac{a_2c_1}{D_3})+a_1\Big]y+a_2z+c_2x-D_1\theta x^{\theta+2}-\frac{D_2\theta}{2}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2}\\&-D_3\theta(\frac{a_1D_2}{4b_1c_1})^{\theta+2}z^{\theta+2}-\frac{\alpha}{4}(\frac{a_1D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2}+B_0+D_1+\frac{1}{2}\sigma_1^2+D_3+\frac{1}{2}\sigma_3^2+\alpha\\\leq&-\frac{1}{\varepsilon}+E\leq-1.\end{aligned}$$

Case 3 : If $(x,y,z,w)\in U_3$, by (3.10) we have

$$L\widetilde{V}\leq-\frac{\alpha y}{w}+E\leq-\frac{\alpha}{\varepsilon}+E\leq-1.$$

Case 4 : If $(x,y,z,w)\in U_4$, according to (3.11), we deduce that

$$L\widetilde{V}\leq-\frac{c_1w}{z}+E\leq-\frac{c_1}{\varepsilon}+E\leq-1.$$

Case 5 : If $(x,y,z,w)\in U_5$, by condition (3.12), we conclude that

$$L\widetilde{V}\leq-\frac{D_2\theta}{4}(\frac{a_1}{b_1})^{\theta+2}y^{\theta+2}+E\leq-\frac{D_2\theta}{4}(\frac{a_1}{b_1\varepsilon})^{\theta+2}+E\leq-1.$$

Case 6 : If $(x,y,z,w)\in U_6$, and the condition (3.13) is satisfied, it follows that

$$L\widetilde{V}\leq-\frac{D_1\theta}{2}x^{\theta+2}+E\leq-\frac{D_1\theta}{2}(\frac{1}{\varepsilon})^{\theta+2}+E\leq-1.$$

Case 7 : If $(x, y, z, w) \in U_7$, in view of (3.14) we get

$$L\widetilde{V} \le -\frac{D_3\theta}{2}(\frac{a_1D_2}{4b_1c_1})^{\theta+2}z^{\theta+2} + E \le -\frac{D_3\theta}{2}(\frac{a_1D_2}{4b_1c_1\varepsilon^3})^{\theta+2} + E \le -1.$$

Case 8 : If $(x, y, z, w) \in U_8$, from (3.15), it is deduced that

$$L\widetilde{V} \le -\frac{\alpha}{8}(\frac{a_1D_2}{2\alpha b_1})^{\theta+2}w^{\theta+2} + E \le -\frac{\alpha}{8}(\frac{a_1D_2}{2\alpha b_1\varepsilon^2})^{\theta+2} + E \le -1.$$

Clearly, there exists a small enough constant ε to make the conclusion

$$L\widetilde{V} \le -1, \text{ for all } (x, y, z, w) \in U_\varepsilon^C.$$

Remark 3.2. In the proof of the above theorem, the construction of V_3 means to obtain the constant E and high-order terms of x, y and z.

4. Probability density function analysis

Let $u_1(t) = \ln x$, $u_2(t) = \ln y$, $u_3(t) = \ln z$ and $u_4(t) = \ln w$. Using the Itô's formula, an equivalent equation of system (1.4) is given as follows

$$\begin{cases} \mathrm{d}u_1 = \Big[e^{-u_1} - a_1e^{u_2} - a_2e^{u_3} - (D_1 + \frac{1}{2}\sigma_1^2)\Big]\mathrm{d}t + \sigma_1\mathrm{d}B_1(t), \\ \mathrm{d}u_2 = \Big[b_1e^{u_1} - (D_2 + \frac{1}{2}\sigma_2^2)\Big]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t), \\ \mathrm{d}u_3 = \Big[c_1e^{u_4-u_3} - c_2e^{u_1} - (D_3 + \frac{1}{2}\sigma_3^2)\Big]\mathrm{d}t + \sigma_3\mathrm{d}B_3(t), \\ \mathrm{d}u_4 = \Big(\alpha e^{u_2-u_4} - \alpha\Big)\mathrm{d}t. \end{cases} \tag{4.1}$$

Assume that $R_0^s > 1$, then, there is the quasi-endemic equilibrium $E^* = (x^*, y^*, z^*, w^*) = (e^{u_1^*}, e^{u_2^*}, e^{u_3^*}, e^{u_4^*})$ where

$$\begin{aligned} x^* &= \frac{D_2 + \frac{1}{2}\sigma_2^2}{b_1}, \\ y^* &= \frac{[b_1(D_3 + \frac{1}{2}\sigma_3^2) + c_2(D_2 + \frac{1}{2}\sigma_2^2)][b_1 - (D_1 + \frac{1}{2}\sigma_1^2)(D_2 + \frac{1}{2}\sigma_2^2)]}{a_2b_1c_1(D_2 + \frac{1}{2}\sigma_2^2) + a_1b_1(D_2 + \frac{1}{2}\sigma_2^2)(D_3 + \frac{1}{2}\sigma_3^2) + a_1c_2(D_2 + \frac{1}{2}\sigma_2^2)^2}, \\ z^* &= \frac{b_1c_1[b_1 - (D_1 + \frac{1}{2}\sigma_1^2)(D_2 + \frac{1}{2}\sigma_2^2)]}{a_2b_1c_1(D_2 + \frac{1}{2}\sigma_2^2) + a_1b_1(D_2 + \frac{1}{2}\sigma_2^2)(D_3 + \frac{1}{2}\sigma_3^2) + a_1c_2(D_2 + \frac{1}{2}\sigma_2^2)^2}, \\ w^* = y^* &= \frac{[b_1(D_3 + \frac{1}{2}\sigma_3^2) + c_2(D_2 + \frac{1}{2}\sigma_2^2)][b_1 - (D_1 + \frac{1}{2}\sigma_1^2)(D_2 + \frac{1}{2}\sigma_2^2)]}{a_2b_1c_1(D_2 + \frac{1}{2}\sigma_2^2) + a_1b_1(D_2 + \frac{1}{2}\sigma_2^2)(D_3 + \frac{1}{2}\sigma_3^2) + a_1c_2(D_2 + \frac{1}{2}\sigma_2^2)^2}. \end{aligned}$$

Before proving Theorem 4.1, we need to firstly introduce two important standard matrices in the following Lemmas 4.1 and 4.2.

Lemma 4.1 ([24]). For the algebraic equation $H_0^2 + B_0\Sigma_0 + \Sigma_0 B_0^T = 0$, where $H_0 = diag(1,0,0,0)$ and Σ_0 is a real symmetric matrix, we have the standard matrix

$$B_0 = \begin{pmatrix} -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If $\gamma_1 > 0,\ \gamma_3 > 0,\ \gamma_4 > 0$ and $\gamma_1\gamma_2\gamma_3 - \gamma_3^2 - \gamma_1^2\gamma_4 > 0$, then Σ_0 is a positive definite matrix, where

$$\Sigma_0 = \begin{pmatrix} \frac{\gamma_2\gamma_3-\gamma_1\gamma_4}{2(\gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4)} & 0 & -\frac{\gamma_3}{2(\gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4)} & 0 \\ 0 & \frac{\gamma_3}{2(\gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4)} & 0 & -\frac{\gamma_1}{2(\gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4)} \\ -\frac{\gamma_3}{2(\gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4)} & 0 & \frac{\gamma_1}{2(\gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4)} & 0 \\ 0 & -\frac{\gamma_1}{2(\gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4)} & 0 & \frac{\gamma_1\gamma_2-\gamma_3}{2\gamma_4(\gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4)} \end{pmatrix}.$$

Here B_0 in this form is called the standard R_1 matrix.

Lemma 4.2 ([24]). For the algebraic equation $H_0^2 + E_0\Omega_0 + \Omega_0 E_0^T = 0$, where $H_0 = diag(1,0,0,0)$ and Ω_0 is a real symmetric matrix, we have the standard matrix

$$E_0 = \begin{pmatrix} -\tau_1 & -\tau_2 & -\tau_3 & -\tau_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \tau_5 \end{pmatrix}.$$

If $\tau_1 > 0,\ \tau_3 > 0$ and $\tau_1\tau_2 - \tau_3 > 0$, then the matrix Ω_0 is a semi-positive definite matrix, which means that

$$\Omega_0 = \begin{pmatrix} \frac{\tau_2}{2(\tau_1\tau_2-\tau_3)} & 0 & -\frac{1}{2(\tau_1\tau_2-\tau_3)} & 0 \\ 0 & \frac{1}{2(\tau_1\tau_2-\tau_3)} & 0 & 0 \\ -\frac{1}{2(\tau_1\tau_2-\tau_3)} & 0 & \frac{\tau_1}{2\tau_3(\tau_1\tau_2-\tau_3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Denote $P = (p_1, p_2, p_3, p_4)^T = (u_1 - u_1^*, u_2 - u_2^*, u_3 - u_3^*, u_4 - u_4^*)^T$ $(k = 1, 2, 3, 4)$. Therefore, the linearized equation system of Eq (4.1) is obtained as follows

$$\begin{cases} \mathrm{d}p_1 = (-b_{11}p_1 - b_{12}p_2 - b_{13}p_3)\mathrm{d}t + \sigma_1\mathrm{d}B_1(t), \\ \mathrm{d}p_2 = b_{21}p_1\mathrm{d}t + \sigma_2\mathrm{d}B_2(t), \\ \mathrm{d}p_3 = (-b_{31}p_1 - b_{33}p_3 + b_{33}p_4)\mathrm{d}t + \sigma_3\mathrm{d}B_3(t), \\ \mathrm{d}p_4 = (\alpha p_2 - \alpha p_4)\mathrm{d}t, \end{cases} \tag{4.2}$$

where

$$b_{11} = \frac{1}{x^*} > b_{12} + b_{13},\ b_{12} = a_1 y^*,\ b_{13} = a_2 z^*,$$
$$b_{21} = b_1 x^*, b_{31} = c_2 x^*,\ b_{33} = c_1\frac{w^*}{z^*} > b_{31}.$$

Theorem 4.1. Assume that $R_0^s > 1$; for any initial value $(x(0), y(0), z(0), w(0)) \in \mathbb{R}_+^4$, then the solution (x, y, z, w) of system (1.4) will have a log-normal probability density $\Phi(x, y, z, w)$ around E^*, which is given by

$$\Phi(x,y,z,w) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(\ln\frac{x}{x^*}, \ln\frac{y}{y^*}, \ln\frac{z}{z^*}, \ln\frac{w}{w^*})\Sigma^{-1}(\ln\frac{x}{x^*}, \ln\frac{y}{y^*}, \ln\frac{z}{z^*}, \ln\frac{w}{w^*})^T}$$

with $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$ as a positive definite matrix; Σ_1, Σ_2 and Σ_3 are defined as follows

$$\Sigma_1 = \begin{cases} (b_{31}b_{33}\sigma_1)^2(T_3T_2T_1)^{-1}\overline{\Omega}_2[(T_3T_2T_1)^{-1}]^T, & \text{if } q_1 = 0, \\ (b_{31}b_{33}q_1\sigma_1)^2(T_4T_2T_1)^{-1}\Omega_1[(T_4T_2T_1)^{-1}]^T, & \text{if } q_1 \neq 0, \end{cases}$$

$$\Sigma_2 = \begin{cases} \dfrac{\alpha^2 b_{13}^2(b_{12}b_{31} + \alpha b_{33})^2\sigma_2^2}{b_{12}^2}(T_7T_6T_5)^{-1}\Omega_1[(T_7T_6T_5)^{-1}]^T, & \text{if } q_2 = 0, \\ (b_{12}b_{31} + b_{33}\alpha)^2\sigma_2^2(T_9T_8T_6T_5)^{-1}\widetilde{\Omega}_2[(T_9T_8T_6T_5)^{-1}]^T, & \text{if } q_2 \neq 0,\ q_3 = 0, \\ (b_{12}b_{31} + b_{33}\alpha)^2 q_3^2\sigma_2^2(T_{10}T_8T_6T_5)^{-1}\Omega_1[(T_{10}T_8T_6T_5)^{-1}]^T, & \text{if } q_2 \neq 0,\ q_3 \neq 0, \end{cases}$$

$$\Sigma_3 = (-b_{13}b_{21}\alpha\sigma_3)^2(T_{12}T_{11})^{-1}\Omega_1[(T_{12}T_{11})^{-1}]^T,$$

where $q_1 = \frac{b_{21}\alpha\left[b_{13}b_{33} - (b_{21}+b_{33})\alpha\right]}{b_{31}^2 b_{33}}b_{33}$, $q_2 = \frac{\alpha(-b_{11}+\alpha)}{b_{12}}$, $q_3 = -\frac{b_{13}\alpha}{b_{12}} - \frac{b_{12}b_{33}q_2}{b_{12}b_{31}+b_{33}\alpha} + \frac{b_{12}q_2(-b_{12}b_{33}q_2 + b_{12}b_{31}\alpha + b_{33}\alpha^2)}{(b_{12}b_{31}+b_{33}\alpha)^2}$, and the matrices $T_1, \dots T_{12}$, $\Omega_1, \overline{\Omega}_2, \widetilde{\Omega}_3$ are defined in the following proof.

Proof. For convenience and simplicity, let $B(t) = (B_1(t), B_2(t), B_3(t), 0)^\tau$, $G = diag(\sigma_1, \sigma_2, \sigma_3, 0)$ and

$$M = \begin{pmatrix} -b_{11} & -b_{12} & -b_{13} & 0 \\ b_{21} & 0 & 0 & 0 \\ -b_{31} & 0 & -b_{33} & b_{33} \\ 0 & \alpha & 0 & -\alpha \end{pmatrix},$$

the linearized system (4.2) can be equivalently rewritten as

$$\mathrm{d}P = MP(t)\mathrm{d}t + G\mathrm{d}B(t). \tag{4.3}$$

Since G is a constant matrix, according to Gardiner [27], we have

$$G^2 + M\Sigma + \Sigma M^T = 0.$$

In view of the independence of Brownian motions $B_1(t), B_2(t)$ and $B_3(t)$, by the principle of finite independent superposition, it is easy to know that Eq (4.4) can be equivalently developed into the sum of the solution to the following three algebraic sub-equations:

$$G_k^2 + M\Sigma_k + \Sigma_k M^T = 0, \quad k = 1, 2, 3,$$

where $G_1^2 = diag(\sigma_1^2, 0, 0, 0), G_2^2 = diag(0, \sigma_2^2, 0, 0), G_3^2 = diag(0, 0, \sigma_3^2, 0)$ and $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$.

Next it will be proved that M has all negative real-part eigenvalues. The characteristic polynomial of M is defined as

$$\Psi_M(\lambda) = \lambda^4 + r_1\lambda^3 + r_2\lambda^2 + r_3\lambda + r_4,$$

where

$$\begin{aligned} r_1 &= b_{11} + b_{33} + \alpha > 0, \\ r_2 &= b_{11}b_{33} - b_{13}b_{31} + b_{12}b_{21} + \alpha(b_{11} + b_{33}), \\ r_3 &= b_{12}b_{21}b_{33} + \alpha(b_{11}b_{33} - b_{13}b_{31} + b_{12}b_{21}), \\ r_4 &= b_{21}b_{31}\alpha(b_{12} + b_{13}) > 0. \end{aligned}$$

By calculation, we can obtain

$$b_{11}b_{33} - b_{13}b_{31} > b_{13}b_{31} - b_{13}b_{31} = 0,$$

which yields that r_2 and $r_3 > 0$. Furthermore, we can verify that $r_1r_2 - r_3 > 0$ and $r_1r_2r_3 - r_3^2 - r_1^2r_4 > 0$ So, the matrix M is a Hurwitz matrix. Next, we will prove the definiteness of Σ by following three steps.

Step 1. Consider the algebraic equation

$$G_1^2 + M\Sigma_1 + \Sigma_1 M^T = 0. \tag{4.4}$$

Let $M_1 = T_1MT_1^{-1}$, where

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{b_{31}}{b_{21}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_1 = \begin{pmatrix} -b_{11} & -b_{12} + \frac{b_{13}b_{31}}{b_{21}} & b_{13} & 0 \\ b_{21} & 0 & 0 & 0 \\ 0 & \frac{b_{31}b_{33}}{b_{21}} & -b_{33} & b_{33} \\ 0 & \alpha & 0 & -\alpha \end{pmatrix}.$$

Take $M_2 = T_2M_1T_2^{-1}$, where

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{b_{21}\alpha}{b_{31}b_{33}} & 1 \end{pmatrix}, M_2 = \begin{pmatrix} -b_{11} & -b_{12} + \frac{b_{13}b_{31}}{b_{21}} & -b_{13} & 0 \\ b_{21} & 0 & 0 & 0 \\ 0 & \frac{b_{31}b_{33}}{b_{21}} & -b_{33} + \frac{b_{21}\alpha}{b_{31}} & b_{33} \\ 0 & 0 & q_1 & -\frac{(b_{21}+b_{31})\alpha}{b_{31}} \end{pmatrix},$$

where $q_1 = \frac{b_{21}\alpha[b_{31}b_{33} - (b_{21}+b_{31})\alpha]}{a_{31}^2 b_{33}}$.

Case 1.1 : If $q_1 = 0$, the standard transformation matrix T_3 is given by

$$T_3 = \begin{pmatrix} b_{31}b_{33} & b_{33}(-\frac{b_{31}b_{33}}{b_{21}} + \alpha) & (b_{33} - \frac{b_{21}\alpha}{b_{31}})^2 & -b_{33}(b_{33} + \alpha) \\ 0 & \frac{b_{31}b_{33}}{b_{21}} & -b_{33} + \frac{b_{21}\alpha}{b_{31}} & b_{33} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we calculate

$$Q_1 = T_3 M_2 T_3^{-1} = \begin{pmatrix} -\tau_{11} & -\tau_{12} & -\tau_{13} & -\tau_{14} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{(b_{21}+b_{31})\alpha}{b_{31}} \end{pmatrix},$$

where

$$\begin{aligned} \tau_{11} &= b_{11} + b_{33} - \frac{b_{21}\alpha}{b_{31}}, \\ \tau_{12} &= b_{12}b_{21} - b_{13}b_{31} + b_{11}b_{33} - \frac{b_{11}b_{21}\alpha}{b_{31}}, \\ \tau_{13} &= -b_{21}(-b_{12}b_{33} - b_{13}\alpha + \frac{b_{12}b_{21}\alpha}{b_{31}}), \\ \tau_{14} &= -b_{33}\left\{b_{12}b_{21} - b_{13}b_{31} + \frac{(b_{21}+b_{31})\alpha\,[-b_{11}b_{31} + (b_{21}+b_{31})\alpha]}{b_{31}^2}\right\}. \end{aligned}$$

Therefore the equivalent equation of (4.4) can be written as $(T_3T_2T_1)G_1^2(T_3T_2T_1)^T + Q_1\left[(T_3T_2T_1)\Sigma_1(T_3T_2T_1)^T\right] + \left[(T_3T_2T_1)\Sigma_1(T_3T_2T_1)^T\right]Q_1^T = 0$. From Lemma 4.2, we have

$$(T_3T_2T_1)\Sigma_1(T_3T_2T_1)^T = (b_{31}b_{33}\sigma_1^2)\overline{\Omega}_2,$$

where

$$\overline{\Omega}_2 = \begin{pmatrix} \frac{\tau_{12}}{2(\tau_{11}\tau_{12}-\tau_{13})} & 0 & -\frac{1}{2(\tau_{11}\tau_{12}-\tau_{13})} & 0 \\ 0 & \frac{1}{2(\tau_{11}\tau_{12}-\tau_{13})} & 0 & 0 \\ -\frac{1}{2(\tau_{11}\tau_{12}-\tau_{13})} & 0 & \frac{\tau_{11}}{2\tau_{13}(\tau_{11}\tau_{12}-\tau_{13})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a symmetric positive semi-definite matrix. Hence,

$$\Sigma_1 = (b_{31}b_{33}\sigma_1)^2(T_3T_2T_1)^{-1}\overline{\Omega}_2\left[(T_3T_2T_1)^{-1}\right]^T.$$

Case 1.2 : If $q_1 \neq 0$, we can find standardized transformation T_4 such that $Q_2 = T_4 M_2 T_4^{-1}$, where

$$T_4 = \begin{pmatrix} b_{31}b_{33}q_1 & j_1 & j_2 & j_3 \\ 0 & \frac{b_{31}b_{33}q_1}{b_{21}} & -q_1(b_{33}+\alpha) & b_{33}q_1 + \frac{(b_{21}+b_{31})^2\alpha^2}{b_{31}^2} \\ 0 & 0 & q_1 & -\frac{(b_{21}+b_{31})\alpha}{b_{31}} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $j_1 = -\frac{b_{31}b_{33}q_1(b_{33}+\alpha)}{b_{21}}$, $j_2 = \frac{q_1\left[b_{31}^2b_{33}(b_{33}+q_1) + b_{31}(-b_{21}+b_{31})b_{33}\alpha + (b_{21}^2+b_{21}b_{31}+b_{31}^2)\alpha^2\right]}{b_{31}^2}$, $j_3 = -b_{33}q_1(b_{33}+\alpha) - \frac{(b_{21}+b_{31})\alpha\left[b_{31}^2b_{33}q_1 + (b_{21}+b_{31})^2\alpha^2\right]}{b_{31}^3}$.

Then, we have

$$Q_2 = \begin{pmatrix} -r_1 & -r_2 & -r_3 & -r_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Likewise, (4.4) can be transformed into the following form: $(T_4T_2T_1)G_1^2(T_4T_2T_1)^T + Q_2\left[(T_4T_2T_1)\Sigma_1(T_4T_2T_1)^T\right] + \left[(T_4T_2T_1)\Sigma_1(T_4T_2T_1)^T\right]Q_2^T = 0$. From Lemma 4.1, we have

$$(T_4T_2T_1)\Sigma_1(T_4T_2T_1)^T = (b_{31}b_{33}q_1\sigma_1)^2\Omega_1,$$

where

$$\Omega_1 = \begin{pmatrix} \frac{r_2r_3-r_1r_4}{2(r_1r_2r_3-r_3^2-r_1^2r_4)} & 0 & -\frac{r_3}{2(r_1r_2r_3-r_3^2-r_1^2r_4)} & 0 \\ 0 & \frac{r_3}{2(r_1r_2r_3-r_3^2-r_1^2r_4)} & 0 & -\frac{r_1}{2(r_1r_2r_3-r_3^2-r_1^2r_4)} \\ -\frac{r_3}{2(r_1r_2r_3-r_3^2-r_1^2r_4)} & 0 & \frac{r_1}{2(r_1r_2r_3-r_3^2-r_1^2r_4)} & 0 \\ 0 & -\frac{r_1}{2(r_1r_2r_3-r_3^2-r_1^2r_4)} & 0 & \frac{r_1r_2-r_3}{2r_4(r_1r_2r_3-r_3^2-r_1^2r_4)} \end{pmatrix}.$$

is a positive definite symmetric matrix; hence,

$$\Sigma_1 = (b_{31}b_{33}q_1\sigma_1)^2(T_4T_2T_1)^{-1}\Omega_1\left[(T_4T_2T_1)^{-1}\right]^T,$$

is also a positive definite matrix.

Step 2 . Consider

$$G_2^2 + M\Sigma_2 + \Sigma_2M^T = 0. \tag{4.5}$$

Let $M_3 = T_5MT_5^{-1}$, where

$$T_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & b_{21} & 0 & 0 \\ -b_{12} & -b_{11} & -b_{13} & 0 \\ 0 & -b_{31} & -b_{33} & b_{33} \\ \alpha & 0 & 0 & -\alpha \end{pmatrix}.$$

Take $M_4 = T_6M_3T_6^{-1}$, where

$$T_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\alpha}{b_{12}} & 0 & 1 \end{pmatrix}, \ M_4 = \begin{pmatrix} 0 & b_{21} & 0 & 0 \\ -b_{12} & -b_{11} & -b_{13} & 0 \\ 0 & -\frac{b_{12}b_{31}+b_{33}\alpha}{b_{12}} & -b_{33} & b_{33} \\ 0 & q_2 & -\frac{b_{13}\alpha}{b_{12}} & -\alpha \end{pmatrix},$$

with $q_2 = \frac{\alpha(-b_{11}+\alpha)}{b_{12}}$.

Case 2.1 : If $q_2 = 0$, we let $Q_3 = T_7M_4T_7^{-1}$, where the standard transformation matrix T_7 is given by

$$T_7 = \begin{pmatrix} -\frac{b_{13}\alpha(b_{12}b_{31}+b_{33}\alpha)}{b_{12}} & j_4 & j_5 & j_6 \\ 0 & \frac{b_{13}\alpha(b_{12}b_{31}+b_{33}\alpha)}{b_{12}^2} & \frac{b_{13}\alpha(b_{33}+\alpha)}{b_{12}} & \alpha(-\frac{b_{13}b_{33}}{b_{12}}+\alpha) \\ 0 & 0 & -\frac{b_{13}\alpha}{b_{12}} & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} -r_1 & -r_2 & -r_3 & -r_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $j_4 = -\frac{b_{13}\alpha(b_{11}+b_{33}+\alpha)(b_{12}b_{31}+b_{33}\alpha)}{b_{12}^2}$, $j_5 = -\frac{b_{13}\alpha(b_{13}b_{31}+b_{33}^2+b_{33}\alpha+\alpha^2)}{b_{12}}$, $j_6 = -\alpha^3 + \frac{b_{13}b_{33}\alpha(b_{33}+2\alpha)}{b_{12}}$. Therefore, we have

$$(T_7T_6T_5)G_2^2(T_7T_6T_5)^T + M\left[(T_7T_6T_5)\Sigma_2(T_7T_6T_5)^T\right] + \left[(T_7T_6T_5)\Sigma_2(T_7T_6T_5)^T\right]M^T = 0,$$

where

$$(T_7T_6T_5)\Sigma_2(T_7T_6T_5)^T = [-\frac{b_{13}\alpha(b_{12}b_{31}+b_{33}\alpha)}{b_{12}}]^2\Omega_1.$$

Hence

$$\Sigma_2 = \frac{\alpha^2 b_{13}^2(b_{12}b_{31}+b_{33}\alpha)^2\sigma_2^2}{b_{12}^2}(T_7T_6T_5)^{-1}\Omega_1\left[(T_7T_6T_5)^{-1})\right]^T,$$

is a positive define matrix.

Case 2.2 : Consider that $q_2 \neq 0$. Let $M_5 = T_8M_4T_8^{-1}$, where

$$T_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{b_{12}q_2}{b_{12}b_{31}+b_{33}\alpha} & 1 \end{pmatrix}, M_5 = \begin{pmatrix} 0 & b_{21} & 0 & 0 \\ -b_{12} & -b_{11} & -b_{13} & 0 \\ 0 & -\frac{b_{12}b_{31}+b_{33}\alpha}{b_{12}} & b_{33}(-1-\frac{b_{12}q_2}{b_{12}b_{31}+b_{33}\alpha}) & b_{33} \\ 0 & 0 & q_3 & -\alpha+\frac{b_{12}b_{33}q_2}{b_{12}b_{31}+b_{33}\alpha} \end{pmatrix},$$

where $q_3 = -\frac{b_{13}\alpha}{b_{12}} - \frac{b_{12}b_{33}q_2}{b_{12}b_{31}+b_{33}\alpha} + \frac{b_{12}q_2(-b_{12}b_{33}q_2+b_{12}b_{31}\alpha+b_{33}\alpha^2)}{(b_{12}b_{31}+b_{33}\alpha)^2}$.

Case 2.2.1. If $q_2 \neq 0$ and $q_3 = 0$, let $Q_4 = T_9M_5T_9^{-1}$, where

$$T_9 = \begin{pmatrix} b_{12}b_{31}+b_{33}\alpha & j_7 & j_8 & j_9 \\ 0 & -\frac{b_{12}b_{31}+b_{33}\alpha}{b_{12}} & b_{33}(-1-\frac{b_{12}q_2}{b_{12}b_{31}+b_{33}\alpha}) & b_{33} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} -\tau_{21} & -\tau_{22} & -\tau_{23} & -\tau_{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\alpha+\frac{b_{12}b_{33}q_2}{b_{12}b_{31}+b_{33}\alpha} \end{pmatrix},$$

where $j_7 = b_{11}b_{31} + b_{33}(b_{31} + q_2) + \frac{b_{33}\alpha(b_{11}+b_{33})}{b_{12}}$, $j_8 = b_{13}(b_{31} + \frac{b_{33}\alpha}{b_{12}}) + (b_{33} + \frac{b_{12}b_{33}q_2}{b_{12}b_{31}+b_{33}\alpha})^2$, $j_9 = -b_{33}(b_{33}+\alpha)$, $\tau_{21} = b_{11}+b_{33}(1+\frac{b_{12}q_2}{b_{12}b_{31}+b_{33}\alpha})$, $\tau_{22} = -(-b_{12}b_{21}+b_{13}b_{31}-b_{11}b_{33}+\frac{b_{13}b_{33}\alpha}{b_{12}}-\frac{b_{11}b_{12}b_{33}q_2}{b_{12}b_{31}+b_{33}\alpha})$, $\tau_{23} = \frac{b_{12}b_{21}b_{33}[b_{12}(b_{31}+q_2)+b_{33}\alpha]}{b_{12}b_{31}+b_{33}\alpha}$, $\tau_{24} = -b_{33}\left\{\alpha(-b_{11}+\alpha) + b_{12}[b_{21} + \frac{b_{33}q_2(b_{11}b_{12}b_{31}+b_{12}b_{33}q_2-2b_{12}b_{31}\alpha+b_{11}b_{33}\alpha-2b_{33}\alpha^2)}{(b_{12}b_{31}+b_{33}\alpha)^2}]\right\}$.
Therefore, we conclude that

$$(T_9T_8T_6T_5)G_2^2(T_9T_8T_6T_5)^T + Q_4\left[(T_9T_8T_6T_5)\Sigma_2(T_9T_8T_6T_5)^T\right] + \left[(T_9T_8T_6T_5)\Sigma_2(T_9T_8T_6T_5)^T\right]Q_4^T = 0.$$

where

$$(T_9T_8T_6T_5)\Sigma_2(T_9T_8T_6T_5)^T = [(b_{12}b_{31}+b_{33}\alpha)\sigma_2]^2\,\widetilde{\Omega_2}.$$

Beside, it follows from Lemmas 4.2 that the specific form of the positive semi-definite matrix $\widetilde{\Omega_2}$ is

$$\widetilde{\Omega_2} = \begin{pmatrix} \frac{\tau_{22}}{2(\tau_{21}\tau_{22}-\tau_{23})} & 0 & -\frac{1}{2(\tau_{21}\tau_{23}-\tau_{23})} & 0 \\ 0 & \frac{1}{2(\tau_{21}\tau_{22}-\tau_{23})} & 0 & 0 \\ -\frac{1}{2(\tau_{21}\tau_{22}-\tau_{23})} & 0 & \frac{\tau_{21}}{2\tau_{23}(\tau_{21}\tau_{22}-\tau_{23})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we conclude that

$$\Sigma_2 = (b_{12}b_{31} + b_{33}\alpha)^2\sigma_2^2(T_9T_8T_6T_5)^{-1}\widetilde{\Omega_2}\left[(T_9T_8T_6T_5)^{-1}\right]^T,$$

is semi-positive definite.

Case 2.2.2. If $q_2 \neq 0, q_3 \neq 0$, we let $Q_5 = T_{10}M_5T_{10}^{-1}$, where

$$T_{10} = \begin{pmatrix} q_3(b_{12}b_{31} + b_{33}\alpha) & j_{10} & j_{11} & j_{12} \\ 0 & -\frac{q_3(b_{12}b_{31}+b_{33}\alpha)}{b_{12}} & -q_3(b_{33} + \alpha) & b_{33}q_3 + (\alpha - \frac{b_{12}b_{33}q_2}{b_{12}b_{31}+b_{33}\alpha})^2 \\ 0 & 0 & q_3 & -\alpha + \frac{b_{12}b_{33}q_2}{b_{12}b_{31}+b_{33}\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Q_5 = \begin{pmatrix} -r_1 & -r_2 & -r_3 & -r_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with

$$j_{10} = \frac{q_3(b_{11} + b_{33} + \alpha)(b_{12}b_{31} + b_{33}\alpha)}{b_{12}},$$

$$j_{11} = q_3\Big\{\alpha^2 + b_{33}(b_{33} + q_3 + \alpha) + b_{13}(b_{31} + \frac{b_{33}\alpha}{b_{12}}) + \frac{b_{12}b_{33}q_2[b_{12}b_{33}(b_{31} + q_2) - b_{12}b_{31}\alpha + b_{33}(b_{33} - \alpha)\alpha]}{(b_{12}b_{31} + b_{33}\alpha)^2}\Big\},$$

$$j_{12} = -b_{33}q_3(b_{33} + \alpha) + (-\alpha + \frac{b_{12}b_{33}q_2}{b_{12}b_{31} + b_{33}\alpha})[b_{33}q_3 + (\alpha - \frac{b_{12}b_{33}q_2}{b_{12}b_{31} + b_{33}\alpha})^2].$$

Therefore, we have

$$(T_{10}T_8T_6T_5)G_2^2(T_{10}T_8T_6T_5)^T + Q_5\left[(T_{10}T_8T_6T_5)\Sigma_2(T_{10}T_8T_6T_5)^T\right] + \left[(T_{10}T_8T_6T_5)\Sigma_2(T_{10}T_8T_6T_5)^T\right]Q_5^T = 0,$$

where

$$(T_{10}T_8T_6T_5)\Sigma_2(T_{10}T_8T_6T_5)^T = q_3^2(b_{12}b_{31} + b_{33}\alpha)^2\sigma_2^2\Omega_1.$$

Hence

$$\Sigma_2 = q_3^2(b_{12}b_{31} + b_{33}\alpha)^2\sigma_2^2(T_{10}T_8T_6T_5)^{-1}\Omega_1\left[(T_{10}T_8T_6T_5)^{-1}\right]^T,$$

is also a positive definite matrix.

Step 3. Consider

$$G_3^2 + M\Sigma_3 + \Sigma_3M^T = 0. \tag{4.6}$$

We let $M_6 = T_{11}MT_{11}^{-1}$, where

$$T_{11} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_6 = \begin{pmatrix} -b_{33} & -b_{31} & 0 & b_{33} \\ -b_{13} & -b_{11} & -b_{12} & 0 \\ 0 & b_{21} & 0 & 0 \\ 0 & 0 & \alpha & -\alpha \end{pmatrix}.$$

Let $Q_6 = T_{12}M_6T_{12}^{-1}$, where the standard transformation matrix T_{12} is given by

$$T_{12} = \begin{pmatrix} -b_{13}b_{21}\alpha & -b_{21}\alpha(b_{11}+\alpha) & -b_{12}b_{21}\alpha+\alpha^3 & -\alpha^3 \\ 0 & b_{21}\alpha & -\alpha^2 & \alpha^2 \\ 0 & 0 & \alpha & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By direct calculation, we obtain that

$$Q_6 = \begin{pmatrix} -r_1 & -r_2 & -r_3 & -r_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus (4.6) can be transformed into the following from:

$$(T_{12}T_{11})G_3^2(T_{12}T_{11})^T + Q_6\left[(T_{12}T_{11})\Sigma_3(T_{12}T_{11})^T\right] + \left[(T_{12}T_{11})\Sigma_3(T_{12}T_{11})^T\right]Q_6^T = 0,$$

where

$$(T_{12}T_{11})\Sigma_3(T_{12}T_{11})^T = (-b_{13}b_{21}\alpha\sigma_3)^2\Omega_1.$$

Thus,

$$\Sigma_3 = (-b_{13}b_{21}\alpha\sigma_3)^2(T_{12}T_{11})^{-1}\Omega_1\left[(T_{12}T_{11})^{-1}\right]^T,$$

is a positive definite matrix.

Summing up the above steps, we have $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$ is a positive definite matrix. The solution $(x(t), y(t), z(t), w(t))$ of system (1.4) has a log-normal probability density function

$$\Phi(x,y,z,w) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(\ln\frac{x}{x^*},\ln\frac{y}{y^*},\ln\frac{z}{z^*},\ln\frac{w}{w^*})\Sigma^{-1}(\ln\frac{x}{x^*},\ln\frac{y}{y^*},\ln\frac{z}{z^*},\ln\frac{w}{w^*})^T}.$$

5. Extinction

Theorem 5.1. Let $(x(t), y(t), z(t), w(t))$ be the solution of the system (1.4) with any initial value $(x(0), y(0), z(0), w(0)) \in \mathbb{R}_+^4$. If $R_0^E = \frac{b_1}{D_1(D_2+\frac{1}{2}\sigma_2^2)} < 1$, then

$$\limsup_{t\to+\infty} \frac{\ln y(t)}{t} \le (D_2 + \frac{1}{2}\sigma_2^2)(R_0^E - 1) < 0, \text{a.s.},$$

which means the extinction of $y(t)$.

Proof. Consider the following SDE:

$$\mathrm{d}\hat{x}(t) = [1 - D_1\hat{x}(t)]\mathrm{d}t + \sigma_1\hat{x}(t)\mathrm{d}B_1(t),$$

with the same initial value $\hat{x}(0) = x(0) > 0$, making use of the stochastic comparison theorem, we have

$$x(t) \leq \hat{x}(t), \text{a.s.}$$

Moreover,

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \hat{x}(s)\mathrm{d}s = \int_0^{+\infty} s \cdot h(s)\mathrm{d}s = \frac{1}{D_1},$$

where $h(s) = \left[(\frac{2}{\sigma_1^2})^{1+\frac{2D_1}{\sigma_1^2}}\Gamma^{-1}(1 + \frac{2D_1}{\sigma_1^2})\right]s^{-2(1+\frac{2D_1}{\sigma_1^2})}e^{-\frac{2}{\sigma_1^2 s}}$, $s > 0$. Applying Itô's formula to $\ln y(t)$, we have

$$\begin{aligned} \mathrm{d}\ln y &= [b_1x - (D_2 + \frac{1}{2}\sigma_2^2)]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t)] \\ &\leq [b_1\hat{x} - (D_2 + \frac{1}{2}\sigma_2^2)]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t)], \end{aligned} \tag{5.1}$$

Integrating (5.1) from 0 to t on both sides, one can see that

$$\frac{\ln y(t)}{t} \leq \frac{\ln y(0)}{t} - b_1\frac{1}{t}\int_0^t \hat{x}(s)\mathrm{d}s - (D_2 + \frac{1}{2}\sigma_2^2) + \frac{\int_0^t \sigma_2\mathrm{d}B_2(s)}{t}. \tag{5.2}$$

Applying the strong law of large numbers yields

$$\lim_{t\to+\infty} \frac{\int_0^t \sigma_2\mathrm{d}B_2(s)}{t} = 0.$$

Next, we take the superior limit on both sides of (5.2)

$$\begin{aligned} \limsup_{t\to+\infty} \frac{\ln y(t)}{t} &\leq b_1 \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \hat{x}(s)\mathrm{d}s - (D_2 + \frac{1}{2}\sigma_2^2) \\ &= \frac{b_1}{D_1} - (D_2 + \frac{1}{2}\sigma_2^2) \\ &= (D_2 + \frac{1}{2}\sigma_2^2)(R_0^E - 1) < 0. \end{aligned}$$

which implies that

$$\lim_{t\to+\infty} y(t) = 0, \quad a.s.$$

6. Numerical simulation

In this section, employing the Milstein higher-order method, we will present numerical simulations to verify the theoretical results. The discretization equations of model (1.4) are given by

$$\begin{cases} x_{k+1} = x_k + (1 - a_1x_ky_k - a_2x_kz_k - D_1x_k)\triangle t + \sigma_1x_k\sqrt{\triangle t}\eta_{1,k} + \dfrac{\sigma_1^2x_k}{2}(\eta_{1,k}^2 - 1)\triangle t, \\ y_{k+1} = y_k + (b_1x_ky_k - D_2y_k)\triangle t + \sigma_2y_k\sqrt{\triangle t}\eta_{2,k} + \dfrac{\sigma_2^2y_k}{2}(\eta_{2,k}^2 - 1)\triangle t, \\ z_{k+1} = z_k + (c_1w_k - c_2x_kz_k - D_3z_k)\triangle t + \sigma_3z_k\sqrt{\triangle t}\eta_{3,k} + \dfrac{\sigma_3^2z_k}{2}(\eta_{3,k}^2 - 1)\triangle t, \\ w_{k+1} = w_k + (\alpha y_k - \alpha w_k)\triangle t, \end{cases} \tag{6.1}$$

where $\eta_{i,k}(i = 1, 2, 3;\ k = 1, 2 \ldots, n)$ denotes independent Gaussian random variables which follow the distribution $N(0, 1)$.

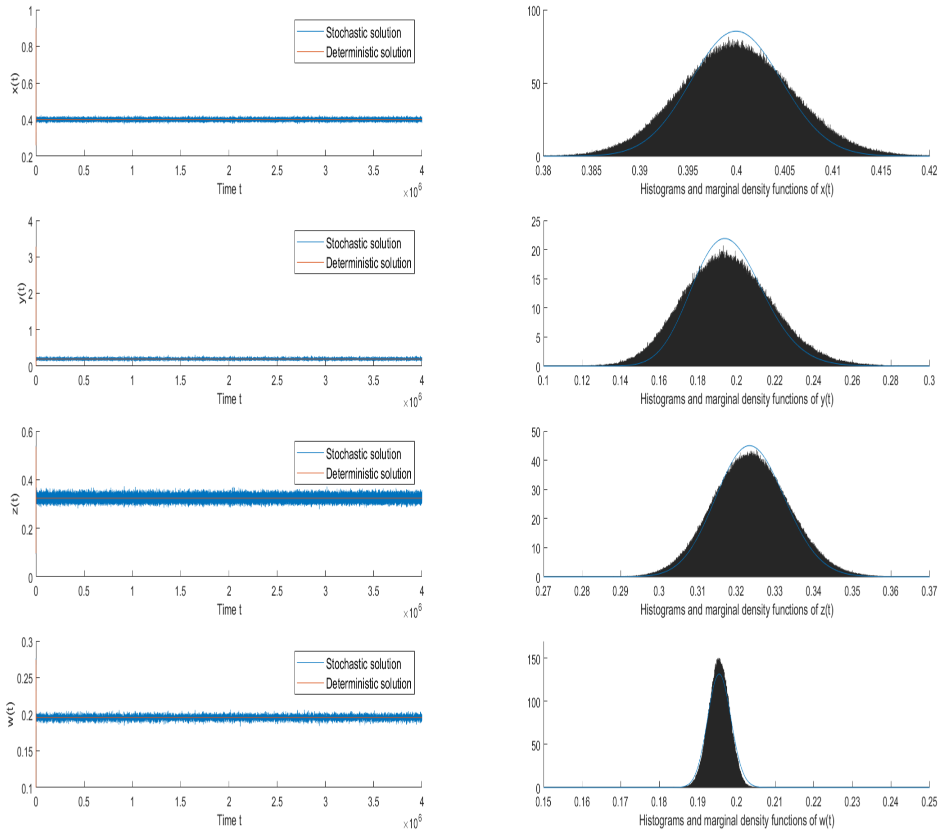

Figure 1. Left-hand column presents the numbers of x, y, z and w for system (6.1) with $(\sigma_1, \sigma_2, \sigma_3) = (0.01, 0.01, 0.05)$, and its deterministic system, respectively. Right-hand columns shows the frequency histograms and corresponding marginal density function curves of x, y, z and w, respectively.

Example 6.1. Let us choose $a_1 = 1$, $a_2 = 4$, $D_1 = 1.01$, $D_2 = 8$, $D_3 = 1.02$, $b_1 = 20, c_1 = 5$, $c_2 = 5$, $\alpha = 0.09$, $\sigma_1 = \sigma_2 = 0.01$, $\sigma_3 = 0.05$ and the initial value $(x(0), y(0), z(0), w(0)) = (0.9, 0.1, 0.1, 0.1)$. We conclude that $R_0 = \frac{b_1}{D_1 D_2} = 2.4752 > 1$ and $R_0^s = \frac{b_1}{(D_1+\frac{1}{2}\sigma_1^2)(D_2+\frac{1}{2}\sigma_2^2)} = 2.4751 > 1$. Theorem 4.1 shows that there exists an ergodic stationary distribution of stochastic model (1.4). Noting that $(x^*, y^*, z^*, w^*) = (0.4, 0.1955, 0.3236, 0.1955)$, we have following the covariance matrix

$$\Sigma = \begin{pmatrix} 1.3643e^{-04} & -4.7814e^{-06} & -2.3417e^{-04} & -1.0541e^{-04} \\ -4.7814e^{-06} & 8.7590e^{-03} & -6.2162e^{-04} & 5.2528e^{-05} \\ -2.3417e^{-04} & -6.2162e^{-04} & 7.5113e^{-04} & 2.3141e^{-04} \\ -1.0541e^{-04} & 5.2528e^{-05} & 2.3141e^{-04} & 2.4051e^{-04} \end{pmatrix}.$$

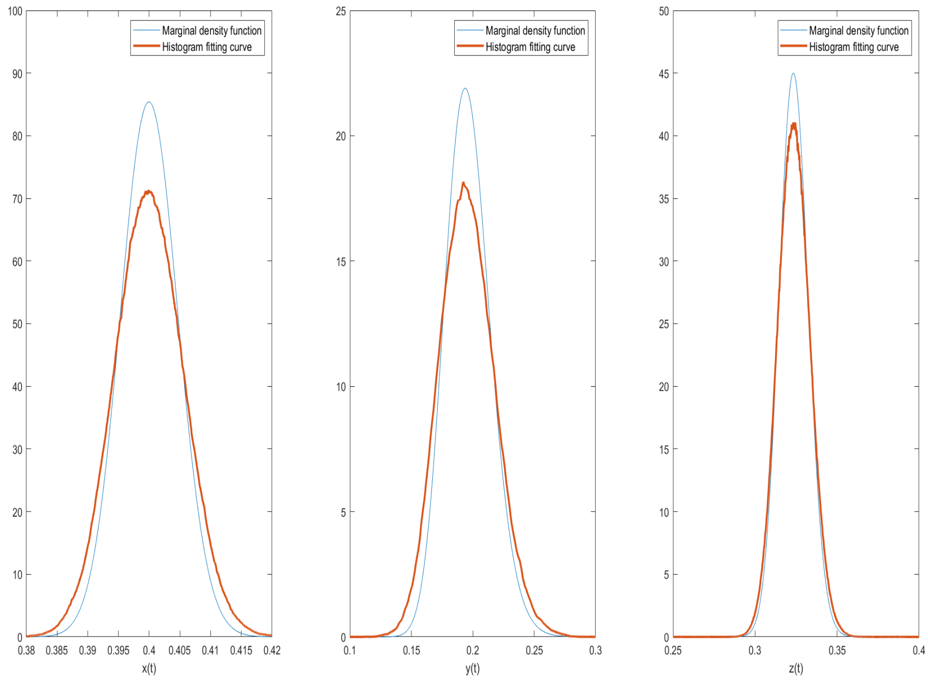

Figure 2. The marginal density functions and frequency fitting curves for x, y and z in system (1.4).

Thus, we have the corresponding probability density function

$$\Phi(x,y,z,w) = 1.0882 \times 10^5 \times e^{-\frac{1}{2}(\ln\frac{x}{2.5},\ln\frac{y}{5.1142},\ln\frac{z}{3.09},\ln\frac{w}{5.1142})\Sigma^{-1}(\ln\frac{x}{2.5},\ln\frac{y}{5.1142},\ln\frac{z}{3.09},\ln\frac{w}{5.1142})^T}.$$

As a result, $\Phi(x,y,z,w)$ has the following four marginal density functions:

$$\frac{\partial\Phi}{\partial x} = \frac{1}{x\sqrt{2\pi w_{11}}}e^{-\frac{(\ln x-\ln x^*)^2}{2w_{11}}} = \frac{1}{34.1553x}e^{-\frac{(\ln x+0.9163)^2}{2.7286e^{-04}}},$$
$$\frac{\partial\Phi}{\partial y} = \frac{1}{y\sqrt{2\pi w_{22}}}e^{-\frac{(\ln y-\ln y^*)^2}{2w_{22}}} = \frac{1}{0.2346y}e^{-\frac{(\ln y+1.632)^2}{0.0175}},$$
$$\frac{\partial\Phi}{\partial z} = \frac{1}{z\sqrt{2\pi w_{33}}}e^{-\frac{(\ln z-\ln z^*)^2}{2w_{33}}} = \frac{1}{0.0687z}e^{-\frac{(\ln z+1.1282)^2}{0.0015}},$$
$$\frac{\partial\Phi}{\partial w} = \frac{1}{w\sqrt{2\pi w_{44}}}e^{-\frac{(\ln w-\ln w^*)^2}{2w_{44}}} = \frac{1}{0.0389w}e^{-\frac{(\ln w+1.632)^2}{4.8101e^{-04}}};$$

we can conclude that system (1.2) admits a global positive stationary solution on $\mathbb{R}^4_+$; see Figure 1 and Figure 2.

Example 6.2. Let us choose $b_1 = 20$, $D_1 = 2.3$, $D_2 = 8$ and $\sigma_2 = 1.8$. Then the condition $R_0^E = 0.693 < 1$ is satisfied. Theorem 5.1 shows that MC-degrading bacteria of system (1.4) will go to extinction with probability one, which is numerically confirmed by Figure 3.

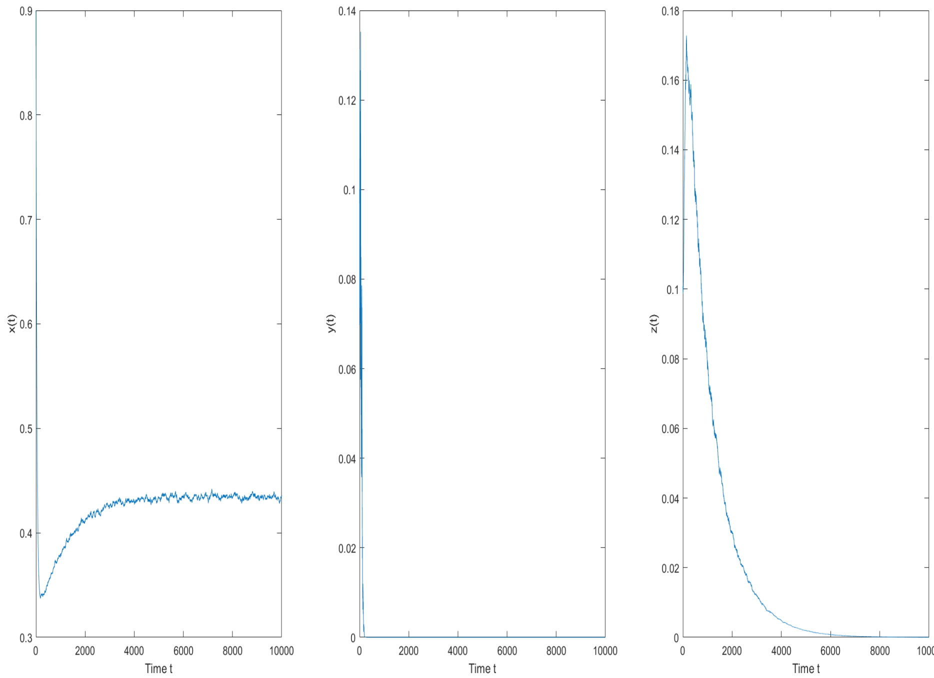

Figure 3. Corresponding numbers for solution $(x(t), y(t), z(t))$ to system (1.4) with random perturbations $(\sigma_1, \sigma_2, \sigma_3) = (0.01, 1.8, 0.05)$ and main parameters $(b_1, D_1, D_2) = (20, 2.3, 8)$.

7. Conclusions

In the current paper, we consider a stochastic Microcystins degradation model with distributed delay. We have established sufficient conditions for the existence of an ergodic stationary distribution of the positive solutions to system (1.4) by constructing a suitable stochastic Lyapunov function. The result shows that a small amount of white noise can guarantee the existence of an ergodic stationary distribution of the positive solutions to system (1.4). In addition, we have obtained the exact probability density function around a quai-equilibrium point.

Some interesting topics deserve further consideration. On the one hand, the paper focuses on the dynamical evolution and stability of the system (1.4). It should be noted that model (1.4) can potentially be solved analytically by using the Lie algebra method [25,26]. On the other hand, because our model is autonomous and only disturbed by white noise, it would be interesting to introduce telegraph noise, such as a continuous-time Markov chain, into system (1.3). Moreover, it would also be interesting to study more complicated MCs degradation models, such as multi-group MCs degradation models. We will leave these problems as our future work.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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