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Research article

Stability of traveling wave solutions for a nonlocal Lotka-Volterra model

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Abstract: In this paper, we studied the stability of traveling wave solutions of a two-species Lotka-Volterra competition model in the form of a coupled system of reaction diffusion equations with nonlocal intraspecific and interspecific competitions in space at times. First, the uniform upper bounds for the solutions of the model was proved. By using the anti-weighted method and the energy estimates, the asymptotic stability of traveling waves with large wave speeds of the system was established.

Keywords: Traveling waves; nonlocal; stability; energy estimates; global boundedness

1. Introduction

This paper is motivated by the following biological question: How do diffusion and nonlocal intraspecific and interspecific competitions affect the competition outcomes of two competing species? It is well known that if we introduced the spatial dispersal into the Lotka-Volterra competition model, traveling wave solutions are possible. Such solutions effected a smooth transition between two steady states of the space independent system, [1–7], but for the models that involve nonlocality, the study of traveling waves is challenging and the properties of the traveling waves becomes more complex. Gourley and Ruan [8] proposed a two-species competition model described by a reaction diffusion system with nonlocal terms. By using linear chain techniques and geometric singular perturbation theory, the existence of traveling waves under some conditions were proved. Some other results about the traveling waves of the Lotka-Volterra system or the similar equations with nonlocal terms can be referred to [9–13].

In this paper we consider the following Lotka-Volterra competition-diffusion system with nonlocal effects [13]:

$$\begin{cases} u_t - u_{xx} = u(1 - (\phi_1 * u) - a_1(\phi_2 * v)), \\ v_t - v_{xx} = rv(1 - (\phi_3 * v) - a_2(\phi_4 * u)), \\ u(0, x) = u_0(x), v(0, t) = v_0(x), \end{cases} \tag{1.1}$$

with

$$\phi_i * u := \int_{\mathbb{R}} \phi_i(x-y)u(y,t)dy, \qquad i = 1,2,3,4.$$

Here the functions $u(x,t)$ and $v(x,t)$ denote the densities of two competing species with respect to location x and time t, respectively. The positive parameter r is the relative growth rate of species v to species u. We assume that the kernels $\phi_i (i = 1,2,3,4)$ are bounded functions and satisfy the following properties, for all $x \in \mathbb{R}$,

(K1) $\phi_i(x) \geq 0$ and $\int_{\mathbb{R}} \phi_i(x)dx = 1$;

(K2) $\int_{\mathbb{R}} \phi_i(y)e^{\lambda y}dy < \infty$ for any $\lambda \in (0, \max\{1, \sqrt{r}\})$;

(K3) $ess \inf_{(-\delta,\delta)} \phi_i > 0$, for some $\delta > 0$.

We propose system (1.1) as an extension of the existing two-species reaction diffusion competition models [1–5]. For these two species, the terms $-u(\phi_i * u), i = 1,3$ represent intraspecific competition for resources. These two terms involve a convolution in space that arises because of the fact that the animals are moving (by diffusion) and have, therefore, not been at the same point in space at times. Thus, intraspecific competition for resources depends not simply on population density at one point in space, but on a weighted average involving values at all points in space. The terms $a_1 u(\phi_2 * v)$ and $a_2 v(\phi_4 * u)$, with a_1 and a_2 positive constants, describe the interspecific competition between these two species for resources, which also involve a convolution in space at times. In this paper, we study the weak competition case with $0 < a_1, a_2 < 1$. It is well known in this case that we have $(u,v)(t) \to (u^*, v^*)$ as $t \to \infty$ in the region $\{u, v > 0\}$.

We are interested in traveling waves of (1.1) in the form of $u(t,x) = \phi(x+ct), v(t,x) = \psi(x+ct)$ which satisfies

$$\begin{cases} c\phi_\xi - \phi_{\xi\xi} = \phi(1 - (\phi_1 * \phi) - a_1(\phi_2 * \psi)), \\ c\psi_\xi - \psi_{\xi\xi} = r\psi(1 - (\phi_3 * \psi) - a_2(\phi_4 * \phi)), \end{cases}$$

where $\xi = x + ct, t > 0, x \in \mathbb{R}$.

Han et al. [13] proved the existence of traveling wave solutions of the system (1.1) connecting the origin to some positive steady state with some minimal wave speed. Besides the existence and uniqueness of traveling waves, the stability of traveling waves is also a central question in the study of traveling waves. In contrast to the studies on the existence on the traveling waves of the nonlocal Lotka-Volterra system, the study about the stability is very minor. Lin and Ruan [14] proved the asymptotic behavior of traveling waves about a Lotka-Volterra competition system with distributed delays by using Schauder's fixed point theorem, and in [1,14], the delay does not need to be sufficiently small. In addition, if $u = 0$ or $v = 0$, the system (1.1) is the Fisher-KPP equation with a nonlocal term in [7,15–17]. Recently, there has been some great progress on traveling waves of the nonlocal Fisher-KPP equation

$$u_t - u_{xx} = \mu u(1 - \phi * u), \quad x \in \mathbb{R}. \tag{1.2}$$

Hamel and Ryzhik [16] proved uniform upper bounds for the solutions of the Cauchy problem of (1.2). After that, Tian et al. [17] proved the asymptotic stability of traveling waves for the system (1.2) with large wave speeds.

Inspired by [13, 15–17], in this paper we study the stability of traveling wave solutions of system (1.1), which describes the scenario when both intraspecific competition and interspecific competition are nonlocal with respect to space. The main mathematical challenge when studying the traveling waves for system (1.1) is that solutions do not obey the maximum principle and the comparison principle cannot be applied to the system. However, we can consider the stability of the zero solution of a perturbation equation about the traveling wave solution and use the anti-weighted method and the energy estimates to reach the expected one. Mei et al. [18] has applied this method in the Nicholsons blowies equation with diffusion, as did [17] in the Fisher-KPP equation with the nonlocal term. For this method, the key step is to establish priori estimates for solutions. Therefore, before presenting the main theorem in this paper, we first give some important preliminaries for the Cauchy problem of system (1.1).

We organize the paper in the following. In section two, we give a global bound of the solutions and some important properties of traveling waves of the system (1.1). The results on the global existence and uniqueness of the perturbation equations about traveling waves are presented in section three. The uniform boundedness for the perturbation equations is given in section four. In section five, we prove the main theorem about the asymptotic stability of traveling waves for the system (1.1). We conclude with a discussion section containing summarization and implications on our findings.

2. Global bounds of the solutions for system (1.1)

In this section, we first consider the global bounds of the solutions for system (1.1), then give some auxiliary statements of traveling waves of system (1.1).

Theorem 2.1. *Assume that the kernel functions $\phi_i, i = 1,2,3,4$ satisfy $(K1)-(K3)$. For every $r > 0$ and every nonnegative initial functions $u_0, v_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, the solution $(u(t,x), v(t,x))$ of (1.1) is globally bounded in time. For all $t > 0, x \in \mathbb{R}, u$ and v satisfy the following estimates*

$$0 \le u(t,x) \le M_{u_0}, \quad 0 \le v(t,x) \le M_{v_0},$$

where

$$M_{u_0} := e \max\{1, C_0\|u_0\|_{L^\infty}, C_0(ess \inf_{(-\delta,\delta)} \phi_1(x))^{-1}\},$$

$$M_{v_0} := e^r \max\{1, C_0\|v_0\|_{L^\infty}, C_0(ess \inf_{(-\delta,\delta)} \phi_3(x))^{-1}\},$$

where C_0 is a constant independent of u_0, v_0.

Proof. By standard parabolic estimates, the solution (u, v) is classical in $(0,+\infty) \times \mathbb{R}$ and we claim that $u(t,x), v(t,x)$ are nonnegative for every $t > 0, x \in \mathbb{R}$. Indeed, if the claim is false, without loss of generality, we assume that for $t \in (0,T]$ where T is some fixed constant, there exist constants $K, \epsilon > 0$ such that $\inf u(T,x) = -\epsilon e^{KT}$ and

$$-\epsilon e^{Kt} < u(t,x) < 0, -\epsilon e^{Kt} < v(t,x).$$

From the system (1.1), for $t \in (0,T]$, it gives

$$u_t - \triangle u = u(1 - \phi_1 * u - \phi_2 * v) \ge u(1 + 2\epsilon e^{KT}).$$

Since $u(0,x)$ is nonnegative, by the maximum principle, it gives that $u(t,x) \ge 0$. This is a contradiction. The claim holds, which gives that $u(t,x), v(t,x)$ satisfy

$$0 \le u(t,x) \le e^t\|u_0\|_{L^\infty(\mathbb{R})}, \quad 0 \le v(t,x) \le e^{rt}\|v_0\|_{L^\infty(\mathbb{R})}, \tag{2.1}$$

for every $t > 0$ and $x \in \mathbb{R}$. Let $\delta > 0$ be defined as in the assumption (K3) and introduce the local average on the scale δ, for $(t, x) \in [0, +\infty) \times \mathbb{R}$,

$$\bar{u}(t,x) = \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} u(t,y)dy, \quad \bar{v}(t,x) = \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} v(t,y)dy.$$

The functions $\bar{u}, \bar{v}$ are of class $C^\infty((0, +\infty) \times \mathbb{R})$, continuous in $[0, +\infty) \times \mathbb{R}$. Furthermore, the functions $\bar{u}, \bar{v}$ obey

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} = \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} u(t,y)(1 - (\phi_1 * u) - a_1(\phi_2 * v))(t,y)dy, \\ \bar{v}_t - \bar{v}_{xx} = r \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} v(t,y)(1 - (\phi_3 * v) - a_2(\phi_4 * u))(t,y)dy, \end{cases}$$

for every $(t, x) \in (0, +\infty) \times \mathbb{R}$. Since the righthand side of the above equations belong to $L^\infty((a, b) \times \mathbb{R})$ for every $0 \le a < b < +\infty$, the functions $||\bar{u}(t, \cdot)||_{L^\infty(\mathbb{R})}$ and $||\bar{v}(t, \cdot)||_{L^\infty(\mathbb{R})}$ are continuous on $[0, +\infty)$.

Owing to the assumption (K3), there exists $\eta > 0$ such that

$$\phi_i \ge \eta > 0 \qquad \text{a.e.} \quad \text{in} \quad (-\delta, \delta), \tag{2.2}$$

and let M be any positive real number such that

$$M = \min\{M_{\bar{u}}, M_{\bar{v}}\} > \max(\delta||u_0||_{L^\infty(\mathbb{R})}, \delta||v_0||_{L^\infty(\mathbb{R})}, \frac{1}{\eta}). \tag{2.3}$$

We now show that $||\bar{u}(t, \cdot)||_{L^\infty(\mathbb{R})} \le M_{\bar{u}}, ||\bar{v}(t, \cdot)||_{L^\infty(\mathbb{R})} \le M_{\bar{v}}$ for all $t > 0$, by contradiction. Assume that this is false. Since $||\bar{u}(t, \cdot)||_{L^\infty(\mathbb{R})}$ is continuous in t on $[0, +\infty)$ and

$$||\bar{u}(0, \cdot)||_{L^\infty(\mathbb{R})} \le \delta||u_0||_{L^\infty(\mathbb{R})} < M_{\bar{u}},$$

there exists $t_0 > 0$ such that $||\bar{u}(t_0, \cdot)||_{L^\infty(\mathbb{R})} = M_{\bar{u}}$ and $||\bar{u}(t, \cdot)||_{L^\infty(\mathbb{R})} < M_{\bar{u}}$ for all $t \in [0, t_0)$. Since $\bar{u}$ is nonnegative, there exists a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ such that $\bar{u}(t_0, x_n) \to M_{\bar{u}}$ as $n \to +\infty$. We define the translations

$$u_n(t, x) = u(t, x + x_n), \qquad \bar{u}_n(t, x) = \bar{u}(t, x + x_n)$$

for $n \in \mathbb{N}$ and $(t, x) \in (0, +\infty)\times\mathbb{R}$. From standard parabolic estimates, the sequences $(u_n)_{n\in\mathbb{N}}$ and $(\bar{u}_n)_{n\in\mathbb{N}}$ are bounded in $C^k_{loc}((0, +\infty) \times \mathbb{R})$ for every $k \in \mathbb{N}$; they converge in these spaces, up to extraction of a subsequence, to some nonnegative functions u_∞ and $\bar{u}_\infty$ of class $C^\infty((0, +\infty) \times \mathbb{R})$, such that

$$\bar{u}_\infty = \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} u_\infty(t,y)dy$$

and

$$(\bar{u}_\infty)_t = (\bar{u}_\infty)_{xx} + \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} u_\infty(t,y)(1 - (\phi_1 * u_\infty)(t,y) - a_1(\phi_2 * v_\infty)(t,y))dy$$

for every $(t, x) \in (0, +\infty) \times \mathbb{R}$. The passage to the limit in the integral terms is possible due to the local uniform convergence of u_n, v_n to u_∞, v_∞ in $(0, +\infty) \times \mathbb{R}$. Furthermore, we have

$$0 \le u_\infty \le M_{\bar{u}},$$

for every $0 < t \leq t_0$ and $x \in \mathbb{R}$, and $\bar{u}_\infty(t_0, 0) = M_{\bar{u}}$. Therefore, we have

$$(\bar{u}_\infty)_t(t_0, 0) \geq 0, \qquad (\bar{u}_\infty)_{xx}(t_0, 0) \leq 0.$$

Hence,

$$\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} u_\infty(t, y)(1 - (\phi_1 * u_\infty)(t, y) - a_1(\phi_2 * v_\infty)(t, y))dy \geq 0.$$

If

$$(\phi_1 * u_\infty)(t_0, \cdot) + a_1(\phi_2 * v_\infty)(t_0, \cdot) > 1 \tag{2.4}$$

everywhere in $[-\delta/2, \delta/2]$, then the continuous function

$$U = u_\infty(t_0, \cdot)(1 - (\phi_1 * u_\infty)(t_0, \cdot) - a_1(\phi_2 * v_\infty)(t_0, \cdot))$$

would be nonpositive on $[-\delta/2, \delta/2]$. Since its integral over $[-\delta/2, \delta/2]$ is nonnegative, the function U would be identically equal to zero on $[-\delta/2, \delta/2]$. Moreover, it follows from (2.3) that $u_\infty(t_0, \cdot) = 0$ on $[-\delta/2, \delta/2]$. Hence $\bar{u}_\infty(t_0, 0) = 0$, which contradicts to the assumption that $\bar{u}_\infty(t_0, 0) = M_{\bar{u}} > 0$. Therefore, there is a real number $y_0 \in [-\delta/2, \delta/2]$ such that

$$(\phi_1 * u_\infty)(t_0, y_0) + a_1(\phi_2 * v_\infty)(t_0, y_0) \leq 1.$$

Since both functions $\phi_i, i = 1, 2$ and u_∞, v_∞ are nonnegative, from (2.1), it gives that

$$\begin{aligned} 1 &\geq (\phi_1 * u_\infty)(t_0, y_0) + a_1(\phi_2 * v_\infty)(t_0, y_0) \geq (\phi_1 * u_\infty)(t_0, y_0) \\ &\geq \int_{-\delta}^{\delta} \phi_1(y)u_\infty(t_0, y_0 - y)dy \geq \eta \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} u_\infty(t_0, y)dy \\ &= \eta\bar{u}_\infty(t_0, 0) = \eta M_{\bar{u}}. \end{aligned}$$

This contradicts to the definition (2.3).

Hence, we obtain that $\|\bar{u}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq M_{\bar{u}}$ for all $t \geq 0$. Since u is nonnegative, this means that

$$0 \leq \int_{x-\delta/2}^{x+\delta/2} u(t, y)dy \leq M_{\bar{u}}, \tag{2.5}$$

for every $t \geq 0$ and $x \in \mathbb{R}$. To gain a global bound for u, we fix an arbitrary time $s \geq 1$ and then for every $x \in \mathbb{R}$, by the maximum principle, it gives that

$$0 \leq u(s, x) \leq w(s, x),$$

where w is the solution of the equation

$$w_t = w_{xx} + w$$

with the initial condition at time $s - 1$ given by $w(s - 1, \cdot) = u(s - 1, \cdot)$. It then follows from (2.5) that, for every $x \in \mathbb{R}$,

$$0 \leq u(s, x) \leq e \int_{-\infty}^{+\infty} \frac{e^{-y^2/4}}{\sqrt{4\pi}} u(s - 1, x - y)dy \leq \frac{2eM_{\bar{u}}}{\sqrt{4\pi}} \sum_{k\in\mathbb{N}} e^{-\delta^2k^2/4} < +\infty,$$

which implies that u is globally bounded. Using the same method, we also prove that v is global bounded. □

Theorem 2.2. *(see [13]) Assume that* $0 < a_1, a_2 < 1$, *and the kernel* $\phi_i, i = 1,2,3,4$ *satisfy (K1)-(K3), then, for any* $c > c^* = \max\{2, 2\sqrt{r}\}$, *there exists a traveling wave solution* (c, ϕ, ψ) *to the following system*

$$\begin{cases} c\phi'(\xi) - \phi''(\xi) = \phi(\xi)(1 - (\phi_1 * \phi)(\xi) - a_1(\phi_2 * \psi)(\xi)), \\ c\psi'(\xi) - \psi''(\xi) = r\psi(\xi)(1 - (\phi_3 * \psi)(\xi) - a_2(\phi_4 * \phi)(\xi)), \\ \phi(-\infty) = \psi(-\infty) = 0, \lim_{\xi\to\infty}(\phi(\xi) + \psi(\xi)) > 0. \end{cases} \tag{2.6}$$

The uniform upper bound of the traveling waves $\phi(\xi), \psi(\xi), \forall c \in (c^*, +\infty)$, *are given by*

$$0 \le \phi(\xi), \psi(\xi) \le \max\left\{\frac{4}{3}\left(\int_{-\sqrt{\frac{1}{2}}}^{0} \phi_1(y)dy\right)^{-1}, \frac{4}{3}\left(\int_{-\sqrt{\frac{1}{2r}}}^{0} \phi_3(y)dy\right)^{-1}\right\} := M_1.$$

Corollary 2.3. *Let* $(\phi(\xi), \psi(\xi))$ *be the traveling wave solution of the system (1.1) with* $c > c^*$ *established by Theorem 2.2, then* $|\phi'(\xi)|, |\psi'(\xi)|$ *are also uniformly bounded.*

Proof. When $c > \max\{2, 2\sqrt{r}\}$, the bounded solutions $\phi(\xi), \psi(\xi)$ satisfy

$$\phi(\xi) = \frac{1}{\lambda_2 - \lambda_1}\int_{\xi}^{\infty}(e^{\lambda_1(\xi-s)} - e^{\lambda_2(\xi-s)})\phi(s)[(\phi_1 * \phi) + a_1(\phi_2 * \psi))](s)ds$$

$$\psi(\xi) = \frac{r}{\lambda_4 - \lambda_3}\int_{\xi}^{\infty}(e^{\lambda_3(\xi-s)} - e^{\lambda_4(\xi-s)})\psi(s)[(\phi_3 * \psi) + a_2(\phi_4 * \phi))](s)ds,$$

where $0 < \lambda_1 < 1 < \lambda_2$ are roots of $\lambda^2 - c\lambda + 1 = 0$ and $0 < \lambda_3 < \sqrt{r} < \lambda_4$ are roots of $\lambda^2 - c\lambda + r = 0$. Hence, we have

$$\phi'(\xi) = \frac{1}{\lambda_2 - \lambda_1}\int_{\xi}^{\infty}(\lambda_1 e^{\lambda_1(\xi-s)} - \lambda_2 e^{\lambda_2(\xi-s)})\phi(s)[(\phi_1 * \phi) + a_1(\phi_2 * \psi))](s)ds$$

$$\psi'(\xi) = \frac{r}{\lambda_4 - \lambda_3}\int_{\xi}^{\infty}(\lambda_3 e^{\lambda_3(\xi-s)} - \lambda_4 e^{\lambda_4(\xi-s)})\psi(s)[(\phi_3 * \psi) + a_2(\phi_4 * \phi))](s)ds,$$

then we get

$$|\phi'(\xi) - \lambda_1\phi(\xi)| = |\int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}\phi(s)[(\phi_1 * \phi) + a_1(\phi_2 * \psi))](s)ds| \le \frac{2M_1^2}{\lambda_2},$$

which indicates that

$$|\phi'(\xi)| \le |\lambda_1\phi(\xi)| + 2M_1^2 \le M_1(1 + 2M_1).$$

Using the same process, we also have

$$|\phi'(\xi)| \le |\lambda_1\phi(\xi)| + 2\sqrt{r}M_1^2 \le \sqrt{r}M_1(1 + 2M_1).$$

□

Finally, from above results, we can assume

$$0 \le \phi(\xi), \psi(\xi) \le M_1, 0 \le |\phi'(\xi)|, |\psi'(\xi) \le M_2 := \max\{M_1(1+2M_1), \sqrt{r}M_1(1+2M_1)\},$$

$$0 \le u \le M_{u_0}, 0 \le v \le M_{v_0},$$

and denote

$$\begin{aligned}
c_{1,u_0,v_0} &= \frac{\lambda_0^2 + 1 + 2(1+a_1)M_{u_0} + a_1 M_{v_0} + \left(\frac{13}{4} + 4a_1\right)M_1 + \frac{1}{2}(1+a_1)M_2}{\lambda_0}, \\
c_{2,u_0,v_0} &= \frac{\lambda_0^2 + r + r[2(1+a_1)M_{v_0} + a_1 M_{u_0} + (\frac{13}{4} + 4a_2)M_1 + \frac{1}{2}(1+a_2)M_2]}{\lambda_0}, \\
c_{3,u_0,v_0} &= \frac{1}{\lambda_0}\{\lambda_0^2 + 1 + \frac{1}{2}[(2 + \lambda_0 + 2\int_{\mathbb{R}} \phi_1(y)e^{-\lambda_0 y}dy + a_1\int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy)M_{u_0} \\
&\quad +2a_1 M_{v_0} + (6 + \lambda_0 + 5a_1)M_1 + 2\int_{\mathbb{R}} \phi_1(y)e^{-\lambda_0 y}dy M_1 \\
&\quad +a_1\int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy M_1 + M_2 + \frac{1}{2}M_2\int_{\mathbb{R}} \phi_1(y)e^{-\lambda_0 y}dy \\
&\quad +a_1 M_2\int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy + \frac{r}{2}a_2 M_1] + \frac{r}{2}a_2 M_1\int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy\}, \\
c_{4,u_0,v_0} &= \frac{1}{\lambda_0}\{\lambda_0^2 + r + \frac{r}{2}[(2 + \lambda_0 + 2\int_{\mathbb{R}} \phi_3(y)e^{-\lambda_0 y}dy + a_2\int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy)M_{v_0} \\
&\quad +2a_2 M_{u_0} + (6 + \lambda_0 + 5a_2)M_1 + 2\int_{\mathbb{R}} \phi_3(y)e^{-\lambda_0 y}dy M_1 \\
&\quad +a_2\int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy M_1 + M_2 + \frac{1}{2}M_2\int_{\mathbb{R}} \phi_3(y)e^{-\lambda_0 y}dy \\
&\quad +a_2 M_2\int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy + \frac{1}{2r}a_1 M_1] + \frac{1}{2}a_1 M_1\int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}\mathrm{dy}\},
\end{aligned} \tag{2.7}$$

which will be used in the next section, λ_0 is defined in (3.2) in the next section.

3. Global existence and uniqueness

This section is devoted to prove the global existence and uniqueness of the solutions for the Cauchy problem (3.1).

Let $p(t,\xi) = u(t,\xi - ct) - \phi(\xi), q(t,\xi) = v(t,\xi - ct) - \psi(\xi)$, then by (1.1) and (2.6), the perturbation system can be written as

$$\begin{cases}
p_t + cp_\xi - p_{\xi\xi} \\
= p - p(\phi_1 * p) - p(\phi_1 * \phi) - \phi(\phi_1 * p) - a_1 p(\phi_2 * q) - a_1 p(\phi_2 * \psi) - a_1\phi(\phi_2 * q), \\
q_t + cq_\xi - q_{\xi\xi} \\
= r(q - q(\phi_3 * q) - q(\phi_3 * \psi) - \psi(\phi_3 * q) - a_2 q(\phi_4 * p) - a_2 q(\phi_4 * \phi) - a_2\psi(\phi_4 * p)).
\end{cases} \tag{3.1}$$

Define a weighted function $w(\xi)$ as the following:

$$w(\xi) = e^{-2\lambda_0\xi}, \quad \xi = x + ct, \quad \lambda_0 \in (0, \sqrt{r}). \tag{3.2}$$

Let

$$\|v\|_{L^2_w} = \Big(\int_{\mathbb{R}} w(x)|v(x)|^2 dx\Big)^{\frac{1}{2}}, \quad \|v\|_{H^k_w} = \Big(\sum_{i=0}^{k}\int_{\mathbb{R}} w(x)\Big|\frac{d^i}{dx^i}v(x)\Big|^2 dx\Big)^{1/2}.$$

Let $\|\cdot\|_C$ denote the supremum norm in $UC(\mathbb{R})$, where $u \in UC(\mathbb{R})$ implies that u is continuous and bounded. Let $0 < T < \infty$ be a number and B be a Banach spcae. We denote by $C([0,T], B)$ the spcae of the B valued continuous functions on $[0,T]$ with the norm

$$\|u\|_{C([0,T],B)} = \max_{t\in[0,T]} \|u(t)\|_B.$$

Similarly, denote $L^2([0,T], B)$ as the space of the B valued L^2- functions on $[0,T]$ with the norm

$$\|u\|^2_{L^2([0,T].B)} = \int_0^T \|u(t)\|_B^2 dt.$$

For $0 < T < \infty$, define $u \in C_{unif}[0,T]$ as follows: $u \in C([0,T]\times\mathbb{R})$ such that $\lim_{x\to+\infty} u(t,x)$ exists uniformly in $t \in [0,T]$ and $\lim_{x\to+\infty} u_x(t,x) = \lim_{x\to+\infty} u_{xx}(t,x) = 0$ uniformly in $t \in [0,T]$. Denote

$$X_0 := \Big\{u_0\Big| u_0 \in H^2_w(\mathbb{R}) \cap UC(\mathbb{R}), \lim_{x\to+\infty} u_0(x) = 0\Big\}$$

with the norm

$$\mathcal{M}^2_{u_0}(0) = \|u_0\|_C^2 + \|\sqrt{w}u_0\|^2_{H^1}.$$

We also denote

$$X(0,T) := \{u|u \in C_{unif}[0,T) \cap C([0,T), UC(\mathbb{R}) \cap H^1_w(\mathbb{R})) \cap L^2([0,T), H^2_w(\mathbb{R}))\},$$

with the norm

$$\mathcal{M}^2_u(T) := \sup_{t\in(0,T)} (\|u(t)\|^2_C + \|\sqrt{w}u(t)\|^2_{H^1}) + \int_0^T \|(\sqrt{w}u)(s)\|^2_{H^2} ds.$$

In particular, for any $T \in (0,+\infty)$, denote $X(0,\infty)$

$$:= \{u|u \in C_{unif}[0,T) \cap C([0,T), UC(\mathbb{R}) \cap H^1_w(\mathbb{R})) \cap L^2([0,+\infty), H^2_w(\mathbb{R})) \cap C([0,+\infty)\times\mathbb{R})\}.$$

Proposition 3.1. *(Global existence) Assume that assumptions (K1)-(K3) hold and* $0 < a_1, a_2 < 1, .$ *Let* $(\phi(x+ct), \psi(x+ct))$ *be a given traveling wave solution of (1.1) with speed* $c > \max\{2, 2\sqrt{r}\}$, *where* (c,ϕ,ψ) *satisfies*

$$\begin{cases} c\phi'(\xi) - \phi''(\xi) = \phi(\xi)(1 - (\phi_1 * \phi)(\xi) - a_1(\phi_2 * \psi)(\xi)), \\ c\psi'(\xi) - \psi''(\xi) = r\psi(\xi)(1 - (\phi_3 * \psi)(\xi) - a_2(\phi_4 * \phi)(\xi)), \\ \phi(-\infty) = \psi(-\infty) = 0, \phi(\infty) = k_1 \ge 0, \psi(\infty)) = k_2 \ge 0. \end{cases}$$

Suppose further that the positive initial value (u_0, v_0) *satisfies* $\max\{c_{1,u_0,v_0}, c_{2,u_0,v_0}, c_{3,u_0,v_0}, c_{4,u_0,v_0}\} < c$ *and the initial perturbation* $(p_0(x), q_0(x)) \in X_0$, *where* $c_{1,u_0,v_0}, c_{2,u_0,v_0}, c_{3,u_0,v_0}$ c_{4,u_0,v_0} *are defined in* (2.7). *System (3.1) has a unique global solution* $(p(t,\xi), q(t,\xi))$, *which belongs to* $X(0,T)$ *for any* $T > 0$ *and satisfies*

$$\mathcal{M}^2_p(T) + \mathcal{M}^2_q(T) \le C_T(\mathcal{M}^2_{p_0}(0) + \mathcal{M}^2_{q_0}(0)),$$

where $C_T > 0$ *is a constant depending on* T.

Proof. We first show the local existence and uniqueness of solutions of the system (3.1). It can be proved by the well-known iteration technique. It is obvious that $p_0, q_0 \in X_0$. For $0 < t_0 \ll 1$, let

$$Y(0,t_0) = \{p, q \in X(0,t_0) | p(0,x) = p_0 \in X_0, q(0,x) = q_0 \in X_0\}.$$

Let $p^0(t,\xi), q^0(t,\xi) \in Y(0,t_0)$, then we define the iteration $(p^{n+1}, q^{n+1}) = \mathcal{T}(p^n, q^n)$ for $n \geq 0$ by

$$\begin{cases} p_t^{n+1} + cp_\xi^{n+1} - p_{\xi\xi}^{n+1} = G(p^n, q^n), \\ q_t^{n+1} + cq_\xi^{n+1} - q_{\xi\xi}^{n+1} = H(p^n, q^n), \\ p^{n+1}(0,\xi) = p_0(\xi), q^{n+1}(0,\xi) = q_0(\xi), \end{cases} \tag{3.3}$$

where

$$G(p^n, q^n) = p^n - p^n(\phi_1 * p^n) - p^n(\phi_1 * \phi) - \phi(\phi_1 * p^n) - a_1 p^n(\phi_2 * q^n) - a_1 p^n(\phi_2 * \psi) - a_1\phi(\phi_2 * q^n),$$

and

$$\begin{aligned} H(p^n, q^n) = r(&q^n - q^n(\phi_3 * q^n) - q^n(\phi_3 * \psi) - \psi(\phi_3 * q^n) - a_2 q^n(\phi_4 * p^n) - a_2 p^n(\phi_4 * \phi) \\ &- a_2\psi(\phi_4 * p^n)). \end{aligned}$$

Thus system (3.3) can be expressed in the integral form

$$\begin{cases} p^{n+1}(t,\xi) = \int_{\mathbb{R}} \Phi(t,\eta) p_0(\xi-\eta) d\eta + \int_0^t \int_{\mathbb{R}} \Phi(t-s,\eta) G(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta ds, \\ q^{n+1}(t,\xi) = \int_{\mathbb{R}} \Phi(t,\eta) q_0(\xi-\eta) d\eta + \int_0^t \int_{\mathbb{R}} \Phi(t-s,\eta) H(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta ds, \end{cases} \tag{3.4}$$

where $\Phi(t,\eta) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(\eta+ct)^2}{4t}}$.

In the following we prove that $p^{n+1}, q^{n+1} \in Y(0,t_0)$. Since $p^n, q^n \in Y(0,t_0)$, then $p^n, q^n \in C_{unif}[0,t_0]$. Thus, $\lim_{\xi\to\infty} p^n(t,\xi)$ and $\lim_{\xi\to\infty} q^n(t,\xi)$ exist uniformly for $t \in [0,t_0]$ and $\lim_{\xi\to\infty} \partial_\xi^k p^n(t,\xi) = 0, \lim_{\xi\to\infty} \partial_\xi^k q^n(t,\xi) = 0$ exist uniformly in t for $k = 1, 2$. Note that $|p^n(t,\xi)| \leq M_{u_0} + M_1, |q^n(t,\xi)| \leq M_{v_0} + M_1$, then we have

$$\begin{aligned} &|G(p^n(t,\xi), q^n(t,\xi))| \leq |p^n(t,\xi)| + |p^n(t,\xi) \int_{\mathbb{R}} \phi_1(y) p^n(t,\xi-y) dy| \\ &+ |p^n(t,\xi) \int_{\mathbb{R}} \phi_1(y)\phi(t,\xi-y) dy| + a_1 |p^n(t,\xi) \int_{\mathbb{R}} \phi_2(y) q^n(t,\xi-y) dy| \\ &+ a_1 |p^n(t,\xi) \int_{\mathbb{R}} \phi_2(y)\psi(t,\xi-y) dy| + a_1 |\phi(t,\xi) \int_{\mathbb{R}} \phi_2(y) q^n(t,\xi-y) dy| \\ &\leq [1 + (1+a_1)M_{u_0} + 2(1+a_1)M_1] |p^n(t,\xi)| + a_1 M_1 | \int_{\mathbb{R}} \phi_2(y) q^n(t,\xi-y) dy|. \end{aligned} \tag{3.5}$$

Similarly,

$$|H(p^n(t,\xi), q^n(t,\xi))|$$

$$\leq r[1 + (1+a_2)M_{v_0} + 2(1+a_2)M_1] |q^n(t,\xi)| + a_2 M_1 | \int_{\mathbb{R}} \phi_4(y) p^n(t,\xi-y) dy|. \tag{3.6}$$

In addition, it follows from (3.4) that

$$\begin{aligned}
&\lim_{\xi\to\infty} p^{n+1}(t,\xi)\\
&=\int_{\mathbb{R}}\lim_{\xi\to\infty}\Phi(t,\eta)p_0(\xi-\eta)d\eta+\int_0^t\int_{\mathbb{R}}\Phi(t-s,\eta)\lim_{\xi\to\infty}G(p^n(s,\xi-\eta),q^n(s,\xi-\eta))d\eta ds\\
&=p_0(\infty)\int_{\mathbb{R}}\Phi(t,\eta)d\eta+\int_0^t G(p^n(s,\infty),q^n(s,\infty))\int_{\mathbb{R}}\Phi(t-s,\eta)d\eta=:p^{n+1}(t,\infty),
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
&\lim_{\xi\to\infty} q^{n+1}(t,\xi)\\
&=\int_{\mathbb{R}}\lim_{\xi\to\infty}\Phi(t,\eta)q_0(\xi-\eta)d\eta+\int_0^t\int_{\mathbb{R}}\Phi(t-s,\eta)\lim_{\xi\to\infty}H(p^n(s,\xi-\eta),q^n(s,\xi-\eta))d\eta ds\\
&=q_0(\infty)\int_{\mathbb{R}}\Phi(t,\eta)d\eta+\int_0^t H(p^n(s,\infty),q^n(s,\infty))\int_{\mathbb{R}}\Phi(t-s,\eta)d\eta=:q^{n+1}(t,\infty),
\end{aligned}\tag{3.8}$$

uniformly with respect to $t\in[0,t_0]$.

Meanwhile, we have

$$\begin{aligned}
&\lim_{\xi\to\infty}\sup_{0\le t\le t_0}|p^{n+1}(t,\xi)-p^{n+1}(t,\infty)|\\
&\le\lim_{\xi\to\infty}\sup_{0\le t\le t_0}\left|\int_{\mathbb{R}}\Phi(t,\eta)p_0(\xi-\eta)d\eta-\int_{\mathbb{R}}\Phi(t,\eta)p_0(\infty)d\eta\right|\\
&+\lim_{\xi\to\infty}\sup_{0\le t\le t_0}\left|\int_0^t\int_{\mathbb{R}}\Phi(t-s,\eta)G(p^n(s,\xi-\eta),q^n(s,\xi-\eta))d\eta ds\right.\\
&\left.-\int_0^t\int_{\mathbb{R}}\Phi(t-s,\eta)G(p^n(s,\infty),q^n(s,\infty))d\eta ds\right|=0,\\
&\lim_{\xi\to\infty}\sup_{0\le t\le t_0}|q^{n+1}(t,\xi)-q^{n+1}(t,\infty)|\\
&\le\lim_{\xi\to\infty}\sup_{0\le t\le t_0}\left|\int_{\mathbb{R}}\Phi(t,\eta)q_0(\xi-\eta)d\eta-\int_{\mathbb{R}}\Phi(t,\eta)q_0(\infty)d\eta\right|\\
&+\lim_{\xi\to\infty}\sup_{0\le t\le t_0}\left|\int_0^t\int_{\mathbb{R}}\Phi(t-s,\eta)H(p^n(s,\xi-\eta),q^n(s,\xi-\eta))d\eta ds\right.\\
&\left.-\int_0^t\int_{\mathbb{R}}\Phi(t-s,\eta)H(p^n(s,\infty),q^n(s,\infty))d\eta ds\right|=0.
\end{aligned}$$

Because

$$\Phi(\eta,t)|_{\pm\infty}=0,\qquad \partial_\eta\Phi(\eta,t)|_{\pm\infty}=0,$$

we can prove that for $k=1,2$

$$\lim_{\xi\to\infty}\partial_\xi^k p^{n+1}(t,\xi):=\int_{\mathbb{R}}\partial_\eta^k\Phi(t,\eta)\lim_{\xi\to\infty}p_0(\xi-\eta)d\eta$$

$$+\int_0^t \int_{\mathbb{R}} \partial_\eta^k \Phi(t-s,\eta) \lim_{\xi\to\infty} G(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta ds$$

$$= 0, \text{uniformly with respect to } t \in (0, t_0]. \tag{3.9}$$

$$\lim_{\xi\to\infty} \partial_\xi^k q^{n+1}(t,\xi) := \int_{\mathbb{R}} \partial_\eta^k \Phi(t,\eta) \lim_{\xi\to\infty} q_0(\xi-\eta) d\eta$$

$$+\int_0^t \int_{\mathbb{R}} \partial_\eta^k \Phi(t-s,\eta) \lim_{\xi\to\infty} H(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta ds$$

$$= 0, \text{uniformly with respect to } t \in (0, t_0]. \tag{3.10}$$

From (3.4)-(3.6) and the property of the heat kernel $\int_{\mathbb{R}} \Phi(t,\eta)d\eta = 1$, we have

$$||p^{n+1}(t)||_C \leq ||p_0||_C + Ct_0 \sup_{t\in(0,t_0]} ||p^n(t)||_C + +Ct_0 \sup_{t\in(0,t_0]} ||q^n(t)||_C, \tag{3.11}$$

$$||q^{n+1}(t)||_C \leq ||q_0||_C + Ct_0 \sup_{t\in(0,t_0]} ||p^n(t)||_C + +Ct_0 \sup_{t\in(0,t_0]} ||q^n(t)||_C. \tag{3.12}$$

By (3.7)-(3.10), it implies that $p^{n+1}, q^{n+1} \in C_{unif}[0, t_0]$.

In the following we show $p^{n+1}, q^{n+1} \in C([0, t_0), UC(\mathbb{R})) \cap H_w^1(\mathbb{R}) \cap L^2([0, t_0), H_w^2(\mathbb{R}))$.

Multiplying the first equation of (3.3) by wp^{n+1} and the second equation of (3.3) by wq^{n+1}, we have,

$$\begin{cases} wp^{n+1}p_t^{n+1} + cwp^{n+1}p_\xi^{n+1} - wp^{n+1}p_{\xi\xi}^{n+1} \\ = wp^{n+1}[p^n - p^n(\phi_1 * p^n) - p^n(\phi_1 * \phi) - \phi(\phi_1 * p^n) - a_1 p^n(\phi_2 * q^n) - a_1 p^n(\phi_2 * \psi) \\ -a_1\phi(\phi_2 * q^n)], \\ wq^{n+1}q_t^{n+1} + cwq^{n+1}q_\xi^{n+1} - wq^{n+1}q_{\xi\xi}^{n+1} \\ = wq^{n+1}[r(q^n - q^n(\phi_3 * q^n) - q^n(\phi_3 * \psi) - \psi(\phi_3 * q^n) - a_2 q^n(\phi_4 * p^n) - a_2 p^n(\phi_4 * \phi) \\ -a_2\psi(\phi_4 * p^n))]. \end{cases} \tag{3.13}$$

Since

$$wp^{n+1}p_t^{n+1} + cwp^{n+1}p_\xi^{n+1} - wp^{n+1}p_{\xi\xi}^{n+1}$$

$$= \{\frac{1}{2}w(p^{n+1})^2\}_t + \{\frac{c}{2}w(p^{n+1})^2\}_\xi - \frac{c}{2}\frac{w'}{w}w(p^{n+1})^2 - \{(wp^{n+1}p_\xi^{n+1})_\xi - w'p^{n+1}p_\xi^{n+1} - w(p_\xi^{n+1})^2\}$$

$$wq^{n+1}q_t^{n+1} + cwq^{n+1}q_\xi^{n+1} - wq^{n+1}q_{\xi\xi}^{n+1}$$

$$= \{\frac{1}{2}w(q^{n+1})^2\}_t + \{\frac{c}{2}w(q^{n+1})^2\}_\xi - \frac{c}{2}\frac{w'}{w}w(q^{n+1})^2 - \{(wq^{n+1}q_\xi^{n+1})_\xi - w'q^{n+1}q_\xi^{n+1} - w(q_\xi^{n+1})^2\},$$

and $\{\frac{c}{2}w(p^{n+1})^2 - wp^{n+1}p_\xi^{n+1}\}|_{-\infty}^{+\infty} = 0, \{\frac{c}{2}w(q^{n+1})^2 - wq^{n+1}q_\xi^{n+1}\}|_{-\infty}^{+\infty} = 0$ because of $p^{n+1}, q^{n+1} \in H_w^2(\mathbb{R})$. Integrating (3.13) with respect to ξ over $\mathbb{R}$ and using the Young inequality,

$$2|w'p^{n+1}p_\xi^{n+1}| \leq 2w(p^{n+1})_\xi^2 + \frac{1}{2}(\frac{w'}{w})^2 w(p^{n+1})^2,$$

then integrating over $[0,t]$ with respect to t, we can get

$$\begin{aligned}
&\| \sqrt{w}p^{n+1}(t)\|_{L^2}^2 + 2c\lambda_0 \int_0^t \int_{\mathbb{R}} w(p^{n+1})^2 d\xi ds - 2\lambda_0^2 \int_0^t \int_{\mathbb{R}} w(p^{n+1})^2 d\xi ds \\
=&\| \sqrt{w}p^{n+1}(t)\|_{L^2}^2 + 2c\lambda_0 \int_0^t \int_{\mathbb{R}} w(p^{n+1})^2 d\xi ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} (w'/w)^2 w(p^{n+1})^2 d\xi ds \\
\leq&\| \sqrt{w}p^{n+1}(t)\|_{L^2}^2 + 2c\lambda_0 \int_0^t \int_{\mathbb{R}} w(p^{n+1})^2 d\xi ds + 2 \int_0^t \int_{\mathbb{R}} w' p^{n+1} p_\xi^{n+1} d\xi ds \\
&+ 2 \int_0^t \int_{\mathbb{R}} w(p_\xi^{n+1})^2 d\xi ds \\
\leq&\| \sqrt{w}p_0\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}} w|p^{n+1}p^n| d\xi ds + 2 \int_0^t \int_{\mathbb{R}} w|p^{n+1}p^n(\phi_1 * p^n)| d\xi ds \\
&+ 2 \int_0^t \int_{\mathbb{R}} w|p^{n+1}p^n(\phi_1 * \phi)| d\xi ds + 2 \int_0^t \int_{\mathbb{R}} w|p^{n+1}\phi(\phi_1 * p^n)| d\xi ds \\
&+ 2a_1 \int_0^t \int_{\mathbb{R}} w|p^{n+1}p^n(\phi_2 * q^n)| d\xi ds + 2a_1 \int_0^t \int_{\mathbb{R}} w|p^{n+1}p^n(\phi_2 * \psi)| d\xi ds \\
&+ 2a_1 \int_0^t \int_{\mathbb{R}} w|p^{n+1}\phi(\phi_2 * q^n)| d\xi \\
\leq&\| \sqrt{w}p_0\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}} w(2(p^{n+1})^2 + \frac{1}{2}(p^n)^2) d\xi ds + (M_{u_0} + M_1) \int_0^t \int_{\mathbb{R}} w(2(p^{n+1})^2 \\
&+ \frac{1}{2}(p^n)^2) d\xi ds + M_1 \int_0^t \int_{\mathbb{R}} w(2(p^{n+1})^2 + \frac{1}{2}(p^n)^2) d\xi ds + M_1 \int_0^t \int_{\mathbb{R}} w(\xi)(2(p^{n+1})^2 \\
&+ \frac{1}{2} \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy (p^n)^2) d\xi ds + a_1(M_{v_0} + M_1) \int_0^t \int_{\mathbb{R}} w(2(p^{n+1})^2 + \frac{1}{2}(p^n)^2) d\xi ds \\
&+ M_1 a_1 \int_0^t \int_{\mathbb{R}} w(2(p^{n+1})^2 + \frac{1}{2}(p^n)^2) d\xi + a_1 M_1 \int_0^t \int_{\mathbb{R}} w(\xi)(2(p^{n+1})^2 \\
&+ \frac{1}{2} \int_{\mathbb{R}} \phi_2(y) e^{-2\lambda_0 y} dy (q^n)^2) d\xi ds \\
\leq&\| \sqrt{w}p_0\|_{L^2}^2 + 2[1 + M_{u_0} + a_1)M_{v_0} + 3(1 + a_1)M_1] \int_0^t \int_{\mathbb{R}} w(p^{n+1})^2 d\xi ds \\
&+ \frac{1}{2}[1 + M_{u_0} + a_1)M_{v_0} + 2(1 + a_1)M_1 + M_1 \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy] \int_0^t \int_{\mathbb{R}} w(p^n)^2 d\xi ds \\
&\qquad + a_1 M_1 \int_{\mathbb{R}} \phi_2(y) e^{-2\lambda_0 y} dy \int_0^t \int_{\mathbb{R}} w(q^n)^2 d\xi ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\| \sqrt{w}p^{n+1}(t)\|_{L^2}^2 + 2\mathcal{A}(c,\lambda_0) \int_0^t \| \sqrt{w}p^{n+1}(s)\|_{L^2}^2 ds \\
\leq&\| \sqrt{w}p_0\|_{L^2}^2 + \frac{1}{2}[1 + M_{u_0} + a_1 M_{v_0} + 2(1 + a_1)M_1 + M_1 \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy] \\
&\cdot \int_0^t \| \sqrt{w}p^n(s)\|_{L^2}^2 ds + a_1 M_1 \int_{\mathbb{R}} \phi_2(y) e^{-2\lambda_0 y} dy \int_0^t \int_{\mathbb{R}} w(q^n)^2 d\xi ds,
\end{aligned}$$

where $\mathcal{A}(c,\lambda_0)=c\lambda_0-\lambda_0^2-1-M_{u_0}-a_1M_{v_0}-3(1+a_1)M_1>0$.

Similarly, we can estimate

$$\begin{aligned}
&\|\sqrt{w}q^{n+1}(t)\|_{L^2}^2+2\mathcal{A}_1(c,\lambda_0)\int_0^t\|\sqrt{w}q^{n+1}(s)\|_{L^2}^2ds\\
&\le\|\sqrt{w}\hat{q}_0\|_{L^2}^2+\frac{r}{2}[1+M_{v_0}+a_2M_{u_0}+2(1+a_2)M_1+M_1\int_{\mathbb{R}}\phi_3(y)e^{-2\lambda_0 y}dy]\\
&\cdot\int_0^t\|\sqrt{w}q^n(s)\|_{L^2}^2ds+a_2rM_1\int_{\mathbb{R}}\phi_4(y)e^{-2\lambda_0 y}dy\int_0^t\|\sqrt{w}p^n(s)\|_{L^2}^2ds\\
&\le\|\sqrt{w}q_0\|_{L^2}^2+C\int_0^t\|\sqrt{w}q^n(s)\|_{L^2}^2ds+a_1M_1\int_{\mathbb{R}}\phi_4(y)e^{-2\lambda_0 y}dy\int_0^t\|\sqrt{w}p^n(s)\|_{L^2}^2ds,
\end{aligned}\tag{3.14}$$

where $\mathcal{A}_1(c,\lambda_0)=c\lambda_0-\lambda_0^2-1-M_{v_0}-a_2M_{u_0}-3(1+a_2)M_1>0$.

Using $2|w'p^{n+1}p_\xi^{n+1}|\le w(p^{n+1})_\xi^2+(\frac{w'}{w})^2w(p^{n+1})^2$, we get

$$\begin{aligned}
&\|\sqrt{w}p^{n+1}(t)\|_{L^2}^2+\int_0^t\|\sqrt{w}p_\xi^{n+1}(s)\|_{L^2}^2ds\\
\le&\|\sqrt{w}p_0\|_{L^2}^2+\frac{1}{2}[1+M_{u_0}+a_1M_{v_0}+2(1+a_1)M_1+M_1\int_{\mathbb{R}}\phi_1(y)e^{-2\lambda_0 y}dy]\\
&\cdot\int_0^t\|\sqrt{w}p^n(s)\|_{L^2}^2ds+|c\lambda_0-2\lambda_0^2-1-M_{u_0}-a_1M_{v_0}-3(1+a_1)M_1|\\
&\cdot\int_0^t\|\sqrt{w}p^{n+1}(s)\|_{L^2}^2ds+a_1M_1\int_{\mathbb{R}}\phi_2(y)e^{-2\lambda_0 y}dy\int_0^t\int_{\mathbb{R}}w(q^n)^2d\xi ds,\\
&\|\sqrt{w}q^{n+1}(t)\|_{L^2}^2+\int_0^t\|\sqrt{w}q_\xi^{n+1}(s)\|_{L^2}^2ds\\
\le&\|\sqrt{w}\hat{q}_0\|_{L^2}^2+\frac{r}{2}[1+M_{v_0}+a_2M_{u_0}+2(1+a_2)M_1+M_1\int_{\mathbb{R}}\phi_3(y)e^{-2\lambda_0 y}dy]\\
&\cdot\int_0^t\|\sqrt{w}q^n(s)\|_{L^2}^2ds+a_2rM_1\int_{\mathbb{R}}\phi_4(y)e^{-2\lambda_0 y}dy\int_0^t\|\sqrt{w}p^n(s)\|_{L^2}^2ds\\
&+|c\lambda_0-2\lambda_0^2-1-M_{v_0}-a_2M_{u_0}-3(1+a_2)M_1|\int_0^t\|\sqrt{w}q^{n+1}(s)\|_{L^2}^2ds\\
&+a_2rM_1\int_{\mathbb{R}}\phi_4(y)e^{-2\lambda_0 y}dy\int_0^t\|\sqrt{w}p^n(s)\|_{L^2}^2ds.
\end{aligned}$$

In order to prove $p^{n+1},q^{n+1}\in L^2([0,t_0];H_w^2(\mathbb{R}))$, we first differentiate the first equation of the system (3.3) with respect to ξ, then multiply it by wp_ξ^{n+1}; that is,

$$\begin{aligned}
&wp_\xi^{n+1}(p_\xi^{n+1})_t+cwp_\xi^{n+1}p_{\xi\xi}^{n+1}-w\hat{p}_\xi p_{\xi\xi\xi}^{n+1}\\
=&\{\frac{1}{2}w(p_\xi^{n+1})^2\}_t+\{\frac{c}{2}w(p_\xi^{n+1})^2\}_\xi-\frac{cw'}{2w}w(p_\xi^{n+1})^2\\
&-\{(wp_\xi^{n+1}p_{\xi\xi}^{n+1})_\xi-w'p_\xi^{n+1}p_{\xi\xi}^{n+1}-w(p_{\xi\xi}^{n+1})^2\}\\
=&wp_\xi^{n+1}[p_\xi^n-p_\xi^n(\phi_1*p^n)-p^n(\phi_1*p^n)_\xi-p_\xi^n(\phi_1*\phi)-p^n(\phi_1*\phi)_\xi\\
&-\phi_\xi(\phi_1*p^n)-\phi(\phi_1*p^n)_\xi-a_1p_\xi^n(\phi_2*q^n)-a_1p^n(\phi_2*q^n)_\xi\\
&-a_1p_\xi^n(\phi_2*\psi)-a_1p^n(\phi_2*\psi)_\xi-a_1\phi_\xi(\phi_2*q^n)-a_1\phi(\phi_2*q^n)_\xi].
\end{aligned}$$

Integrating the above equation with respect to ξ over $\mathbb{R}$, by the Young inequality, $2|w' p_{\xi}^{n+1} p_{\xi\xi}^{n+1}| \leq 2w(p_{\xi\xi}^{n+1})^2 + \frac{1}{2}(\frac{w'}{w})^2 w(p_{\xi}^{n+1})^2$, then integrating over $[0,t]$ with respect to t, we have

$$\begin{aligned}
&\| \sqrt{w} p_{\xi}^{n+1}(t)\|_{L^2}^2 + 2c\lambda_0 \int_0^t \int_{\mathbb{R}} w(p_{\xi}^{n+1})^2 d\xi ds - 2\lambda_0^2 \int_0^t \int_{\mathbb{R}} w(p_{\xi}^{n+1})^2 d\xi \\
=&\| \sqrt{w} p_{\xi}^{n+1}(t)\|_{L^2}^2 + 2c\lambda_0 \int_0^t \int_{\mathbb{R}} w(p_{\xi}^{n+1})^2 d\xi ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} (w'/w)^2 w(p_{\xi}^{n+1})^2 d\xi \\
\leq&\| \sqrt{w} p_{0\xi}\|_{L^2}^2 + \int_0^t w(2(p_{\xi}^{n+1})^2 + \frac{1}{2}(p_{\xi}^{n})^2) d\xi ds + (M_{u_0} + M_1) \int_0^t w(2(p_{\xi}^{n+1})^2 \\
&+ \frac{1}{2}(p_{\xi}^{n})^2) d\xi ds + (M_{u_0} + M_1) \int_0^t w(2(p_{\xi}^{n+1})^2 + \frac{1}{2}(p_{\xi}^{n})^2 \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy) d\xi ds \\
&+ M_1 \int_0^t w(2(p_{\xi}^{n+1})^2 + \frac{1}{2}(p_{\xi}^{n})^2) d\xi ds + M_2 \int_0^t w(\frac{1}{2}(p_{\xi}^{n+1})^2 + 2(p^n)^2) d\xi ds \\
&+ M_2 \int_0^t w(\frac{1}{2}(p_{\xi}^{n+1})^2 + 2(p^n)^2 \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy) d\xi ds + M_1 \int_0^t w(\frac{1}{2}(p_{\xi}^{n+1})^2
\end{aligned}$$

$$\begin{aligned}
&+ 2(p_{\xi}^{n})^2 \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy) d\xi ds + a_1(M_{v_0} + M_1) \int_0^t w(2(p_{\xi}^{n+1})^2 + \frac{1}{2}(p_{\xi}^{n})^2) d\xi ds \\
&+ a_1(M_{u_0} + M_1) \int_0^t w(2(p_{\xi}^{n+1})^2 + \frac{1}{2}(q_{\xi}^{n})^2 \int_{\mathbb{R}} \phi_2(y) e^{-2\lambda_0 y} dy) d\xi ds \\
&+ a_1 M_1 \int_0^t w(2(p_{\xi}^{n+1})^2 + \frac{1}{2}(p_{\xi}^{n})^2) d\xi ds + a_1 M_2 \int_0^t w(\frac{1}{2}(p_{\xi}^{n+1})^2 + 2(p^n)^2) d\xi ds \\
&+ a_1 M_2 \int_0^t w(\frac{1}{2}(p_{\xi}^{n+1})^2 + 2(q^n)^2 \int_{\mathbb{R}} \phi_2(y) e^{-2\lambda_0 y} dy) d\xi ds + a_1 M_1 \int_0^t w(2\hat{p}_{\xi}^2 \\
&+ \frac{1}{2} q_{\xi}^2 \int_{\mathbb{R}} \phi_2(y) e^{-2\lambda_0 y} dy) d\xi ds
\end{aligned}$$

$$\begin{aligned}
\leq&\| \sqrt{w} p_{0\xi}\|_{L^2}^2 + 2[1 + 2(1 + a_1) M_{u_0} + a_1 M_{v_0} + \left(\frac{13}{4} + 4a_1\right) M_1 \\
&+ \frac{1}{2}(1 + a_1) M_2] \int_0^t \| \sqrt{w} p_{\xi}^{n+1}\|_{L^2}^2 ds + \frac{1}{2}[1 + (1 + \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy) M_{u_0} + a_1 M_{v_0} \\
&+ (2 + 2a_1 + 5 \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy) M_1] \int_0^t \| \sqrt{w} p_{\xi}^{n}\|_{L^2}^2 ds + 2M_2(1 + a_1 + \int_{\mathbb{R}} \phi_1(y) e^{-2\lambda_0 y} dy) \\
&\cdot \int_0^t \| \sqrt{w} p^n\|_{L^2}^2 ds + 2a_1 M_2 \int_{\mathbb{R}} \phi_2(y) e^{-2\lambda_0 y} dy \int_0^t \| \sqrt{w} q^n\|_{L^2}^2 ds \\
&+ \frac{1}{2} a_1 M_1 \int_{\mathbb{R}} \phi_2(y) e^{-2\lambda_0 y} dy \int_0^t \| \sqrt{w} q_{\xi}^{n}\|_{L^2}^2 ds,
\end{aligned}$$

then

$$\begin{aligned}
&\| \sqrt{w} p_\xi^{n+1}(t)\|_{L^2}^2 + 2\mathcal{A}_2(c,\lambda_0)\int_0^t \| \sqrt{w} p_\xi^{n+1}\|_{L^2}^2 ds \\
\leq &\| \sqrt{w} p_{0\xi}\|_{L^2}^2 + 2M_2(1 + a_1 + \int_{\mathbb{R}} \phi_1(y)e^{-2\lambda_0 y}dy)\int_0^t \| \sqrt{w} p^n\|_{L^2}^2 ds \\
&+ \frac{1}{2}[1 + (1 + \int_{\mathbb{R}} \phi_1(y)e^{-2\lambda_0 y}dy)M_{u_0} + a_1 M_{v_0} + (2 + 2a_1 + 5\int_{\mathbb{R}} \phi_1(y)e^{-2\lambda_0 y}dy)M_1] \\
&\cdot \int_0^t \| \sqrt{w} p_\xi^n\|_{L^2}^2 ds \phi_2(y)e^{-2\lambda_0 y}dy \int_0^t \| \sqrt{w} q_\xi^n\|_{L^2}^2 ds,
\end{aligned} \tag{3.15}$$

where

$$\mathcal{A}_2(c,\lambda_0) = c\lambda_0 - \lambda_0^2 - 1 - 2(1 + a_1)M_{u_0} - a_1 M_{v_0} - \left(\frac{13}{4} + 4a_1\right)M_1 - \frac{1}{2}(1 + a_1)M_2 > 0.$$

In addition, by a series of calculation as above, we can get

$$\begin{aligned}
&\| \sqrt{w} p_\xi^{n+1}(t)\|_{L^2}^2 + \int_0^t \| \sqrt{w} p_{\xi\xi}^{n+1}(s)\|_{L^2}^2 ds \\
\leq &\| \sqrt{w} p_{0\xi}\|_{L^2}^2 + 2M_2(1 + a_1 + \int_{\mathbb{R}} \phi_1(y)e^{-2\lambda_0 y}dy)\int_0^t \| \sqrt{w} p^n\|_{L^2}^2 ds \\
&+ \frac{1}{2}[1 + (1 + \int_{\mathbb{R}} \phi_1(y)e^{-2\lambda_0 y}dy)M_{u_0} + a_1 M_{v_0} + (2 + 2a_1 + 5\int_{\mathbb{R}} \phi_1(y)e^{-2\lambda_0 y}dy)M_1] \\
&\cdot \int_0^t \| \sqrt{w} p_\xi^n\|_{L^2}^2 ds + 2a_1 M_2 \int_{\mathbb{R}} \phi_2(y)e^{-2\lambda_0 y}dy \int_0^t \| \sqrt{w} q^n\|_{L^2}^2 ds \\
&+ \frac{1}{2}a_1 M_1 \int_{\mathbb{R}} \phi_2(y)e^{-2\lambda_0 y}dy \int_0^t \| \sqrt{w} q_\xi^n\|_{L^2}^2 ds + 2|c\lambda_0 - 2\lambda_0^2 - 1 - 2(1 + a_1)M_{u_0} \\
&- a_1 M_{v_0} - \left(\frac{13}{4} + 4a_1\right)M_1 - \frac{1}{2}(1 + a_1)M_2| \int_0^t \| \sqrt{w} p_\xi^{n+1}\|_{L^2}^2 ds \\
\leq &\| \sqrt{w} p_{0\xi}\|_{L^2}^2 + 2\| \sqrt{w} p_0\|_{L^2}^2 + C_1 \int_0^t \| \sqrt{w} p^n\|_{L^2}^2 ds + C_2 \int_0^t \| \sqrt{w} p_\xi^n\|_{L^2}^2 ds \\
&+ C_3 \int_0^t \| \sqrt{w} q^n\|_{L^2}^2 ds + C_4 \int_0^t \| \sqrt{w} q_\xi^n\|_{L^2}^2 ds.
\end{aligned} \tag{3.16}$$

Using the same process as above, we estimate q^{n+1} as follows:

$$\begin{aligned}
&\| \sqrt{w} q_\xi^{n+1}(t)\|_{L^2}^2 + \int_0^t \| \sqrt{w} q_{\xi\xi}^{n+1}(s)\|_{L^2}^2 ds \\
\leq &\| \sqrt{w} q_{0\xi}\|_{L^2}^2 + 2\| \sqrt{w} q_0\|_{L^2}^2 + C_5 \int_0^t \| \sqrt{w} q^n\|_{L^2}^2 ds + C_6 \int_0^t \| \sqrt{w} q_\xi^n\|_{L^2}^2 ds \\
&+ C_7 \int_0^t \| \sqrt{w} p^n\|_{L^2}^2 ds + C_8 \int_0^t \| \sqrt{w} p_\xi^n\|_{L^2}^2 ds
\end{aligned} \tag{3.17}$$

Combining (3.11), (3.12), (3.16), and (3.17), we have

$$
\begin{aligned}
&\|p^{n+1}(t)\|_C^2 + \|\sqrt{w}p^{n+1}\|_{H^1}^2 + \int_0^t \|\sqrt{w}p^{n+1}\|_{H^2}^2 ds \\
\le&\|p_0\|_C^2 + \|\sqrt{w}p_0\|_{H^1}^2 + Ct_0^2 \sup_{t\in(0,t_0]} \|p^n(t)\|_C^2 + Ct_0^2 \sup_{t\in(0,t_0]} \|q^n(t)\|_C^2 \\
&+ Ct_0 \sup_{t\in(0,t_0]} \|\sqrt{w}p^n\|_{H^1}^2 + Ct_0 \sup_{t\in(0,t_0]} \|\sqrt{w}q^n\|_{H^1}^2
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
&\|q^{n+1}(t)\|_C^2 + \|\sqrt{w}q^{n+1}\|_{H^1}^2 + \int_0^t \|\sqrt{w}q^{n+1}\|_{H^2}^2 ds \\
\le&\|q_0\|_C^2 + \|\sqrt{w}q_0\|_{H^1}^2 + Ct_0^2 \sup_{t\in(0,t_0]} \|p^n(t)\|_C^2 + Ct_0^2 \sup_{t\in(0,t_0]} \|q^n(t)\|_C^2 \\
&+ Ct_0 \sup_{t\in(0,t_0]} \|\sqrt{w}p^n\|_{H^1}^2 + Ct_0 \sup_{t\in(0,t_0]} \|\sqrt{w}q^n\|_{H^1}^2.
\end{aligned}
\tag{3.19}
$$

The estimates (3.18)-(3.19) imply that $p^{n+1}, q^{n+1} \in UC(\mathbb{R}) \cap H_w^1(\mathbb{R})$ and $p^{n+1}, q^{n+1} \in L^2([0,t_0), H_w^2(\mathbb{R}))$.

In the following, we show that $p^{n+1}, q^{n+1} \in UC(\mathbb{R}) \cap H_w^1(\mathbb{R})$ is continuous with respect to $t \in [0, t_0]$. As above, by a series calculations as that of (3.14),

$$
\begin{aligned}
&\frac{d}{dt}\|\sqrt{w}p^{n+1}(t)\|_{L^2}^2 + 2\mathcal{A}(c,\lambda_0)\|\sqrt{w}p^{n+1}(s)\|_{L^2}^2 \\
\le&\frac{1}{2}[1 + M_{u_0} + a_1 M_{v_0} + 2(1+a_1)M_1 + M_1 \int_{\mathbb{R}} \phi_1(y)e^{-2\lambda_0 y}dy]\|\sqrt{w}p^n(s)\|_{L^2}^2 \\
&+ a_1 M_1 \int_{\mathbb{R}} \phi_2(y)e^{-2\lambda_0 y}dy\|\sqrt{w}q^n(s)\|_{L^2}^2, \\
&\frac{d}{dt}\|\sqrt{w}q^{n+1}(t)\|_{L^2}^2 + 2\mathcal{A}_1(c,\lambda_0)\|\sqrt{w}q^{n+1}(s)\|_{L^2}^2 \\
\le&\frac{r}{2}[1 + M_{v_0} + a_2 M_{u_0} + 2(1+a_2)M_1 + M_1 \int_{\mathbb{R}} \phi_3(y)e^{-2\lambda_0 y}dy]\|\sqrt{w}q(s)\|_{L^2}^2 \\
&+ a_2 r M_1 \int_{\mathbb{R}} \phi_4(y)e^{-2\lambda_0 y}dy\|\sqrt{w}p^n(s)\|_{L^2}^2.
\end{aligned}
$$

Integrating the above inequality with respect to t over $[0,t]$, we obtain

$$
\begin{aligned}
&\int_0^t \frac{d}{dt}\|\sqrt{w}p^{n+1}(t)\|_{L^2}^2 ds \\
\le&\frac{1}{2}[1 + M_{u_0} + a_1 M_{v_0} + 2(1+a_1)M_1 + M_1 \int_{\mathbb{R}} \phi_1(y)e^{-2\lambda_0 y}dy] \\
&\cdot \int_0^t \|\sqrt{w}p^n(s)\|_{L^2}^2 ds + a_1 M_1 \int_{\mathbb{R}} \phi_2(y)e^{-2\lambda_0 y}dy \int_0^t \|\sqrt{w}q^n(s)\|_{L^2}^2 ds,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t \frac{d}{dt}\|\sqrt{w}q^{n+1}(t)\|_{L^2}^2 ds \\
\le&\frac{r}{2}[1 + M_{v_0} + a_2 M_{u_0} + 2(1+a_2)M_1 + M_1 \int_{\mathbb{R}} \phi_3(y)e^{-2\lambda_0 y}dy] \int_0^t \|\sqrt{w}q^n(s)\|_{L^2}^2 ds \\
&+ a_2 r M_1 \int_{\mathbb{R}} \phi_4(y)e^{-2\lambda_0 y}dy \int_0^t \|\sqrt{w}p^n(s)\|_{L^2}^2 ds,
\end{aligned}
$$

which means that $(p^{n+1})'(t), (q^{n+1})'(t) \in L^2([0,t_0), L^2_w(\mathbb{R}))$, then we have

$$p^{n+1}(t), q^{n+1}(t) \in C([0,t_0), L^2_w(\mathbb{R})). \tag{3.20}$$

Similarly,

$$p^{n+1}_\xi(t), q^{n+1}_\xi(t) \in C([0,t_0), L^2_w(\mathbb{R})). \tag{3.21}$$

Therefore, (3.20) and (3.21) imply that

$$p^{n+1}(t), q^{n+1}(t) \in C([0,t_0), H^1_w(\mathbb{R})).$$

In the following we prove $p^{n+1}(t), q^{n+1}(t) \in C([0,t_0), UC(\mathbb{R}))$. Indeed, for any $0 \le t_1 < t_2 \le t_0$, let $\epsilon > 0$ and choose $\delta > 0$ such that $0 < t_2 - t_1 < \delta$, and

$$\left| \int_{\mathbb{R}} (\Phi(t_1,\eta) - \Phi(t_2,\eta)) d\eta \right| < \epsilon.$$

Set $\delta' = \min\{\epsilon, \delta\}$ and let $0 < t_2 - t_1 < \delta'$, then, we have the following two cases.

Case 1: If $t_1 \le \epsilon$ and $0 < t_2 - t_1 < \delta'$, then

$$\begin{aligned}
&|p^{n+1}(t_1,\xi) - p^{n+1}(t_2,\xi)| \\
\le&\left| \int_R \Phi(t_1,\eta) p_0(\xi-\eta) d\eta - \int_R \Phi(t_2,\eta) p_0(\xi-\eta) d\eta \right| \\
&+ \left| \int_0^{t_1} \int_{\mathbb{R}} \Phi(t_1 - s,\eta) G(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta ds \right. \\
&\left. - \int_0^{t_2} \int_{\mathbb{R}} \Phi(t_2 - s,\eta) G(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta ds \right| \\
\le&\left| \int_R (\Phi(t_1,\eta) - \Phi(t_2,\eta)) p_0(\xi-\eta) d\eta \right| \\
&+ \int_0^{t_1} \left| \int_{\mathbb{R}} \Phi(t_1 - s,\eta) G(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta \right| ds \\
&+ \int_0^{t_2} \left| \int_{\mathbb{R}} \Phi(t_2 - s,\eta) G(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta \right| ds \\
&\le \epsilon \|p_0\|_C + \epsilon \max_{p^n,q^n} |G(p^n,q^n)| + 2\epsilon \max_{p^n,q^n} |G(p^n,q^n)|.
\end{aligned}$$

Case 2: If $t_1 > \epsilon$ and $0 < t_2 - t_1 < \delta'$, then

$$\begin{aligned}
&|p^{n+1}(t_1,\xi) - p^{n+1}(t_2,\xi)| \\
\le&\left| \int_R \Phi(t_1,\eta) p_0(\xi-\eta) d\eta - \int_R \Phi(t_2,\eta) p_0(\xi-\eta) d\eta \right| \\
&+ \left| \left(\int_0^{t_1-\epsilon} + \int_{t_1-\epsilon}^{t_1} \right) \int_{\mathbb{R}} \Phi(t_1 - s,\eta) G(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta ds \right. \\
&\left. - \left(\int_0^{t_1-\epsilon} + \int_{t_1-\epsilon}^{t_1} + \int_{t_1}^{t_2} \right) \int_{\mathbb{R}} \Phi(t_1 - s,\eta) G(p^n(s,\xi-\eta), q^n(s,\xi-\eta)) d\eta ds \right| \\
\le&\epsilon \|p_0\|_C + 2\epsilon \max_{p^n,q^n} |G(p^n,q^n)| + \int_0^{t_1-\epsilon} \left| \Phi(t_1 - s,\eta) - \Phi(t_2 - s,\eta) d\eta \right| ds \max_{p^n,q^n} |G(p^n,q^n)| \\
\le&\epsilon \|p_0\|_C + 2\epsilon \max_{p^n,q^n} |G(p^n,q^n)| + (t_1 - \epsilon)\epsilon \max_{p^n,q^n} |G(p^n,q^n)|.
\end{aligned}$$

Thus, we have $|p^{n+1}(t_1,\xi)-p^{n+1}(t_2,\xi)|\to 0$ as $|t_2-t_1|\to 0$. Using the same process, we also show that $|q^{n+1}(t_1,\xi)-q^{n+1}(t_2,\xi)|\to 0$ as $|t_2-t_1|\to 0$, so we have

$$p^{n+1},q^{n+1}\in C([0,t_0),UC(\mathbb{R})).$$

Up to now, we proved that $p^{n+1},q^{n+1}\in Y(0,t_0)$.

Next we prove that $\mathcal{T}$ is a contraction mapping on $Y(0,t_0)$. For any $p^{n-1},q^{n-1},\in Y(0,t_0)$, define $(p^{n+1},q^{n+1})=\mathcal{T}(p^n,q^n),(p^n,q^n)=\mathcal{T}(p^{n-1},q^{n-1})$. By a series of calculations similar to (3.18)-(3.19), there is exists C^* such that

$$\begin{aligned}
&\|(p^{n+1},q^{n+1})-(p^n,q^n)\|_{Y(0,t_0)}=\|p^{n+1}-p^n\|_{Y(0,t_0)}+\|q^{n+1}-q^n\|_{Y(0,t_0)}\\
&=\|\mathcal{T}(p^n,q^n)-\mathcal{T}(p^{n-1},q^{n-1})\|_{Y(0,t_0)}\le C^*t_0(\|p^n-p^{n-1}\|_{Y(0,t_0)}+\|q^n-q^{n-1}\|_{Y(0,t_0)})\\
&=C^*t_0\|(p^n,q^n)-(p^{n-1},q^{n-1})\|_{Y(0,t_0)}.
\end{aligned}$$

Taking $0<t_0<\frac{1}{C^*}$,

$$\begin{aligned}
\|(p^{n+1},q^{n+1})-(p^n,q^n)\|_{Y(0,t_0)}&=\|\mathcal{T}(p^n,q^n)-\mathcal{T}(p^{n-1},q^{n-1})\|_{Y(0,t_0)}\\
&\le\tau\|(p^n,q^n)-(p^{n-1},q^{n-1})\|_{Y(0,t_0)}
\end{aligned}$$

where $0<\tau<1$. Hence, we prove that $(p^{n+1},q^{n+1})=\mathcal{T}(p^n,q^n)$ defined by (3.3) is a contraction mapping in $Y(0,t_0)$ if $0<t_0\ll 1$. By the Banach fixed point theorem, we can prove the local existence of the solution in $Y(0,t_0)$. In addition, by the similar calculation as (3.18)-(3.19), we have

$$\begin{aligned}
&\|p^{n+1}(t)\|_C^2+\|\sqrt{w}p^{n+1}\|_{H^1}^2+\int_0^t\|\sqrt{w}p^{n+1}\|_{H^2}^2ds\\
&+\|q^{n+1}(t)\|_C^2+\|\sqrt{w}q^{n+1}\|_{H^1}^2+\int_0^t\|\sqrt{w}q^{n+1}\|_{H^2}^2ds\\
&\le\frac{1}{1-Ct_0}[\|p_0\|_C^2+\|\sqrt{w}p_0\|_{H^1}^2+\|q_0\|_C^2+\|\sqrt{w}q_0\|_{H^1}^2].
\end{aligned}$$

When $t\in[t_0,2t_0]$, choosing the initial data $p(s,\xi),q(s,\xi)$ for $s\in[0,t_0]$ and repeating the above procedure, we can prove that $p,q\in Y(t_0,2t_0)$ uniquely exists and satisfies for $t\in[t_0,2t_0]$,

$$\begin{aligned}
&\|p^{n+1}(t)\|_C^2+\|\sqrt{w}p^{n+1}\|_{H^1}^2+\int_0^t\|\sqrt{w}p^{n+1}\|_{H^2}^2ds\\
&+\|q^{n+1}(t)\|_C^2+\|\sqrt{w}q^{n+1}\|_{H^1}^2+\int_0^t\|\sqrt{w}q^{n+1}\|_{H^2}^2ds\\
&\le\frac{1}{(1-Ct_0)(1-Ct_0)}[\|p_0\|_C^2+\|\sqrt{w}p_0\|_{H^1}^2+\|q_0\|_C^2+\|\sqrt{w}q_0\|_{H^1}^2].
\end{aligned}$$

Step by step, finally, we get that $u\in Y(0,T)$ uniquely exists for any $T>0$ and satisfies

$$\begin{aligned}
&\|p^{n+1}(t)\|_C^2+\|\sqrt{w}p^{n+1}\|_{H^1}^2+\int_0^t\|\sqrt{w}p^{n+1}\|_{H^2}^2ds\\
&+\|q^{n+1}(t)\|_C^2+\|\sqrt{w}q^{n+1}\|_{H^1}^2+\int_0^t\|\sqrt{w}q^{n+1}\|_{H^2}^2ds\\
&\le C_T[\|p_0\|_C^2+\|\sqrt{w}p_0\|_{H^1}^2+\|q_0\|_C^2+\|\sqrt{w}q_0\|_{H^1}^2].
\end{aligned}$$

4. The uniform boundedness

In this section, we show the uniform boundedness of the solutions of system (3.1). For the global solution of system (3.1), $p, q \in X(0,T)$ for any fixed $T > 0$, when the initial perturbation $p_0, q_0 \in X_0$, we prove $u \in X(0,\infty)$ by deriving the uniform boundedness. As stated before, here we adopt the so-called anti-weighted method [17, 18]. For this, define the following transform:

$$\bar{p}(t,\xi) = \sqrt{w(\xi)}p(t,\xi), \quad \bar{q}(t,\xi) = \sqrt{w(\xi)}q(t,\xi),$$

and it yields that

$$\begin{cases} \bar{p}_t - \bar{p}_{\xi\xi} + (c-2\lambda_0)\bar{p}_\xi + (c\lambda_0 - \lambda_0^2 - 1)\bar{p} \\ = -\bar{p}(\phi_1 * (e^{\lambda_0\xi}\bar{p})) - \bar{p}(\phi_1 * \phi) - \phi(\phi_1 * \bar{p}) \\ -a_1\bar{p}(\phi_1 * (e^{\lambda_0\xi}\bar{q})) - a_1\bar{p}(\phi_2 * \psi) - a_1\phi(\phi_2 * \bar{q}), \\ \bar{q}_t - \bar{q}_{\xi\xi} + (c-2\lambda_0)\bar{q}_\xi + (c\lambda_0 - \lambda_0^2 - r)\bar{q} \\ = r[-\bar{q}(\phi_3 * (e^{\lambda_0\xi}\bar{q})) - \bar{q}(\phi_3 * \psi) - \psi(\phi_3 * \bar{q}) \\ -a_2\bar{q}(\phi_4 * (e^{\lambda_0\xi}\bar{p})) - a_2\bar{q}(\phi_4 * \phi) - a_2\psi(\phi_4 * \bar{p})]. \end{cases} \tag{4.1}$$

Theorem 4.1. *Suppose that the assumptions of Proposition 3.1 hold, then the solution* $(p(t,\xi), q(t,\xi))$ *of system (3.1) belongs to* $X(0,\infty)$ *and there exists a positive constant C, which is independent of t such that*

$$\|p(t)\|_C^2 + \|\sqrt{w}p\|_{H^1}^2 + \int_0^\infty \|\sqrt{w}p\|_{H^2}^2 ds + \|q(t)\|_C^2 + \|\sqrt{w}q\|_{H^1}^2 + \int_0^\infty \|\sqrt{w}q\|_{H^2}^2 ds$$

$$\leq C[\|p_0\|_C^2 + \|\sqrt{w}p_0\|_{H^1}^2 + \|q_0\|_C^2 + \|\sqrt{w}q_0\|_{H^1}^2]. \tag{4.2}$$

Proof. The proof of this Theorem will be accomplished in the following three steps.

Step 1. We claim that the following inequality holds.

$$\|\bar{p}(t)\|_{L^2}^2 + 2\int_0^t \|\bar{p}_\xi\|_{L^2}^2 ds + \int_0^t \|\bar{p}\|_{L^2}^2 ds + \|\bar{q}(t)\|_{L^2}^2 + 2\int_0^t \|\bar{q}_\xi\|_{L^2}^2 ds + \int_0^t \|\bar{q}\|_{L^2}^2 ds$$

$$\leq \|p_0\|_C^2 + \|\sqrt{w}p_0\|_{H^1}^2 + \|q_0\|_C^2 + \|\sqrt{w}q_0\|_{H^1}^2, \quad \forall t \in [0,T], \tag{4.3}$$

where $T > 0$ is a given constant.

Multiplying the first equation of (4.1) by $\bar{p}$ and the second equation by $\bar{q}$, then integrating them over

$\mathbb{R}\times[0,t]$ with respect to ξ and t, we get

$$
\begin{aligned}
&||\bar{p}||_{L^2}^2+2\int_0^t||\bar{p}_\xi||_{L^2}^2ds+2(c\lambda_0-\lambda_0^2-1)\int_0^t||\bar{p}||_{L^2}^2ds\\
\le&||\bar{p}_0||_{L^2}^2+2\int_0^t\int_{\mathbb{R}}|\bar{p}^2(\phi_1*(e^{\lambda_0\xi}\bar{p})+\bar{p}^2(\phi_1*\phi)+\bar{p}\phi\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}\bar{p}(s,\xi-y)dy|d\xi ds\\
&+2a_1\int_0^t\int_{\mathbb{R}}|\bar{p}^2(\phi_2*(e^{\lambda_0\xi}\bar{q})+\bar{p}^2(\phi_2*\psi)+\bar{p}\phi\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}\bar{q}(s,\xi-y)dy|d\xi ds\\
\le&||\bar{p}_0||_{L^2}^2+2\int_0^t\int_{\mathbb{R}}|\bar{p}^2\int_{\mathbb{R}}\phi_1(y)e^{\lambda_0(\xi-y)}\bar{p}(s,\xi-y)dy+\bar{p}^2(\phi_1*\phi)+\bar{p}\phi\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}\\
&\cdot\bar{p}(s,\xi-y)dy|d\xi ds+2a_1\int_0^t\int_{\mathbb{R}}|\bar{p}^2\int_{\mathbb{R}}\phi_1(y)e^{\lambda_0(\xi-y)}\bar{q}(s,\xi-y)dy\\
&+\bar{p}^2(\phi_2*\psi)+\bar{p}\phi\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}\bar{q}(s,\xi-y)dy|d\xi ds\\
\le&||\bar{p}_0||_{L^2}^2+2(M_{u_0}+2M_1)\int_0^t\int_{\mathbb{R}}|\bar{p}|^2d\xi ds+2M_1\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}dy\int_0^t\int_{\mathbb{R}}|\bar{p}|^2d\xi ds\\
&+2a_1(M_{v_0}+2M_1)\int_0^t\int_{\mathbb{R}}|\bar{p}|^2d\xi ds+a_1M_1\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}dy\int_0^t\int_{\mathbb{R}}(|\bar{p}|^2+|\bar{q}|^2)d\xi ds\\
\le&||\bar{p}_0||_{L^2}^2+2[M_{u_0}+a_1M_{v_0}+2(1+a_1)M_1+2M_1\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}dy\\
&+a_1M_1\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}dy]\int_0^t\int_{\mathbb{R}}|\bar{p}|^2d\xi ds+a_1M_1\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}dy\int_0^t\int_{\mathbb{R}}|\bar{q}|^2d\xi ds
\end{aligned}
\tag{4.4}
$$

where we use $p(t,\xi)\le u(t,\xi-ct)+\phi(\xi)\le M_{u_0}+M_1, q(t,\xi)\le v(t,\xi-ct)+\psi(\xi)\le M_{v_0}+M_1$ and

$$
\begin{aligned}
&2\int_0^t\int_{\mathbb{R}}|\bar{p}\phi\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}\bar{p}(s,\xi-y)dy|d\xi ds\\
\le&\ 2M_1\int_0^t\int_{\mathbb{R}}|\bar{p}|\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}|\bar{p}(s,\xi-y)|dyd\xi ds\\
\le&\ M_1\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}(|\bar{p}|^2+|\bar{p}(s,\xi-y)|^2)dyd\xi ds\\
\le&\ 2M_1\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}|\bar{p}|^2dyd\xi ds.
\end{aligned}
$$

By the similar arguments, we also have

$$
\begin{aligned}
&||\bar{q}||_{L^2}^2+2\int_0^t||\bar{q}_\xi||_{L^2}^2ds+2(c\lambda_0-\lambda_0^2-r)\int_0^t||\bar{q}||_{L^2}^2ds\\
\le&\ ||\bar{q}_0||_{L^2}^2+2[M_{v_0}+a_2M_{u_0}+2(1+a_2)M_1+2M_1\int_{\mathbb{R}}\phi_3(y)e^{-\lambda_0 y}dy\\
&+a_2M_1\int_{\mathbb{R}}\phi_4(y)e^{-\lambda_0 y}dy]\int_0^t\int_{\mathbb{R}}|\bar{q}|^2d\xi ds+a_2M_1\int_{\mathbb{R}}\phi_4(y)e^{-\lambda_0 y}dy\int_0^t\int_{\mathbb{R}}|\bar{p}|^2d\xi ds.
\end{aligned}
\tag{4.5}
$$

From (4.4)-(4.5), we have

$$\|\bar{p}\|_{L^2}^2 + \|\bar{q}\|_{L^2}^2 + 2\int_0^t \|\bar{p}_\xi\|_{L^2}^2 ds$$
$$+2\int_0^t \|\bar{q}_\xi\|_{L^2}^2 ds + 2\mathcal{A}_4(c,\lambda_0)\int_0^t \|\bar{p}\|_{L^2}^2 ds + 2\mathcal{A}_5(c,\lambda_0)\int_0^t \|\bar{q}\|_{L^2}^2 ds$$
$$\le \|\bar{p}_0\|_{L^2}^2 + \|\bar{q}_0\|_{L^2}^2,$$

where

$$\mathcal{A}_4(c,\lambda_0)$$
$$= c\lambda_0 - \lambda_0^2 - 1 - M_{u_0} - a_1 M_{v_0} - 2(1+a_1)M_1 - 2M_1\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}dy - a_1 M_1\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}dy$$
$$-\frac{1}{2}a_2 M_1\int_{\mathbb{R}}\phi_4(y)e^{-\lambda_0 y}dy > 0$$

and

$$\mathcal{A}_5(c,\lambda_0)$$
$$= c\lambda_0 - \lambda_0^2 - r - M_{v_0} - a_2 M_{u_0} - 2(1+a_2)M_1 - 2M_1\int_{\mathbb{R}}\phi_3(y)e^{-\lambda_0 y}dy - a_2 M_1\int_{\mathbb{R}}\phi_4(y)e^{-\lambda_0 y}dy$$
$$-\frac{1}{2}a_1 M_1\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}dy > 0.$$

Step 2. We show

$$\|\bar{p}_\xi(t)\|_{L^2}^2 + 2\int_0^t \|\bar{p}_{\xi\xi}\|_{L^2}^2 ds + \int_0^t \|\bar{p}_\xi\|_{L^2}^2 ds + \|\bar{q}_\xi(t)\|_{L^2}^2 + 2\int_0^t \|\bar{q}_{\xi\xi}\|_{L^2}^2 ds + \int_0^t \|\bar{q}_\xi\|_{L^2}^2 ds$$
$$\le \|p_0\|_C^2 + \|\sqrt{w}p_0\|_{H^1}^2 + \|q_0\|_C^2 + \|\sqrt{w}q_0\|_{H^1}^2 \tag{4.6}$$

and

$$\int_0^t \left|\frac{d}{ds}\|\bar{p}_\xi(s)\|_{L^2}^2\right| ds + \int_0^t \left|\frac{d}{ds}\|\bar{q}_\xi(s)\|_{L^2}^2\right| ds$$
$$\le \|p_0\|_C^2 + \|\sqrt{w}p_0\|_{H^1}^2 + \|q_0\|_C^2 + \|\sqrt{w}q_0\|_{H^1}^2, \forall t \in [0,T], \tag{4.7}$$

where $T > 0$ is a given constant and C is a positive constant which is independent of T.

Differentiating the equations of (4.1) with respect to ξ and multiplying the first equation of (4.1) by $\bar{p}_\xi$ and the second equation by $\bar{q}_\xi$, we get

$$\begin{cases} \bar{p}_\xi(\bar{p}_\xi)_t - \bar{p}_\xi\bar{p}_{\xi\xi\xi} + (c-2\lambda_0)\bar{p}_\xi\bar{p}_{\xi\xi} + (c\lambda_0 - \lambda_0^2 - 1)\bar{p}_\xi\bar{p}_\xi \\ = \bar{p}_\xi[-\bar{p}_\xi(\phi_1 * (e^{\lambda_0\xi}\bar{p})) - \bar{p}(\phi_1 * (e^{\lambda_0\xi}\bar{p}))_\xi - \bar{p}_\xi(\phi_1 * \phi) - \bar{p}(\phi_1 * \phi)_\xi - \phi_\xi(\phi_1 * \bar{p}) \\ -\phi(\phi_1 * \bar{p})_\xi - a_1\bar{p}_\xi(\phi_1 * (e^{\lambda_0\xi}\bar{q})) - a_1\bar{p}(\phi_1 * (e^{\lambda_0\xi}\bar{q}))_\xi - a_1\bar{p}_\xi(\phi_2 * \psi) \\ -a_1\bar{p}(\phi_2 * \psi)_\xi - a_1\phi_\xi(\phi_2 * \bar{q}) - a_1\phi(\phi_2 * \bar{q})_\xi] \\ \bar{q}_\xi(\bar{q}_\xi)_t - \bar{q}_\xi\bar{q}_{\xi\xi\xi} + (c-2\lambda_0)\bar{q}_\xi\bar{q}_{\xi\xi} + (c\lambda_0 - \lambda_0^2 - r)\bar{q}_\xi\bar{q}_\xi \\ = r[-\bar{q}_\xi(\phi_3 * (e^{\lambda_0\xi}\bar{q})) - \bar{q}(\phi_3 * (e^{\lambda_0\xi}\bar{q}))_\xi - \bar{q}_\xi(\phi_3 * \psi) - \bar{q}(\phi_3 * \psi)_\xi - \psi_\xi(\phi_3 * \bar{q}) \\ -\psi(\phi_3 * \bar{q})_\xi - a_2\bar{q}_\xi(\phi_4 * (e^{\lambda_0\xi}\bar{p})) - a_2\bar{q}(\phi_4 * (e^{\lambda_0\xi}\bar{p}))_\xi - a_2\bar{q}_\xi(\phi_4 * \phi) \\ -a_2\bar{q}(\phi_4 * \phi)_\xi - a_2\psi_\xi(\phi_4 * \bar{p}) - a_2\psi(\phi_4 * \bar{p})_\xi]. \end{cases} \tag{4.8}$$

Since

$$\bar{p}_\xi(\bar{p}_\xi)_t - \bar{p}_\xi\bar{p}_{\xi\xi\xi} + (c-2\lambda_0)\bar{p}_\xi\bar{p}_{\xi\xi} + (c\lambda_0 - \lambda_0^2 - 1)\bar{p}_\xi\bar{p}_\xi$$

$$= \{\frac{1}{2}\bar{p}_\xi^2\}_t - [(\bar{p}_\xi\bar{p}_{\xi\xi})_\xi - \bar{p}_{\xi\xi}^2)] + (c-2\lambda_0)\{\bar{p}_\xi^2\}_\xi + (c\lambda_0 - \lambda_0^2 - 1)\bar{p}_\xi^2$$

$$\bar{q}_\xi(\bar{q}_\xi)_t - \bar{q}_\xi\bar{q}_{\xi\xi\xi} + (c-2\lambda_0)\bar{q}_\xi\bar{q}_{\xi\xi} + (c\lambda_0 - \lambda_0^2 - r)\bar{q}_\xi\bar{q}_\xi$$

$$= \{\frac{1}{2}\bar{q}_\xi^2\}_t - [(\bar{q}_\xi\bar{q}_{\xi\xi})_\xi - \bar{q}_{\xi\xi}^2)] + (c-2\lambda_0)\{\bar{q}_\xi^2\}_\xi + (c\lambda_0 - \lambda_0^2 - r)\bar{q}_\xi^2,$$

integrating the first equation of (4.8) with respect to ξ over $\mathbb{R}$, we have

$$\frac{d}{dt}||\bar{p}_\xi||_{L^2}^2 + 2||\bar{p}_{\xi\xi}||_{L^2}^2 + 2(c\lambda_0 - \lambda_0^2 - 1)||\bar{p}_\xi||_{L^2}^2$$

$$\leq 2\int_{\mathbb{R}} |\bar{p}_\xi^2(\phi_1 * (e^{\lambda_0\xi}\bar{p}))|d\xi + 2\int_{\mathbb{R}} |\bar{p}_\xi\bar{p}(\phi_1 * (e^{\lambda_0\xi}\bar{p}))_\xi|d\xi + 2\int_{\mathbb{R}} |\bar{p}_\xi^2(\phi_1 * \phi)|d\xi$$

$$+2\int_{\mathbb{R}} |\bar{p}_\xi\bar{p}(\phi_1 * \phi)_\xi|d\xi + 2\int_{\mathbb{R}} |\bar{p}_\xi\phi_\xi(\phi_1 * \bar{p})|d\xi + 2\int_{\mathbb{R}} |\bar{p}_\xi\phi(\phi_1 * \bar{p})_\xi|d\xi$$

$$+2a_1\int_{\mathbb{R}} |\bar{p}_\xi^2(\phi_1 * (e^{\lambda_0\xi}\bar{q}))|d\xi + 2a_1\int_{\mathbb{R}} |\bar{p}_\xi\bar{p}(\phi_1 * (e^{\lambda_0\xi}\bar{q}))_\xi|d\xi + 2a_1\int_{\mathbb{R}} |\bar{p}_\xi^2(\phi_1 * \psi)|d\xi$$

$$+2a_1\int_{\mathbb{R}} |\bar{p}_\xi\bar{p}(\phi_1 * \psi)_\xi|d\xi + 2a_1\int_{\mathbb{R}} |\bar{p}_\xi\phi_\xi(\phi_1 * \bar{q})|d\xi + 2a_1\int_{\mathbb{R}} |\bar{p}_\xi\phi(\phi_1 * \bar{q})_\xi|d\xi. \qquad (4.9)$$

Next, integrating (4.9) with respect to t over $[0, t]$, we have

$$||\bar{p}_\xi||_{L^2}^2 + 2\int_0^t ||\bar{p}_{\xi\xi}||_{L^2}^2 ds + 2(c\lambda_0 - \lambda_0^2 - 1)\int_0^t ||\bar{p}_\xi||_{L^2}^2 ds$$

$$
\begin{aligned}
&\le \|\bar p_\xi(0)\|_{L^2}^2 + 2\int_0^t\int_{\mathbb{R}} |\bar p_\xi^2(\phi_1 * (e^{\lambda_0\xi}\bar p))| d\xi ds + 2\int_0^t\int_{\mathbb{R}} |\bar p_\xi \bar p(\phi_1 * (e^{\lambda_0\xi}\bar p))_\xi| d\xi ds \\
&\quad + 2\int_0^t\int_{\mathbb{R}} |\bar p_\xi^2(\phi_1 * \phi)| d\xi ds + 2\int_0^t\int_{\mathbb{R}} |\bar p_\xi \bar p(\phi_1 * \phi)_\xi| d\xi ds + 2\int_0^t\int_{\mathbb{R}} |\bar p_\xi \phi_\xi(\phi_1 * \bar p)| d\xi ds \\
&\quad + 2\int_0^t\int_{\mathbb{R}} |\bar p_\xi \phi(\phi_1 * \bar p)_\xi| d\xi ds + 2a_1[\int_0^t\int_{\mathbb{R}} |\bar p_\xi^2(\phi_2 * (e^{\lambda_0\xi}\bar q))| d\xi ds \\
&\quad + \int_0^t\int_{\mathbb{R}} |\bar p_\xi \bar p(\phi_2 * (e^{\lambda_0\xi}\bar q))_\xi| d\xi ds + \int_0^t\int_{\mathbb{R}} |\bar p_\xi^2(\phi_2 * \psi)| d\xi ds] \\
&\quad + 2a_1[\int_0^t\int_{\mathbb{R}} |\bar p_\xi \bar p(\phi_2 * \psi)_\xi| d\xi ds + \int_0^t\int_{\mathbb{R}} |\bar p_\xi \phi_\xi(\phi_2 * \bar q)| d\xi ds \\
&\quad + \int_0^t\int_{\mathbb{R}} |\bar p_\xi \phi(\phi_2 * \bar q)_\xi| d\xi ds] \\
&\le \|\bar p_\xi(0)\|_{L^2}^2 + 2(M_{u_0} + M_1)\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds + 2\lambda_0(M_{u_0} + M_1)\int_0^t\int_{\mathbb{R}} |\bar p \bar p_\xi| d\xi ds \\
&\quad + 2\int_0^t\int_{\mathbb{R}} |\bar p_\xi \bar p \int_{\mathbb{R}} \phi_1(y) e^{\lambda_0(\xi - y)} \bar p_\xi(s, \xi - y) dy| d\xi ds + 2M_1\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds \\
&\quad + 2M_2\int_0^t\int_{\mathbb{R}} |\bar p_\xi \bar p| d\xi ds + 2M_2\int_0^t\int_{\mathbb{R}} |\bar p_\xi(\phi_1 * \bar p)| d\xi ds + 2M_1\int_0^t\int_{\mathbb{R}} |\bar p_\xi(\phi_1 * \bar p)_\xi| d\xi ds \\
&\quad + 2a_1(M_{v_0} + M_1)\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds \\
&\quad + 2a_1\int_0^t\int_{\mathbb{R}} |\bar p_\xi \bar p \int_{\mathbb{R}} \phi_2(y) e^{\lambda_0(\xi - y)} \bar q_\xi(s, \xi - y) dy| d\xi ds + 2a_1M_2\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds \\
&\quad + 2a_1M_2\int_0^t\int_{\mathbb{R}} |\bar p_\xi \bar p| d\xi ds + 2a_1M_2\int_0^t\int_{\mathbb{R}} |\bar p_\xi(\phi_2 * \bar q)| d\xi ds \\
&\quad + 2a_1M_1\int_0^t\int_{\mathbb{R}} |\bar p_\xi(\phi_2 * \bar q)_\xi| d\xi ds \\
&\le \|\bar p_\xi(0)\|_{L^2}^2 + 2(M_{u_0} + M_1)\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds + \lambda_0(M_{u_0} + M_1)\int_0^t\int_{\mathbb{R}} (\bar p^2 + \bar p_\xi^2) d\xi ds \\
&\quad + 2(M_{u_0} + M_1)\int_{\mathbb{R}} \phi_1(y) e^{-\lambda_0 y} dy \int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds + 2M_1\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds \\
&\quad + M_2\int_0^t\int_{\mathbb{R}} (\bar p^2 + \bar p_\xi^2) d\xi ds + M_2\int_{\mathbb{R}} \phi_1(y) e^{-\lambda_0 y} dy \int_0^t\int_{\mathbb{R}} (2\bar p^2 + \frac{1}{2}\bar p_\xi^2) d\xi ds \\
&\quad + 2M_1\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds + 2a_1(M_{v_0} + M_1)\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds + a_1\lambda_0(M_{v_0} + M_1)\int_0^t\int_{\mathbb{R}} (\bar p^2 \\
&\quad + \bar p_\xi^2) d\xi ds + a_1(M_{u_0} + M_1)\int_{\mathbb{R}} \phi_2(y) e^{-\lambda_0 y} dy \int_0^t\int_{\mathbb{R}} (\bar p_\xi^2 + \bar q_\xi^2) d\xi ds \\
&\quad + 2a_1M_1\int_0^t\int_{\mathbb{R}} |\bar p_\xi|^2 d\xi ds + a_1M_2\int_{\mathbb{R}} \phi_2(y) e^{-\lambda_0 y} dy \int_0^t\int_{\mathbb{R}} (\frac{1}{2}\bar p_\xi^2 + 2\bar q^2) d\xi ds \\
&\quad + a_1M_2\int_{\mathbb{R}} \phi_2(y) e^{-\lambda_0 y} dy \int_0^t\int_{\mathbb{R}} (2\bar p^2 + \frac{1}{2}\bar p_\xi^2) d\xi ds + a_1M_1\int_0^t\int_{\mathbb{R}} (\bar p_\xi^2 + \bar q_\xi^2) d\xi ds \\
&\le \|\bar p_\xi(0)\|_{L^2}^2 + [(2 + \lambda_0 + 2\int_{\mathbb{R}} \phi_1(y) e^{-\lambda_0 y} dy + a_1\int_{\mathbb{R}} \phi_2(y) e^{-\lambda_0 y} dy) M_{u_0} + 2a_1M_{v_0} \\
&\quad + (6 + \lambda_0 + 5a_1)M_1 + 2\int_{\mathbb{R}} \phi_1(y) e^{-\lambda_0 y} dy M_1 + a_1\int_{\mathbb{R}} \phi_2(y) e^{-\lambda_0 y} dy M_1
\end{aligned}
$$

$$+ M_2 + \frac{1}{2}M_2 \int_{\mathbb{R}} \phi_1(y)e^{-\lambda_0 y}dy + a_1 M_2 \int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy] \int_0^t \int_{\mathbb{R}} \bar{p}_\xi^2 d\xi ds$$
$$+ [\lambda_0 M_{u_0} + M_2 + 2M_2 \int_{\mathbb{R}} \phi_1(y)e^{-\lambda_0 y}dy + 2a_1 M_2 \int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy] \int_0^t \int_{\mathbb{R}} \bar{p}^2 d\xi ds \tag{4.10}$$
$$+ 2a_1 M_2 \int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy \int_{\mathbb{R}} \bar{q}^2 d\xi ds + a_1 M_1 \int_0^t \int_{\mathbb{R}} \bar{q}_\xi^2 d\xi ds.$$

Similarly, we also have

$$\|\bar{q}_\xi\|_{L^2}^2 + 2\int_0^t \|\bar{q}_{\xi\xi}\|_{L^2}^2 ds + 2(c\lambda_0 - \lambda_0^2 - r)\int_0^t \|\bar{q}_\xi\|_{L^2}^2 ds$$
$$\leq \|\bar{q}_\xi(0)\|_{L^2}^2 + r[(2 + \lambda_0 + 2\int_{\mathbb{R}} \phi_3(y)e^{-\lambda_0 y}dy + a_2 \int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy)M_{v_0} + 2a_2 M_{u_0}$$
$$+(6 + \lambda_0 + 5a_2)M_1 + 2\int_{\mathbb{R}} \phi_3(y)e^{-\lambda_0 y}dy M_1 + a_2 \int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy M_1$$
$$+M_2 + \frac{1}{2}M_2 \int_{\mathbb{R}} \phi_3(y)e^{-\lambda_0 y}dy + a_2 M_2 \int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy] \int_0^t \int_{\mathbb{R}} \bar{q}_\xi^2 d\xi ds$$
$$+r[\lambda_0 M_{v_0} + M_2 + 2M_2 \int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy + 2a_2 M_2 \int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy] \int_0^t \int_{\mathbb{R}} \bar{q}^2 d\xi ds$$
$$+2ra_2 M_2 \int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy \int_0^t \int_{\mathbb{R}} \bar{p}^2 d\xi ds + ra_2 M_1 \int_0^t \int_{\mathbb{R}} \bar{p}_\xi^2 d\xi ds. \tag{4.11}$$

From (4.10) and (4.11), it follows

$$\|\bar{p}_\xi\|_{L^2}^2 + 2\int_0^t \|\bar{p}_{\xi\xi}\|_{L^2}^2 ds + 2\mathcal{A}_6(c, \lambda_0)\int_0^t \|\bar{p}_\xi\|_{L^2}^2 ds$$
$$+\|\bar{q}_\xi\|_{L^2}^2 + 2\int_0^t \|\bar{q}_{\xi\xi}\|_{L^2}^2 ds + 2\mathcal{A}_7(c, \lambda_0)\int_0^t \|\bar{q}_\xi\|_{L^2}^2 ds$$
$$\leq \|\bar{p}_\xi(0)\|_{L^2}^2 + \|\bar{q}_\xi(0)\|_{L^2}^2 + [\lambda_0 M_{u_0} + M_2 + 2M_2 \int_{\mathbb{R}} \phi_1(y)e^{-\lambda_0 y}dy$$
$$+2a_1 M_2 \int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy + 2ra_2 M_2 \int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy] \int_0^t \int_{\mathbb{R}} \bar{p}^2 d\xi ds$$
$$+r[\lambda_0 M_{v_0} + M_2 + 2M_2 \int_{\mathbb{R}} \phi_3(y)e^{-\lambda_0 y}dy + 2a_2 M_2 \int_{\mathbb{R}} \phi_4(y)e^{-\lambda_0 y}dy$$
$$+\frac{2a_1 M_2}{r} \int_{\mathbb{R}} \phi_2(y)e^{-\lambda_0 y}dy] \int_0^t \int_{\mathbb{R}} \bar{q}^2 d\xi ds$$
$$\leq \|\bar{p}_\xi(0)\|_{L^2}^2 + \|\bar{q}_\xi(0)\|_{L^2}^2 + C\|\bar{p}(0)\|_{L^2}^2 + C\|\bar{q}(0)\|_{L^2}^2$$

where

$$\mathcal{A}_6(c,\lambda_0) = c\lambda_0 - \lambda_0^2 - 1 - \frac{1}{2}[(2+\lambda_0+2\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}dy + a_1\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}dy)M_{u_0} + 2a_1M_{v_0}$$
$$+(6+\lambda_0+5a_1)M_1 + 2\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}dyM_1 + a_1\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}dyM_1$$
$$+M_2 + \frac{1}{2}M_2\int_{\mathbb{R}}\phi_1(y)e^{-\lambda_0 y}dy + a_1M_2\int_{\mathbb{R}}\phi_2(y)e^{-\lambda_0 y}dy + \frac{r}{2}a_2M_1] > 0,$$

and

$$\mathcal{A}_7(c,\lambda_0) = c\lambda_0 - \lambda_0^2 - r - \frac{r}{2}[(2+\lambda_0+2\int_{\mathbb{R}}\phi_3(y)e^{-\lambda_0 y}dy + a_2\int_{\mathbb{R}}\phi_4(y)e^{-\lambda_0 y}dy)M_{v_0} + 2a_2M_{u_0}$$
$$+(6+\lambda_0+5a_2)M_1 + 2\int_{\mathbb{R}}\phi_3(y)e^{-\lambda_0 y}dyM_1 + a_2\int_{\mathbb{R}}\phi_4(y)e^{-\lambda_0 y}dyM_1$$
$$+M_2 + \frac{1}{2}M_2\int_{\mathbb{R}}\phi_3(y)e^{-\lambda_0 y}dy + a_2M_2\int_{\mathbb{R}}\phi_4(y)e^{-\lambda_0 y}dy + \frac{1}{2r}a_1M_1] > 0,$$

Similarly, inequality (4.7) holds by (4.4), (4.5), and (4.9). The details are omitted for simplicity.

Step 3. We show that

$$||p(t)||_C + ||q(t)||_C \le C[||p_\xi(0)||_{L^2}^2 + ||q_\xi(0)||_{L^2}^2 + ||p(0)||_{L^2}^2 + ||q(0)||_{L^2}^2], \quad \forall t \in [0,T],$$

where C is a positive constant which is independent of T.

Indeed, due to $p, q \in C_{unif}[0,T]$, we find that

$$\lim_{\xi\to+\infty} p(t,\xi) = p(t,\infty) =: p_1(t), \lim_{\xi\to+\infty} q(t,\xi) = q(t,\infty) =: q_1(t)$$

exists uniformly for $t \in [0,T]$. Let us take the limit to (3.1) as $\xi \to \infty$, then

$$\begin{cases} p_1'(t) = (1-2k_1-a_1k_2)p_1(t) - p_1^2(t) - a_1p_1(t)p_2(t) - a_1k_1p_2(t), \\ p_2'(t) = r(1-2k_2-a_2k_1)p_2(t) - rp_2^2(t) - ra_2p_2(t)p_1(t) - ra_2k_2p_1(t), \\ p_1(0) = p_2(0) = 0. \end{cases} \tag{4.12}$$

By the theory of order differential equations, we have

$$p_1(t) = p_2(t) = 0.$$

Thus we can get, for any given $\epsilon_0 > 0$, there exists a large number $\xi_0(\epsilon_0) \gg 1$ independent of $t \in [0,+\infty)$ such that

$$|p(t,\xi)| < \epsilon_0, \quad |q(t,\xi)| < \epsilon_0, \qquad \xi \in [\xi_0, \infty).$$

Therefore,

$$\sup_{\xi\in[\xi_0,\infty)} |p(t,\xi)| \le \epsilon_0 < C[||p_\xi(0)||_{L^2}^2 + ||q_\xi(0)||_{L^2}^2 + ||p(0)||_{L^2}^2 + ||q(0)||_{L^2}^2],$$
$$\sup_{\xi\in[\xi_0,\infty)} |q(t,\xi)| \le \epsilon_0 < C[||p_\xi(0)||_{L^2}^2 + ||q_\xi(0)||_{L^2}^2 + ||p(0)||_{L^2}^2 + ||q(0)||_{L^2}^2].$$

For $\xi \in (-\infty, \xi_0)$, $\sqrt{w(\xi)} = e^{-\lambda_0 \xi} \geq e^{-\lambda_0 \xi_0}$, and the Sobolev inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, we obtain

$$\sup_{\xi\in(-\infty,\xi_0)} |p(t,\xi)| \leq \sup_{\xi\in(-\infty,\xi_0)} \left|\frac{\sqrt{w(\xi)}}{e^{-\lambda_0\xi_0}} p(t,\xi)\right| = e^{-\lambda_0\xi_0} \sup_{\xi\in(-\infty,\xi_0)} |\sqrt{w(\xi)} p(t,\xi)|$$
$$\leq C\|\sqrt{w}p(t)\|_{H^1},$$
$$\sup_{\xi\in(-\infty,\xi_0)} |q(t,\xi)| \leq \sup_{\xi\in(-\infty,\xi_0)} \left|\frac{\sqrt{w(\xi)}}{e^{-\lambda_0\xi_0}} q(t,\xi)\right| = e^{-\lambda_0\xi_0} \sup_{\xi\in(-\infty,\xi_0)} |\sqrt{w(\xi)} q(t,\xi)|$$
$$\leq C\|\sqrt{w}q(t)\|_{H^1}.$$

From (4.6), we have

$$\|p(t)\|_C + \|q(t)\|_C \leq C[\|\bar{p}_\xi(0)\|_{L^2}^2 + \|\bar{q}_\xi(0)\|_{L^2}^2 + \|\bar{p}(0)\|_{L^2}^2 + \|\bar{q}(0)\|_{L^2}^2].$$

The proof of this theorem is finished.

5. The main theorem

In this section, the stability of all traveling wave solutions with sufficiently large wave speed of system (1.1) is proved.

Theorem 5.1. *Under the assumptions of Proposition 3.1, we have*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| = 0, \quad \lim_{t\to\infty} \sup_{x\in\mathbb{R}} |v(t,x) - \psi(x+ct)| = 0. \tag{5.1}$$

Proof. From Theorem 4.1, we have

$$\|p(t)\|_C + \|q(t)\|_C + \|\sqrt{w}p(t)\|_{H^1}^2 + \|\sqrt{w}q(t)\|_{H^1}^2 + \int_0^t \|\sqrt{w}p(t)\|_{H^2}^2 ds + \int_0^t \|\sqrt{w}q(t)\|_{H^2}^2 ds$$
$$+ \int_0^t |\frac{d}{ds}\|\partial_\xi(\sqrt{w}p)(s)\|_{L^2}^2| ds + \int_0^t |\frac{d}{ds}\|\partial_\xi(\sqrt{w}q)(s)\|_{L^2}^2| ds$$
$$\leq C[\|\bar{p}(0)\|_{H^1}^2 + \|\bar{q}(0)\|_{H^1}^2 + \|p(0)\|_C^2 + \|q(0)\|_C^2], \quad t \in [0,\infty). \tag{5.2}$$

Set

$$P(t) = \|\partial_\xi(\sqrt{w}p(t))\|_{L^2}^2, \quad Q(t) = \|\partial_\xi(\sqrt{w}q(t))\|_{L^2}^2.$$

By (5.2), we get

$$0 \leq P(t), Q(t) \leq C[\|\bar{p}(0)\|_{H^1}^2 + \|\bar{q}(0)\|_{H^1}^2 + \|p(0)\|_C^2 + \|q(0)\|_C^2], \quad t \in [0,\infty),$$
$$\int_0^\infty P(s)ds, \int_0^\infty Q(s)ds \leq C[\|\bar{p}(0)\|_{H^1}^2 + \|\bar{q}(0)\|_{H^1}^2 + \|p(0)\|_C^2 + \|q(0)\|_C^2], \quad t \in [0,\infty),$$
$$\int_0^\infty |P'(s)|ds, \int_0^\infty |Q'(s)|ds \leq C[\|\bar{p}(0)\|_{H^1}^2 + \|\bar{q}(0)\|_{H^1}^2 + \|p(0)\|_C^2 + \|q(0)\|_C^2], \quad t \in [0,\infty),$$

which implies that

$$\lim_{t\to\infty} P(t) = 0, i.e. \lim_{t\to\infty} \|\bar{p}_\xi\|_{L^2}^2 = 0; \lim_{t\to\infty} Q(t) = 0, i.e. \lim_{t\to\infty} \|\bar{q}_\xi\|_{L^2}^2 = 0. \tag{5.3}$$

Using the interpolation inequality, we get

$$\|\bar{p}(t)\|_C \le C\|\bar{p}(t)\|_{L^2}^{\frac{1}{2}}\|\bar{p}_\xi(t)\|_{L^2}^{\frac{1}{2}}, \quad \|\bar{q}(t)\|_C \le C\|\bar{q}(t)\|_{L^2}^{\frac{1}{2}}\|\bar{q}_\xi(t)\|_{L^2}^{\frac{1}{2}}.$$

Since $\|\bar{p}(t)\|_{L^2}, \|\bar{q}(t)\|_{L^2}$ are bounded, from (5.3), it holds

$$\lim_{t\to\infty} \|\bar{p}(t)\|_C = \lim_{t\to\infty} \|\bar{q}(t)\|_C = 0. \tag{5.4}$$

In the following we focus on the long time behavior of $p(t,\xi), q(t,\xi)$. Since $|p(t,\infty)| = |q(t,\infty)| = 0$, then

$$|p(t,\infty)| = |q(t,\infty)| \le \min\{e^{-t}, e^{-rt}\}, \quad t \in (0,\infty). \tag{5.5}$$

By system (3.1), it holds

$$\begin{cases} p(t,\xi) := e^{-\frac{1}{2}t}\int_{\mathbb{R}} \Phi(t,\eta)p_0(\xi-\eta)d\eta + \int_0^t e^{-\frac{t-s}{2}}\int_{\mathbb{R}}\Phi(t-s,\eta)[\frac{3}{2}p \\ -p(\phi_1 * p) - p(\phi_1 * \phi) - \phi(\phi_1 * p) - a_1 p(\phi_2 * q) - a_1 p(\phi_2 * \psi) - a_1\phi(\phi_2 * q)]d\eta ds, \\ q(t,\xi) := e^{-\frac{r}{2}t}\int_{\mathbb{R}} \Phi(t,\eta)q_0(\xi-\eta)d\eta + \int_0^t e^{-\frac{r(t-s)}{2}}\int_{\mathbb{R}}\Phi(t-s,\eta)r[\frac{3}{2}q \\ -q(\phi_3 * q) - q(\phi_3 * \psi) - \psi(\phi_3 * q) - a_2 q(\phi_4 * p) - a_2 q(\phi_4 * \phi) - a_2\psi(\phi_4 * p)]d\eta ds. \end{cases} \tag{5.6}$$

Multiplying the first equation of (5.6) by $e^{\tau t}$, where $0 < \tau < \min\{1/2, r/2\}$, by the property of the heat kernel and the expression (5.5), we have

$$\lim_{\xi\to\infty} e^{\tau t}|p(t,\xi)| \le e^{-(\frac{1}{2}-\tau)t}\int_{\mathbb{R}} \Phi(t,\eta)\lim_{\xi\to\infty}|p_0(\xi-\eta)|d\eta + e^{\tau t}\int_0^t e^{-\frac{t-s}{2}}\int_{\mathbb{R}}\Phi(t-s,\eta)\lim_{\xi\to\infty}|[\frac{3}{2}p$$

$$-p(\phi_1 * p) - p(\phi_1 * \phi) - \phi(\phi_1 * p) - a_1 p(\phi_2 * q) - a_1 p(\phi_2 * \psi) - a_1\phi(\phi_2 * q)]|d\eta ds$$

$$\le \frac{11}{2}e^{\tau t}\int_0^t e^{-\frac{1}{2}(t-s)}e^{-s}ds + 2e^{\tau t}\int_0^t e^{-\frac{1}{2}(t-s)}e^{-2s}ds$$

$$= 2\cdot\frac{11}{2}e^{-(1/2-\tau)t}[1-e^{-1/2t}] + 2e^{-(1/2-\tau)t}\cdot\frac{2}{3}[1-e^{-3/2t}] < \infty, \qquad t > 0. \tag{5.7}$$

It follows from (5.7) that, there exists a number $\zeta \gg 1$ such that

$$\sup_{\xi\in[\zeta,\infty)} |p(t,\xi) \le Ce^{-\tau t}, \qquad t > 0,$$

then we have,

$$\lim_{t\to\infty}\sup_{\xi\in[\zeta,\infty)} |p(t,\xi)| = 0.$$

For $\xi \in (-\infty, \zeta)$, since $\sqrt{w(\xi)} = e^{-\lambda_0 \xi} \geq e^{-\lambda_0 \zeta}$, from (5.4), it holds that

$$\lim_{t\to\infty} \sup_{\xi\in(-\infty,\zeta)} |p(t,\xi)| \leq \lim_{t\to\infty} \sup_{\xi\in(-\infty,\zeta)} |\frac{\sqrt{w(\xi)}}{e^{-\lambda_0\zeta}} p(t,\xi)|$$

$$\leq e^{-\lambda_0\zeta} \lim_{t\to\infty} \sup_{\xi\in(-\infty,\zeta)} | \sqrt{w(\xi)} p(t,\xi)| = 0.$$

Similarly, we obtain

$$\lim_{t\to\infty} \sup_{\xi\in(-\infty,\infty)} |q(t,\xi)| = 0.$$

The proof is completed.

6. Discussion

This paper was motivated by the biological question of how diffusion and nonlocal intraspecific and interspecific competitions affect the competition outcomes of two competing species. This may provide us with insights of how species learn to compete and point out species evolution directions. The model (1.1) is a two-species Lotka-Volterra competition model in the form of a coupled system of reaction diffusion equations with nonlocal intraspecific and interspecific competitions in space at times. Han et al. [13] has proved the existence of traveling wave solutions of the system (1.1) connecting the origin to some positive steady state with some minimal wave speed. Following their steps, we studied the stability of these traveling wave solutions. The main mathematical challenge to study the traveling waves for system (1.1) was that solutions do not obey the maximum principle and the comparison principle cannot be applied to the system. We considered the stability of the zero solution of a perturbation equation about the traveling wave solution and used the anti-weighted method and the energy estimates to reach the expected one. The stability of traveling wave solutions with large enough wave speed of system (1.1) was proved.

The existence, stability, and wave speed of traveling wave solutions could help us to understand for phenomenons such as the movement of the hybrid zone. Hybrid zones are locations where hybrids between species, subspecies, or races are found. Climate change has been implicated as driving shifts of hybridizing species' range limits. However, Hunter et al. [19] found that fitness is also linked to both climatic conditions and movement of hybrid zones. These Lotka-Volterra competition models with advection, diffusion, and nonlocal effects can be used to describe the dynamics of species' range [20] and estimate the movement of the hybrid zone under different assumptions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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