Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(8): 14438–14463.
DOI: 10.3934/mbe.2023646
Received: 19 April 2023
Revised: 04 June 2023
Accepted: 11 June 2023
Published: 30 June 2023

Research article

Optimal control and stability analysis of an age-structured SEIRV model with imperfect vaccination

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Abstract: Vaccination programs are crucial for reducing the prevalence of infectious diseases and ultimately eradicating them. A new age-structured SEIRV (S-Susceptible, E-Exposed, I-Infected, R-Recovered, V-Vaccinated) model with imperfect vaccination is proposed. After formulating our model, we show the existence and uniqueness of the solution using semigroup of operators. For stability analysis, we obtain a threshold parameter R_0. Through rigorous analysis, we show that if $R_0 < 1$, then the disease-free equilibrium point is stable. The optimal control strategy is also discussed, with the vaccination rate as the control variable. We derive the optimality conditions, and the form of the optimal control is obtained using the adjoint system and sensitivity equations. We also prove the uniqueness of the optimal controller. To visually illustrate our theoretical results, we also solve the model numerically.

Keywords: age-structured model; imperfect vaccination; optimal control

1. Introduction

Imperfect or leaky vaccines are those that do not provide complete immunity to the disease in their hosts and are incapable of preventing transmission to others. It is known that leaky vaccines can reduce the symptoms of the disease but do not offer full protection against infection or the spread of the pathogen. Therefore, studying the impact of imperfect vaccines on the human population becomes crucial. Additionally, for certain infectious diseases, it has been observed that the effectiveness of the vaccine may diminish over time. In the recent COVID-19 pandemic, it has been noted that most vaccines remain effective for a specific duration before individuals become susceptible to the virus once again.

During the course of the follow-up period of six months, Stephen et al. [1] studied the vaccine

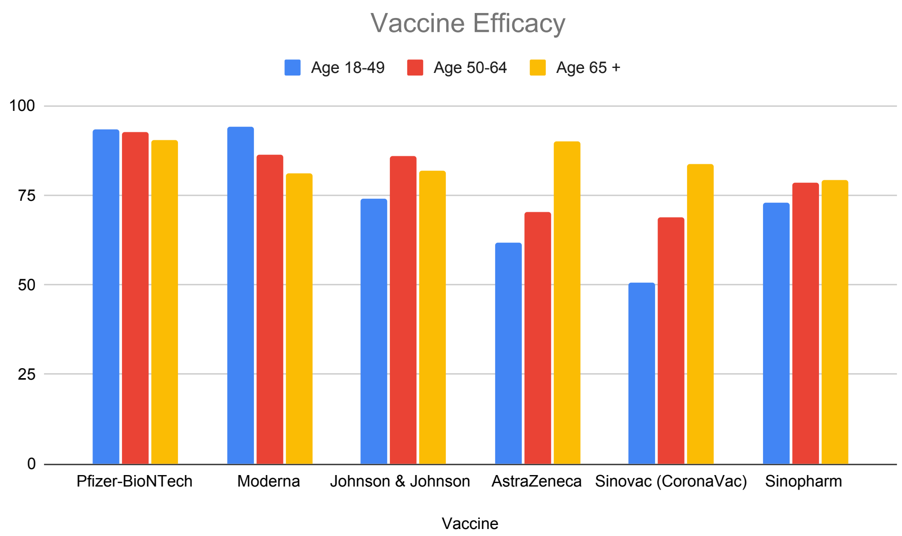


Figure 1. Age-dependent vaccine efficacy of different vaccines [1,4].

efficacy of different vaccines (shown in Figure 1). Menni et al. [2] showed that in the case of COVID-19, the vaccine's efficacy remained high after five months among those under the age of 55. They also showed that the booster dose can restore the effectiveness of the vaccine. Jake et al. [3] also studied the vaccine efficacy and demonstrated that the evidence of waning immunity is overstated. They demonstrated that antibody levels can be boosted in the general population and robust clinical data are required to assess the need for additional doses. This naturally motivates us to consider a compartmental model for infectious diseases with a vaccine-waning rate.

Kermack and Mckendrick [5–7] developed ordinary differential equations (ODEs) based models to study the transmission of infectious diseases. These models are formulated based on the assumption that all individuals have identical vital rates regardless of age or size. Enough literature is available on ODE-based infectious disease models; more details can be found in [8–12]. In epidemic models, the "disease clock" is important for studying the long-term dynamics of an infectious disease. Moreover, in transmitting infectious diseases, the age of infection or the time since infection starts plays a vital role. Age-structured population models characterize individuals based on their age, which can be class-age or demographic age. Class age means the time since infection, and demographic age means the chronological age of individuals.

Iannelli [13] discussed age-structured population models and their role in describing the growth and interaction of populations. Thieme and Castilo [14] examined the role of infection age in the HIV viral dynamics. More details about age and size-structured population models and their applications in biology and epidemiology are given in [16]. For the first time, Arino et al. [17] discussed the role of leaky vaccines with a general waning function. They consider three classes S, I, and V (without age structure), corresponding to susceptible, infected, and vaccinated individuals. They showed that sub-threshold equilibria are possible, which may play an important role while designing vaccine strategies. Optimal control strategies for age-structured compartmental models are discussed by [18,19]. Semi-random epidemic network models can also describe the dynamics of many infectious diseases [20]. Awel et al. [19] incorporated imperfect vaccination in their model for Malaria disease, and Wei et al. [21] studied imperfect vaccination on scale-free networks. Many authors have also considered delay terms in fractional-order population models with or without structure variables. For more details, we refer to [22–25]. There are many interesting works on age and size-structured models for infectious diseases (see, for example, [26–32]).

We propose an age-structured SEIRV model with imperfect vaccination and vaccine-waning rates. After formulation of our model, we show the existence and uniqueness of the solution. We use semigroup of operators to show the existence and uniqueness of the solution. We find steady-state solutions to our model and check the stability of the disease-free steady state. We also study the optimal control problem with vaccination rate as a control variable. Our objective is to minimize the cost of vaccination and to reduce the number of infected individuals. Using adjoint equations, we derive the form of the optimal control variable. We show that the optimal control variable depends on the vaccine waning rate, infection rate, and the cost of vaccination. We also solve our model numerically to validate our theoretical findings. Our study is mathematical, and we study the qualitative properties of the solution. We obtain an explicit form for the optimal control variable, which involves important parameters of the proposed model. Due to the complex partial differential equations-based model, it is difficult to get the data, so we do not validate our model with real data. But the model results are interesting and involve realistic assumptions.

Our work is divided in the following manner. In Section 2, we formulated our model, and using semigroup of operators, we show the existence and uniqueness of the solution. Section 3 is devoted to the stability analysis of disease-free equilibrium points. In Section 4, we study the optimal control problem. In Section 5, we solve our model numerically, and the last section is devoted to the concluding remarks.

2. Model formulation

In this section, we formulate our model. We also show the existence and uniqueness of the solution to the model under consideration. We convert our problem into the abstract Cauchy problem to show the existence of the solution.

Let $S(a,t), E(a,t), I(a,t)$ and $R(a,t)$ denote the age and time-dependent population densities of individuals in the susceptible, exposed, infected, and recovered class. Whenever there is no confusion in the notation, we use the notation S, E, I and R for $S(a,t), E(a,t), I(a,t)$ and $R(a,t)$ respectively. We denote the age-dependent transmission coefficient by $c(a,b)$, which describes the interaction between susceptible and infected class, that is, $c(a_1,a_2)S(a_1,t)I(a_2,t)da_1da_2$ is the number of susceptible individuals aged in $(a_1, a_1 + da_1)$ that contract the disease after interacting with infected individuals aged in $(a_2, a_2 + da_2)$. We assume that the functional form of the force of infection is given by

$$\varphi(a,t) = \int_0^{a_m} c(a,\sigma)I(\sigma,t)d\sigma,$$

where a_m is the maximum age which an individual can attain. Then the following system of partial differential equations (PDEs) describes the spread of disease

$$\begin{aligned}
&\frac{\partial S}{\partial t}+\frac{\partial S}{\partial a}=\Lambda-\varphi(a,t)S-m(a)S-v(a)S+w_0(a)V\\
&\frac{\partial E}{\partial t}+\frac{\partial E}{\partial a}=\varphi(a,t)S-m_1E-m(a)E\\
&\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a}=m_1E-\gamma I-m(a)I+\beta\sigma(a)IV\\
&\frac{\partial R}{\partial t}+\frac{\partial R}{\partial a}=\gamma I-m(a)R\\
&\frac{\partial V}{\partial t}+\frac{\partial V}{\partial a}=-(m(a)+w_0(a)+\beta\sigma(a)I)V\\
&S(0,t)=S_1,\ E(0,t)=I(0,t)=R(0,t)=0,V(0,t)=\int_0^{a_m}v(a)S(a,t)da\\
&S(a,0)=S_0(a),E(a,0)=E_0(a),I(a,0)=I_0(a),\\
&R(a,0)=R_0(a)\text{ and }V(a,0)=V_0(a),
\end{aligned}\tag{2.1}$$

where $\varphi(a,t)=\int_0^{a_m}c(a,\sigma)I(\sigma,t)d\sigma$.

$m(a)$	Mortality rate of age a individuals
m_1	Rate at which individuals move from exposure to onset of symptoms
$v(a)$	Age-dependent Vaccination policy
$w_0(a)$	Vaccine wanes rate
γ	Recovery rate of infected population
$\sigma(a)$	Infection ratio of vaccinated individuals
$\beta\sigma(a)$	Rate at which vaccinated individuals can be infected

Let $X=\mathbb{R}^4\times L^1((0,a_m),\mathbb{R})\times L^1((0,a_m),\mathbb{R})\times L^1((0,a_m),\mathbb{R})\times L^1((0,a_m),\mathbb{R})$ and let us define the linear operator

$$A\begin{pmatrix}0\\0\\0\\0\\x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}x(0)\\y(0)\\z(0)\\w(0)\\-x'-(m(a)+v(a))x\\-y'-(m_1+m(a))y\\-z'-(\gamma+m(a))z\\-w'-(m(a)+w_0(a))w\end{pmatrix}$$

with

$$\mathrm{D}(A)=\{0\}\times\{0\}\times\{0\}\times\{0\}\times W^{1,1}((0,a_m),\mathbb{R})\times W^{1,1}((0,a_m),\mathbb{R})\times W^{1,1}((0,a_m),\mathbb{R})\times W^{1,1}((0,a_m),\mathbb{R}).$$

Here it is clear that $\overline{D(A)}$ is not dense in X. Let us also define the nonlinear operator in the following manner

$$F\begin{pmatrix}0\\0\\0\\0\\x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}S_1\\0\\0\\\int_0^{a_m} v(a)x(a)da\\\Lambda - x(a)\int_0^{a_m} c(a)z(a)da + w_0(a)w\\x(a)\int_0^{a_m} c(a)z(a)da\\m_1 y+\beta\sigma(a)z(a)w(a)\\-\beta\sigma(a)z(a)w(a)\end{pmatrix}.$$

Our model in abstract form can be written as

$$\begin{cases}\frac{d}{dt}v(t) = Av(t) + F(v(t)), & t \geq 0\\ v(0) = v_0,\end{cases} \tag{2.2}$$

where $v(t) = (0, 0, 0, 0, S(\cdot, t), E(\cdot, t), I(\cdot, t), V(\cdot, t))$ and $v_0 = (0, 0, 0, 0, S_0, E_0, I_0, V_0)$. Instead of solving (2.2), we consider the following integral equation

$$v(t) = v_0 + A\int_0^t v(s)ds + \int_0^t F(v(s))ds. \tag{2.3}$$

We have the following assumptions:

(i) m, w_0, v, σ are positive functions and lies in $L_+^\infty((0, a_m), \mathbb{R})$.
(ii) For $0 \leq a \leq a_m$, we assume that $0 < \sigma(a) < 1$.

Now, our task is to prove that A is a Hille-Yosida operator and therefore it generates a C_0 semigroup on the closure of $D(A)$. Linearized system of (2.2) can be written as

$$\begin{cases}\frac{d}{dt}v(t) = Av(t) + DF(v^*)v(t), & t \geq 0\\ v(0) = v_0 - v^*,\end{cases} \tag{2.4}$$

where v^* is steady state solution of (2.2) and

$$DF(v^*)\begin{pmatrix}0\\0\\0\\0\\x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}S_1\\0\\0\\\int_0^{a_m} v(a)x(a)da\\-x^*(a)\int_0^{a_m} c(a)z(a,t)da - x(a,t)\int_0^{a_m} c(a)z^*(a)da + w_0(a)w(a,t)\\x^*(a)\int_0^{a_m} c(a)z(a,t)da + x(a,t)\int_0^{a_m} c(a)z^*(a)da\\m_1 y+\beta\sigma(a)z^*(a)w(a,t)+\beta\sigma(a)z(a,t)w^*(a)\\-\beta\sigma(a)z^*(a)w(a,t)-\beta\sigma(a)z(a,t)w^*(a)\end{pmatrix}.$$

Here, $DF(v^*)$ is a compact bounded linear operator on X. Let

$$\Omega = \{\lambda \in \mathbb{C} : Re\lambda > -w_0\},$$

where

$$w_0 = \min\{\inf_a(m(a) + v(a)), \inf_a(m_1 + m(a)), \inf_a(\gamma + m(a)), \inf_a(m(a) + w_0(a))\}.$$

Now, our task is to prove that $(A, D(A))$ is a Hille-Yosida operator. For

$$(\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot), \phi_4(\cdot), \phi_5(\cdot), \phi_6(\cdot), \phi_7(\cdot), \phi_8(\cdot)) \in X \text{ and } (0, 0, 0, 0, \xi_1(\cdot), \xi_2(\cdot), \xi_3(\cdot), \xi_4(\cdot))$$

we have

$$(\lambda I - A)^{-1}\begin{pmatrix}\phi_1(\cdot)\\ \phi_2(\cdot)\\ \phi_3(\cdot)\\ \phi_4(\cdot)\\ \phi_5(\cdot)\\ \phi_6(\cdot)\\ \phi_7(\cdot)\\ \phi_8(\cdot)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\\ 0\\ \xi_1(\cdot)\\ \xi_2(\cdot)\\ \xi_3(\cdot)\\ \xi_4(\cdot)\end{pmatrix}.$$

This implies

$$\begin{aligned}
\xi_1' + (\lambda + m(a) + v(a))\xi_1(a) &= \phi_5(a)\\
\xi_2' + (\lambda + m_1 + m(a))\xi_2(a) &= \phi_6(a)\\
\xi_3' + (\lambda + \gamma + m(a))\xi_3(a) &= \phi_7(a)\\
\xi_4' + (\lambda + m(a) + w_0(a))\xi_4(a) &= \phi_8(a)\\
\xi_1(0) &= \phi_1\\
\xi_2(0) &= \phi_2\\
\xi_3(0) &= \phi_3\\
\xi_4(0) &= \phi_4.
\end{aligned}$$

Solving we get,

$$\begin{aligned}
\xi_1(a) &= \phi_1 e^{-\int_0^a(\lambda+m(s)+v(s))ds} + \int_0^a \phi_5(s)e^{-\int_s^a(\lambda+m(\tau)+v(\tau))d\tau}ds\\
\xi_2(a) &= \phi_2 e^{-\int_0^a(\lambda+m_1+m(s))ds} + \int_0^a \phi_6(s)e^{-\int_s^a(\lambda+m_1+m(\tau))d\tau}ds\\
\xi_3(a) &= \phi_3 e^{-\int_0^a(\lambda+\gamma+m(s))ds} + \int_0^a \phi_7(s)e^{-\int_s^a(\lambda+\gamma+m(\tau))d\tau}ds\\
\xi_4(a) &= \phi_4 e^{-\int_0^a(\lambda+m(s)+w_0(s))ds} + \int_0^a \phi_8(s)e^{-\int_s^a(\lambda+m(\tau)+w_0(\tau))d\tau}ds.
\end{aligned}$$

Now, taking L^1 norm we get

$$||\xi_1||_{L^1}+||\xi_2||_{L^1}+||\xi_3||_{L^1}+||\xi_4||_{L^1} \le \frac{1}{\lambda + w_0}(||\xi_1||_{L^1}+||\xi_2||_{L^1}+||\xi_3||_{L^1}+||\xi_4||_{L^1}).$$

Therefore, we have

$$||(\lambda I - A)^{-1}|| \le \frac{1}{\lambda + w_0} \text{ for all } \lambda \in \Omega.$$

So, we conclude that $(A, D(A))$ is a Hille-Yosida operator.

Now, let us recall the following results.

Lemma 2.1. *[Pazy [33], 1983] Let X be a Banach space. Let operator A with domain D(A) be a Hille-Yosida operator on X and B be a bounded linear operator on X. Then A + B is also a Hille-Yosida operator.*

Lemma 2.2. *[Pazy [33], 1983] Let* $(A, D(A))$ *be a Hille-Yosida operator, then its part* $(A_0, D(A_0))$ *generates a* C_0 *semigroup* $\{T_0(t) \mid t \geq 0\}$ *on* X_0, *where* $X_0 = (\overline{D(A)}, \|\cdot\|)$.

Using Lemmas 2.1 and 2.2, and proof given above, we conclude that the operator $A + DF(v^*)$ is a Hille-Yosida operator and the part of $(A, D(A)$ and the part of $(A + DF(u^*))$ generate C_0 semigroups on X_0. So, the proof given above and Lemma 2.1, Lemma 2.2 prove the existence and uniqueness of mild solution to our model (for more details we refer to Pazy [33]). So, we have the following theorem:

Theorem 2.3. *For any* $v_0 \in D_+ = \{0\} \times \{0\} \times \{0\} \times \{0\} \times W^{1,1}((0, a_m), \mathbb{R}_+) \times W^{1,1}((0, a_m), \mathbb{R}_+) \times W^{1,1}((0, a_m), \mathbb{R}_+) \times W^{1,1}((0, a_m), \mathbb{R}_+)$ *system 2.4 has a unique continuous solution with initial data in* D_+. *Moreover, the function defined by* $\Psi(t, v_0) = v(t, v_0)$ *is strongly continuous semigroup.*

Remark 2.4. *Since* $\Psi(t, v_0)$ *is strongly continuous semigroup, there exists* $\mathcal{M} \geq 1$, $\omega \geq 0$ *such that* $\|\Psi(t)\| \leq \mathcal{M}e^{\omega t}$. *So, if we assume* $t \in [0, T]$ *for some finite* $T > 0$, *then we have the boundedness of solution to the abstract Cauchy problem 2.2.*

3. Stability analysis

In this section, we discuss the stability analysis of disease-free steady state. Steady-state solutions play an important role to study the qualitative properties of the solution when the explicit form of the solution is not known. Steady-state equations for our model are given by

$$\begin{aligned}
\frac{dS}{da} &= \Lambda - \varphi(a)IS - m(a)S - v(a)S + w_0(a)V \\
\frac{dE}{da} &= \varphi(a)IS - m_1E - m(a)E \\
\frac{dI}{da} &= m_1E - \gamma I - m(a)I + \beta\sigma(a)IV \\
\frac{dR}{da} &= \gamma I - m(a)R \\
\frac{dV}{da} &= -(m(a) + w_0(a) + \beta\sigma(a)I)V \\
S(0) &= S_1, E(0) = I(0) = R(0) = 0 \\
V(0) &= \int_0^{a_m} v(a)S(a)da.
\end{aligned} \tag{3.1}$$

Disease free steady state $(S^0(a), 0, 0, 0, V^0(a))$ is given by

$$S^0(a) = S_1 e^{-\int_0^a (m(\tau)+v(\tau))d\tau} + \int_0^a (\Lambda - w_0(s)V^0(s))e^{-\int_a^s (m(\tau)+v(\tau))d\tau} ds \tag{3.2}$$

$$V^0(a) = \left(\int_0^{a_m} v(\tau)S^0(\tau)d\tau\right)e^{-\int_0^a (m(s)+w_0(s))ds}. \tag{3.3}$$

Now, our task is to derive expressions for S^0 which is independent of V^0 and similarly for V^0. Let

$$\Theta = \int_0^{a_m} v(\tau)S^0(\tau)d\tau. \tag{3.4}$$

Now, Eqs (3.2) and (3.3) can be written as

$$S^0(a) = S_1 e^{-\int_0^a (m(\tau)+v(\tau))d\tau} + \int_0^a \left(\Lambda - w_0(s)\Theta e^{-\int_0^s (m(\tau)+w_0(\tau))d\tau}\right)e^{-\int_a^s (m(\tau)+v(\tau))d\tau}ds \tag{3.5}$$

$$V^0(a) = \Theta e^{-\int_0^a (m(s)+w_0(s))ds}. \tag{3.6}$$

Now, substituting value of S^0 from (3.5) into (3.4), we get

$$\Theta = \frac{\int_0^a S_1 v(a)e^{-\int_0^a (m(\tau)+v(\tau))d\tau}da + \int_0^a \Lambda e^{-\int_a^s (m(\tau)+v(\tau))d\tau}da}{1 + \int_0^{a_m} v(a)\left(\int_0^a w_0(s)e^{-\int_0^s (m(\tau)+w_0(\tau))d\tau}e^{-\int_a^s (m(\tau)+v(\tau))d\tau}ds\right)da} \tag{3.7}$$

So, we have steady-state solutions in explicit form. From these solutions, it is clear that both S^0 and V^0 are non-negative. Let us define the threshold number in the following manner (for more details we refer to [27]):

$$R_0 = m_1 \int_0^{a_m} S^0(z)\varphi(z)e^{-\int_z^a (m_1+m(\tau))d\tau}\left(\int_0^z e^{-\int_s^z (\gamma+m(\tau)-\beta\sigma(\tau)V^0(\tau))d\tau}ds\right)dz. \tag{3.8}$$

The above defined threshold parameter may not be biologically relevant but plays an important role to study the stability of disease-free steady state. We can state the following result which relates the stability of disease-free steady state with the threshold parameter R_0.

Theorem 3.1. *The disease free steady state is stable if* $R_0 < 1$ *and unstable if* $R_0 > 1$.

Proof. For the local stability of the disease-free equilibrium point, firstly we will linearize our system around the disease-free steady state. Linearizing system (2.1) around the disease-free equilibrium, we get

$$\begin{aligned}
\frac{\partial S}{\partial t}+\frac{\partial S}{\partial a} &= -\varphi(a)S^0(a)I - m(a)S - v(a)S + w_0(a)V\\
\frac{\partial E}{\partial t}+\frac{\partial E}{\partial a} &= \varphi(a)S^0(a)I - m_1E - m(a)E\\
\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a} &= m_1E - \gamma I - m(a)I + \beta\sigma(a)V^0(a)I\\
\frac{\partial R}{\partial t}+\frac{\partial R}{\partial a} &= \gamma I - m(a)R\\
\frac{\partial V}{\partial t}+\frac{\partial V}{\partial a} &= -(m(a)+w_0(a))V - \beta\sigma(a)V^0(a)I\\
S(0,t) &= S_1, E(0,t) = I(0,t) = R(0,t) = 0\\
V(0,t) &= \int_0^{a_m} v(a)S(a,t)da\\
S(a,0) &= S_0(a), E(a,0) = E_0(a), I(a,0) = I_0(a)\\
R(a,0) &= R_0(a) \text{ and } V(a,0) = V_0(a).
\end{aligned} \tag{3.9}$$

We are looking for solutions of the form

$$S(a,t) = \bar{S}(a)e^{\lambda t}, E(a,t) = \bar{E}(a)e^{\lambda t}, I(a,t) = \bar{I}(a)e^{\lambda t}, R(a,t) = \bar{R}(a)e^{\lambda t}, V(a,t) = \bar{V}(a)e^{\lambda t}.$$

Therefore, system (3.9) becomes

$$\begin{aligned}
\frac{d\bar{S}}{da}+\lambda\bar{S} &= -\varphi(a)S^0(a)\bar{I} - m(a)\bar{S} - v(a)\bar{S} + w_0(a)\bar{V}\\
\frac{d\bar{E}}{da}+\lambda\bar{E} &= \varphi(a)S^0(a)\bar{I} - m_1\bar{E} - m(a)\bar{E}\\
\frac{d\bar{I}}{da}+\lambda\bar{I} &= m_1\bar{E} - \gamma\bar{I} - m(a)\bar{I} + \beta\sigma(a)V^0(a)\bar{I}\\
\frac{d\bar{R}}{da}+\lambda\bar{R} &= \gamma\bar{I} - m(a)\bar{R}\\
\frac{d\bar{V}}{da}+\lambda\bar{V} &= -(m(a)+w_0(a))\bar{V} - \beta\sigma(a)V^0(a)\bar{I}
\end{aligned} \tag{3.10}$$

Solving these equations, we get

$$\begin{aligned}
\bar{S}(a) &= S_1e^{-\int_0^a(\lambda+m(\tau)+v(\tau))d\tau} + \int_0^a\left(w_0(z)\bar{V}(z) - S^0(z)\varphi(z)\bar{I}(z)\right)e^{-\int_z^a(\lambda+m(\tau)+v(\tau))d\tau}dz\\
\bar{E}(a) &= \int_0^a S^0(z)\varphi(z)\bar{I}(z)e^{-\int_z^a(\lambda+m_1+m(\tau))d\tau}dz\\
\bar{I}(a) &= \int_0^a m_1\bar{E}(z)e^{-\int_z^a(\lambda+\gamma+m(\tau)-\beta\sigma(\tau)V^0(\tau))d\tau}dz\\
\bar{R}(a) &= \int_0^a \gamma\bar{I}(z)e^{-\int_z^a(\lambda+m(\tau))d\tau}dz
\end{aligned}$$

$$\bar{V}(a) = e^{-\int_0^a(\lambda+m(\tau)+w_0(\tau))d\tau}\int_0^{a_m} v(a)\bar{S}(a)da - \int_0^a \beta\sigma(z)V^0(z)\bar{I}(z)e^{-\int_z^a(\lambda+m(\tau)+w_0(\tau))d\tau}dz.$$

Substituting value of $\bar{E}(a)$ in $\bar{I}(a)$, we get

$$\bar{I}(a) = \int_0^a m_1\left(\int_0^z \left(S^0(\eta)\varphi(\eta)\bar{I}(\eta)\right)e^{-\int_\eta^z(\lambda+m_1+m(\tau))d\tau}d\eta\right)e^{-\int_z^a(\lambda+\gamma+m(\tau)-\beta\sigma(\tau)V^0(\tau))d\tau}dz$$

Observe that

$$\frac{d\bar{I}}{da} + \lambda\bar{I}(a) = m_1\left(\int_0^a S^0(\eta)\varphi(\eta)\bar{I}(\eta)e^{-\int_\eta^a(\lambda+m_1+m(\tau))d\tau}d\eta\right) - \gamma\bar{I}(a) - m(a)\bar{I}(a) + \beta\sigma(a)V^0(a)\bar{I}(a).$$

In simplified form, it can be written as

$$\frac{d\bar{I}}{da} + (\lambda+\gamma+m(a)-\beta\sigma(a)V^0(a))\bar{I}(a) = m_1\left(\int_0^a S^0(z)\varphi(z)\bar{I}(z)e^{-\int_z^a(\lambda+m_1+m(\tau))d\tau}dz\right). \tag{3.11}$$

Using (3.11), the characteristic equation can be written as

$$G(\lambda) = 1,$$

where

$$G(\lambda) = m_1\int_0^a S^0(z)\varphi(z)e^{-\int_z^a(\lambda+m_1+m(\tau))d\tau}\left(\int_0^z e^{-\int_s^z(\lambda+\gamma+m(\tau)-\beta\sigma(\tau)V^0(\tau))d\tau}ds\right)dz \tag{3.12}$$

It is easy to see that $G(0) = R_0$. G can also be written as

$$G(\lambda) = m_1\int_0^a e^{-\lambda(a-z)}S^0(z)\varphi(z)e^{-\int_z^a(m_1+m(\tau))d\tau}\left(\int_0^z e^{-\lambda(z-s)}e^{-\int_s^z(\gamma+m(\tau)-\beta\sigma(\tau)V^0(\tau))d\tau}ds\right)dz \tag{3.13}$$

Taking derivative of G with respect to λ, we get

$$\begin{aligned}G'(\lambda) = &-m_1\int_0^a (a-z)e^{-\lambda(a-z)}S^0(z)\varphi(z)e^{-\int_z^a(m_1+m(\tau))d\tau}\left(\int_0^z e^{-\lambda(z-s)}e^{-\int_s^z(\gamma+m(\tau)-\beta\sigma(\tau)V^0(\tau))d\tau}ds\right)dz\\ &-m_1\int_0^a e^{-\lambda(a-z)}S^0(z)\varphi(z)e^{-\int_z^a(m_1+m(\tau))d\tau}\left(\int_0^z (z-s)e^{-\lambda(z-s)}e^{-\int_s^z(\gamma+m(\tau)-\beta\sigma(\tau)V^0(\tau))d\tau}ds\right)dz.\end{aligned}$$

Observe that G is a decreasing function of λ as $G'(\lambda) < 0$ and $\lim_{\lambda\to\infty} G(\lambda) = 0$. Let us assume that $\lambda = x_1 + ix_2$ is a root of equation $G(\lambda) = 1$. Then for $x_1 \geq 0$, we have

$$1 = |G(\lambda)| \leq |G(0)| \leq R_0.$$

Thus the real part of eigenvalue λ is negative if threshold parameter $R_0 < 1$. Therefore, the disease free equilibrium point is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. □

4. Optimal control problem

In this section, we discuss the optimal control strategy for our problem. Optimal control theory plays an important role in epidemiological models because it is very important to control the spread of disease in an optimal manner. Here, our control variable is vaccination effort and we want to minimize the cost of vaccination and also our aim is to reduce the number of infected individuals. We assume that $\varphi(a,t) = \psi(a,t)I(a,t)$. For this form of $\varphi(a,t)$, it is easy to construct operators A and F to show the existence and uniqueness of the solution. Also, we assume that v depends on both age variable a and time t. We consider the following optimal control problem:

$$\begin{aligned}
\frac{\partial S}{\partial t} + \frac{\partial S}{\partial a} &= \Lambda - \psi(a,t)IS - m(a)S - v(a,t)S + w_0(a)V \\
\frac{\partial E}{\partial t} + \frac{\partial E}{\partial a} &= \psi(a,t)IS - m_1 E - m(a)E \\
\frac{\partial I}{\partial t} + \frac{\partial I}{\partial a} &= m_1 E - \gamma I - m(a)I + \beta\sigma(a)IV \\
\frac{\partial R}{\partial t} + \frac{\partial R}{\partial a} &= \gamma I - m(a)R \\
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial a} &= -(m(a) + w_0(a) + \beta\sigma(a)I)V \\
S(0,t) &= S_1, E(0,t) = I(0,t) = R(0,t) = 0 \\
V(0,t) &= \int_0^{a_m} v(a,t)S(a,t)da \\
S(a,0) &= S_0(a), E(a,0) = E_0(a), I(a,0) = I_0(a) \\
R(a,0) &= R_0(a) \text{ and } V(a,0) = V_0(a).
\end{aligned} \tag{4.1}$$

$v(a,t)$ is the control variable and our aim is to minimize the number of infected individuals and the cost of implementing control.

Theorem 4.1. *Let $(S^{v_1}, E^{v_1}, I^{v_1}, R^{v_1}, V^{v_1}), (S^{v_2}, E^{v_2}, I^{v_2}, R^{v_2}, V^{v_2})$ be solutions of system 4.1 corresponding to the control variables v_1 and v_2, respectively. Then for sufficiently small $T > 0$, the following estimates hold*

$$\int_Q (|S^{v_1} - S^{v_2}| + |E^{v_1} - E^{v_2}| + |I^{v_1} - I^{v_2}| + |R^{v_1} - R^{v_2}| + |V^{v_1} - V^{v_2}|)dadt \le C_{01} \int_Q (|v_1 - v_2|)dadt,$$

where C_{01} is a positive constant which depends on the bound of $m, m_1, \gamma, \beta, \sigma, \psi, \omega_0$, and initial data. Moreover,

$$\begin{aligned}
||S^{v_1} - S^{v_2}||_{L^\infty(Q)} + ||E^{v_1} - E^{v_2}||_{L^\infty(Q)} + ||I^{v_1} - I^{v_2}||_{L^\infty(Q)} + ||R^{v_1} - R^{v_2}||_{L^\infty(Q)} + ||V^{v_1} - V^{v_2}||_{L^\infty(Q)} \\
\le C_{02}||v_1 - v_2||_{L^\infty(Q)},
\end{aligned}$$

where C_{02} is a positive contant.

Proof. Using the method of characteristics, the system 4.1 can be solved as:

$$S(a,t)=\begin{cases}-\int_0^t\left[m(\tau+a-t)+\nu(\tau+a-t)\right]S(\tau+a-t,\tau)d\tau\\ -\int_0^t\psi(\tau+a-t)I(\tau+a-t,\tau)S(\tau+a-t,\tau)d\tau\\ +\int_0^t(\Lambda+\omega_0(\tau+a-t)V(\tau+a-t,\tau)d\tau+S_0(a-t)\\ \text{if } a>t,\\ -\int_{t-a}^t\left[m(\tau+a-t)+\nu(\tau+a-t)\right]S(\tau+a-t,\tau)d\tau\\ -\int_0^t\psi(\tau+a-t)I(\tau+a-t,\tau)S(\tau+a-t,\tau)d\tau\\ +\int_{t-a}^t(\Lambda+\omega_0(\tau+a-t)V(\tau+a-t,\tau)d\tau+S_1\\ \text{if } a<t\end{cases}\tag{4.2}$$

$$E(a,t)=\begin{cases}-\int_0^t\left[m_1+m(\tau+a-t)\right]E(\tau+a-t,\tau)d\tau\\ -\int_0^t\psi(\tau+a-t)I(\tau+a-t,\tau)S(\tau+a-t,\tau)d\tau+E_0(a-t)\\ \text{if } a>t,\\ -\int_{t-a}^t\left[m_1+m(\tau+a-t)\right]E(\tau+a-t,\tau)d\tau\\ -\int_{t-a}^t\psi(\tau+a-t)I(\tau+a-t,\tau)S(\tau+a-t,\tau)d\tau\\ \text{if } a<t\end{cases}\tag{4.3}$$

$$I(a,t)=\begin{cases}-\int_0^t\left[\gamma+m(\tau+a-t)\right]I(\tau+a-t,\tau)d\tau\\ +\int_0^t\beta\sigma(\tau+a-t)I(\tau+a-t,\tau)V(\tau+a-t,\tau)d\tau\\ +\int_0^t m_1E(\tau+a-t,\tau)d\tau+I_0(a-t)\\ \text{if } a>t,\\ -\int_{t-a}^t\left[\gamma+m(\tau+a-t)\right]I(\tau+a-t,\tau)d\tau\\ +\int_{t-a}^t\beta\sigma(\tau+a-t)I(\tau+a-t,\tau)V(\tau+a-t,\tau)d\tau\\ +\int_{t-a}^t m_1E(\tau+a-t,\tau)d\tau\\ \text{if } a<t\end{cases}\tag{4.4}$$

$$R(a,t)=\begin{cases}-\int_0^t m(\tau+a-t)R(\tau+a-t,\tau)d\tau+\int_0^t\gamma I(\tau+a-t,\tau)d\tau+R_0(a-t)\\ \text{if } a>t,\\ -\int_{t-a}^t m(\tau+a-t)R(\tau+a-t,\tau)d\tau\\ +\int_{t-a}^t\gamma I(\tau+a-t,\tau)d\tau\\ \text{if } a<t\end{cases}\tag{4.5}$$

$$V(a,t)=\begin{cases}-\int_0^t\left[m(\tau+a-t)+\omega_0(\tau+a-t)+\beta\sigma(\tau+a-t)I(\tau+a-t,\tau)\right]V(\tau+a-t,\tau)d\tau\\ +V_0(a-t)\\ \text{ if } a>t,\\ -\int_{t-a}^t\left[m(\tau+a-t)+\omega_0(\tau+a-t)+\beta\sigma(\tau+a-t)I(\tau+a-t,\tau)\right]V(\tau+a-t,\tau)d\tau\\ +\int_0^{a_m}v(a)S(a,t-a)da\\ \text{ if } a<t.\end{cases} \tag{4.6}$$

Let $Q=(0,T)\times(0,a_m)$, and assume that $(S^{v_1},E^{v_1},I^{v_1},R^{v_1},V^{v_1}),(S^{v_2},E^{v_2},I^{v_2},R^{v_2},V^{v_2})$ be solution of system (4.1) corresponding to the control variables v_1 and v_2 respectively. Then using the solution given by (4.2), we get

$$\int_{Q\cap\{a<t\}}|S^{v_1}-S^{v_2}|dadt\le C_{11}\int_Q(|v_1-v_2|+|S^{v_1}-S^{v_2}|)dadt$$
$$+C_{22}\int_Q(|S^{v_1}-S^{v_2}|+|I^{v_1}-I^{v_2}|)dadt+C_{33}\int_Q|V^{v_1}-V^{v_2}|dadt,$$

where C_{11},C_{22},C_{33} are constants which depends on the bound of m,ψ, and ω_0. Similarly, we can consider the case when $a>t$, and we will obtain similar types of estimates. So, we can write

$$\int_Q|S^{v_1}-S^{v_2}|dadt\le C_{11}\int_Q(|v_1-v_2|+|S^{v_1}-S^{v_2}|)dadt$$
$$+C_{22}\int_Q(|S^{v_1}-S^{v_2}|+|I^{v_1}-I^{v_2}|)dadt+C_{33}\int_Q|V^{v_1}-V^{v_2}|dadt,$$

Using the same procedure for other state variables, we get

$$\int_Q(|S^{v_1}-S^{v_2}|+|E^{v_1}-E^{v_2}|+|I^{v_1}-I^{v_2}|+|R^{v_1}-R^{v_2}|+|V^{v_1}-V^{v_2}|)dadt\le C_{01}\int_Q(|v_1-v_2|)dadt,$$

where C_{01} is a positive constant. Now, estimating the integral in the age variable only, we will get

$$\|S^{v_1}-S^{v_2}\|_{L^\infty(Q)}+\|E^{v_1}-E^{v_2}\|_{L^\infty(Q)}+\|I^{v_1}-I^{v_2}\|_{L^\infty(Q)}+\|R^{v_1}-R^{v_2}\|_{L^\infty(Q)}+\|V^{v_1}-V^{v_2}\|_{L^\infty(Q)}$$
$$\le C_{02}\|v_1-v_2\|_{L^\infty(Q)},$$

where C_{02} is a positive constant that depends on model parameters and initial population distributions. More details about these types of estimates can be found in [31, 32]. □

Let us define the following objective functional:

$$J(v)=\int_0^T\int_0^{a_m}\left(C_1I(a,t)-C_2S(a,t)+C_3v(a,t)^2\right)dadt. \tag{4.7}$$

Here, C_1, C_2 and C_3 are weight factors and our aim is to minimize $J(v)$. We assume that the admissible control variable lies in the set $\mathcal{V}$ defined by

$$\mathcal{V} = \{u : [0, a_m] \times [0, T] \mapsto [0, v_m]\}, \tag{4.8}$$

where we assume that $v \in L^\infty((0, a_m) \times (0, T))$ and $0 \leq v \leq v_m$. Now, our next result establish the existence of optimal control.

Theorem 4.2. *Let $J : L^1(0, T) \to (-\infty, \infty]$ be defined by*

$$J(v) = \begin{cases} \int_0^T \int_0^{a_m} \left(C_1 I(a,t) - C_2 S(a,t) + C_3 v(a,t)^2\right) dadt & \text{if } v \in \mathcal{V} \\ \infty & \text{otherwise}. \end{cases} \tag{4.9}$$

Then to the control problem (4.1), there exists an optimal control u^* such that

$$\min_{v \in \mathcal{V}} J(v) = J(v^*).$$

Proof. We have already shown the existence of unique solution to the system (4.1) which is uniformly bounded. Also control variable v is uniformly bounded in $(0, a_m)$. Therefore, there exist a sequence $v_n \in V$ such that

$$\lim_{n \to \infty} J(u_n) = \liminf_{v \in \mathcal{V}} J(v).$$

Because the function $C_3 v^2$ is a lower semi-continuous function which is also convex, we have

$$\int_0^{a_m} C_3 (v^*)^2 da \leq \liminf_{n \to \infty} \int_0^{a_m} C_3 (v_n)^2 da. \tag{4.10}$$

Also, using the fact that the bound of S, E, I, R and V depends on the control variable v, we have $S_n = S(v_n), E_n = E(v_n), I_n = I(v_n), R_n = R(v_n), V_n = V(v_n)$. Because S_n, E_n, I_n, R_n and V_n are uniformly bounded for each n. We have $S^* = S(v^*), E^* = E(v^*), I^* = I(v^*), R^* = R(v^*)$ and $V^* = V(v^*)$. Now,

$$\begin{aligned} J(v^*) &= \int_0^T \int_0^{a_m} [C_1 I^* - C_2 S^* + C_3 (v^*)^2] dadt \\ &\leq \liminf_{n \to \infty} \int_0^T \int_0^{a_m} [C_1 I_n - C_2 S_n + C_3 (v_n)^2] dadt \\ &= \lim_{n \to \infty} \int_0^T \int_0^{a_m} [C_1 I_n - C_2 S_n + C_3 (v_n)^2] dadt \\ &= \inf_{v \in \mathcal{V}} J(v). \end{aligned}$$

From the relation $J(v^*) \leq \inf_{v \in \mathcal{V}} J(v)$, we conclude that v^* is optimal control which minimizes the objective functional. □

Now, our next task is to obtain the necessary conditions for our optimal control problem. Let us define $v^\epsilon(a, t) = v(a, t) + \epsilon g(a)$, where $0 < \epsilon < 1$, $g \in L^\infty(0, a_m)$ and also called the variation function. Let us denote $S^\epsilon = S(v^\epsilon), E^\epsilon = E(v^\epsilon), I^\epsilon = I(v^\epsilon), R^\epsilon = R(v^\epsilon)$ and $V^\epsilon = V(v^\epsilon)$. Then we have the following system

$$
\begin{aligned}
\frac{\partial S^{\epsilon}}{\partial t}+\frac{\partial S^{\epsilon}}{\partial a} &= \Lambda-\psi(a,t)S^{\epsilon}I^{\epsilon}-m(a)S^{\epsilon}-(v(a,t)+\epsilon g(a))S^{\epsilon}+w_0(a)V^{\epsilon}\\
\frac{\partial E^{\epsilon}}{\partial t}+\frac{\partial E^{\epsilon}}{\partial a} &= \psi(a,t)S^{\epsilon}I^{\epsilon}-m_1E^{\epsilon}-m(a)E^{\epsilon}\\
\frac{\partial I^{\epsilon}}{\partial t}+\frac{\partial I^{\epsilon}}{\partial a} &= m_1E^{\epsilon}-\gamma I^{\epsilon}-m(a)I^{\epsilon}+\beta\sigma(a)I^{\epsilon}V^{\epsilon}\\
\frac{\partial R^{\epsilon}}{\partial t}+\frac{\partial R^{\epsilon}}{\partial a} &= \gamma I^{\epsilon}-m(a)R^{\epsilon}\\
\frac{\partial V^{\epsilon}}{\partial t}+\frac{\partial V^{\epsilon}}{\partial a} &= -(m(a)+w_0(a)+\beta\sigma(a)I^{\epsilon})V^{\epsilon}\\
S^{\epsilon}(0,t) &= S_1, E^{\epsilon}(0,t)=I^{\epsilon}(0,t)=R^{\epsilon}(0,t)=0\\
V^{\epsilon}(0,t) &= \int_0^{a_m}(v(a,t)+\epsilon g(a))S(a,t)da\\
S^{\epsilon}(a,0) &= S_0(a), E^{\epsilon}(a,0)=E_0(a), I^{\epsilon}(a,0)=I_0(a)\\
R^{\epsilon}(a,0) &= R_0(a) \text{ and } V^{\epsilon}(a,0)=V_0(a).
\end{aligned}
\tag{4.11}
$$

Let us set the following notations for weak derivatives of S, E, I, R and V:

$\bar{S}(a,t)=\lim_{\epsilon\to 0}\frac{S(v+\epsilon g)-S(v)}{\epsilon}$, $\bar{E}(a,t)=\lim_{\epsilon\to 0}\frac{E(v+\epsilon g)-E(v)}{\epsilon}$, $\bar{I}(a,t)=\lim_{\epsilon\to 0}\frac{I(v+\epsilon g)-I(v)}{\epsilon}$, $\bar{R}(a,t)=\lim_{\epsilon\to 0}\frac{R(v+\epsilon g)-R(v)}{\epsilon}$, $\bar{V}(a,t)=\lim_{\epsilon\to 0}\frac{V(v+\epsilon g)-V(v)}{\epsilon}$. Then we have the following system of differential equations

$$
\begin{aligned}
\frac{\partial \bar{S}}{\partial t}+\frac{\partial \bar{S}}{\partial a} &= -\psi(a,t)\left(\bar{S}I+\bar{I}S\right)-m(a)\bar{S}-v(a,t)\bar{S}+w_0(a)\bar{V}-g(a)S\\
\frac{\partial \bar{E}}{\partial t}+\frac{\partial \bar{E}}{\partial a} &= \psi(a,t)\left(\bar{S}I+\bar{I}S\right)-m_1\bar{E}-m(a)\bar{E}\\
\frac{\partial \bar{I}}{\partial t}+\frac{\partial \bar{I}}{\partial a} &= m_1\bar{E}-\gamma\bar{I}-m(a)\bar{I}+\beta\sigma(a)(\bar{I}V+I\bar{V})\\
\frac{\partial \bar{R}}{\partial t}+\frac{\partial \bar{R}}{\partial a} &= \gamma\bar{I}-m(a)\bar{R}\\
\frac{\partial \bar{V}}{\partial t}+\frac{\partial \bar{V}}{\partial a} &= -(m(a)+w_0(a))\bar{V}-\beta\sigma(a)(\bar{I}V+I(a,t)\bar{V})\\
\bar{S}(0,t) &= 0, \bar{E}(0,t)=\bar{I}(0,t)=\bar{R}(0,t)=0\\
\bar{V}(0,t) &= \int_0^{a_m}\left(v(a,t)\bar{S}(a,t)+g(a)S(a,t)\right)da\\
\bar{S}(a,0) &= \bar{E}(a,0)=\bar{I}(a,0)=\bar{R}(a,0)=0 \text{ and } \bar{V}(a,0)=0.
\end{aligned}
\tag{4.12}
$$

The directional derivative of J with respect to v in the direction of g is given by

$$
J'(v)=\frac{J(v^{\epsilon})-J(v)}{\epsilon}=\int_0^T\int_0^{a_m}\left[C_1\bar{I}(a,t)-C_2\bar{S}(a,t)+2C_3v(a,t)g(a)\right]dadt. \tag{4.13}
$$

Now, our next task is to find the adjoint system. Let us define

$$\langle f, g\rangle = \int_0^T \int_0^\infty f(a,t)g(a,t)dadt.$$

Observe that

$$\begin{aligned}
0 &= \left\langle \frac{\partial \bar{S}}{\partial t} + \frac{\partial \bar{S}}{\partial a} + \psi(a,t)\left(\bar{S}I + \bar{I}S\right)\right. \\
&\quad \left. + m(a)\bar{S} + v(a,t)\bar{S} - w_0(a)\bar{V} + g(a)S, \lambda_1(a,t)\right\rangle \\
0 &= \left\langle \bar{S}(a,t), -\frac{\partial}{\partial t}\lambda_1 - \frac{\partial}{\partial a}\lambda_1 + (m(a)+v(a,t))\lambda_1 + \psi(a,t)\lambda_1 I\right\rangle \qquad (4.14) \\
&\quad + \int_0^T \int_0^{a_m} \left(\psi(a,t)\bar{I}(a,t) + g(a)\right) S(a,t)\lambda_1(a,t)dadt - \int_0^T \int_0^{a_m} w_0(a)\bar{V}(a,t)\lambda_1(a,t)dadt
\end{aligned}$$

with the following conditions

$$\lambda_1(t, a_m) = \lambda_1(T, a) = 0.$$

Similarly we have

$$\begin{aligned}
0 &= \left\langle \frac{\partial \bar{E}}{\partial t} + \frac{\partial \bar{E}}{\partial a} - \psi(a,t)\left(\bar{S}I - \bar{I}S\right) + m_1\bar{E} + m(a)\bar{E}, \lambda_2(a,t)\right\rangle \\
0 &= \left\langle \bar{E}(a,t), -\frac{\partial}{\partial t}\lambda_2 - \frac{\partial}{\partial a}\lambda_2 + (m_1 + m(a))\lambda_2\right\rangle \qquad (4.15) \\
&\quad - \int_0^T \int_0^{a_m} \psi(a,t)\lambda_2(a,t)\left(S(a,t)\bar{I}(a,t) + I(a,t)\bar{S}(a,t)\right) dadt
\end{aligned}$$

$$\begin{aligned}
0 &= \left\langle \frac{\partial \bar{I}}{\partial t} + \frac{\partial \bar{I}}{\partial a} - m_1\bar{E} + \gamma\bar{I} + m(a)\bar{I} - \beta\sigma(a)(\bar{I}V - I\bar{V}), \lambda_3(a,t)\right\rangle \\
0 &= \left\langle \bar{I}(a,t), -\frac{\partial}{\partial t}\lambda_3 - \frac{\partial}{\partial a}\lambda_3 - m_1\lambda_3\right\rangle \qquad (4.16) \\
&\quad - \int_0^T \int_0^{a_m} (\gamma + m(a) - \beta\sigma(a)V(a,t))\bar{I}(a,t)\lambda_3(a,t)dadt - \int_0^T \int_0^{a_m} \lambda_3(a,t)I(a,t)\bar{I}(a,t)dadt
\end{aligned}$$

$$\begin{aligned}
0 &= \left\langle \frac{\partial \bar{R}}{\partial t} + \frac{\partial \bar{R}}{\partial a} - \gamma\bar{I} + m(a)\bar{R}, \lambda_4(a,t)\right\rangle \\
0 &= \left\langle \bar{R}(a,t), -\frac{\partial}{\partial t}\lambda_4 - \frac{\partial}{\partial a}\lambda_4 - m(a)\lambda_4\right\rangle \qquad (4.17) \\
&\quad - \int_0^T \int_0^{a_m} \gamma\lambda_4(a,t)\bar{I}(a,t)dadt
\end{aligned}$$

$$0 = \left\langle \frac{\partial \bar{V}}{\partial t} + \frac{\partial \bar{V}}{\partial a} + (m(a) + w_0(a))\bar{V} + \beta\sigma(a)(\bar{I}V + I\bar{V}), \lambda_5(a,t)\right\rangle$$

$$0 = \left\langle \bar{V}(a,t), -\frac{\partial}{\partial t}\lambda_5 - \frac{\partial}{\partial a}\lambda_5 + (m(a) + w_0(a))\lambda_5 + \beta\sigma(a)I\lambda_5 \right\rangle \quad (4.18)$$
$$+ \int_0^T \int_0^{a_m} \beta\sigma(a)\bar{I}(a,t)\lambda_5(a,t)dadt - \int_0^T \int_0^{a_m} (v(a,t)\bar{S}(a,t) + g(a)S(a,t))\lambda_5(0,t)dadt$$

Now, we get the following adjoint system

$$\begin{aligned}
\frac{\partial\lambda_1}{\partial t} + \frac{\partial\lambda_1}{\partial a} &= (m(a) + v(a,t))\lambda_1 + \psi(a,t)\lambda_1 I - \psi(a,t)\lambda_2 I - v(a,t)\lambda_5(0,t) - C_2 \\
\frac{\partial\lambda_2}{\partial t} + \frac{\partial\lambda_2}{\partial a} &= (m_1 + m(a))\lambda_2 \\
\frac{\partial\lambda_3}{\partial t} + \frac{\partial\lambda_3}{\partial a} &= -m_1\lambda_3 - (\gamma + m(a) - \beta\sigma(a)V - I)\lambda_3 - \gamma\lambda_4 \\
&\quad + \beta\sigma(a)\lambda_5 - \psi(a,t)\lambda_2(a,t)S + C_1 \\
\frac{\partial\lambda_4}{\partial t} + \frac{\partial\lambda_4}{\partial a} &= -m(a)\lambda_4 \\
\frac{\partial\lambda_5}{\partial t} + \frac{\partial\lambda_5}{\partial a} &= (m(a) + w_0(a) + \beta\sigma(a)I)\lambda_5.
\end{aligned} \quad (4.19)$$

Also, we have the following conditions

$$\lambda_1(a,T) = \lambda_2(a,T) = \lambda_3(a,T) = \lambda_4(a,T) = \lambda_5(a,T) = 0.$$

$$\lambda_1(a_m,t) = \lambda_2(a_m,t) = \lambda_3(a_m,t) = \lambda_4(a_m,t) = \lambda_5(a_m,t) = 0.$$

Using the same steps as we followed in 4.1, we can state the following theorem

Theorem 4.3. *Let $(\lambda_{11}, \lambda_{21}, \lambda_{31}, \lambda_{41}, \lambda_{51})$ and $(\lambda_{12}, \lambda_{22}, \lambda_{32}, \lambda_{42}, \lambda_{52})$ be solution of adjoint system (4.19) with control variable v_1 and v_2 respectivey. Then*

$$\|\lambda_{11} - \lambda_{12}\|_{L^\infty(Q)} + \|\lambda_{21} - \lambda_{22}\|_{L^\infty(Q)} + \|\lambda_{31} - \lambda_{32}\|_{L^\infty(Q)} + \|\lambda_{41} - \lambda_{42}\|_{L^\infty(Q)} + \|\lambda_{51} - \lambda_{52}\|_{L^\infty(Q)} \leq C_{11}\|v_1 - v_2\|_{L^\infty(Q)},$$

where C_{11} is a positive constant.

Now, we state and prove the following result, which gives the explicit form of optimal control.

Theorem 4.4. *Let $(S^*, E^*, I^*, R^*, V^*)$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be state solutions and solutions to corresponding adjoint system respectively. Also, let $v^* \in \mathcal{V}$ be optimal control which minimizes (4.7). Then optimal control will be of the form*

$$v^* = \max\left\{0, \min\left\{v_m, \frac{\lambda_1(a,t)(w_0(a)V^*(a,t) - \psi(a,t)I^*(a,t))}{2C_3 g(a))} - \frac{(\lambda_1(a,t) - \lambda_5(0,t))S^*(a,t)}{2C_3}\right\}\right\}.$$

Proof. We know that

$$J'(v) = \int_0^T \int_0^{a_m} \left[C_1\bar{I}(a,t) - C_2\bar{S}(a,t) + 2C_3 v(a,t)g(a)\right] dadt. \quad (4.20)$$

Now using the adjoint system, we have

$$\begin{aligned} J'(v) &= \int_0^T\int_0^{a_m}\bar{I}(a,t)\left[\frac{\partial\lambda_3}{\partial t}+\frac{\partial\lambda_3}{\partial a}+m_1\lambda_3(a,t)+(\gamma+m(a)-\beta\sigma(a)V(a,t)-I(a,t))\lambda_3(a,t)\right]datdt \\ &+\int_0^T\int_0^{a_m}\bar{I}(a,t)\left[\gamma\lambda_4(a,t)-\beta\sigma(a)\lambda_5(a,t)+\psi(a,t)\lambda_2(a,t)S(a,t)\right]dadt \\ &-\int_0^T\int_0^{a_m}\bar{S}(a,t)\left[-\frac{\partial\lambda_1}{\partial t}-\frac{\partial\lambda_1}{\partial a}\right]dadt+\int_0^T\int_0^{a_m}\bar{S}(a,t)\left[(m(a)+v(a,t))\lambda_1(a,t)\right]dadt \\ &+\int_0^T\int_0^{a_m}\bar{S}(a,t)\left[\psi(a,t)\lambda_1(a,t)I(a,t)-\psi(a,t)\lambda_2(a,t)I(a,t)-v(a,t)\lambda_5(0,t)\right]dadt \\ &-\int_0^T\int_0^{a_m}\bar{E}(a,t)\left[\frac{\partial\lambda_2}{\partial t}+\frac{\partial\lambda_2}{\partial a}-(m_1+m(a))\lambda_2(a,t)\right]dadt \\ &-\int_0^T\int_0^{a_m}\bar{R}(a,t)\left[\frac{\partial\lambda_4}{\partial t}+\frac{\partial\lambda_4}{\partial a}+m(a)\lambda_4(a,t)\right]dadt \\ &-\int_0^T\int_0^{a_m}\bar{V}(a,t)\left[-\frac{\partial\lambda_5}{\partial t}-\frac{\partial\lambda_5}{\partial a}+(m(a)+w_0(a)+\beta\sigma(a)I(a,t))\lambda_5(a,t)\right]dadt \end{aligned}$$

Using (4.14, 4.15, 4.16, 4.17) and (4.18), we have

$$\begin{aligned} J'(v) &= \int_0^T\int_0^{a_m}2C_3v(a,t)g(a)dadt \qquad (4.21)\\ &+\int_0^T\int_0^{a_m}(\lambda_1(a,t)-\lambda_5(0,t))S^*(a,t)dadt-\int_0^T\int_0^{a_m}\lambda_1(a,t)(w_0(a)V^*(a,t)-\psi(a,t)I^*(a,t))dadt. \end{aligned}$$

Because $J'(v)\geq 0$, so this implies in this case

$$v(a,t)=\frac{\lambda_1(a,t)(w_0(a)V^*(a,t)-\psi(a,t)I^*(a,t))}{2C_3g(a))}-\frac{(\lambda_1(a,t)-\lambda_5(0,t))S^*(a,t)}{2C_3}$$

By using the upper and lower bounds of control variable, we have

$$v^*=\max\left\{0,\min\left\{v_m,\frac{\lambda_1(a,t)(w_0(a)V^*(a,t)-\psi(a,t)I^*(a,t))}{2C_3g(a))}-\frac{(\lambda_1(a,t)-\lambda_5(0,t))S^*(a,t)}{2C_3}\right\}\right\}.$$

□

Remark 4.5. *It is clear from the characterization of optimal control that it depends on the susceptible, infected, and vaccinated class. Although it depends on function g, we can always fix a particular choice of g.*

Theorem 4.6. *Let M be a generic positive constant that depends on C_{02}, C_{11} and the bound of vaccine vaning rate ω_0, and transmission coefficient ψ. Then for sufficiently small $\frac{M}{2C_3}$, there exists a unique optimal controller v^*.*

Proof. Let $(S^{v_1},E^{v_1},I^{v_1},R^{v_1},V^{v_1}),(S^{v_2},E^{v_2},I^{v_2},R^{v_2},V^{v_2})$ be solution of system (4.1) corresponding to the control variables v_1 and v_2 respectively, and $(\lambda_{11},\lambda_{21},\lambda_{31},\lambda_{41},\lambda_{51}),(\lambda_{12},\lambda_{22},\lambda_{32},\lambda_{42},\lambda_{52})$ be solution of adjoint system 4.19 with control variable v_1 and v_2 respectivey. Define the map $L:\mathcal{U}\to\mathcal{U}$ by

$$L(v)=\max\left\{0,\min\left\{v_m,\frac{\lambda_1(a,t)(w_0(a)V(a,t)-\psi(a,t)I(a,t))}{2C_3g(a))}-\frac{(\lambda_1(a,t)-\lambda_5(0,t))S(a,t)}{2C_3}\right\}\right\}.$$

Then

$$
\begin{aligned}
\|L(v_1) - L(v_2)\| \le & \left\| \frac{\lambda_{11}(a,t)(w_0(a)V^{v_1}(a,t) - \psi(a,t)I^{v_1}(a,t))}{2C_3 g(a))} - \frac{(\lambda_{11}(a,t) - \lambda_{51}(0,t))S^{v_1}(a,t)}{2C_3} \right. \\
& \left. - \frac{\lambda_{12}(a,t)(w_0(a)V^{v_2}(a,t) - \psi(a,t)I^{v_2}(a,t))}{2C_3 g(a))} + \frac{(\lambda_{12}(a,t) - \lambda_{52}(0,t))S^{v_2}(a,t)}{2C_3} \right\|_{L^\infty(Q)} \\
\le & \frac{1}{2C_3} \left[\frac{\omega_0(a)}{g(a)} \|\lambda_{11} V^{v_1} - \lambda_{12} V^{v_2}\|_{L^\infty(Q)} + \frac{\psi(a,t)}{g(a)} \|\lambda_{11} I^{v_1} - \lambda_{12} I^{v_2}\|_{L^\infty(Q)} \right. \\
& \left. + \|\lambda_{11} S^{v_1} - \lambda_{12} S^{v_2}\|_{L^\infty(Q)} + \|\lambda_{51}(0,t) S^{v_1} - \lambda_{52}(0,t) S^{v_2}\|_{L^\infty(Q)} \right] \\
\le & \frac{M}{2C_3} [\|\lambda_{11} V^{v_1} - \lambda_{12} V^{v_2}\|_{L^\infty(Q)} + \|\lambda_{11} I^{v_1} - \lambda_{12} I^{v_2}\|_{L^\infty(Q)} \\
& + \|\lambda_{11} S^{v_1} - \lambda_{12} S^{v_2}\|_{L^\infty(Q)} + \|\lambda_{51}(0,t) S^{v_1} - \lambda_{52}(0,t) S^{v_2}\|_{L^\infty(Q)}],
\end{aligned}
$$

where M is a positive generic constant which depends on the bound of ω_0, g and ψ. Therefore, we have

$$
\|L(v_1) - L(v_2)\| \le \frac{M}{2C_3} \|v_1 - v_2\|_{L^\infty(Q)},
$$

where M is a positive generic constant which also depends on C_{01}, C_{02} and T. So, if $\frac{M}{2C_3} < 1$, then we have the existence of unique fixed point v^*. □

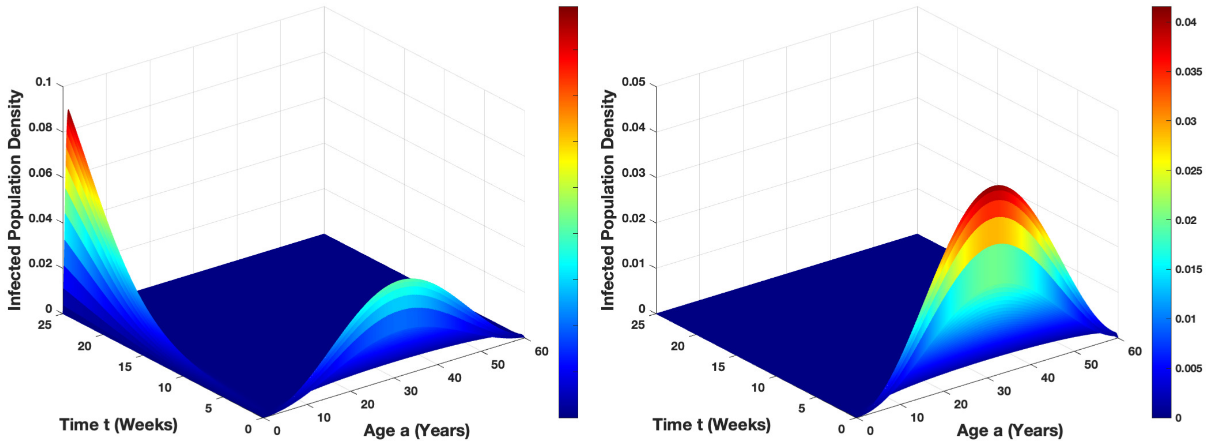
5. Numerical simulation

In this section, we discuss numerical simulation results. We applied an explicit finite difference scheme to solve our model numerically. Since the explicit finite difference scheme is highly sensitive to the discretization size of independent variables, the discretization size is chosen in such a way that the scheme is convergent. A finite difference scheme is applied to solve the system of differential equations and for the adjoint system, we applied a backward finite difference scheme. For the optimal control problem, we applied the forward-backward sweep method. The forward-backward sweep method for optimal control problems in PDEs combines the forward integration of the state equations with the backward integration of the adjoint equations. By iteratively adjusting the control variables based on the computed gradient, the method aims to find the optimal control that minimizes the cost functional while satisfying the given system of PDEs with initial and boundary data. We use the following initial age distribution in our model:

$$
\begin{aligned}
S_0(a) &= \exp(-0.09a) + 0.7\sin(0.05a^2) \\
E_0(a) &= 0.0001\exp(-a^2) + 0.05\sin(0.05a^2) \\
I_0(a) &= 0.0001\exp(-a^2) + 0.02\sin(0.05a^2) \\
R_0(a) &= 0.0001\exp(-0.5a) + 0.01\sin(0.05a^2) \\
V_0(a) &= 0.0001\exp(-0.5a) + 0.001\sin(0.05a^2).
\end{aligned}
$$

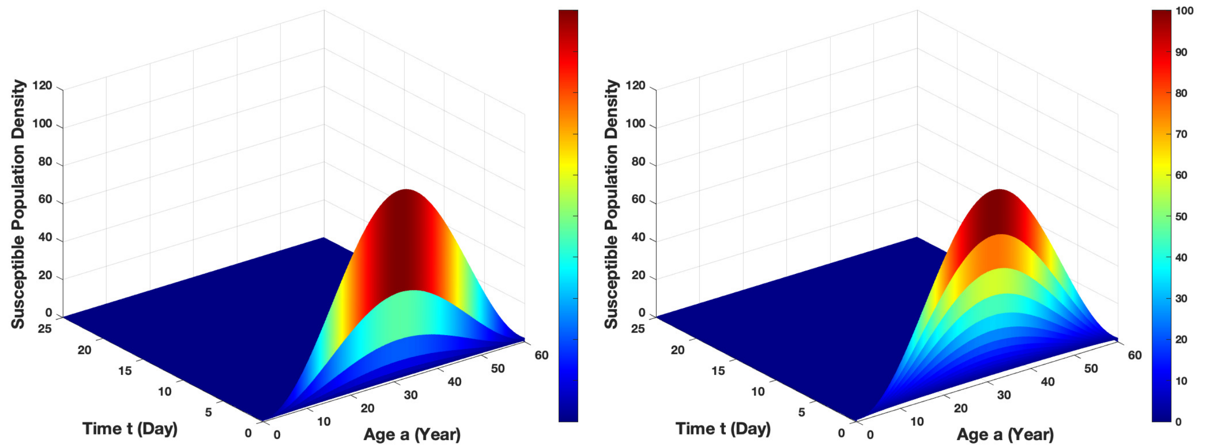
Mortality rate is taken $\mu(a) = \frac{0.01}{a+0.0001}, \psi(a,t) = 0.02e^{-a}$. Other parameters taken are $m_1 = 0.2, q_1 = 0.4, \gamma_1 = 0.4, \gamma_2 = 0.6, \gamma = 0.02, \beta = 1, \Lambda = 0.9$ and $g = 1$.

For the first two plots Figures 2(a),(b), we assume that a fixed proportion of individuals are vaccinated and observe the dynamics of infected individuals. So, for these figures we do not use the optimal vaccination strategy.



(a) (b)

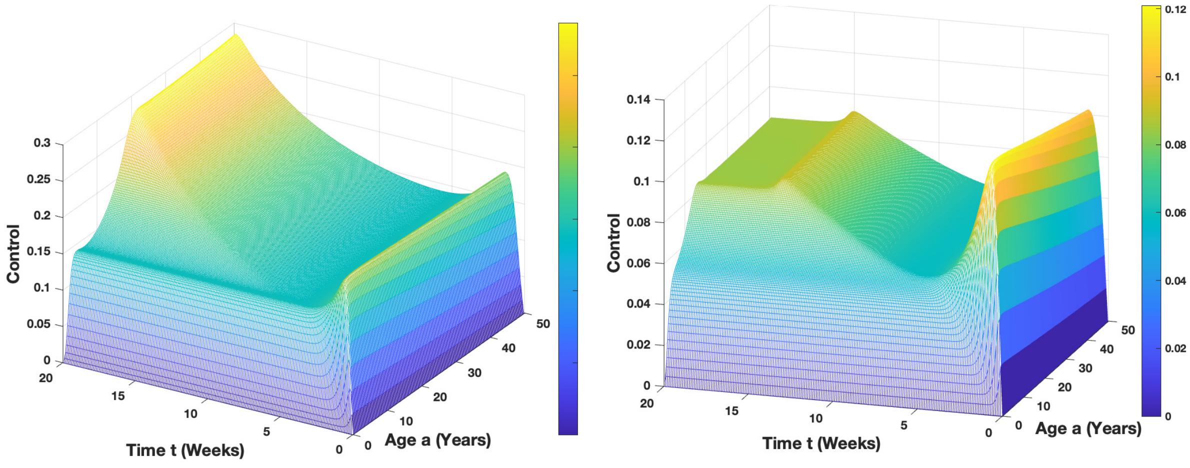
Figure 2. Figure shows the dynamics of infected individuals for different vaccination rates. The vaccination rate is taken 0.108 in Figure 2(a) and 0.2 in Figure 2(b).



(a) (b)

Figure 3. Figure shows the dynamics of susceptible individuals for different vaccine waning rate.

From Figure 2(a),(b), it is clear that increasing the vaccination rate will reduce the proportion of infected individuals. It is clear that increasing the vaccination waning rate will increase the population density of individuals who again become susceptible to disease. In Figure 3(a),(b), we have taken vaccine waning rates as 0 and 0.3 respectively. So, even if we are vaccinating a population, still there will be an increase in the population density of susceptible individuals due to imperfect vaccines.



(a) **(b)**

Figure 4. Figure shows the dynamics of control variable. Vaccine waning rate is taken 0.1 in Figure 4(a) and 0.01 in Figure 4(b).

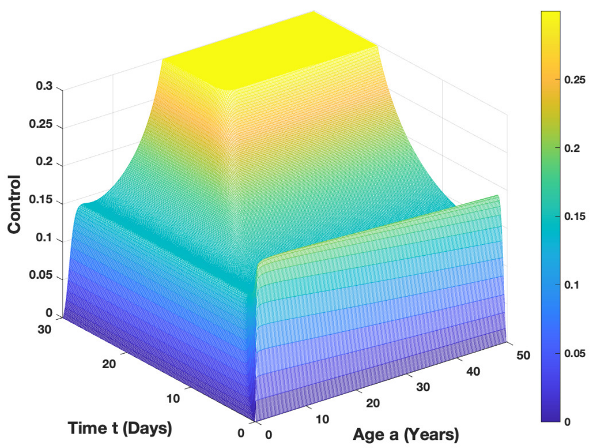
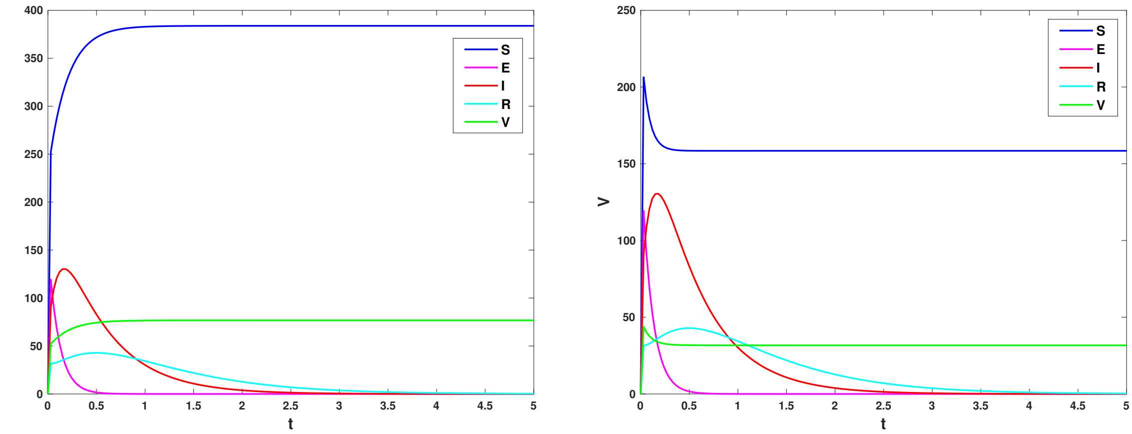


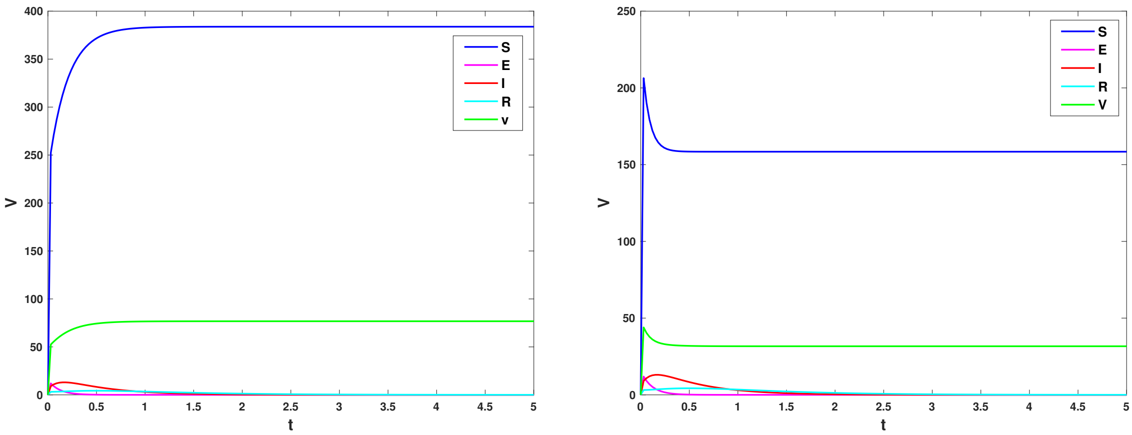
Figure 5. The dynamics of optimal control function with vaccine waning rate 0.1. The figure shows that with the passage of time, the controller acquires stable behavior.

Figure 4(a),(b) and Figure 5 show the dynamics of control variable for vaccine waning rate 0.1 and 0.01. As time progresses, the profile of the control variable attains a stable behavior. Initially, the control will be high and as time progresses, the control will take its minimum value.



(a) **(b)**

Figure 6. For $R_0 < 1$, figure shows the stability of disease free equilibrium point. Also $w_0(a)$ is taken 0.3 in Figure 6(a) and 0.1 in Figure 6(b).



(a) **(b)**

Figure 7. Stability of disease free equilibrium point is shown for $w_0(a) = 0.3$ and $w_0(a) = 0.1$ respectively.

In Section 3, we already prove that if threshold parameter R_0 is less than 1, then the steady state in which there are no individuals in the exposed or infected class is stable. In Figures 6 and 7, we have taken age between 10 and 50 and studied the stability of disease-free equilibrium point. We have chosen the parameters in such a way that the threshold number $R_0 < 1$. These figures show that if threshold parameter $R_0 < 1$, then the disease-free equilibrium point is stable. So, numerical simulations validate our theoretical results.

6. Conclusions

Age-structured modeling is important when studying imperfect vaccination because it allows a more realistic representation of how infectious diseases spread in a population. Imperfect vaccination refers to situations where not everyone in a population is fully protected by a vaccine due to vaccine effectiveness or a decline in immunity over time.

In this work, we proposed a new age-structured SEIRV model, where we incorporate vaccine waning rate and imperfect vaccination in our model. The aim was to study the effect of leaky vaccines and to investigate the optimal approach to contain the disease spread. The Well-posedness of the proposed model is shown using the semigroup of operators. We convert our model into an abstract Cauchy problem and then show that the operator in a homogeneous problem is the Hille-Yosida operator. Finally, the basic reproduction number is defined as a threshold parameter to study the stability analysis of the disease-free equilibrium point.

Vaccination strategy and optimal control problems with vaccination effort as a control variable are also discussed. The optimality conditions are derived for the optimal control problem. With the help of the adjoint system, optimal control is characterized. Numerical simulations are also added to establish theoretical results. An explicit finite difference scheme is used to solve the model numerically. The discretization size is chosen so that the scheme is convergent. The stability of disease-free steady state is shown using numerical simulation. Using the forward-backward sweep method, we plot the optimal control variable with respect to age and time variables.

Age-structured compartmental models are a valuable tool for understanding and analyzing the effects of imperfect vaccination. Although getting real data for verification is difficult, the model assumptions and outcomes are quite realistic. This study can help the public health policymakers optimally vaccinate a population and reduce the impact of the disease with a low cost of vaccination. The threshold number R_0 involves some parameters, and by increasing or decreasing these parameters, accordingly, we can eradicate the infectious disease. By choosing different vaccine waning rates, the dynamics of individuals in different compartments can be studied using the numerical simulation results. We also encourage other researchers to explore this area. Moreover, the age-structured SEIR model with stochastic perturbation can be investigated. More general size-structured variables can also be considered for epidemiological models. Finally, It is also natural to consider the delay in age and size-structured population models as it represents the maturation period. Therefore, age-structured compartmental models with delay can also be explored.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgement

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions that helped improve this manuscript's quality.

A. Tridane is supported by UAEU UPAR grant number 12S125.

Conflict of interest

The authors declare there is no conflict of interest.

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