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Research article

Bifurcation analysis and exact solutions for a class of generalized time-space fractional nonlinear Schrödinger equations

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Abstract: In this work, we focus on a class of generalized time-space fractional nonlinear Schrödinger equations arising in mathematical physics. After utilizing the general mapping deformation method and theory of planar dynamical systems with the aid of symbolic computation, abundant new exact complex doubly periodic solutions, solitary wave solutions and rational function solutions are obtained. Some of them are found for the first time and can be degenerated to trigonometric function solutions. Furthermore, by applying the bifurcation theory method, the periodic wave solutions and traveling wave solutions with the corresponding phase orbits are easily obtained. Moreover, some numerical simulations of these solutions are portrayed, showing the novelty and visibility of the dynamical structure and propagation behavior of this model.

Keywords: time-space fractional nonlinear Schrödinger equation; general mapping deformation method; Caputo fractional derivative; bifurcation; exact solutions

1. Introduction

In recent years, due to the rapid development and wide applications in nonlinear science of fractional calculus theory, many problems of mathematical physics and engineering have been successfully modeled by fractional differential equations (FDEs), such as materials [1], plasma physics [2], chaotic oscillations [3], chemistry and biochemistry [4], hydrology [5] and so on [6–9]. To better understand the physical meanings of these models, people have constructed many efficient methods for finding the exact explicit solutions of these FDEs, including the Bäcklund transformation [10], Darboux transformation [11] and Hirota bilinear method [12], which can be used to find N-soliton solutions. Furthermore, the general algebraic method [13], projective Riccati equations method [14], Jacobi elliptic function expansion method [15], G'/G-expansion method [16], sine-Gordon method [17], new Kudryashov method [18], fractional sub-equation method [19], fractional Hirota bilinear method [20], Riemann-Hilbert method [21], complex method [22], Bernoulli G'/G-expansion method [23], etc. [24–29] can be used to find doubly periodic solutions, solitary wave solutions and trigonometric solutions of these models.

As we all know, the nonlinear Schrödinger equation is highly focused in nonlinear science, and it describes many phenomena, including plasma [30], electromagnetic wave propagation [31], quantum mechanics [32], optics of nonlinear media [33], underwater acoustics [34], etc. [35]. Hence, solving this equation is highly important for researchers.

In this article, let us consider the following generalized time-space fractional nonlinear Schrödinger equation (GTSFNLS) mentioned in [36–46]:

$$iD_t^{\alpha}u + aD_x^{2\beta}u + \gamma|u|^{2s}u + vu = 0, \quad t > 0, 0 < \alpha, \beta \le 1, s \ge 1. \tag{1}$$

$D_x^{2\beta}u = D_x^{\beta}(D_x^{\beta}u), u = u(x,t), i = \sqrt{-1}$, and a, γ, v are real parameters. When $a = 1, s = 1, \; v = \gamma = 2\lambda$, Eq (1) turns into an unstable nonlinear Schrödinger equation describing the bilayer baroclinic instability of some long waves [36]. When $a = 1, s = 1, v = 0$, Eq (1) occurs in various fields of physics, including optical fiber communication, quantum mechanics, fluid mechanics, superconductivity, plasma physics, etc. [37–41]. The authors studied its approximate solution by Adomian expansion in [37]. When $a = 1, \gamma = \frac{1}{2}\mu, \; s = 1, v = 0$, some exact solutions were obtained by direct method in [38]. When $\alpha = \beta = 1, \; a = 1, s = 1, v = 0$, it translates into the classical nonlinear Schrödinger equation [39–41]. In addition, Eq (1) has many special cases, and related studies can be found in [42–48]. The main purpose of this paper is to find the new exact solution of Eq (1) under the famous Caputo fractional derivative definition by using the general mapping deformation method and to study the structure of these solutions by using the theory of planar dynamical systems. Next, let us review some definitions about classical fractional calculus [49–51].

Definition 1. For a function $f(t):[0,\infty) \to R$, we define the Riemann-Liouville fractional integral operator of order $\alpha > 0$ as [49–51]

$$J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} f(\xi)d\xi, \alpha > 0, t > 0, J_t^0 f(t) = f(t).$$

It admits the following properties:

$$J^{\alpha}J^{\beta}f(t) = J^{\alpha+\beta}f(t), J^{\alpha}J^{\beta}f(t) = J^{\beta}J^{\alpha}f(t), J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}.$$

Definition 2. For $\alpha > 0$, the Caputo fractional derivative operator of order α is defined as [49–51]

$$D^{\alpha} f(t) = J^{n-\alpha} D^{n} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, n-1 < \alpha < n, n \in N, \\ \dfrac{d^{(n)} f(t)}{dt^n}, \alpha = n \in N. \end{cases}$$

Moreover, we have the following properties:

$$D^{\alpha} t^{\gamma} = \begin{cases} \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \gamma > \alpha - 1, \\ 0, \gamma \le \alpha - 1. \end{cases}$$

This article is organized as follows: The general mapping deformation method is described in Section 2. In Section 3, some new exact solutions and bifurcation structures of the GTSFNLS are found by utilizing the proposed method and the planar dynamic system theory method. Finally, the conclusion is presented in Section 4.

2. The general mapping deformation method (GMDM)

Consider the following partial differential equation:

$$E(u, u_t, u_x, u_{xx}, \cdots) = 0. \tag{2}$$

We assume Eq (2) has solutions as follows:

$$u(\eta) = \sum_{i=0}^{n} A_i F^i(\eta) + \sum_{i=1}^{n} A_{-i} F^{-i}(\eta). \tag{3}$$

n is a balance number, and the coefficients A_i , A_{-i} and variable function $\eta = \eta(x,t)$ are determined later. $F(\eta)$ satisfies the following auxiliary equation:

$$F'^2(\eta) = \sum_{i=0}^{4} a_i F^i(\eta). \tag{4}$$

Substituting Eqs (3) and (4) into Eq (2), collecting the coefficients of $F^j(\eta)\sqrt{\sum_{i=0}^{4} a_i F^i(\eta)}$ and $F^i(\eta)(i, j = 0, \pm 1, \pm 2, \cdots)$ to zero yields algebraic equations (AEs) for $a_0, a_1, a_2, a_3, a_4,$ A_i, A_{-i} and η. Utilizing mathematical software to solve the AEs and substituting each $F(\eta)$ into Eq (3), we obtain the solutions of Eq (2). For finding some new general solutions of Eq (4), we assume

$$F(\eta) = b_0 + b_1 e + b_2 f + b_3 g + b_4 h + b_{-1} e^{-1} + b_{-2} f^{-1} + b_{-3} g^{-1} + b_{-4} h^{-1}. \tag{5}$$

$b_i (i = 0, \pm 1, \cdots, \pm 4)$ are undetermined coefficients, and the functions $e = e(\eta), f = f(\eta)$, $g = g(\eta), h = h(\eta)$ are constructed as below [15,52,53]:

$$e = \frac{1}{p + qsn\eta + rcn\eta + ldn\eta}, f = \frac{sn\eta}{p + qsn\eta + rcn\eta + ldn\eta}, g = \frac{cn\eta}{p + qsn\eta + rcn\eta + ldn\eta}, h = \frac{dn\eta}{p + qsn\eta + rcn\eta + ldn\eta}.$$

p, q, r, l are undetermined coefficients, and e, f, g, h satisfy the nexus (4) and (5a–5d) mentioned in [15,52,53].

Remark 1. Our method proposed here can be used to extend many other traditional methods such as the generalized Jacobi elliptic functions expansion method [15,53], the extended projective Riccati equations method [14], many other algebra expansion methods [18,19,38,46], etc.

3. Bifurcation and exact solutions for the GTSFNLS

3.1. Exact solutions

If we let $D_t^{\alpha} u = \frac{\partial^{\alpha} u}{\partial t^{\alpha}}, D_x^{2\beta} u = \frac{\partial^{2\beta} u}{\partial x^{2\beta}}$, Eq (1) can be rewritten as follows:

$$i\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + a\frac{\partial^{2\beta} u}{\partial x^{2\beta}} + \gamma u |u|^{2s} + vu = 0, \quad t > 0, 0 < \alpha, \beta \le 1, s \ge 1. \tag{6}$$

Let us give a functional transformation [46,54,55]

$$u = \varphi(\eta) e^{i\xi}, \tag{7}$$

$$\xi = \frac{k_1 x^{\beta}}{\Gamma(1+\beta)} + \frac{c_1 t^{\alpha}}{\Gamma(1+\alpha)}, \eta = \frac{k_2 x^{\beta}}{\Gamma(1+\beta)} + \frac{c_2 t^{\alpha}}{\Gamma(1+\alpha)}. \tag{8}$$

k_1, k_2, c_1, c_2 are parameters to be determined later. Substituting Eqs (7) and (8) into Eq (6), separating the real part and the imaginary part, we obtain

$$\begin{cases} ak_2^2 \varphi_{\eta\eta}(\eta) + (v - c_1 - ak_1^2)\varphi(\eta) + \gamma \varphi^{2s+1}(\eta) = 0, & (9.1) \\ (c_2 + 2ak_1 k_2)\varphi_{\eta}(\eta) = 0. & (9.2) \end{cases} \tag{9}$$

Here, $\varphi_{\eta\eta}(\eta)=\dfrac{d^2\varphi(\eta)}{d\eta^2},\varphi_\eta(\xi)=\dfrac{d\varphi(\eta)}{d\eta}$.

Using the transformation $\psi=\psi(\eta)=\varphi^s(\eta)=\varphi^s$ for (9.1) yields

$$ak_2^2(1-s)(\psi_\eta)^2+ak_2^2s\psi\psi_{\eta\eta}+(v-c_1-ak_1^2)s^2\psi^2+\gamma s^2\psi^4=0. \tag{10}$$

Clearly, the balance number $n=1$, and we assume solutions of Eq (10) have the form

$$\psi=\psi(\eta)=A_0+A_1F(\eta)+A_{-1}F^{-1}(\eta)=A_0+A_1F+A_{-1}F^{-1}, \tag{11}$$

where $F'^2=\sum_{i=0}^{4}a_iF^i, F=F(\eta)$ and $F'=\dfrac{dF(\eta)}{d\eta}$.

Substituting Eq (4) and (11) into Eq (10) and by utilizing the GMDM with the aid of mathematical software, we get the following solutions:

Family 1. $\varphi(\eta)=C$.

We find the trivial solution of Eq (6)

$$u_0=\sqrt[2s]{\frac{c_1+ak_1^2-v}{\gamma}}e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{c_1t^\alpha}{\Gamma(1+\alpha)})},(\gamma\neq 0).$$

Remark 2. If we select $c_1=c_0^{2s}a+v,k_1=c_0^s,\gamma=2a,\alpha=\beta=1$ in u_0, we have the solution $u_{01}=c_0e^{i(k_1x+c_1t)}$, which can be obtained by many authors by using approximate methods such as VIM or HAM [37,39].

Family 2. $c_2+2ak_1k_2=0.$

Case 1

$$s=1,A_0=0,A_1=\pm\sqrt{\frac{-2aa_4}{\gamma}}k_2,A_{-1}=0,c_2=-2ak_1k_2,c_1=aa_2k_2^2-ak_1^2+v,a_1=a_3=0.$$

Case 2

$$s=1,A_0=0,A_1=\pm\sqrt{\frac{-2aa_4}{\gamma}}k_2,A_{-1}=\pm\sqrt{\frac{-2aa_0}{\gamma}}k_2,c_2=-2ak_1k_2,$$
$$c_1=a(a_2-6\sqrt{a_0a_4})k_2^2-ak_1^2+v,a_1=a_3=0.$$

We obtain two types of solutions for Eq (6):

$$u_1 = \pm\sqrt{\frac{-2aa_4}{\gamma}}k_2F(\frac{k_2x^\beta}{\Gamma(1+\beta)} - \frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(aa_2k_2^2-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})},$$

$$u_2 = [\pm\sqrt{\frac{-2aa_4}{\gamma}}k_2F(\frac{k_2x^\beta}{\Gamma(1+\beta)} - \frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)}) \pm \sqrt{\frac{-2aa_0}{\gamma}}k_2F^{-1}(\frac{k_2x^\beta}{\Gamma(1+\beta)} - \frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})]E,$$
$$E = e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(a(a_2-6\sqrt{a_0a_4})k_2^2-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})}.$$

F_i is an arbitrary solution of the auxiliary equation $F_i'^2 = a_0 + a_2F_i^2 + a_4F_i^4$ in u_1, u_2, and the coefficients a_0, a_2, a_4 are arbitrary constants. Many types of F_i have been found in a large number of papers, such as [53,56,57]. Let us choose $a_0 = 1-m^2, a_2 = 2m^2-1, \quad a_4 = -m^2, F_1 = cn\xi$. Thus,

$$u_{1.1} = \pm\sqrt{\frac{2am^2}{\gamma}}k_2cn(\frac{k_2x^\beta}{\Gamma(1+\beta)} - \frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(a(2m^2-1)k_2^2-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})},$$

and the solution $u_{1.1}$ is translated into a bell-soliton solution when $\alpha = \beta = 1, m = 1$.

$$u_{1.2} = \pm\sqrt{\frac{2a}{\gamma}}k_2 \sec h(k_2x - 2ak_1k_2t)e^{i(k_1x+(ak_2^2-ak_1^2+v)t)}.$$

If we let $a_0 = 1, a_2 = -m^2-1, a_4 = m^2,\ F_0 = sn\xi$, then

$$u_{2.1} = [\pm\sqrt{\frac{-2am^2}{\gamma}}k_2sn(\frac{k_2x^\beta}{\Gamma(1+\beta)} - \frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)}) \pm \sqrt{\frac{-2a}{\gamma}}k_2ns(\frac{k_2x^\beta}{\Gamma(1+\beta)} - \frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})]E,$$
$$E = e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(a(-1-m^2-6m)k_2^2-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})}.$$

Case 3

$$s = 1, A_0 = 0, A_1 = \pm\sqrt{\frac{-2a}{\gamma}}k_2q, A_{-1} = 0, c_2 = -2ak_1k_2, c_1 = -ak_1^2 + v, a_0 = a_1 = a_3 = 0, a_4 = q^2.$$

Case 4

$$s = 1, A_0 = \mp\sqrt{\frac{-a}{2\gamma}}k_2, A_1 = 0, A_{-1} = \pm\sqrt{\frac{-a}{2\gamma}}k_2, c_2 = -2ak_1k_2, c_1 = -a(1+m^2)k_2^2 - ak_1^2 + v,$$
$$a_0 = \frac{1}{4}, a_1 = -1, a_2 = \frac{1}{2} - m^2, a_3 = 1 + 2m^2, a_4 = -\frac{3}{4} + 3m^2.$$

Case 5

$$s=1, A_0=\mp\sqrt{\frac{-2a}{\gamma}}k_2, A_1=0, A_{-1}=\pm\sqrt{\frac{-2a}{\gamma}}k_2, c_2=-2ak_1k_2, c_1=2a(1+m^2)k_2^2-ak_1^2+v,$$
$$a_0=1, a_1=-4, a_2=8+2m^2, a_3=-8-4m^2, a_4=4+m^4.$$

We find the following solutions of Eq (6), respectively:

$$u_3=\frac{\pm\sqrt{\frac{-2a}{\gamma}}k_2q}{1+q(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})}e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})},$$

$$u_4=\mp\sqrt{\frac{-a}{2\gamma}}k_2[-\frac{1\pm cn(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})}{sn(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})}-\frac{sn(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})}{1\pm cn(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})}]e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(-a(1+m^2)k_2^2-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})},$$

$$u_5=\pm\sqrt{\frac{-2a}{\gamma}}k_2cs(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})dn(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(2a(1+m^2)k_2^2-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})},$$

If we let $m=1$ or $m=0$, solution u_5 is degenerated to the following form:

$$u_{5.1}=\pm\sqrt{\frac{-2a}{\gamma}}k_2\csc h(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})\sec h(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(4ak_2^2-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})},$$

$$u_{5.2}=\pm\sqrt{\frac{-2a}{\gamma}}k_2\cot(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)})e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(2ak_2^2-ak_1^2+v)t^\alpha}{\Gamma(1+\alpha)})}.$$

Case 6

$$A_0=0, A_1=\pm\sqrt{\frac{(1+s)(c_1+ak_1^2-v)}{\gamma}}, A_{-1}=0, k_2=\pm s\sqrt{\frac{c_1+ak_1^2-v}{a}}, c_2=-2ak_1k_2,$$
$$a_0=a_1=a_3=0, a_2=1, a_4=-1.$$

Case 7

$$A_0=0, A_1=\pm\frac{1}{s}\sqrt{-\frac{(1+s)ak_2^2}{\gamma}}, A_{-1}=0, c_1=ak_1^2+\frac{ak_2^2}{s^2}+v, c_2=-2ak_1k_2,$$
$$a_0=a_1=a_3=0, a_2=1, a_4=1.$$

Case 8

$$A_0=0, A_1=\pm\frac{1}{s}\sqrt{-\frac{(1+s)ak_2^2}{\gamma}}, A_{-1}=0, c_1=-ak_1^2-\frac{ak_2^2}{s^2}+v, c_2=-2ak_1k_2,$$
$$a_0=a_1=a_3=0, a_2=-1, a_4=1.$$

With the same process, we obtain

$$u_6=\pm\sqrt[2s]{\frac{(1+s)(c_1+ak_1^2-v)}{\gamma}}\sec h^{\frac{1}{s}}\left(\frac{\pm s\sqrt{\frac{c_1+ak_1^2-v}{a}}x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)}\right)e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{c_1t^\alpha}{\Gamma(1+\alpha)})},$$

$$u_7=\pm\sqrt[2s]{-\frac{(1+s)ak_2^2}{\gamma s^2}}csch^{\frac{1}{s}}\left(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)}\right)e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(ak_1^2+\frac{ak_2^2}{s^2}+v)t^\alpha}{\Gamma(1+\alpha)})},$$

$$u_{8.1}=\pm\sqrt[2s]{-\frac{(1+s)ak_2^2}{\gamma s^2}}\sec^{\frac{1}{s}}\left(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)}\right)e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(-ak_1^2-\frac{ak_2^2}{s^2}+v)t^\alpha}{\Gamma(1+\alpha)})},$$

$$u_{8.2}=\pm\sqrt[2s]{-\frac{(1+s)ak_2^2}{\gamma s^2}}csc^{\frac{1}{s}}\left(\frac{k_2x^\beta}{\Gamma(1+\beta)}-\frac{2ak_1k_2t^\alpha}{\Gamma(1+\alpha)}\right)e^{i(\frac{k_1x^\beta}{\Gamma(1+\beta)}+\frac{(-ak_1^2-\frac{ak_2^2}{s^2}+v)t^\alpha}{\Gamma(1+\alpha)})}.$$

If selecting $k_1=0, v=0$ in u_6, we get

$$u_{6.1}=\sqrt[2s]{\frac{(1+s)c_1}{\gamma}}\sec h^{\frac{1}{s}}\left[\sqrt{\frac{c_1}{a}}\frac{sx^\beta}{\Gamma(1+\beta)}\right]e^{i\frac{c_1t^\alpha}{\Gamma(1+\alpha)}},$$

Remark 3. If we select $k_1=0, v=0$, $\alpha=\beta=1, a=1, \gamma=1, k_2=1, c_1=\omega, s=\frac{p-1}{2}$, u_6 turns into the following solution mentioned in [58].

$$u_{scipio}=\left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}\omega^{\frac{1}{p-1}}\sec h^{\frac{2}{p-1}}\left[\frac{p-1}{2}\sqrt{\omega}x\right]e^{i\omega t}.$$

We simulate some structures of the periodic and solitary solutions for Eq (1) below. Some optical waves of the GTSFNLS are propagated by a periodic wave pattern in Figures 1 and 2 or a bright-soliton wave pattern in Figure 3 and blow-up wave pattern in Figure 4 with the fractional order $\alpha=0.4, \beta=0.8$. The density plots of $\mathrm{Re}\,u_4, \mathrm{Im}\,u_5, |u_6|$ and $|u_{8.1}|$ are shown in Figure 5.

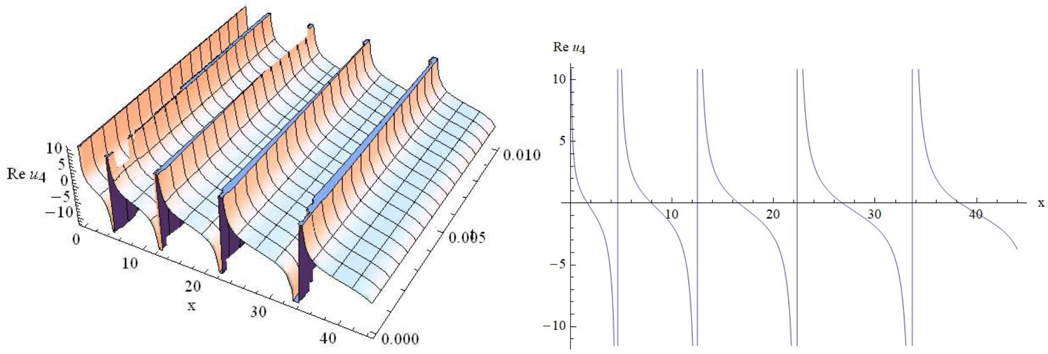


Figure 1. The real part of u_4 at $k_1 = k_2 = 1, a = -1, v = -2, \gamma = 0.5, m = 0.1, \alpha = 0.6, \beta = 0.7$ and $t = 0.01$.

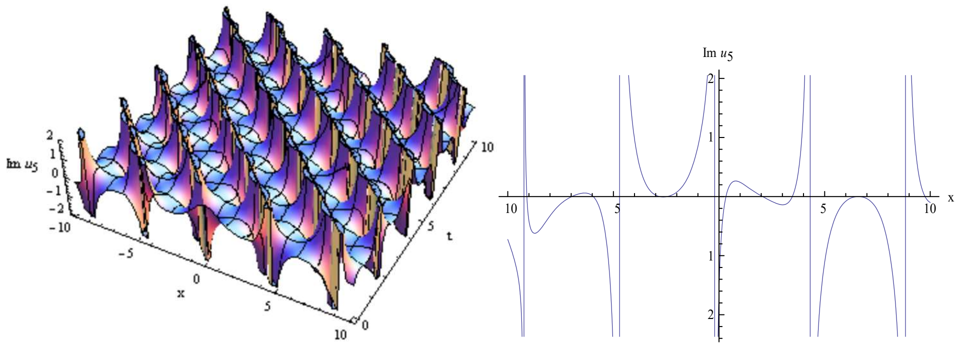


Figure 2. The plot of Im u_5 at $\alpha = \beta = k_1 = k_2 = 1, a = -1, v = -2, \gamma = 2, m = 0.1$ and $t = 0.05$.

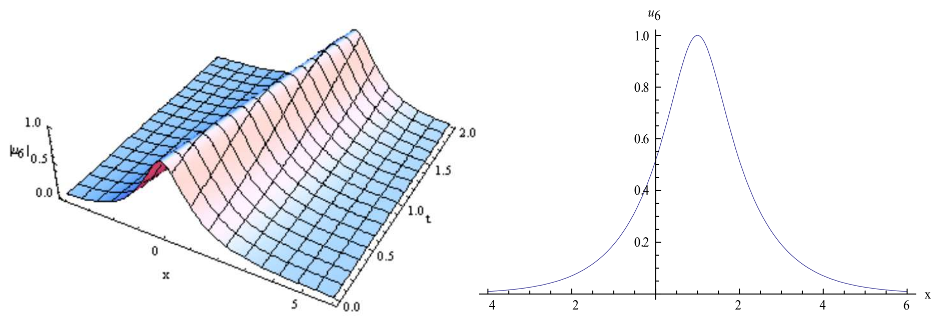


Figure 3. The modulus plot of u_6 at $s = 2, \alpha = \beta = k_1 = k_2 = a = c_1 = v = 1, \gamma = 3, m = 1$ and $t = 1$.

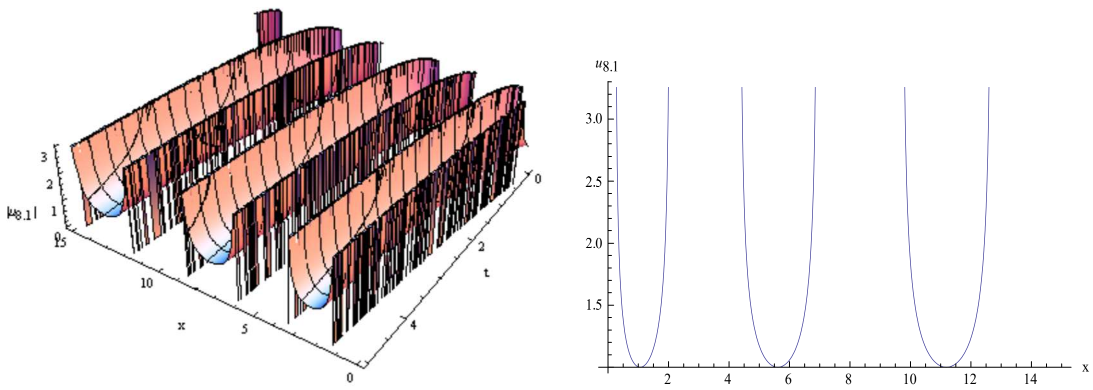


Figure 4. The modulus plot of $u_{8.1}$ at $s=2, k_1=k_2=a=c_1=v=1,$ $\gamma=-0.75, \alpha=0.4, \beta=0.8,$ and $t=1$.

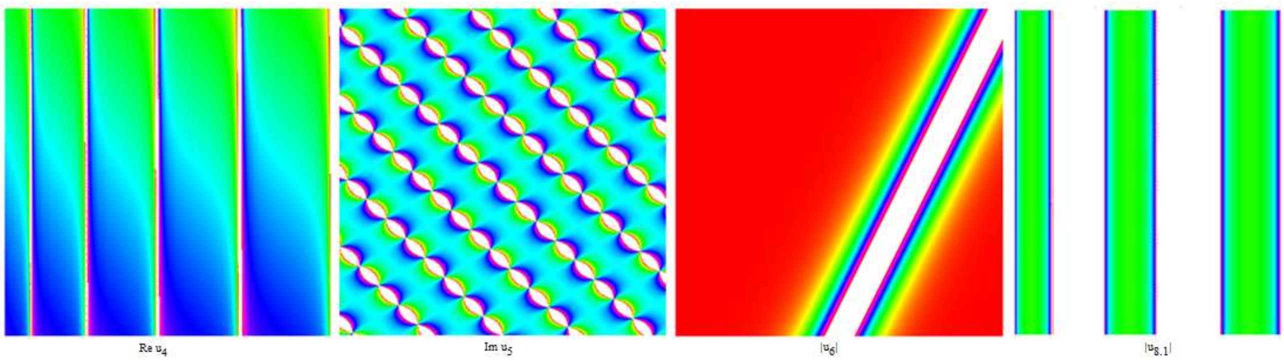


Figure 5. The density plots of $\operatorname{Re} u_4, \operatorname{Im} u_5, |u_6|$ and $|u_{8.1}|$.

3.2. Bifurcation analysis

Below, let us discuss the plane phase portrait structure of Eq (9) by using the plane dynamic system theory [59,60]. Without loss of generality, select $k_1=k_2=1,\quad a=0.5, c_2=-1$. Thus,

$$\varphi_{\eta\eta}(\eta)-[1+2(c_1-v)]\varphi(\eta)+2\gamma\varphi^{2s+1}(\eta)=0\,. \tag{12}$$

Let $\dfrac{d\varphi}{d\eta}=y$, and clearly, Eq (12) is equivalent to the following regular system:

$$\begin{cases}\dfrac{d\varphi}{d\eta}=y,\\[2mm] \dfrac{dy}{d\eta}=[1+2(c_1-v)]\varphi-2\gamma\varphi^{2s+1}.\end{cases} \tag{13}$$

We can get the following Hamiltonian of Eq (13):

$$H = H(\varphi, y) = y^2 - [1 + 2(c_1 - v)]\varphi^2 + \frac{2\gamma}{1+s}\varphi^{2s+2} = h, \qquad h \in R. \tag{14}$$

Indeed, system (13) has three equilibrium points $P(\varphi, y) = P(\varphi, \varphi')$ on the φ-axis:

$$P_0(0,0), P_1\left(\sqrt[2s]{\frac{1+2(c_1 - v)}{2\gamma}}, 0\right), P_2\left(-\sqrt[2s]{\frac{1+2(c_1 - v)}{2\gamma}}, 0\right).$$

The coefficient matrix of system (13) is defined as $M(\varphi, y)$, and $J(P_i) = \det M(\varphi_i, y_i), i = 0,1,2$ is the determinant of $M(\varphi, y)$ about P_i.

$$J(P) = \begin{vmatrix} 0 & 1 \\ 1 + 2(c_1 - v) - 2\gamma(2s+1)\varphi^{2s} & 0 \end{vmatrix} = -[1 + 2(c_1 - v)] + 2\gamma(2s+1)\varphi^{2s}.$$

By the dynamical bifurcation theory of planar systems, as we all know, the equilibrium point P_i of system (13) is a center if $J(P_i) > 0$, it is a saddle if $J(P_i) < 0$, and it is a cusp if $J(P_i) = 0$. Let us analyze the bifurcation structures of system (13) by using mathematical software and the above facts.

Case 1. $c_1 - v = 1, \gamma = \frac{1}{2}, s \in Z^+$.

Notice that $J(P_0) = -3 < 0, J(P_1) = J(P_2) = 6s > 0$. Hence, P_1, P_2 are center points, and the origin P_0 is a saddle point. Let us discuss three situations of Eq (14) with $s = 1$ and

$$(\varphi_\eta)^2 = -\frac{1}{2}\varphi^4 + 3\varphi^2 + h.$$

(i) When $h < 0, h \in (-4.5, 0)$, we can find two clusters of doubly periodic solutions for the periodic orbits of Eq (13) defined by the following integral equation (see Figure 6(a)):

$$\frac{d\varphi}{d\eta} = \pm\sqrt{-\frac{1}{2}\varphi^4 + 3\varphi^2 + h} = \pm\sqrt{\frac{1}{2}(\varphi^2 - \varphi_1^2)(-\varphi^2 + \varphi_2^2)},$$

Integrate this equation along the periodic orbits thus:

$$\int_\varphi^{\varphi_2} \frac{d\varphi}{\sqrt{-\frac{1}{2}\varphi^4 + 3\varphi^2 + h}} = \int_\varphi^{\varphi_2} \frac{d\varphi}{\sqrt{(\varphi^2 - \varphi_1^2)(-\varphi^2 + \varphi_2^2)}} = \pm\sqrt{\frac{1}{2}}\eta.$$

We get the following smooth doubly periodic solutions:

$$\varphi_{h<0} = \pm\varphi_2 dn[\varphi_2\sqrt{\frac{1}{2}}\eta, \frac{\sqrt{\varphi_2^2-\varphi_1^2}}{\varphi_2}],$$

$$u_{1.h<0} = \pm\varphi_2 dn[\varphi_2\sqrt{\frac{1}{2}}(\frac{x^\beta}{\Gamma(1+\beta)}-\frac{t^\alpha}{\Gamma(1+\alpha)}), \frac{\sqrt{\varphi_2^2-\varphi_1^2}}{\varphi_2}]e^{i(\frac{x^\beta}{\Gamma(1+\beta)}+\frac{(1+v)t^\alpha}{\Gamma(1+\alpha)})},$$

where $\varphi_1 = \sqrt{3-\sqrt{9+2h}}, \varphi_2 = \sqrt{3+\sqrt{9+2h}}$.

(ii) When $h=0$, there exist two clusters of bell-soliton wave solutions for the symmetric homoclinic orbits of system (13) with the following form:

$$\varphi_{h=0} = \pm\sqrt{6}\sec h(\sqrt{3}\eta),$$

$$u_{1.h=0} = \pm\sqrt{6}\sec h[\sqrt{3}(\frac{x^\beta}{\Gamma(1+\beta)}-\frac{t^\alpha}{\Gamma(1+\alpha)})]e^{i(\frac{x^\beta}{\Gamma(1+\beta)}+\frac{(1+v)t^\alpha}{\Gamma(1+\alpha)})}.$$

(iii) When $h>0$, we obtain the Jacobi cosine wave solutions for the periodic orbit of system (13) with the following form:

$$\varphi_{h>0} = \varphi_2 cn[\sqrt{\frac{\varphi_1^2+\varphi_2^2}{2}}\eta, \sqrt{\frac{\varphi_2^2}{\varphi_1^2+\varphi_2^2}}],$$

$$u_{1.h>0} = \varphi_2 cn[\sqrt{\frac{\varphi_1^2+\varphi_2^2}{2}}(\frac{x^\beta}{\Gamma(1+\beta)}-\frac{t^\alpha}{\Gamma(1+\alpha)}), \sqrt{\frac{\varphi_2^2}{\varphi_1^2+\varphi_2^2}}]e^{i(\frac{x^\beta}{\Gamma(1+\beta)}+\frac{(1+v)t^\alpha}{\Gamma(1+\alpha)})},$$

where $\varphi_1^2 = \sqrt{9+2h}-3, \varphi_2^2 = \sqrt{9+2h}+3$.

Case 2. $c_1 - v = -1, \gamma = -\frac{1}{2}, s \in Z^+$.

Clearly, $J(P_0)=1>0$, $J(P_1)=J(P_2)=-2s<0$. Hence, P_1, P_2 are saddle points, and P_0 is a center point (see Figure 6(b)). Let $s=1, (\varphi_\eta)^2 = \frac{1}{2}\varphi^4 - \varphi^2 + h$, and we can discuss the solutions of Eq (12) with the same process. Due to space limitations, we only give the solutions. When $h \in (0, 0.5)$, we can obtain two clusters of Jacobi sine wave solutions for the periodic orbits. When $h=0.5$, there exist two kink and antikink solutions corresponding to the solutions of two symmetric heteroclinic orbits. When $h<0$, there exist two blow-up wave solutions for the corresponding orbits.

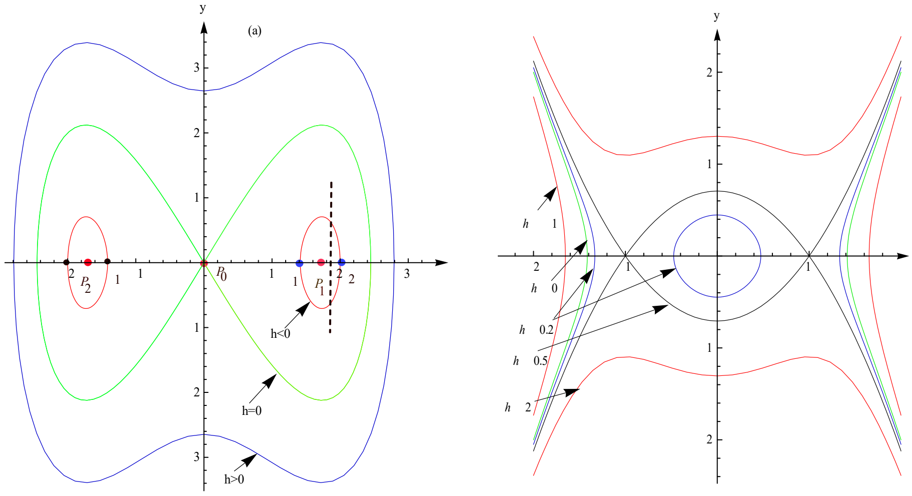
$$u_{2.h>0} = \pm\varphi_1 sn[\pm\varphi_2\sqrt{\frac{1}{2}}(\frac{x^\beta}{\Gamma(1+\beta)}-\frac{t^\alpha}{\Gamma(1+\alpha)}), \frac{\varphi_1}{\varphi_2}]e^{i(\frac{x^\beta}{\Gamma(1+\beta)}+\frac{(v-1)t^\alpha}{\Gamma(1+\alpha)})}, h \in (0, 0.5),$$

where $\varphi_1 = \sqrt{1-\sqrt{1-2h}}, \varphi_2 = \sqrt{1+\sqrt{1-2h}}$.

$$u_{2.h=0.5} = \pm \tanh[\pm\sqrt{\frac{1}{2}}(\frac{x^{\beta}}{\Gamma(1+\beta)} - \frac{t^{\alpha}}{\Gamma(1+\alpha)})]e^{i(\frac{x^{\beta}}{\Gamma(1+\beta)}+\frac{(v-1)t^{\alpha}}{\Gamma(1+\alpha)})},$$

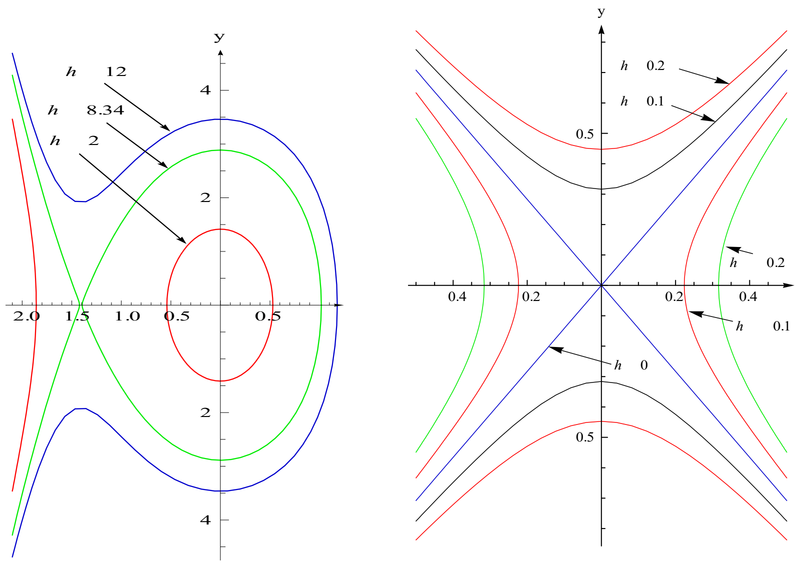
$$u_{2.h<0} = \pm\varphi_2 nc[\pm\sqrt{\frac{\varphi_1^2+\varphi_2^2}{2}}(\frac{x^{\beta}}{\Gamma(1+\beta)} - \frac{t^{\alpha}}{\Gamma(1+\alpha)}), \sqrt{\frac{\varphi_1^2}{\varphi_1^2+\varphi_2^2}}]e^{i(\frac{x^{\beta}}{\Gamma(1+\beta)}+\frac{(v-1)t^{\alpha}}{\Gamma(1+\alpha)})},$$

where $\varphi_1^2 = \sqrt{1-2h} - 1, \varphi_2^2 = \sqrt{1-2h} + 1$.



(a) (b)

Figure 6. The phase graphs of system (13) for case 1 and case 2.



(a) (b)

Figure 7. The phase graphs of system (13) when $s \neq 1$.

Some planar phase graphs of system (13) with the parameter $s \neq 1$ are shown in Figure 7. From Figure 7(a), ($\gamma = 1.25, c_1 - v = -4, s = 1.5$), and we can find that there exist a periodic solution and a bell-soliton solution, while there are not these two types of solutions in Figure 7(b) ($\gamma = -2.5, c_1 - v = 0.5, s = 4$).

Remark 4. It is notable that with the increase of h, the periodic wave solution is degenerated into a solitary wave solution and then into another periodic solution in Figure 6(a). The periodic wave solution is transformed into a kink or antikink wave solution and then into an unbounded solution in Figure 6(b).

4. Conclusions

In brief, many types of new exact solutions for the GTSFNLS have been found after utilizing the GMDM. Some dynamic behaviors of these solutions are discussed using bifurcation theory. We simulate the 3D plots, 2D plots, density plots and phase graphs of the partial solutions in Figures 1–7, which show that these doubly periodic wave solutions, solitary wave solutions and single periodic solutions can be mutually transformed along with the concomitant energy constant and its corresponding orbits. These efficient and significant two methods can be used for many other nonlinear models, such as the Korteweg–de Vries (KdV) equation, Ginzburg-Landau equation, Burgers-BBM (Benjamin-Bona-Mahony) equation, etc.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of Interest

The author declare that he has no competing interests in this paper.

References

1. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
2. R. Hilfer, *Applications of Fractional Calculus in Physics*, Academic Press, Orlando, 1999.
3. M. S. Tavazoei, M. Haeri, S. Jafari, S. Bolouki, M. Siami, Some applications of fractional calculus in suppression of chaotic oscillations, *IEEE Trans. Ind. Electron.*, **55** (2008), 4094–4101. https://doi.org/10.1109/TIE.2008.925774
4. L. Acedo, S. B. Yuste, K. Lindenberg, Reaction front in an a+b→c reaction-subdiffusion process, *Phys. Rev. E*, **69** (2004), 136–144. https://doi.org/10.1103/PhysRevE.69.036126

5. D. A. Benson, S. W. Wheatcraft, M. M. Meerschaert, The fractional-order governing equation of lévy motion, *Water Resour. Res.*, **36** (2000), 1413–1423. https://doi.org/10.1029/2000WR900032
6. J. H. He, Fractal calculus and its geometrical explanation, *Results Phys.*, **10** (2018), 272–276. https://doi.org/10.1016/j.rinp.2018.06.011
7. J. H. He, A modified Li-He's variational principle for plasma, *Int. J. Numer. Methods Heat Fluid Flow*, **31** (2021), 1369–1372. https://doi.org/10.1108/HFF-06-2019-0523
8. D. D. Dai, T. T. Ban, Y. L. Wang, W. Zhang, The piecewise reproducing kernel method for the time variable fractional order advection-reaction-diffusion equations, *Therm. Sci.*, **25** (2021), 1261–1268. https://doi.org/10.2298/TSCI200302021D
9. Q. T. Ain, J. H. He, N. Anjum, M. Ali, The Fractional complex transform: A novel approach to the time fractional Schrödinger equation, *Fractals*, **28** (2020), 2050141. https://doi.org/10.1142/S0218348X20501418
10. D. C. Lu, B. J. Hong, Bäcklund transformation and n-soliton-like solutions to the combined KdV-Burgers equation with variable coefficients, *Int. J. Nonlinear Sci.*, **1** (2006), 3–10.
11. V. A. Matveev, M. A. Salle, *Darboux Transformations and Solitons*, Heidelberg: Springer Verlag, Berlin, 1991. https://doi.org/10.1007/978-3-662-00922-2
12. J. J. Fang, D. S. Mou, H. C. Zhang, Y. Y. Wang, Discrete fractional soliton dynamics of the fractional Ablowitz-Ladik model, *Optik*, **228** (2021), 166186. https://doi.org/10.1016/j.ijleo.2020.166186
13. B. J. Hong, D. C. Lu, New exact Jacobi elliptic function solutions for the coupled Schrödinger-Boussinesq equations, *J. Appl. Math.*, **2013** (2013), 170835. https://doi.org/10.1155/2013/170835
14. D. C. Lu, B. J. Hong, L. X. Tian, New explicit exact solutions for the generalized coupled Hirota-Satsuma KdV system, *Comput. Math. Appl.*, **53** (2007), 1181–1190. https://doi.org/10.1016/j.camwa.2006.08.047
15. B. J. Hong, New Jacobi elliptic functions solutions for the variable-coefficient mKdV equation, *Appl. Math. Comput.*, **215** (2009), 2908–2913. https://doi.org/10.1016/j.amc.2009.09.035
16. S. K. Mohanty, O. V. Kravchenko, A. N. Dev, Exact traveling wave solutions of the Schamel Burgers equation by using generalized-improved and generalized G'/G -expansion methods, *Results Phys.*, **33** (2022), 105124. https://doi.org/10.1016/j.rinp.2021.105124
17. P. R. Kundu, M. R. A. Fahim, M. E. Islam, M. A. Akbar, The sine-Gordon expansion method for higher-dimensional NLEEs and parametric analysis, *Heliyon*, **7** (2021), e06459. https://doi.org/10.1016/j.heliyon.2021.e06459
18. S. M. Mirhosseini-Alizamini, N. Ullah, J. Sabi'u, H. Rezazadeh, M. Inc, New exact solutions for nonlinear Atangana conformable Boussinesq-like equations by new Kudryashov method, *Int. J. Modern Phys. B*, **35** (2021), 2150163. https://doi.org/10.1142/S0217979221501630
19. S. Zhang, H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Phys. Lett. A*, **375** (2011), 1069–1073. https://doi.org/10.1016/j.physleta.2011.01.029
20. B. Xu, Y. F. Zhang, S. Zhang, Line soliton interactions for shallow ocean-waves and novel solutions with peakon, ring, conical, columnar and lump structures based on fractional KP equation, *Adv. Math. Phys.*, **2021** (2021), 6664039. https://doi.org/10.1155/2021/6664039
21. B. Xu, S. Zhang, Riemann-Hilbert approach for constructing analytical solutions and conservation laws of a local time-fractional nonlinear Schrödinger equation, *Symmetry*, **13** (2021), 13091593. https://doi.org/10.3390/sym13091593

22. Y. Y. Gu, L. W. Liao, Closed form solutions of Gerdjikov-Ivanov equation in nonlinear fiber optics involving the beta derivatives, *Int. J. Modern Phys. B*, **36** (2022), 2250116. https://doi.org/10.1142/S0217979222501168
23. Y. Y. Gu, N. Aminakbari, Bernoulli (G'/G)-expansion method for nonlinear Schrödinger equation with third-order dispersion, *Modern Phys. Lett. B*, **36** (2022), 2250028. https://doi.org/10.1007/s11082-021-02807-0
24. S. Zhang, Y. Y. Wei, B. Xu, Fractional soliton dynamics and spectral transform of time-fractional nonlinear systems: an concrete example, *Complexity*, **2019** (2019), 7952871. https://doi.org/10.1155/2019/7952871
25. K. L. Geng, D. S. Mou, C. Q. Dai, Nondegenerate solitons of 2-coupled mixed derivative nonlinear Schrödinger equations, *Nonlinear Dyn.*, **111** (2023), 603–617. https://doi.org/10.1007/s11071-022-07833-5
26. W. B. Bo, R. R. Wang, Y. Fang, Y. Y. Wang, C. Q. Dai, Prediction and dynamical evolution of multipole soliton families in fractional Schrödinger equation with the PT-symmetric potential and saturable nonlinearity, *Nonlinear Dyn.*, **111** (2023), 1577–1588. https://doi.org/10.1007/s11071-022-07884-8
27. R. R. Wang, Y. Y. Wang, C. Q. Dai, Influence of higher-order nonlinear effects on optical solitons of the complex Swift-Hohenberg model in the mode-locked fiber laser, *Opt. Laser Technol.*, **152** (2022), 108103. https://doi.org/10.1016/j.optlastec.2022.108103
28. S. L. He, B. A. Malomed, D. Mihalache, X. Peng, X. Yu, Y. J. He, et al., Propagation dynamics of abruptly autofocusing circular Airy Gaussian vortex beams in the fractional Schrödinger equation, *Chaos, Solitons Fractals*, **142** (2021), 110470. https://doi.org/10.1016/j.chaos.2020.110470
29. S. L. He, Z. W. Mo, J. L. Tu, Z. L. Lu, Y. Zhang, X. Peng, et al., Chirped Lommel Gaussian vortex beams in strongly nonlocal nonlinear fractional Schrödinger equations, *Results Phys.*, **42** (2022), 106014. https://doi.org/10.1016/j.rinp.2022.106014
30. R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, H. C. Morris, *Solitons and Nonlinear Wave Equations*, Academic Press, London-New York, 1982.
31. M. Levy, *Parabolic Equation Methods for Electromagnetic Wave Propagation*, Institution of Electrical Engineers (IEE), London, 2000.
32. A. Arnold, Numerically absorbing boundary conditions for quantum evolution equations, *VLSI Des.*, **6** (1998), 38298. https://doi.org/10.1155/1998/38298
33. Y. S. Kivshar, G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press, New York, 2003.
34. F. D. Tappert, *The Parabolic Approximation Method*, Springer, Berlin. https://doi.org/10.1007/3-540-08527-0_5
35. M. Hosseininia, M. H. Heydari, C. Cattani, N. Khanna, A wavelet method for nonlinear variable-order time fractional 2D Schrödinger equation, *Discrete Contin. Dyn. Syst. Ser. S*, **14** (2021), 2273–2295. https://doi.org/10.1016/j.camwa.2019.06.008
36. A. Somayeh, N. Mohammad, Exact solitary wave solutions of the complex nonlinear Schrödinger equations, *Opt. Int. J. Light Electron. Opt.*, **127** (2016), 4682–4688. https://doi.org/10.1016/j.ijleo.2016.02.008

37. M. A. E. Herzallah, K. A. Gepreel, Approximate solution to the time–space fractional cubic nonlinear Schrödinger equation, *Appl. Math. Modell.*, **36** (2012), 5678–5685. https://doi.org/10.1016/j.apm.2012.01.012
38. H. Gündodu, M. F. Gzükzl, Cubic nonlinear fractional Schrödinger equation with conformable derivative and its new travelling wave solutions, *J. Appl. Math. Comput. Mech.*, **20** (2021), 29–41. https://doi.org/10.17512/jamcm.2021.2.03
39. A. M. Wazwaz, A study on linear and nonlinear Schrödinger equations by the variational iteration method, *Chaos Solitons Fractals*, **37** (2008), 1136–1142. https://doi.org/10.1016/j.chaos.2006.10.009
40. A. Hasegawa, F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. II. Normal dispersion, *Appl. Phys. Lett.*, **23** (1973), 171–172. https://doi.org/10.1063/1.1654836
41. A. Ebaid, S. M. Khaled, New types of exact solutions for nonlinear Schrödinger equation with cubic nonlinearity, *J. Comput. Appl. Math.*, **235** (2011), 1984–1992. https://doi.org/10.1016/j.cam.2010.09.024
42. A. Demir, M. A. Bayrak, E. Ozbilge, New approaches for the solution of space-time fractional Schrödinger equation, *Adv. Differ. Equations*, **2020** (2020), 1–21. https://doi.org/10.1186/s13662-020-02581-5
43. N. A. Kudryashov, Method for finding optical solitons of generalized nonlinear Schrödinger equations, *Opt. Int. J. Light Electron. Opt.*, **261** (2022), 169163. https://doi.org/10.1016/j.ijleo.2022.169163
44. B. Ghanbari, K. S. Nisar, M. Aldhaifallah, Abundant solitary wave solutions to an extended nonlinear Schrödinger equation with conformable derivative using an efficient integration method, *Adv. Differ. Equations*, **328** (2020), 1–25. https://doi.org/10.1186/s13662-020-02787-7
45. M. T. Darvishi, M. Najafi, A. M. Wazwaz, Conformable space-time fractional nonlinear (1+1)-dimensional Schrödinger-type models and their traveling wave solutions, *Chaos Solitons Fractals*, **150** (2021), 111187. https://doi.org/10.1016/j.chaos.2021.111187
46. T. Mathanaranjan, Optical singular and dark solitons to the (2+1)-dimensional time-space fractional nonlinear Schrödinger equation, *Results Phys.*, **22** (2021), 103870. https://doi.org/10.1016/j.rinp.2021.103870
47. Y. X. Chen, X. Xiao, Z. L. Mei, Optical soliton solutions of the (1+1)-dimensional space-time fractional single and coupled nonlinear Schrödinger equations, *Results Phys.*, **18** (2020), 103211. https://doi.org/10.1016/j.rinp.2020.103211
48. E. K. Jaradat, O. Alomari, M. Abudayah, A. Al-Faqih, An approximate analytical solution of the nonlinear Schrödinger equation with harmonic oscillator using homotopy perturbation method and Laplace-Adomian decomposition method, *Adv. Math. Phys.*, **2018** (2018), 1–11. https://doi.org/10.1155/2018/6765021
49. M. G. Sakar, F. Erdogan, A. Yzldzrzm, Variational iteration method for the time-fractional Fornberg-Whitham equation, *Comput. Math. Appl.*, **63** (2012), 1382–1388. https://doi.org/10.1016/j.camwa.2012.01.031
50. G. Jumarie, Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions, *Appl. Math. Lett.*, **22** (2009), 378–385. https://doi.org/10.1016/j.aml.2008.06.003

51. B. J. Hong, D. C. Lu, W. Chen, Exact and approximate solutions for the fractional Schrödinger equation with variable coefficients, *Adv. Differ. Equations*, **2019** (2019), 1–10. https://doi.org/10.1186/s13662-019-2313-z
52. B. J. Hong, D. C. Lu, New exact solutions for the generalized variable-coefficient Gardner equation with forcing term, *Appl. Math. Comput.*, **219** (2012), 2732–2738 https://doi.org/10.1016/j.amc.2012.08.104
53. B. J. Hong, New exact Jacobi elliptic functions solutions for the generalized coupled Hirota-Satsuma KdV system, *Appl. Math. Comput.*, **217** (2010), 472–479. https://doi.org/10.1016/j.amc.2010.05.079
54. J. H. He, S. K. Elagan, Z. B. Li, Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus, *Phys. Lett. A*, **376** (2012), 257–259. https://doi.org/10.1016/j.physleta.2011.11.030
55. Z. B. Li, J. H. He, Fractional complex transformation for fractional differential equation, *Math. Comput. Appl.*, **15** (2010), 970–973. https://doi.org/10.3390/mca15050970
56. A. Ebaid, E. H. Aly, Exact solutions for the transformed reduced Ostrovsky equation via the F-expansion method in terms of Weierstrass-elliptic and Jacobian-elliptic functions, *Wave Motion*, **49** (2012), 296–308. https://doi.org/10.1016/j.wavemoti.2011.11.003
57. M. H. Bashar, S. M. R. Islam, Exact solutions to the (2+1)-Dimensional Heisenberg ferromagnetic spin chain equation by using modified simple equation and improve F-expansion methods, *Phys. Open*, **5** (2020), 100027. https://doi.org/10.1016/j.physo.2020.100027
58. C. Scipio, An invariant set in energy space for supercritical NLS in 1D, *J. Math. Anal. Appl.*, **352** (2009), 634–644. https://doi.org/10.1016/j.jmaa.2008.11.023
59. L. J. Zhang, P. Y. Yuan, J. L. Fu, C. M. Khalique, Bifurcations and exact traveling wave solutions of the Zakharov-Rubenchik equation, *Discrete Contin. Dyn. Syst. S*, **13** (2018), 2927–2939. https://doi.org/10.3934/dcdss.2020214
60. G. A. Xu, Y. Zhan, J. B. Li, Exact solitary wave and periodic-peakon solutions of the complex Ginzburg-Landau equation: Dynamical system approach, *Math. Comput. Simul.*, **191** (2022), 157–167. https://doi.org/10.1016/j.matcom.2021.08.007