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Research article

The growth or decay estimates for nonlinear wave equations with damping and source terms

Peng Zeng, Dandan Li and Yuanfei Li*

School of Data Science, Guangzhou Huashang College, Guangdong 511300, China

* **Correspondence:** Email: liqfd@163.com.

Abstract: The spatial decay or growth behavior of a coupled nonlinear wave equation with damping and source terms is considered. By defining the wave equations in a cylinder or an exterior region, the spatial growth and decay estimates for the solutions are obtained by assuming that the boundary conditions satisfy certain conditions. We also show that the growth or decay rates are faster than those obtained by relevant literature. This kind of spatial behavior can be extended to a nonlinear system of viscoelastic type. In the case of decay, we also prove that the total energy can be bounded by known data.

Keywords: wave equations; spatial behavior; viscoelastic type; total energies

1. Introduction

In the present paper, we study the following coupled nonlinear wave equations (see [1])

$$u_{tt} - div\Big(\rho\Big(|\nabla u|^2\Big)\nabla u\Big) + |u_t|^{m-1}u_t + f_1(u,v) = 0, \tag{1.1}$$

$$v_{tt} - div\Big(\rho\Big(|\nabla v|^2\Big)\nabla v\Big) + |v_t|^{n-1}v_t + f_2(u,v) = 0, \tag{1.2}$$

where $u_t = \frac{\partial u}{\partial t}, v_t = \frac{\partial v}{\partial t}$, $m,n > 1$, ∇ is the gradient operator, div is the divergence operator and $f_i(.,.) : \mathbb{R}^2 \to \mathbb{R}, i = 1,2$ are known functions. For arbitrary solutions of (1.1) and (1.2), the function ρ is supposed to satisfy one or the other of the two conditions:

A1). $0 < \rho^2(q^2)q^2 \le m_1\widetilde{\rho}(q^2)$,

or

A2). $0 < \frac{1}{K_1} < \rho^{-1}(q^2) \le K_2\widetilde{\rho}(q^2)\frac{1}{q^2}$,

where $\widetilde{\rho}(q^2) = \int_0^{q^2} \rho(\zeta)d\zeta,\ q^2 = |\nabla u|^2, m_1, K_1, K_2 > 0$.

Quintanilla [2] imposed condition A1 and obtained the growth or decay estimates of the solution to the type III heat conduction. The condition A1 can be satisfied easily, e.g., $\rho(q^2) = \frac{1}{\sqrt{1+q^2}}$ or

$\rho(q^2) = (1 + \frac{b}{p}q^2)^{p-1}, b > 0, 0 < p \leq 1$. The similar condition as A2 was also considered by many papers (see [3,4]). $\rho(q^2) = \sqrt{1 + q^2}$ satisfies the condition A2.

In addition, we introduce a function $F(u, v)$ which is defined as $\frac{\partial F}{\partial u} = f_1(u, v)$, $\frac{\partial F}{\partial v} = f_2(u, v)$, where $F(0, 0) = 0$.

The wave equations have attracted many attentions of scholars due to their wide application, and a large number of achievements have been made in the existence of solutions (see [1, 5–12]). The fuzzy inference method is used to solve this problem. The algebraic formulation of fuzzy relation is studied in [13, 14]. In this paper, we study the Phragmén-Lindelöf type alternative property of solutions of wave equations (1.1)–(1.2). It is proved that the solution of the equations either grows exponentially (polynomially) or decays exponentially (polynomially) when the space variable tends to infinite. In the case of decay, people usually expect a fast decay rate. The Phragmén-Lindelöf type alternative research on partial differential equations has lasted for a long time (see [2, 15–23]).

It is worth emphasizing that Quintanilla [2] considered an exterior or cone-like region. Under some appropriate conditions, the growth/decay estimates of some parabolic problems are obtained. Inspired by [2], we extends the research of to the nonlinear wave model in this paper. However, different from [2], in addition to condition A1 and condition A2, we also consider a special condition of ρ. The appropriate energy function is established, and the nonlinear differential inequality about the energy function is derived. By solving this differential inequality, the Phragmén-Lindelöf type alternative results of the solution are obtained. Our model is much more complex than [2], so the methods used in [2] can not be applied to our model directly. Finally, a nonlinear system of viscoelastic type is also studied when the system is defined in an exterior or cone-like regions and the growth or decay rates are also obtained.

2. The Phragmén-Lindelöf type alternative result under A1

Letting that $\Omega(r)$ denotes a cone-like region, i.e.,

$$\Omega(r) = \Big\{\boldsymbol{x}\Big||\boldsymbol{x}|^2 \geq r^2, r \geq R_0 > 0\Big\},$$

and that $B(r)$ denotes the exterior surface to the sphere, i.e.,

$$B(r) = \Big\{\boldsymbol{x}\Big||\boldsymbol{x}|^2 = r^2, r \geq R_0 > 0\Big\},$$

Equations (1.1) and (1.2) also have the following initial-boundary conditions

$$u(\boldsymbol{x}, 0) = v(\boldsymbol{x}, 0) = 0, \; in \; \Omega, \tag{2.1}$$

$$u(\boldsymbol{x}, t) = g_1(\boldsymbol{x}, t), \; v(\boldsymbol{x}, t) = g_2(\boldsymbol{x}, t), \; in \; B(R_0) \times (0, \tau), \tag{2.2}$$

where g_1 and g_2 are positive known functions, $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\tau > 0$.

Now, we establish an energy function

$$\begin{aligned}\mathcal{E}(r, t) = &\int_0^t \int_{B(r)} e^{-\omega\eta}\rho\Big(|\nabla u|^2\Big)\nabla u \cdot \frac{\boldsymbol{x}}{r}u_\eta dS\, d\eta + \int_0^t \int_{B(r)} e^{-\omega\eta}\rho\Big(|\nabla v|^2\Big)\nabla v \cdot \frac{\boldsymbol{x}}{r}v_\eta dS\, d\eta \\ \doteq &\, \mathcal{E}_1(r, t) + \mathcal{E}_2(r, t).\end{aligned} \tag{2.3}$$

Let r_0 be a positive constant which satisfies $r > r_0 \geq R_0$. Integrating $\mathcal{E}(z,t)$ from r_0 to r, using the divergence theorem and Eqs (1.1) and (1.2), (2.1) and (2.2), we have

$$\begin{aligned}
\mathcal{E}(r,t) - \mathcal{E}(r_0,t) = & \int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta} \nabla \cdot \Big[\rho\Big(|\nabla u|^2\Big)\nabla u u_\eta\Big] ds d\xi d\eta \\
& + \int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta} \nabla \cdot \Big[\rho\Big(|\nabla v|^2\Big)\nabla v v_\eta\Big] ds d\xi d\eta \\
& + \int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}\Big[u_{\eta\eta} + |u_\eta|^{m+1} + f_1(u,v)\Big]u_\eta ds d\xi d\eta \\
& + \int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}\Big[v_{\eta\eta} + |v_\eta|^{n+1} + f_2(u,v)\Big]v_\eta ds d\xi d\eta \\
& + \frac{1}{2}\int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}\frac{\partial}{\partial\eta}\widetilde{\rho}\Big(|\nabla u|^2\Big) ds d\xi d\eta + \frac{1}{2}\int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}\frac{\partial}{\partial\eta}\widetilde{\rho}\Big(|\nabla v|^2\Big) ds d\xi d\eta \\
& = \frac{1}{2}e^{-\omega t}\int_{r_0}^r \int_{B(\xi)} \Big[|u_t|^2 + |v_t|^2 + \widetilde{\rho}\Big(|\nabla u|^2\Big) + \widetilde{\rho}\Big(|\nabla v|^2\Big) + 2F(u,v)\Big] ds d\xi \\
& + \frac{1}{2}\omega\int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}\Big[|u_\eta|^2 + |v_\eta|^2 + \widetilde{\rho}\Big(|\nabla u|^2\Big) + \widetilde{\rho}\Big(|\nabla v|^2\Big) + 2F(u,v)\Big] dS\, d\xi d\eta \\
& + \int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}\Big[|u_\eta|^{m+1} + |v_\eta|^{n+1}\Big] ds d\xi d\eta, \qquad (2.4)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\frac{\partial}{\partial r}\mathcal{E}(r,t) = & \frac{1}{2}e^{-\omega t}\int_{B(r)} \Big[|u_t|^2 + |v_t|^2 + \widetilde{\rho}\Big(|\nabla u|^2\Big) + \widetilde{\rho}\Big(|\nabla v|^2\Big) + 2F(u,v)\Big] ds \\
& + \frac{1}{2}\omega\int_0^t \int_{B(r)} e^{-\omega\eta}\Big[|u_\eta|^2 + |v_\eta|^2 + \widetilde{\rho}\Big(|\nabla u|^2\Big) + \widetilde{\rho}\Big(|\nabla v|^2\Big) + 2F(u,v)\Big] dS\, d\eta \\
& + \int_0^t \int_{B(r)} e^{-\omega\eta}\Big[|u_\eta|^{m+1} + |v_\eta|^{n+1}\Big] ds d\eta, \qquad (2.5)
\end{aligned}$$

where ω is positive constant.

Now, we show how to bound $\mathcal{E}(r,t)$ by $\frac{\partial}{\partial r}\mathcal{E}(r,t)$. We use the Hölder inequality, the Young inequality and the condition A1 to have

$$\begin{aligned}
\Big|\mathcal{E}_1(r,t)\Big| & \leq \Big[\int_0^t \int_{B(r)} e^{-\omega\eta}\rho^2\Big(|\nabla u|^2\Big)|\nabla u|^2 ds d\eta \cdot \int_0^t \int_{B(r)} e^{-\omega\eta}u_\eta^2 ds d\eta\Big]^{\frac{1}{2}} \\
& \leq \sqrt{m_1}\Big[\int_0^t \int_{B(r)} e^{-\omega\eta}\widetilde{\rho}\Big(|\nabla u|^2\Big) ds d\eta \cdot \int_0^t \int_{B(r)} e^{-\omega\eta}u_\eta^2 ds d\eta\Big]^{\frac{1}{2}} \\
& \leq \frac{\sqrt{m_1}}{2}\Big[\int_0^t \int_{B(r)} e^{-\omega\eta}\widetilde{\rho}\Big(|\nabla u|^2\Big) ds d\eta + \int_0^t \int_{B(r)} e^{-\omega\eta}u_\eta^2 ds d\eta\Big], \qquad (2.6)
\end{aligned}$$

and

$$\Big|\mathcal{E}_2(r,t)\Big| \leq \frac{\sqrt{m_1}}{2}\Big[\int_0^t \int_{B(r)} e^{-\omega\eta}\widetilde{\rho}\Big(|\nabla v|^2\Big) ds d\eta + \int_0^t \int_{B(r)} e^{-\omega\eta}v_\eta^2 ds d\eta\Big]. \qquad (2.7)$$

Inserting (2.6) and (2.7) into (2.3) and combining (2.5), we have

$$\Big|\mathcal{E}(r,t)\Big| \le \frac{\sqrt{m_1}}{\omega}\Big[\frac{\partial}{\partial r}\mathcal{E}(r,t)\Big]. \tag{2.8}$$

We consider inequality (2.8) for two cases.

I. If $\exists r_0 > R_0$ such that $\mathcal{E}(r_0,t) \ge 0$. From (2.5), we obtain $\mathcal{E}(r,t) \ge \mathcal{E}(r_0,t) \ge 0, r \ge r_0$. Therefore, from (2.8) we have

$$\mathcal{E}(r,t) \le \frac{\sqrt{m_1}}{\omega}\Big[\frac{\partial}{\partial r}\mathcal{E}(r,t)\Big], r \ge r_0. \tag{2.9}$$

Integrating (2.9) from r_0 to r, we have

$$\mathcal{E}(r,t) \ge \Big[\mathcal{E}(r_0,t)\Big]e^{\frac{\omega}{\sqrt{m_1}}(r-r_0)}, r \ge r_0. \tag{2.10}$$

Combing (2.4) and (2.10), we have

$$\begin{aligned}
&\lim_{r\to\infty}\Big\{e^{-\frac{\omega}{\sqrt{m_1}}r}\Big[\frac{1}{2}e^{-\omega t}\int_{r_0}^{r}\int_{B(\xi)}\Big[|u_t|^2+|v_t|^2+\widetilde{\rho}\Big(|\nabla u|^2\Big)+\widetilde{\rho}\Big(|\nabla v|^2\Big)+2F(u,v)\Big]dsd\xi\\
&+\frac{1}{2}\omega\int_0^t\int_{r_0}^{r}\int_{B(\xi)}e^{-\omega\eta}\Big[|u_\eta|^2+|v_\eta|^2+\widetilde{\rho}\Big(|\nabla u|^2\Big)+\widetilde{\rho}\Big(|\nabla v|^2\Big)+2F(u,v)\Big]dsd\xi d\eta\\
&+\int_0^t\int_{r_0}^{r}\int_{B(\xi)}e^{-\omega\eta}\Big[|u_\eta|^{m+1}+|v_\eta|^{n+1}\Big]dsd\xi d\eta\Big]\Big\}\\
&\ge \mathcal{E}(R_0,t)e^{-\frac{\omega}{\sqrt{m_1}}R_0}.
\end{aligned} \tag{2.11}$$

II. If $\forall r > R_0$ such that $\mathcal{E}(r,t) < 0$. The inequality (2.8) can be rewritten as

$$-\mathcal{E}(r,t) \le \frac{\sqrt{m_1}}{\omega}\Big[\frac{\partial}{\partial r}\mathcal{E}(r,t)\Big], r \ge R_0. \tag{2.12}$$

Integrating (2.12) from r_0 to r, we have

$$-\mathcal{E}(r,t) \ge \Big[-\mathcal{E}(R_0,t)\Big]e^{-\frac{\omega}{\sqrt{m_1}}(r-R_0)}, r \ge R_0. \tag{2.13}$$

Inequality (2.13) shows that $\lim_{r\to\infty}\Big[-\mathcal{E}(r,t)\Big]=0$. Integrating (2.5) from r to ∞ and combining (2.13), we obtain

$$\begin{aligned}
&\frac{1}{2}e^{-\omega t}\int_{r}^{\infty}\int_{B(\xi)}\Big[|u_t|^2+|v_t|^2+\widetilde{\rho}\Big(|\nabla u|^2\Big)+\widetilde{\rho}\Big(|\nabla v|^2\Big)+2F(u,v)\Big]dsd\xi\\
&+\frac{1}{2}\omega\int_0^t\int_{r}^{\infty}\int_{B(\xi)}e^{-\omega\eta}\Big[|u_\eta|^2+|v_\eta|^2+\widetilde{\rho}\Big(|\nabla u|^2\Big)+\widetilde{\rho}\Big(|\nabla v|^2\Big)+2F(u,v)\Big]dsd\xi d\eta\\
&+\int_0^t\int_{r}^{\infty}\int_{B(\xi)}e^{-\omega\eta}\Big[|u_\eta|^{m+1}+|v_\eta|^{n+1}\Big]dsd\xi d\eta\\
&\le \Big[-\mathcal{E}(r_0,t)\Big]e^{-\frac{\omega}{\sqrt{m_1}}(r-R_0)}.
\end{aligned} \tag{2.14}$$

We summarize the above result as the following theorem.

Theorem 2.1. Let (u,v) be solution of the (1.1), (1.2), (2.1), (2.2) in $\Omega(R_0)$, and ρ satisfies condition A1. Then for fixed t, (u,v) either grows exponentially or decays exponentially. Specifically, either (2.11) holds or (2.14) holds.

3. The Phragmén-Lindelöf type alternative result under A2

If the function ρ satisfies the condition A2, we recompute the inequality (2.6) and (2.7). Therefore

$$\begin{aligned}\left|\mathcal{E}_1(r,t)\right| &\leq \Big[\int_0^t \int_{B(r)} e^{-\omega\eta}\rho^2\big(|\nabla u|^2\big)|\nabla u|^2 ds d\eta \cdot \int_0^t \int_{B(r)} e^{-\omega\eta} u_\eta^2 ds d\eta\Big]^{\frac{1}{2}} \\ &\leq K_1\sqrt{K_1}\Big[\int_0^t \int_{B(r)} e^{-\omega\eta}\rho^{-1}\big(|\nabla u|^2\big) ds d\eta \cdot \int_0^t \int_{B(r)} e^{-\omega\eta} u_\eta^2 ds d\eta\Big]^{\frac{1}{2}} \\ &\leq \frac{K_1\sqrt{K_1K_2}}{2}\Big[\int_0^t \int_{B(r)} e^{-\omega\eta}\widetilde{\rho}\big(|\nabla u|^2\big) ds d\eta + \int_0^t \int_{B(r)} e^{-\omega\eta} u_\eta^2 ds d\eta\Big], \end{aligned} \tag{3.1}$$

and

$$\left|\mathcal{E}_2(r,t)\right| \leq \frac{K_1\sqrt{K_1K_2}}{2}\Big[\int_0^t \int_{B(r)} e^{-\omega\eta}\widetilde{\rho}\big(|\nabla v|^2\big) ds d\eta + \int_0^t \int_{B(r)} e^{-\omega\eta} v_\eta^2 ds d\eta\Big]. \tag{3.2}$$

Inserting (3.1) and (3.2) into (2.3) and combining (2.5), we have

$$\left|\mathcal{E}(r,t)\right| \leq \frac{K_1\sqrt{K_1K_2}}{\omega}\Big[\frac{\partial}{\partial r}\mathcal{E}(r,t)\Big]. \tag{3.3}$$

By following a similar method to that used in Section 2, we can obtain the Phragmén-Lindelöf type alternative result.

Theorem 3.1. Let (u, v) be solution of the (1.1), (1.2), (2.1), (2.2) in $\Omega(R_0)$, and ρ satisfies condition A1. Then for fixed t, (u, v) either grows exponentially or decays exponentially.

Remark 3.1. Clearly, the rates of growth or decay obtained in Theorems 2.1 and 3.1 depend on ω. Because ω can be chosen large enough, the rates of growth or decay of the solutions can become large as we want.

Remark 3.2. The analysis in Sections 2 and 3 can be adapted to the single-wave equation

$$u_{tt} - div\Big(\rho\big(|\nabla u|^2\big)\nabla u\Big) + |u_t|^{m-1}u_t + f(u) = 0 \tag{3.4}$$

and the heat conduction at low temperature

$$au_{tt} + bu_t - c\Delta u + \Delta u_t = 0, \tag{3.5}$$

where $a, b, c > 0$.

4. The Phragmén-Lindelöf type alternative result when $\rho(q^2) = b_1 + b_2 q^{2\beta}$

In this section, we suppose that ρ satisfies $\rho(q^2) = b_1 + b_2 q^{2\beta}$, where b_1, b_2 and β are positive constants. Clearly, $\rho(q^2) = b_1 + b_2 q^{2\beta}$ can not satisfy A1 or A2. In this case, we define an "energy" function

$$\mathcal{F}(r,t) = \int_0^t \int_{B(r)} e^{-\omega\eta}\big(b_1 + b_2|\nabla u|^{2\beta}\big)\nabla u \cdot \frac{\boldsymbol{x}}{r} u_\eta ds d\eta$$

$$
\begin{aligned}
&+ \int_0^t \int_{B(r)} e^{-\omega\eta}\Big(b_1 + b_2|\nabla v|^{2\beta}\Big)\nabla v \cdot \frac{\boldsymbol{x}}{r} v_\eta ds d\eta \\
&\doteq \mathcal{F}_1(r,t) + \mathcal{F}_2(r,t).
\end{aligned} \tag{4.1}
$$

Computing as that in (2.4) and (2.5), we can get

$$
\begin{aligned}
\mathcal{F}(r,t) = \mathcal{F}(r_0,t) &+ \frac{1}{2}e^{-\omega t}\int_{r_0}^r \int_{B(\xi)} \Big[|u_t|^2 + b_1|\nabla u|^2 + \frac{1}{\beta+1}b_2|\nabla u|^{2(\beta+1)}\Big] ds d\xi \\
&+ \frac{1}{2}e^{-\omega t}\int_{r_0}^r \int_{B(\xi)} \Big[|v_t|^2 + b_1|\nabla v|^2 + \frac{1}{\beta+1}b_2|\nabla v|^{2(\beta+1)} + F(u,v)\Big] ds d\xi \\
&+ \frac{1}{2}\omega \int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}\Big[|u_\eta|^2 + b_1|\nabla u|^2 + \frac{1}{\beta+1}b_2|\nabla u|^{2(\beta+1)}\Big] ds d\xi d\eta \\
&+ \frac{1}{2}\omega \int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}\Big[|v_\eta|^2 + b_1|\nabla v|^2 + \frac{1}{\beta+1}b_2|\nabla v|^{2(\beta+1)} + F(u,v)\Big] ds d\xi d\eta \\
&+ \int_0^t \int_{r_0}^r \int_{B(\xi)} e^{-\omega\eta}|u_\eta|^{m+1} ds d\xi d\eta + \int_0^t \int_{B(r)} e^{-\omega\eta}|v_\eta|^{n+1} ds d\xi d\eta,
\end{aligned} \tag{4.2}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial r}\mathcal{F}(r,t) = &\frac{1}{2}e^{-\omega t}\int_{B(r)} \Big[|u_t|^2 + b_1|\nabla u|^2 + \frac{1}{\beta+1}b_2|\nabla u|^{2(\beta+1)}\Big] ds \\
&+ \frac{1}{2}e^{-\omega t}\int_{B(r)} \Big[|v_t|^2 + b_1|\nabla v|^2 + \frac{1}{\beta+1}b_2|\nabla v|^{2(\beta+1)} + F(u,v)\Big] ds \\
&+ \frac{1}{2}\omega \int_0^t \int_{B(r)} e^{-\omega\eta}\Big[|u_\eta|^2 + b_1|\nabla u|^2 + \frac{1}{\beta+1}b_2|\nabla u|^{2(\beta+1)}\Big] ds d\eta \\
&+ \frac{1}{2}\omega \int_0^t \int_{B(r)} e^{-\omega\eta}\Big[|v_\eta|^2 + b_1|\nabla v|^2 + \frac{1}{\beta+1}b_2|\nabla v|^{2(\beta+1)} + F(u,v)\Big] ds d\eta \\
&+ \int_0^t \int_{B(r)} e^{-\omega\eta}|u_\eta|^{m+1} ds d\eta + \int_0^t \int_{B(r)} e^{-\omega\eta}|v_\eta|^{n+1} ds d\eta.
\end{aligned} \tag{4.3}
$$

Using the Hölder inequality and Young's inequality, we have

$$
\begin{aligned}
\Big|\mathcal{F}_1(r,t)\Big| \le\ & b_1\Big[\int_0^t \int_{B(r)} e^{-\omega\eta}|\nabla u|^2 ds d\eta \int_0^t \int_{B(r)} e^{-\omega\eta}u_\eta^2 ds d\eta\Big]^{\frac{1}{2}} \\
&+ b_2\Big[\int_0^t \int_{B(r)} e^{-\omega\eta}|\nabla u|^{2(\beta+1)} ds d\eta\Big]^{\frac{2\beta+1}{2(\beta+1)}} \int_0^t \int_{B(r)} e^{-\omega\eta}|u_\eta|^{m+1} ds d\eta\Big]^{\frac{1}{m+1}} |t2\pi r|^{\frac{1}{2(\beta+1)}-\frac{1}{m+1}} \\
\le\ & \frac{\sqrt{b_1}}{R_0\omega}\Big[\frac{b_1}{2}r\omega \int_0^t \int_{B(r)} e^{-\omega\eta}|\nabla u|^2 ds d\eta + \frac{1}{2}r\omega \int_0^t \int_{B(r)} e^{-\omega\eta}u_\eta^2 ds d\eta\Big] \\
&+ b_2|2t\pi|^{\frac{1}{2(\beta+1)}-\frac{1}{m+1}} R_0^{-\frac{\beta}{\beta+1}-\frac{2}{m+1}}\Big[\frac{\frac{2\beta+1}{2(\beta+1)}r}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}} \int_0^t \int_{B(r)} e^{-\omega\eta}|\nabla u|^{2(\beta+1)} ds d\eta \\
&+ \frac{\frac{1}{m+1}r}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}} \int_0^t \int_{B(r)} e^{-\omega\eta}|u_\eta|^{m+1} ds d\eta\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}},
\end{aligned} \tag{4.4}
$$

and

$$\Big|\mathcal{F}_2(r,t)\Big| \le \frac{\sqrt{b_1}}{R_0\omega}\Big[\frac{b_1}{2}r\omega\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla v|^2 dsd\eta + \frac{1}{2}r\omega\int_0^t\int_{B(r)} e^{-\omega\eta}v_\eta^2 dsd\eta\Big]$$
$$+ b_2|2t\pi|^{\frac{1}{2(\beta+1)}-\frac{1}{n+1}}R_0^{-\frac{\beta}{\beta+1}-\frac{2}{n+1}}\Big[\frac{\frac{2\beta+1}{2(\beta+1)}r}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{n+1}}\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla v|^{2(\beta+1)}dsd\eta$$
$$+\frac{\frac{1}{n+1}r}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{n+1}}\int_0^t\int_{B(r)} e^{-\omega\eta}|v_\eta|^{n+1}dsd\eta\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{n+1}}, \tag{4.5}$$

where we have chosen $\frac{1}{2(\beta+1)} > \frac{1}{n+1}$. Inserting (4.4) and (4.5) into (4.1), we obtain

$$\Big|\mathcal{F}(r,t)\Big| \le c_1\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big] + c_2\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}} + c_3\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{n+1}}, \tag{4.6}$$

where $c_1 = \frac{\sqrt{b_1}}{R_0\omega}, c_2 = b_2|2t\pi|^{\frac{1}{2(\beta+1)}-\frac{1}{n+1}}R_0^{-\frac{\beta}{\beta+1}-\frac{2}{m+1}}, c_3 = b_2|2t\pi|^{\frac{1}{2(\beta+1)}-\frac{1}{n+1}}R_0^{-\frac{\beta}{\beta+1}-\frac{2}{n+1}}$.

Next, we will analyze Eq (4.6) in two cases

I. If $\exists r_0 \ge R_0$ such that $\mathcal{F}(r_0,t) \ge 0$, then $\mathcal{F}(r,t) \ge \mathcal{F}(r_0,t) \ge 0, r \ge r_0$. Therefore, (4.6) can be rewritten as

$$\mathcal{F}(r,t) \le c_1\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big] + c_2\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}$$
$$+ c_3\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{n+1}}, \qquad r \ge r_0. \tag{4.7}$$

Using Young's inequality, we have

$$\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}} \le \Big(\frac{4\beta+3}{4(\beta+1)}+\frac{1}{2(m+1)}\Big)\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{1}{2}}$$
$$+\Big(\frac{1}{4(\beta+1)}-\frac{1}{2(m+1)}\Big)\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big], \tag{4.8}$$
$$\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{n+1}} \le \Big(\frac{4\beta+3}{4(\beta+1)}+\frac{1}{2(n+1)}\Big)\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{1}{2}}$$
$$+\Big(\frac{1}{4(\beta+1)}-\frac{1}{2(n+1)}\Big)\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]. \tag{4.9}$$

Inserting (4.8) and (4.9) into (4.7), we have

$$\mathcal{F}(r,t) \le c_4\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{1}{2}} + c_5\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big], \quad r \ge r_0. \tag{4.10}$$

where $c_4 = c_2\Big(\frac{4\beta+3}{4(\beta+1)}+\frac{1}{2(m+1)}\Big)+c_3\Big(\frac{4\beta+3}{4(\beta+1)}+\frac{1}{2(n+1)}\Big)$ and $c_5 = c_1 + c_2\Big(\frac{1}{4(\beta+1)}-\frac{1}{2(m+1)}\Big)+c_3\Big(\frac{1}{4(\beta+1)}-\frac{1}{2(n+1)}\Big)$. From (4.10) we have

$$\frac{\mathcal{F}(r,t)}{c_5} \le \Big[\sqrt{r\frac{\partial}{\partial r}\mathcal{F}(r,t)}+\frac{c_4}{2c_5}\Big]^2 - \frac{c_4^2}{4c_5^2}, \quad r \ge r_0$$

or

$$\frac{\frac{\partial}{\partial r}\mathcal{F}(r,t)}{\Big[\sqrt{\frac{\mathcal{F}(r,t)}{c_5}+\frac{c_4^2}{4c_5^2}}-\frac{c_4}{2c_5}\Big]^2}\geq\frac{1}{r},\quad r\geq r_0 \tag{4.11}$$

Integrating (4.11) from r_0 to r, we get

$$\begin{aligned}&2c_5\Big[ln\Big(\sqrt{\frac{\mathcal{F}(r,t)}{c_5}+\frac{c_4^2}{4c_5^2}}-\frac{c_4}{2c_5}\Big)-ln\Big(\sqrt{\frac{\mathcal{F}(r_0,t)}{c_5}+\frac{c_4^2}{4c_5^2}}-\frac{c_4}{2c_5}\Big)\Big]\\&-c_4\Big\{\Big[\sqrt{\frac{\mathcal{F}(r,t)}{c_5}+\frac{c_4^2}{4c_5^2}}-\frac{c_4}{2c_5}\Big]^{-1}-\Big[\sqrt{\frac{\mathcal{F}(r_0,t)}{c_5}+\frac{c_4^2}{4c_5^2}}-\frac{c_4}{2c_5}\Big]^{-1}\Big\}\\&\qquad\geq ln\Big(\frac{r}{r_0}\Big),\quad r\geq r_0\end{aligned} \tag{4.12}$$

Dropping the third term on the left of (4.12), we have

$$\sqrt{\frac{\mathcal{F}(r,t)}{c_5}+\frac{c_4^2}{4c_5^2}}-\frac{c_4}{2c_5}\geq Q_1(r_0,t)\Big(\frac{r}{r_0}\Big)^{\frac{1}{2c_5}}, \tag{4.13}$$

where $Q_1(r_0,t)=\Big(\sqrt{\frac{\mathcal{F}(r_0,t)}{c_5}+\frac{c_4^2}{4c_5^2}}-\frac{c_4}{2c_5}\Big)\exp\Big\{-\frac{c_4}{2c_5}\Big[\sqrt{\frac{\mathcal{F}(r_0,t)}{c_5}+\frac{c_4^2}{4c_5^2}}-\frac{c_4}{2c_5}\Big]^{-1}\Big\}$.

In view of

$$\sqrt{\frac{\mathcal{F}(r,t)}{c_5}+\frac{c_4^2}{4c_5^2}}\leq\sqrt{\frac{\mathcal{F}(r,t)}{c_5}}+\frac{c_4}{2c_5},$$

we have from (4.13)

$$\mathcal{F}(r,t)\geq c_5Q_1^2(r_0,t)\Big(\frac{r}{r_0}\Big)^{\frac{1}{c_5}}. \tag{4.14}$$

Combining (4.2) and (4.14), we have

$$\begin{aligned}&\lim_{r\to\infty}\Big\{r^{-\frac{1}{c_5}}\Big[\frac{1}{2}e^{-\omega t}\int_{r_0}^{r}\int_{B(\xi)}\Big[|u_t|^2+b_1|\nabla u|^2+\frac{1}{\beta+1}b_2|\nabla u|^{2(\beta+1)}\Big]dsd\xi\\&+\frac{1}{2}e^{-\omega t}\int_{r_0}^{r}\int_{B(\xi)}\Big[|v_t|^2+b_1|\nabla v|^2+\frac{1}{\beta+1}b_2|\nabla v|^{2(\beta+1)}+F(u,v)\Big]dsd\xi\\&+\frac{1}{2}\omega\int_0^t\int_{r_0}^{r}\int_{B(\xi)}e^{-\omega\eta}\Big[|u_\eta|^2+b_1|\nabla u|^2+\frac{1}{\beta+1}b_2|\nabla u|^{2(\beta+1)}\Big]dsd\xi d\eta\\&+\frac{1}{2}\omega\int_0^t\int_{r_0}^{r}\int_{B(\xi)}e^{-\omega\eta}\Big[|v_\eta|^2+b_1|\nabla v|^2+\frac{1}{\beta+1}b_2|\nabla v|^{2(\beta+1)}+F(u,v)\Big]dsd\xi d\eta\\&+\int_0^t\int_{r_0}^{r}\int_{B(\xi)}e^{-\omega\eta}|u_\eta|^{m+1}dsd\xi d\eta+\int_0^t\int_{B(r)}e^{-\omega\eta}|v_\eta|^{n+1}dsd\xi d\eta\Big]\Big\}\\&\geq c_5Q_1^2(r_0,t)r_0^{-\frac{1}{c_5}}.\end{aligned} \tag{4.15}$$

II. If $\forall r \geq R_0$ such that $\mathcal{F}(r,t) < 0$, then (4.6) can be rewritten as

$$-\mathcal{F}(r,t) \leq c_1\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big] + c_2\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}} + c_3\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{n+1}}, \qquad r \geq R_0. \tag{4.16}$$

Without losing generality, we suppose that $m > n > 1$.

$$\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{n+1}} \leq \frac{\frac{1}{n+1}-\frac{1}{m+1}}{\frac{2\beta+1}{2(\beta+1)}-\frac{1}{m+1}}\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big] + \frac{\frac{2\beta+1}{2(\beta+1)}-\frac{1}{m+1}}{\frac{2\beta+1}{2(\beta+1)}-\frac{1}{m+1}}\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}, \tag{4.17}$$

$$\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big] \leq \frac{2\beta(m+3)+4}{(2\beta+1)(m+2)+1}\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}} + \frac{m+1-2(\beta+1)}{(2\beta+1)(m+2)+1}\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{2\Big[\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}\Big]}. \tag{4.18}$$

Inserting (4.17) and (4.18) into (4.16), we get

$$-\mathcal{F}(r,t) \leq c_6\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}} + c_7\Big[r\frac{\partial}{\partial r}\mathcal{F}(r,t)\Big]^{2\Big[\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}\Big]},\ r \geq R_0, \tag{4.19}$$

where $c_6 = \Big[c_1 + c_3\frac{\frac{1}{n+1}-\frac{1}{m+1}}{\frac{2\beta+1}{2(\beta+1)}-\frac{1}{m+1}}\Big]\frac{2\beta(m+3)+4}{(2\beta+1)(m+2)+1} + c_3\frac{\frac{2\beta+1}{2(\beta+1)}-\frac{1}{m+1}}{\frac{2\beta+1}{2(\beta+1)}-\frac{1}{m+1}}, c_7 = c_3\frac{m+1-2(\beta+1)}{(2\beta+1)(m+2)+1}$. From (4.19) we obtain

$$r\frac{\partial}{\partial r}\mathcal{F}(r,t) \geq \Big[\sqrt{\frac{-\mathcal{F}(r,t)}{c_7}+\frac{c_6^2}{4c_7^2}} - \frac{c_6}{2c_7}\Big]^{\frac{1}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}},\ r \geq R_0$$

or

$$2c_7\frac{\sqrt{\frac{-\mathcal{F}(r,t)}{c_7}+\frac{c_6^2}{4c_7^2}}}{\Big[\sqrt{\frac{-\mathcal{F}(r,t)}{c_7}+\frac{c_6^2}{4c_7^2}} - \frac{c_6}{2c_7}\Big]^{\frac{1}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}}}d\Big\{\sqrt{\frac{-\mathcal{F}(r,t)}{c_7}+\frac{c_6^2}{4c_7^2}} - \frac{c_6}{2c_7}\Big\} \leq -\frac{1}{r},\ r \geq R_0 \tag{4.20}$$

Integrating (4.20) from R_0 to r, we obtain

$$\frac{2c_7\Big(\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}\Big)}{\frac{\beta}{2(\beta+1)}+\frac{2}{m+1}}\Big[\sqrt{\frac{-\mathcal{F}(r,t)}{c_7}+\frac{c_6^2}{4c_7^2}} - \frac{c_6}{2c_7}\Big]^{\frac{\frac{\beta}{2(\beta+1)}+\frac{2}{m+1}}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}} - \frac{2c_7\Big(\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}\Big)}{\frac{\beta}{2(\beta+1)}+\frac{2}{m+1}}\Big[\sqrt{\frac{-\mathcal{F}(R_0,t)}{c_7}+\frac{c_6^2}{4c_7^2}} - \frac{c_6}{2c_7}\Big]^{\frac{\frac{\beta}{2(\beta+1)}+\frac{2}{m+1}}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}} - \frac{c_6\Big(\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}\Big)}{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}\Big[\sqrt{\frac{-\mathcal{F}(r,t)}{c_7}+\frac{c_6^2}{4c_7^2}} - \frac{c_6}{2c_7}\Big]^{-\frac{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}}$$

$$+\frac{c_6\Big(\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}\Big)}{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}\Big[\sqrt{\frac{-\mathcal{F}(R_0,t)}{c_7}+\frac{c_6^2}{4c_7^2}}-\frac{c_6}{2c_7}\Big]^{-\frac{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}}$$
$$\leq ln\Big(\frac{R_0}{r}\Big). \tag{4.21}$$

Dropping the first and fourth terms on the left of (4.21), we obtain

$$\sqrt{\frac{-\mathcal{F}(r,t)}{c_7}+\frac{c_6^2}{4c_7^2}}\leq\Big[c_8 ln\Big(\frac{r}{R_0}\Big)-Q_2(R_0,t)\Big]^{-\frac{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}}+\frac{c_6}{2c_7}, \tag{4.22}$$

where $Q_2(R_0,t)=\frac{2c_7\Big(\frac{1}{2(\beta+1)}-\frac{1}{m+1}\Big)}{c_6\Big(\frac{\beta}{2(\beta+1)}+\frac{2}{m+1}\Big)}\Big[\sqrt{\frac{-\mathcal{F}(R_0,t)}{c_7}+\frac{c_6^2}{4c_7^2}}-\frac{c_6}{2c_7}\Big]^{\frac{\frac{\beta}{2(\beta+1)}+\frac{2}{m+1}}{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}}$ and $c_8=\frac{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}{c_6\Big(\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}\Big)}$.

Squaring (4.22) we have

$$-\mathcal{F}(r,t)\leq c_7\Big[c_8 ln\Big(\frac{r}{R_0}\Big)-Q_2(R_0,t)\Big]^{-\frac{\frac{2\beta+1}{\beta+1}+\frac{2}{m+1}}{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}}$$
$$+c_6\Big[c_8 ln\Big(\frac{r}{R_0}\Big)-Q_2(R_0,t)\Big]^{-\frac{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}}. \tag{4.23}$$

From (4.15) and (4.23) we can obtain the following theorem.

Theorem 4.1. Let (u,v) be the solution of (1.1), (1.2), (2.1) and (2.2) with $\rho(q^2)=b_1+b_2q^{2\beta}$, where $\frac{1}{2(\beta+1)}>\max\Big\{\frac{1}{m+1},\frac{1}{n+1}\Big\}$. Then for fixed t, when $r\to\infty$, (u,v) either grows algebraically or decays logarithmically. The growth rate is at least as fast as $z^{\frac{1}{c_5}}$ and the decay rate is at least as fast as $\Big(lnr\Big)^{-\frac{\frac{2\beta+1}{2(\beta+1)}+\frac{1}{m+1}}{\frac{1}{2(\beta+1)}-\frac{1}{m+1}}}$.

Remark 4.1. Obviously, in this case of $\rho(q^2)=b_1+b_2q^{2\beta}$, the decay rate obtained by Theorem 4.1 is slower than that obtained by Theorem 2.1 and Theorem 3.1.

5. A nonlinear system of viscoelastic type

In this section, we concern with a system of two coupled viscoelastic equations

$$u_{tt}-\Delta u+\int_0^t h_1(t-\eta)\Delta u(\eta)d\eta+f_1(u,v)=0, \tag{5.1}$$
$$v_{tt}-\Delta v+\int_0^t h_2(t-\eta)\Delta v(\eta)d\eta+f_2(u,v)=0, \tag{5.2}$$

which describes the interaction between two different fields arising in viscoelasticity. In (5.1) and (5.2), $0<t<T$ and h_1,h_2 are differentiable functions satisfying $h_1(0),h_2(0)>0$ and

$$2\Big(\int_0^T h_1^2(T-\eta)d\eta\Big),\ \frac{4T}{h_1(0)}\Big[\int_0^T\Big(h_1'(T-\eta)-\omega h_1(T-\eta)\Big)^2 d\eta\Big]\leq h_1(0), \tag{5.3}$$

$$2\Big(\int_0^T h_2^2(T-\eta)d\eta\Big),\ \frac{4T}{h_2(0)}\Big[\int_0^T \Big(h_2'(T-\eta)-\omega h_2(T-\eta)\Big)^2 d\eta\Big] \le h_2(0). \tag{5.4}$$

Messaoudi and Tatar [24] considered the system (5.1) and (5.2) in a bounded domain and proved the uniform decay for the solution when $t \to \infty$. For more special cases, one can refer to [25–27]. They mainly concerned the well-posedness of the solutions and proved that the solutions decayed uniformly under some suitable conditions. However, the present paper extends the previous results to Eqs (5.1) and (5.2) in an exterior region. We consider Eqs (5.1) and (5.2) with the initial-boundary conditions (2.1) and (2.2) in Ω.

We define two functions

$$\begin{aligned} G_1(r,t) &= \int_0^t \int_{B(r)} e^{-\omega\eta}\nabla u\cdot\frac{\boldsymbol{x}}{r}u_\eta ds d\eta - \int_0^t \int_{B(r)} e^{-\omega\eta}\Big(\int_0^\eta h_1(\eta-s)\nabla u ds\Big)\cdot\frac{\boldsymbol{x}}{r}u_\eta ds d\eta \\ &\doteq I_1 + I_2, \end{aligned} \tag{5.5}$$

$$\begin{aligned} G_2(r,t) &= \int_0^t \int_{B(r)} e^{-\omega\eta}\nabla v\cdot\frac{\boldsymbol{x}}{r}v_\eta ds d\eta - \int_0^t \int_{B(r)} e^{-\omega\eta}\Big(\int_0^\eta h_2(\eta-s)\nabla v ds\Big)\cdot\frac{\boldsymbol{x}}{r}v_\eta ds d\eta \\ &\doteq J_1 + J_2. \end{aligned} \tag{5.6}$$

Integrating (5.5) from r_0 to r and using (5.1), (5.2), (2.1), (2.2) and the divergence theorem, we have

$$\begin{aligned} G_1(r,t) &= G_1(r_0,t) + \frac{1}{2}\int_{r_0}^r\int_{B(\xi)} e^{-\omega t}\Big[|u_t|^2+|\nabla u|^2\Big]ds d\xi + \frac{1}{2}\omega\int_0^t\int_{r_0}^r\int_{B(\xi)} e^{-\omega\eta}|u_\eta|^2 ds d\xi d\eta \\ &+ \frac{1}{2}\omega\int_0^t\int_{r_0}^r\int_{B(\xi)} e^{-\omega\eta}|\nabla u|^2 ds d\xi d\eta + \int_0^t\int_{r_0}^r\int_{B(\xi)} e^{-\omega\eta}h_1(0)|\nabla u|^2 ds d\xi d\eta \\ &- \int_{r_0}^r\int_{B(\xi)} e^{-\omega t}\Big(\int_0^t h_1(t-\tau)\nabla u d\tau\Big)\cdot\nabla u ds d\xi \\ &+ \int_0^t\int_{r_0}^r\int_{B(\xi)} e^{-\omega\eta}\Big[\int_0^\eta \Big(h_1'(\eta-\tau)-\omega h_1(\eta-\tau)\Big)\nabla u d\tau\Big]\cdot\nabla u ds d\xi d\eta \\ &+ \int_0^t\int_{r_0}^r\int_{B(\xi)} e^{-\omega\eta} f_1(u,v)u_\eta ds d\xi d\eta. \end{aligned} \tag{5.7}$$

From (5.7) it follows that

$$\begin{aligned} \frac{\partial}{\partial z}G_1(r,t) &= \frac{1}{2}\int_{B(r)} e^{-\omega t}\Big[|u_t|^2+|\nabla u|^2\Big]ds + \frac{1}{2}\omega\int_0^t\int_{B(r)} e^{-\omega\eta}|u_\eta|^2 ds d\eta \\ &+ \frac{1}{2}\omega\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla u|^2 ds d\eta + \int_0^t\int_{B(r)} e^{-\omega\eta}h_1(0)|\nabla u|^2 ds d\eta \\ &- \int_{B(r)} e^{-\omega t}\Big(\int_0^t h_1(t-\tau)\nabla u d\tau\Big)\cdot\nabla u ds \\ &+ \int_0^t\int_{B(r)} e^{-\omega\eta}\Big[\int_0^\eta \Big(h_1'(\eta-\tau)-\omega h_1(\eta-\tau)\Big)\nabla u d\tau\Big]\cdot\nabla u ds d\eta \\ &+ \int_0^t\int_{B(r)} e^{-\omega\eta} f_1(u,v)u_\eta ds d\eta. \end{aligned} \tag{5.8}$$

By the Young inequality and the Hölder inequality, we have

$$\begin{aligned}&\Big|-\int_{B(r)} e^{-\omega t}\Big(\int_0^t h_1(\eta-\tau)\nabla u d\tau\Big)\cdot\nabla u ds\Big|\\&\le\int_{B(r)} e^{-\omega t}\Big[\Big(\int_0^t h_1(t-s)\nabla u(s)ds\Big)^2+\frac{1}{4}|\nabla u|^2\Big]ds\\&\le\Big(\int_0^t h_1^2(t-\tau)d\tau\Big)\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla u|^2 dsd\eta+\frac{1}{4}\int_{B(r)} e^{-\omega t}|\nabla u|^2 ds,\end{aligned}\tag{5.9}$$

and

$$\begin{aligned}&\Big|\int_0^t\int_{B(r)} e^{-\omega\eta}\Big[\int_0^\eta\Big(h_1'(\eta-\tau)-\omega h_1(\eta-\tau)\Big)\nabla u d\tau\Big]\cdot\nabla u dsd\eta\Big|\\&\le\frac{t}{h_1(0)}\Big[\int_0^t\Big(h_1'(\eta-\tau)-\omega h_1(\eta-\tau)\Big)^2 d\tau\Big]\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla u|^2 dsd\eta\\&+\frac{1}{4}\int_0^t\int_{B(r)} e^{-\omega\eta}h_1(0)|\nabla u|^2 dsd\eta.\end{aligned}\tag{5.10}$$

Inserting (5.9) and (5.10) into (5.8) and using (5.3), we have

$$\begin{aligned}\frac{\partial}{\partial z}G_1(r,t)\ge&\frac{1}{2}\int_{B(r)} e^{-\omega t}\Big[|u_t|^2+\frac{1}{2}|\nabla u|^2\Big]ds+\frac{1}{2}\omega\int_0^t\int_{B(r)} e^{-\omega\eta}|u_\eta|^2 dsd\eta\\&+\frac{1}{2}\omega\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla u|^2 dsd\eta+\int_0^t\int_{B(r)} e^{-\omega\eta}f_1(u,v)u_\eta dsd\eta.\end{aligned}\tag{5.11}$$

Similar to (5.11), we also have for $G_2(r,t)$

$$\begin{aligned}\frac{\partial}{\partial z}G_2(r,t)\ge&\frac{1}{2}\int_{B(r)} e^{-\omega t}\Big[|v_t|^2+\frac{1}{2}|\nabla v|^2\Big]ds+\frac{1}{2}\omega\int_0^t\int_{B(r)} e^{-\omega\eta}|v_\eta|^2 dsd\eta\\&+\frac{1}{2}\omega\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla v|^2 dsd\eta+\int_0^t\int_{B(r)} e^{-\omega\eta}f_2(u,v)v_\eta dsd\eta.\end{aligned}\tag{5.12}$$

If we define

$$G(r,t)=G_1(r,t)+G_2(r,t),$$

then by (5.11) and (5.12) we have

$$\begin{aligned}\frac{\partial}{\partial z}G(r,t)\ge&\frac{1}{2}\int_{B(r)} e^{-\omega t}\Big[|u_t|^2+|v_t|^2+\frac{1}{2}|\nabla u|^2+\frac{1}{2}|\nabla v|^2+2F(u,v)\Big]ds\\&+\frac{1}{2}\omega\int_0^t\int_{B(r)} e^{-\omega\eta}\Big[|u_\eta|^2+|v_\eta|^2+|\nabla u|^2+|\nabla v|^2+2F(u,v)\Big]dsd\eta.\end{aligned}\tag{5.13}$$

On the other hand, we bound $G(r,t)$ by $\frac{\partial}{\partial r}G(r,t)$. Using the Hölder inequality, the AG mean inequality, (5.3) and combining (5.13), we have

$$\Big|I_1\Big|+\Big|J_1\Big|\le\Big(\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla u|^2 dAd\eta\cdot\int_0^t\int_{B(r)} e^{-\omega\eta}|u_\eta|^2 dsd\eta\Big)^{\frac{1}{2}}$$

$$
\begin{aligned}
&+\Big(\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla v|^2 dAd\eta\cdot\int_0^t\int_{B(r)} e^{-\omega\eta}|v_\eta|^2 dsd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{1}{\omega}\Big[\frac{\partial}{\partial r}G(r,t)\Big],
\end{aligned}
\tag{5.14}
$$

and

$$
\begin{aligned}
\Big|I_2\Big| &\le \Big(\int_0^t\int_{B(r)} e^{-\omega\eta}\Big|\int_0^\eta h_1(\eta-\tau)\nabla u d\tau\Big|^2 dsd\eta\cdot\int_0^t\int_{B(r)} e^{-\omega\eta}|u_\eta|^2 dsd\eta\Big)^{\frac{1}{2}}\\
&\le \Big(\int_0^t\int_{B(r)} e^{-\omega\eta}\Big(\int_0^\eta h_1^2(\eta-\tau)d\tau\Big)\Big(\int_0^\eta|\nabla u|^2 d\tau\Big)dsd\eta\cdot\int_0^t\int_{B(r)} e^{-\omega\eta}|u_\eta|^2 dsd\eta\Big)^{\frac{1}{2}}\\
&\le t\Big(\int_0^t h_1^2(\eta-\tau)d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla u|^2 dsd\eta\cdot\int_0^t\int_{B(r)} e^{-\omega\eta}|u_\eta|^2 dsd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{t}{2}\Big(\int_0^t h_1^2(\eta-\tau)d\tau\Big)^{\frac{1}{2}}\Big[\int_0^t\int_{B(r)} e^{-\omega\eta}|\nabla u|^2 dsd\eta+\int_0^t\int_{B(r)} e^{-\omega\eta}|u_\eta|^2 dsd\eta\Big]\\
&\le \frac{T}{2\omega}\sqrt{h_1(0)}\Big[\frac{\partial}{\partial r}G(r,t)\Big].
\end{aligned}
\tag{5.15}
$$

Similar to (5.15), we have

$$
\Big|J_2\Big| \le \frac{T}{2\omega}\sqrt{h_2(0)}\Big[\frac{\partial}{\partial r}G(r,t)\Big].
\tag{5.16}
$$

Inserting (5.14)–(5.16) into (5.5) and (5.6), we have

$$
\Big|G(r,t)\Big| \le c_9\frac{1}{\omega}\Big[\frac{\partial}{\partial r}G(r,t)\Big],
\tag{5.17}
$$

where $c_9=\frac{T}{2}(\sqrt{h_1(0)}+\sqrt{h_2(0)})+2$.

We can follow the similar arguments given in the previous sections to obtain the following theorem.

Theorem 5.1. Let (u,v) be the solution of (5.1), (5.2), (2.1) and (2.2) in Ω, and (5.3) and (5.4) hold. For fixed t,

(1) If $\exists\, R_0\ge r_0$, $G(R_0,t)\ge 0$, then

$$
G(r,t)\ge G(R_0,t)e^{\frac{\omega}{b_9}(r-R_0)}.
$$

(2) If $\forall\, r\ge r_0$, $G(r,t)<0$, then

$$
-G(r,t)\le\Big[-G(r_0,t)\Big]e^{-\frac{\omega}{b_9}(r-r_0)}.
$$

Again, the rate of growth or decay obtained in this case is arbitrarily large

Remark 5.1. It is clear that the above analysis can be adapted without difficulties to the equation (see [28, 29])

$$
u_{tt}-k_0\triangle u+\int_0^t div[a(x)h(t-s)\nabla u(s)]ds+b(x)g(u_t)+f(u)=0
$$

and the equation (see [30])

$$
|u_t|^\sigma u_{tt}-k_0\triangle u-\Delta u_{tt}+\int_0^t h(t-s)\Delta u(s)]ds-\gamma\Delta u_t=0
$$

with some suitable g and $a(x)+b(x)\ge b_{10}>0$ and $k_0,\sigma,\gamma>0$.

6. Conclusions

In this paper, we have considered several situations where the solutions of Eqs (1.1) and (1.2) either grow or decay exponentially or polynomially. We emphasize that the Poincaré inequality on the cross sections is not used in this paper. Thus, our results also hold for the two-dimensional case. On the other hand, there are some deeper problems to be studied in this paper. We can continue to study the continuous dependence of coefficients in the equation as that in [31]. These are the issues we will continue to study in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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