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Research article

A soil water indicator for a dynamic model of crop and soil water interaction

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Abstract: Water scarcity is a critical issue in agriculture, and the development of reliable methods for determining soil water content is crucial for effective water management. This study proposes a novel, theoretical, non-physiological indicator of soil water content obtained by applying the next-generation matrix method, which reflects the water-soil-crop dynamics and identifies the minimum viable value of soil water content for crop growth. The development of this indicator is based on a two-dimensional, nonlinear dynamic that considers two different irrigation scenarios: the first scenario involves constant irrigation, and the second scenario irrigates in regular periods by assuming each irrigation as an impulse in the system. The analysis considers the study of the local stability of the system by incorporating parameters involved in the water-soil-crop dynamics. We established a criterion for identifying the minimum viable value of soil water content for crop growth over time. Finally, the model was calibrated and validated using data from an independent field study on apple orchards and a tomato crop obtained from a previous field study. Our results suggest the advantages of using this theoretical approach in modeling the plants' conditions under water scarcity as the first step before an empirical model. The proposed indicator has some limitations, suggesting the need for future studies that consider other factors that affect soil water content.

Keywords: indicator; impulsive irrigation; mathematical model; vegetative growth; water scarcity

1. Introduction

The increasing temperatures resulting from global climate change are affecting the fragile equilibrium of ecosystems worldwide [1]. Mediterranean regions are affected by heat waves and decreased rainfall, which affects the water supply and food production systems [2]. In Mediterranean and semi-arid climate areas, artificial irrigation is necessary to maintain global agriculture production during the growing season (spring to summer). The main source of irrigation water is either reservoirs that usually fill with rain or snowmelt during the fall and winter seasons. In recent years, the lack of precipitation has substantially decreased water availability for seasonal irrigation. Dry soils and low water availability reduce the plants' evapotranspiration fluxes, which generates a physiological response, called water stress, due to water scarcity [3]. When plants experience water stress, a physiological response is induced that reduces their photosynthetic rates. This negatively impacts the growth of the plant [4–7], compromising the balance between water availability in the soil and the crop. Farmers were forced to adjust irrigation practices to improve irrigation water productivity.

Theoretically, the implementation of an irrigation strategy should focus on the quantity and timing of irrigation to meet two criteria: (I) supplying the plants' water needs of evapotranspiration and (II) keeping the available soil water content within a certain range [8]. However, in the field, irrigation decisions are generally empirical by observing signs of water stress in plants [9].

There are different technical recommendations for establishing an irrigation strategy to reduce total water consumption during the growing season without negatively affecting the fruit yield and quality.

Among the different technical recommendations, one highly recommended strategy is regulated deficit irrigation (RDI). RDI consists of irrigating only when the soil water level is minimal or until the physiological response of the plant is not affected, producing controlled levels of water stress [10]. Several field studies suggest their implementation to optimize water use without affecting the soil-plant dynamics [11–14]. However, due to the empirical nature of RDI, one needs to set-up the experimental plots before their application in the entire field, requiring time and costs.

The construction of mathematical models facilitates our understanding of agricultural phenomena and offers advantages for simulating different scenarios of dryness by considering the complexes involved in determining how dryness affects the plant's production. These models can be used to analyze the possible effects of implementing RDI strategies, thereby providing the theoretical behavior of plant-water dynamics in the soil. Mathematical modeling offers an excellent alternative to simulate the interaction between the plant and water in the soil. Mathematical models applied to agricultural systems incorporate mass balance equations, and cause-effect relationships between variables, among others [15]. Recently, some models for the growth of crops under water deficit have been presented [4, 16], considering advantages such as simulating multiple scenarios without affecting live systems and reducing costs associated with experimental field studies.

Indicators are numerical variables that provide simple and relevant information [17], which can be environmental, physical, physiological, or biological. In agricultural systems, indicators have been used for decision-making concerning particular phenomena or conditions subject to a specific value or range of values. However, the construction of these indicators is subject to multiple variables that contemplate the dynamics of the phenomenon studied.

For example, physiological indicators such as the stem water potential (Ψ), stomatal conductance (g_s), and the Crop Water Stress Index (CWSI), among others, have been used to monitor the water status of plants and to determine when to irrigate [18–22]. These indicators allow the producers to verify whether or not the plants are suffering from water stress [23, 24]. However, measuring these indicators in the field can be limited by their cost, time cunsumption, and representativeness, mainly because they are obtained from sentinel plants.

Based on those mentioned above, the construction of a non-physiological indicator to decide the moment of irrigation could be an important proposal for simulation, thereby reflecting the interaction among the soil water content and the plant water behavior. Indicators are widely used in mathematical modeling, providing a theoretical basis for decision-making, considering simulated situations. The next generation matrix method developed by [25] is a mathematical method used in dynamic and biological systems to calculate the local stability of a system of ordinary differential equations. This method has been extended and developed in epidemiology by [26] to determine a threshold parameter known as the basic reproduction number. This parameter is used to establish either the persistence or absence of a disease, considering compartmentalized mathematical models [26]. In biology, this kind of model has been applied in several ways, such as calculating the source-sink dynamics in marine meta populations [27] and the evolutionary invasion analysis [28]. Additionally, the next-generation method has been applied in agricultural systems to study the behavior of plant diseases [29–34].

This work proposes the development of a soil water indicator based on the next-generation matrix method to determine the minimum soil water amount required for optimal plant growth and fruit production. As far as the author's know, while the next-generation matrix method has been mainly used in biotic systems, its application to abiotic phenomena such as the water fluxes dynamic between soil and plants has not been explored. It is hypothesized that this method could provide an effective theoretical indicator for irrigation decisions.

This work presents the construction of the indicator based on a theoretical mathematical model that considers two irrigation scenarios: a constant rate of irrigation without water limitations and an impulsive irrigation strategy with cycles of irrigation and water cut.

2. Mathematical modeling

The construction of a mathematical model that describes the soil-crop dynamics implies identifying relevant variables and parameters, such as genotype, soil water availability, energy, soil type, crop management conditions, fertilization, depth roots, and environmental factors, among other variables [4, 35–38].

To approach the mathematical model describing soil moisture-crop dynamics carried out through the relationship between the amount of water in the soil and the growth of the fruit tree, we consider the following assumptions presented in [4, 39]:

i. Crop growth dynamics are influenced by the interaction between energy, water, and vegetative growth variables.
ii. The crops respond immediately to irrigation application.
iii. Adult crop and a suitable soil for crop growth.
iv. Optimal agronomic management conditions, including pest control and fertilizer management.
v. The system´s energy is constant.
vi. The system only considers the application of irrigation as a source of water; other possible sources, such as rainfall or groundwater, as not been considered in the model construction.

Table 1 provides a summary of the parameters that affect the growth dynamics of plants that produce fruits, according to our model.

We denote $W = W(t)$ as the water amount in the soil in time t and $C = C(t)$ as the concentration of biomass in the plant that produces fruits at time t, both state variables. The variation of the amount of water in the soil concerning to time denoted $W'(t)$ is determined by the relationship between environmental and physiological variables. That is, there is a loss due to the energy (q) - water (W) interaction (Evapotranspiration) at a β rate. In addition, an alternative amount of water is used for the growth of the plant that produces fruit at a rate of r. Additionally, the variation of the biomass concentration concerning to time, denoted $C'(t)$, is positively affected by the contribution of water to the growth of the plant that produces fruits at the rate of r. An entry is presented in the variation of the biomass concentration due to the photosynthetic contribution to the growth of the plant given by the relation energy (q) - water W and vegetative growth C at a rate γ and is considered an output given by the term ωC, which represents a loss of biomass concentration due to natural death at a rate of ω.

Table 1. Parameters used in the plants that produce fruits growth model.

Parameter	Meaning
q	Accumulated energy constant
r	Intrinsic growth rate of plants that produce fruits
N	Fruit tree carrying capacity
p	Rainfall contribution rate
I	Irrigation amount
β	Evapotranspiration rate
γ	Photosynthetic contribution rate
ω	Mortality rate of plants that produce fruits

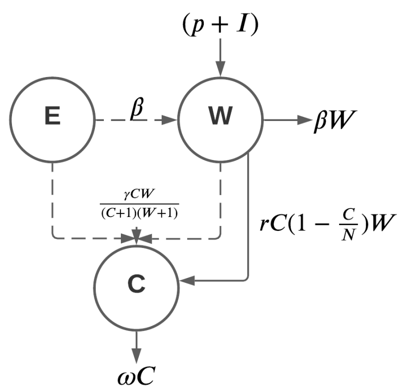


Figure 1. Dynamics flow. where E = energy; W = water amount in soil; C = crop biomass amount; r = intrinsic growth rate of crop; N = crop carrying capacity; β = evapotranspiration rate; γ = photosynthetic contribution rate; ω = mortality rate of crops; p = rainfall contribution rate and I = irrigation amount. Modified from [4].

The growth dynamics of the trees that produce fruits are represented in Figure 1, where E = energy; W = water amount in soil; C = crop biomass amount; r = intrinsic growth rate of crops; N = crop carrying capacity; β = evapotranspiration rate; γ = photosynthetic contribution rate; ω = mortality rate of crops; p = rainfall contribution rate; and I = irrigation amount.

The system of differential equations given by (2.1) presents the theoretical mathematical model that describes the growth of crops with continuous irrigation, as mentioned before:

$$\begin{cases} W'(t) &= (p + I(t)) - \beta q W(t) - rC(t)(1 - \frac{C(t)}{N})W(t) \\ C'(t) &= rC(t)(1 - \frac{C(t)}{N})W(t) + \frac{\gamma q C(t) W(t)}{(C+1)(W+1)} - \omega C(t) \end{cases} \tag{2.1}$$

where $I(t)$ is non-negative, smooth, and bounded function. The qualitative behavior of this model with $p = 0$ and $I(t) = 0$ has been studied in [4], and the results show the local and global stability of the system. The results show that a scenario of water deficit can compromise the growth and development rates of the crop; if this scenario continues, the persistence of the crop is compromised. Therefore, it is important to find an indicator that identifies the point at which the amount of water in the soil does not contribute to the growth of the crops.

Definition 2.1. *The soil water indicator w_d is a numerical value based on model parameters that indicate the minimum amount of water that the crop requires for their growth.*

The soil water indicator will be important to determine the water requirements in a crop and the levels at which the water resource should be managed. The indicator (w_d) is calculated for system (2.1) following the next generation matrix methodology established in [26].

Let $I(t)$ be a smooth, monotonous, and bounded function, in the absence of crop the system (2.1) admit the solution $(W^*, 0)$, where $W^* = \lim_{t\to\infty} e^{-\beta q t}\left[\int_0^t I(s)e^{\beta q s} ds\right]$.

Additionally,

$$\mathcal{F} = \begin{pmatrix} rC(1 - \frac{C}{N})W + \gamma q \frac{WC}{(C+1)(W+1)} \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} \omega C \\ \beta q W + rC(1 - \frac{C}{N})W - I(t) \end{pmatrix},$$

the matrices F and V are the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$, respectively, and have been defined in [26].

$$F(W^*, 0) = \begin{pmatrix} rW^* + \frac{\gamma q W^*}{W^*+1} & 0 \\ 0 & 0 \end{pmatrix}, \quad V(W^*, 0) = \begin{pmatrix} \omega & 0 \\ rW^* & \beta q \end{pmatrix},$$

$$V^{-1}(W^*, 0) = \frac{1}{\omega\beta q} \begin{pmatrix} \beta q & 0 \\ -rW^* & \omega \end{pmatrix}.$$

The next-generation matrix is described as

$$FV^{-1}(W^*, 0) = \begin{pmatrix} rW^* + \frac{\gamma q W^*}{W^*+1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\omega} & 0 \\ \frac{-rW^*}{\omega\beta q} & \frac{1}{\beta q} \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}(rW^* + \frac{pW^*}{W^*+1}) & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of matrix FV^{-1} are $\lambda_1 = \frac{1}{\omega}(rW^* + \frac{\gamma q W^*}{W^*+1}), \quad \lambda_2 = 0.$

$$\max\{\lambda_1, \lambda_2\} = \frac{1}{\omega}(rW^* + \frac{\gamma q W^*}{W^*+1}).$$

Therefore, the soil water indication (w_d) is:

$$w_d = \frac{1}{\omega}\left(rW^* + \frac{\gamma q W^*}{W^* + 1}\right). \tag{2.2}$$

In what follows, the water soil indicator will be obtained for two irrigation scenarios, and based on this indicator, a criterion will be defined to determine the minimum viable amount for crop growth based on the local stability of the system solutions.

The calculation of the soil water indicator is hypothetically illustrated in this example. It is not a common scenario in practice, as it is inefficient to consider unlimited water resources. However, despite being unrealistic, this theoretical case serves as an example to analyze the indicator and its relevant parameters. Let us assume that I(t) = I constant. The system (2.1) can be solved using this assumption.

2.1. Constant irrigation case

This example is a hypothetical illustration for the calculation of the soil water indicator. This case is not common in practice because it is not very efficient to consider unlimited water resources. However, despite being unrealistic, this theoretical case provides an example to analyze the indicator and its relevant parameters. Suppose $I(t) = I$ constant, the system (2.1) admits the solution $(W^*, 0)$, where $W^* = \frac{I}{m}$, with $m = \beta q$.
Additionally,

$$\mathcal{F} = \begin{pmatrix} rC(1-\frac{C}{N})W + \gamma q \frac{WC}{(C+1)(W+1)} \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} \omega C \\ mW + rC(1-\frac{C}{N})W - I \end{pmatrix},$$

the matrices F and V are determined as

$$F(W^*, 0) = \begin{pmatrix} rW^* + \frac{pW^*}{W^*+1} & 0 \\ 0 & 0 \end{pmatrix}, \quad V(W^*, 0) = \begin{pmatrix} \omega & 0 \\ rW^* & m \end{pmatrix},$$

$$V^{-1}(W^*, 0) = \frac{1}{\omega m} \begin{pmatrix} m & 0 \\ -rW^* & \omega \end{pmatrix},$$

$$FV^{-1}(W^*, 0) = \begin{pmatrix} rW^* + \frac{pW^*}{W^*+1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\omega} & 0 \\ \frac{-rW^*}{\omega m} & \frac{1}{m} \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}(rW^* + \frac{pW^*}{W^*+1}) & 0 \\ 0 & 0 \end{pmatrix}$$

with $p = \gamma q$, the eigenvalues of matrix FV^{-1} are $\lambda_1 = \frac{1}{\omega}(rW^* + \frac{pW^*}{W^*+1}), \quad \lambda_2 = 0.$

$$\max\{\lambda_1, \lambda_2\} = \frac{1}{\omega}(rW^* + \frac{pW^*}{W^*+1}).$$

Therefore, the soil water indication (w_d) is:

$$w_d = \frac{I}{\omega}\left(\frac{r}{m} + \frac{p}{I+m}\right). \tag{2.3}$$

Lemma 1. *If $w_d < 1$ the solution $(W^*, 0)$ of system (2.1) is asymptotically stable.*

Proof. The Jacobian matrix of the system (2.1) is

$$J = \begin{pmatrix} -rW + 2rW\frac{C}{N} & -m - rC(1-\frac{C}{N}) \\ rW - 2rW\frac{C}{N} + p\frac{W}{(C+1)^2(W+1)} - \omega & rC(1-\frac{C}{N}) + p\frac{C}{(W+1)^2(C+1)} \end{pmatrix}$$

The Jacobian matrix of (2.1) evaluated in the point $(W^*, 0)$ is

$$J(W^*, 0) = \begin{pmatrix} -rW^* & -m \\ rW^* + p\frac{W^*}{(W^*+1)} - \omega & 0 \end{pmatrix}$$

and characteristic polynomial given by:

$$P(\lambda) = \lambda^2 + rW^*\lambda + m(rW^* + p\frac{W^*}{(W^*+1)} - \omega)$$

The Routh-Hurwitz criterion [40] for a quadratic equation states that $P(\lambda)$ has roots with negative real part if $rW^* > 0$ and $m(rW^* + p\frac{W^*}{(W^*+1)} - \omega) > 0$, with $W^* = \frac{I}{m}$ and $w_d = \frac{I}{\omega}\left(\frac{r}{m} + \frac{p}{I+m}\right)$. This is $r\frac{I}{m} > 0$ and $mw_d > 0$ if and only if $w_d < 1$. Therefore $(W^*, 0)$ is asymptotically stable. □

In terms of model interpretation, the equilibrium stability implies that the crop tends to decrease until it inevitably disappears when the indicator of water in the soil is less than one. Thus, this result can be interpreted as a criterion that allows defining a threshold for the application of irrigation, and this situation should be avoided.

The studied scenario provides relevant information on the parameters required to obtain the indicator. In order to establish what are the most relevant parameters in the variation of the value of this indicator, a sensitivity analysis was carried out (see appendix for details). However, it is necessary to consider other irrigation scenarios that reflect the application of this methodology in the field.

2.2. Impulsive irrigation case

A more realistic scenario for describing the growth dynamics of crops is to carry out irrigation applications at specific points in time. This scenario has been studied from numerical simulations by [41], who proposes the impulsive model described in the equation (2.4)

$$\begin{cases} \left.\begin{array}{rcl} W'(t) &=& -\beta q W(t) - rC(t)(1-\frac{C(t)}{N})W(t) \\ C'(t) &=& rC(t)(1-\frac{C(t)}{N})W(t) + \frac{\gamma q C(t)W(t)}{(C(t)+1)(W(t)+1)} - \omega C(t) \end{array}\right\} & \text{if} \quad t \neq nT, \\ \\ \left.\begin{array}{rcl} W(t^+) &=& W(t) + I \\ C(t^+) &=& C(t) \end{array}\right\} \quad \text{if} \quad t = nT, & \end{cases} \tag{2.4}$$

where $W = W(t), C = C(t)$ are nonnegative. This model is defined in space

$$\Omega = \left\{(W, C) \in \mathbb{R}^2 \, : \, W \geq 0, \, 0 \leq C < N\right\},$$

and the parameters are all positive in the space $\rho = \left\{(q, r, N, I, \beta, \gamma, \omega) \in \mathbb{R}^7_+\right\}$.

It is important to consider the previously proposed system as a realistic model of the phenomenon to be studied to prove that its solutions are bounded (i.e. there is α depending only on the initial conditions such that for all $t > t_0$, $\|(W(t), C(t))\| \leq \alpha$).

Lemma 2. *The solutions of system (2.4) are bounded.*

Proof. We denote $t_n = nT$, $n \in \mathbb{Z}$. Clearly $0 < t_1 < t_2 < ... < t_n < ...$, $\inf\{t_n - t_{n-1} : n \geq 2\} \geq 0$, and $\lim_{n\to\infty} t_n = \infty$. We define the set $\sigma_n = \{(t, W, C) \in [0, \infty] \times \mathbb{R}^2 : t = t_n\}$. At most, the integral curves of the system (2.4) meets the hypersurface σ_n once. If I is constant, it is possible to consider a continuous function for each choice of n. From the construction of the model, it is clear that the functions that define the system (2.4) are continuously differentiable and locally Lipschitz. Moreover, they satisfy the theorem of existence and uniqueness in each time interval $[nT, (n + 1)T]$. If the solution $(W(t), C(t))$ is defined for any fixed initial condition in the interval $[t_0, t]$ with $t > 0$ constant, from the above conditions, the result follows from Lemma 1 in [42]. □

To determine the soil water indicator for (2.4), we follow the methodology outlined in [26,43]. System (2.4) admits a periodic solution $(W^*, 0)$, where

$$W^* = \frac{I}{1-e^{-mT}} e^{-m(t-nT)}; \ t \in (nT, (n+1)T). \tag{2.5}$$

Additionally,

$$\mathcal{F} = \begin{pmatrix} rC(1-\frac{C}{N})W + \gamma q \frac{WC}{(C+1)(W+1)} \\ 0 \end{pmatrix}, \ \mathcal{V} = \begin{pmatrix} \omega C \\ \beta q W + rC(1-\frac{C}{N})W \end{pmatrix},$$

$$\begin{pmatrix} C(nT^+) \\ W(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C(nT) \\ W(nT) \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

According to Theorem 2.1 in [43], the soil water indicator is the solution of $\rho(U(T,0,\chi)) = 1$, where $\rho(U(t,s,\chi))$ is the evolution operator of the system

$$\begin{cases} S'(t) = \left(-\omega + \dfrac{rW^* + \gamma q \frac{W^*}{(W^*+1)}}{\chi}\right) S \Big\} \quad \text{if} \quad t \neq nT, \\ S(nT^+) = S(nT) \Big\} \quad \text{if} \quad t = nT, \end{cases} \tag{2.6}$$

Then $\rho(U(T,0,\chi)) = \exp\left(\int_0^T \left(-\omega + \dfrac{rW^* + \frac{\gamma q W^*}{(W^*+1)}}{\chi}\right) dt\right)$. Solving the polynomial about χ,

$$exp\left(\int_0^T \left(-\omega + \frac{rW^* + \frac{\gamma q W^*}{(W^*+1)}}{\chi}\right) dt\right) = 1,$$

$$\int_0^T \left(-\omega + \frac{rW^* + \frac{\gamma q W^*}{(W^*+1)}}{\chi}\right) dt = 0,$$

$$\int_0^T -\omega dt + \int_0^T \left(\frac{rW^* + \frac{\gamma q W^*}{(W^*+1)}}{\chi}\right) dt = 0,$$

$$\int_0^T \left(rW^* + \frac{\gamma q W^*}{(W^*+1)}\right) dt = \omega T \chi,$$

$$\int_0^T rW^* dt + \int_0^T \frac{\gamma q W^*}{(W^*+1)} dt = \omega T \chi.$$

Solving each integral we have:

Step 1:

$$\int_0^T rW^* dt = \frac{rI}{m}.$$

Step 2:

$$\int_0^T \frac{\gamma q W^*}{(W^*+1)} dt = -\frac{\gamma q}{m}\left[\ln\left(\frac{Ke^{-mT}+1}{(K+1)}\right)\right], \text{ with } K = \frac{I}{1-e^{-mT}}.$$

Therefore, the soil water indicator for the model (2.4) is:

$$w_d = \frac{1}{\omega m T}\left[rI - \gamma q \ln\left(\frac{Ke^{-mT}+1}{K+1}\right)\right]. \tag{2.7}$$

Equation (2.7) shows the relationship between the parameters considered in the dynamics. Based on this relationship, it is posible to establish criteria for the efficiency of the water-soil-crop dynamics that guarantees crop growth.

Lemma 3. *If* $w_d < 1$ *the periodic solution* $(W^*, 0)$ *of system (2.4) is asymptotically stable.*

Proof. The linearization of system (2.4) evaluated in $(W^*, 0)$ is given by

$$\begin{cases} X'(t) = \begin{pmatrix} rW^* + p\frac{W^*}{(W^*+1)} - \omega & 0 \\ 0 & -m \end{pmatrix} X(t) & \text{if } t \neq nT, \\ X(nT^+) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X(nT) & \text{if } t = nT, \end{cases} \tag{2.8}$$

the monodromy matrix of the impulsive system (2.8) is

$$Q\Phi_M(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Phi_{F-V}(T) & 0 \\ 0 & \Phi_{-m}(T) \end{pmatrix} = \begin{pmatrix} \exp\left(\int_0^T \left(rW^* + p\frac{W^*}{(W^*+1)} - \omega\right) dt\right) & 0 \\ 0 & \exp\left(\int_0^T -m\, dt\right) \end{pmatrix}$$

Since $w_d = \exp\left(\int_0^T \left(rW^* + p\frac{W^*}{(W^*+1)} - \omega\right) dt\right) < 1$, we have $\rho(Q\Phi_M(T)) < 1$. Therefore the periodic solution $(W^*, 0)$ is asymptotically stable. □

3. Calibration and validation

3.1. Modelling of soil-moisture patterns

After creating a theoretical mathematical model, validation is necessary to assess its robustness in characterizing the examined phenomenon. This study did not have field data for the model's validation. Thus, this process was performed by reusing published data. In this case, these data were extracted from the works of Ooi et al. [44] and Filippucci et al. [45]. The process consisted of hand-selecting data from some of the original works using the WebPlotDigitizer https://automeris.io/WebPlotDigitizer/, accessed on December 15, 2021 and March 20, 2023, respectively). After their selection, they were extracted and exported to Comma Separated Values (CSV) files for post-processing.

Regarding the work of Ooi et al. [44], the authors presented the results of the automation of an apple tree orchard irrigation experiment using wireless sensor network technologies in Australia. In this case, for calibration and validation of the proposed models, data was extracted from Figure 2, which describes the measured soil moisture dynamics in the apple orchard, considering irrigation and water-cut cycles. The work of Filippucci et al. [45] presented the results of an irrigation experiment on irrigated tomatoes in Italy. In this case, the trend data for soil moisture was extracted from Figure 3. Please revise their original works for more details about Ooi et al. [44] and Filippucci et al. [45].

The soil moisture database obtained from Ooi et al. [44] consisted of 653 pairs of data. In the case of the data from Filippucci et al. [45], 258 pairs of data were record for the soil moisture content (percent). To ensure data independence for calibration and validation, these databases were split into two parts (9 and 642 pairs of data from Ooi et al. [44], and 9 and 249 pairs of data from Filippucci et al. [45].

The parameterization of the proposed model was carried out using a non-linear least-squares curve fitting method [46], using a specific script written in Matlab©R2019a (Mathworks Inc., Natick, MA, USA). Considering the soil-moisture dynamic, the calibration allowed us to solve the system (2.1) numerically, for both irrigation (water refill) and no-irrigation events (soil-water depletion between each irrigation), adjusting the output W(t) for the soil water content, considering I(t) = 0 (non-irrigation case).

The calibration allowed for the parameter's adjusting of Table 2 to minimize the difference between the simulated and measured data. The calibrated impulsive models were then validated using the separate database mentioned above. For this purpose, the actual data of soil moisture were compared against those simulated using the classical linear regression method suggested by Mayer and Butler [47]. In this case, the statistical deviance parameters used were the Root Mean Square Error (RMSE), Mean Absolute Error (MAE), and Mean Absolute Percent Error (MAPE). Additionally, the index of agreement (d) [48,49] and Lin's concordance index (ρ) [50] was added to this analysis (Eq (3.1)).

Table 2. Fitted parameters of the model.

Parameters	From database 1	From database 2
β	0.0358	0.0004
q	0.7541	0.0081
r	0.0002	0.0675
γ	0.0021	0.0032
ω	0.0569	0.0037

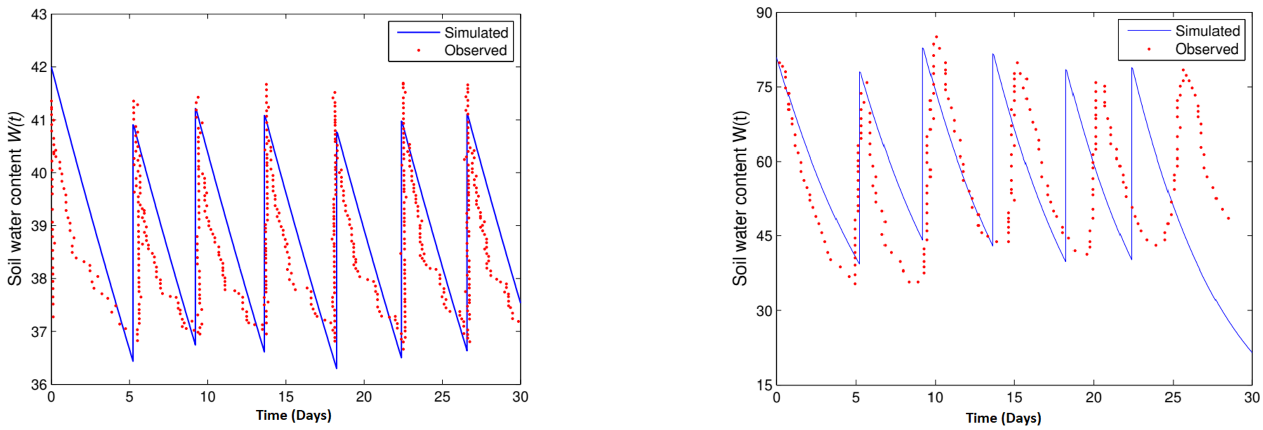
where data were extracted from[1] Ooi et al. [44] and [2] Filippucci et al. [45].

$$\begin{aligned} MAE &= \frac{\sum |obs-est|}{n} \\ RMSE &= \sqrt{\frac{\sum (obs-est)^2}{n}} \\ MAPE &= 100\frac{[\sum(|obs-est|/|obs|)]}{n} \\ d &= 1-\frac{\sum[(est-\overline{obs})-(obs-\overline{obs})]^2}{\sum(|est-\overline{obs}|+|obs-\overline{obs}|)^2} \\ \rho &= \frac{2s_{obsest}}{s_{obs}^2+s_{est}^2+(\overline{obs}-\overline{est})^2} \end{aligned} \tag{3.1}$$

where obs: observed data; est: estimated data; n: the number of pairs [47]; $\overline{obs}$: mean of observed; $\overline{est}$: mean of estimate data; s_{obs}^2: variance of observed data and s_{est}^2: variance of estimated data [48–50].

Figure 2 shows that the trends of the dynamics of filling and emptying of water in the soil were imitated by the proposed impulsive model (blue line), compared with the real values (points), for the examples obtained from the works from Ooi et al. [44] and from Filippucci et al. [45]. (Figures 2(a) and (b), respectively) In both cases, the observed data trends showed some regularity in the irrigations (irrigation frequency), fulfilling the main assumptions necessary to apply the proposed model.

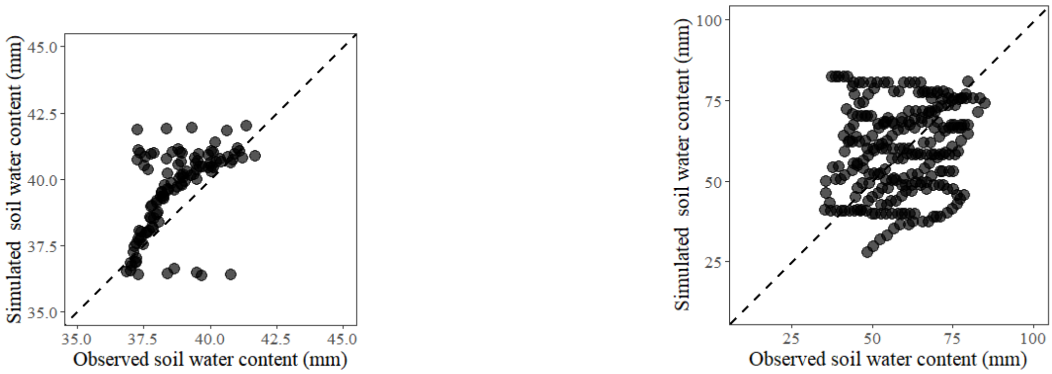
On the other hand, the trends of the impulsive model proposed (Eq (2.4)) contrasted against the observed data and had limitations on properly reflecting the soil-moisture dynamics. Figure 2(a) shows that the modeled values could not reach the highest soil moisture values, presenting a difference close to 1% at the maximum peaks. Regarding the lowest soil moisture values, the model imitated them well. In Figure 2(a), the trends of the soil water content observed showed values in the range of 36 to 42 %. In the case of Figure 2b, there was a lag between the observed and modeled soil moisture values, particularly from days 10 to 25. The range of observed values is between 42 and 84 mm, with a difference between the minimum and maximum achievable values of 12 mm. When comparing the observed values against the modeled values using the linear regression method, it was observed that in the first database, there was a trend of the cloud of points to stay above the 1:1 line. In this case, the statistician analysis (Table 3) indicated an MAE, RMSE, MAPE, d, and ρ of 1.19, 1.57, 0.03, 0.70, and 0.46 mm, respectively. Regarding the second database, it shows a uniform cloud of points around the 1:1 line. The statistical analysis indicated a poor performance compared to the first one (MAE = 13.28 mm, RMSE = 16.56 mm, MAPE = 0.24, d = 0.52, and ρ = 0.17). These contrasting results show that the performance of the proposed impulsive model will be highly dependent on the assumptions used to calibrate the proposed parameters (Table 2). According to [16, 51], these kinds of models are generalist, explaining the studied phenomenon. However, their performance could be poor compared to empirical models specifically calibrated from the data. As indicated herein, compared to Filippucci et al. [45], the database from Ooi et al. [44] presented regular patterns of irrigation events, despite the original data presented in Filippucci et al. [45], which was irregular. This second scenario did not agree with the main assumption of the proposed theoretical model, which was reflected in their poorest performance after the calibration.



(a) database 1

(b) database 2

Figure 2. Model fit from soil moisture data (data extracted from [1] Ooi et al. [44] and [2] Filippucci et al. [45]).



(a) database 1: Extracted from Ooi et al. [44] (b) database 2: Extracted from Filippucci et al. [45]

Figure 3. Model performance evaluation.

Table 3. Simulation performance.

	MAE	**RMSE**	**MAPE**	**d**	ρ
Database 1	1.19 mm	1.57 mm	0.03	0.70	0.46
Database 2	13.28 mm	16.56 mm	0.24	0.52	0.17

where: Mean Absolute Error (MAE); Root Mean Square Error (RMSE); Mean Absolute Percent Error (MAPE); Index of agreement (d); Lin's concordance index (ρ).

3.2. Indicator of water in the soil based on the next-generation matrix method

After carrying out the calibration and validation of the impulsive model, the values established in Table 2 were used to calculate the soil water indicator associated with the equation of the impulsive model (2.4). Initially, we calculate the indicator with the parameters set for database 1, this is:

$$\begin{aligned} w_d &= \frac{1}{\omega mT}\left[rI - \gamma q \ln\left(\frac{Ke^{-mT}+1}{K+1}\right)\right] \\ &= \frac{1}{0.003}(0.0036 - 0.0015(-0.0233)) \\ &= 1.21. \end{aligned}$$

In this case, the parameters involved in the indicator came from a study where adjustment with the observed data is acceptable. Therefore, the value of the indicator provides a reliable numerical value that implies that the amount of water applied in the period of time presented in Figure 2 (a) favors the water-in-soil-crop interaction.

The soil water indicator for the parameters associated with database 2 is given by:

$$\begin{aligned} w_d &= \frac{1}{\omega mT}\left[rI - \gamma q \ln\left(\frac{Ke^{-mT}+1}{K+1}\right)\right] \\ &= \frac{1}{16683}(0.00877 - 0.00002(-0.00001)) \\ &= 146. \end{aligned}$$

In the case of database 2, the indicator is $w_d > 1$, which implies that the amount of water supplied in the time described in Figure 2(b) favors the interaction water- cultivation in the soil. However, in this case, the parameters used to calculate this indicator are the result of poor validation, which implies that this numerical value is not reliable for this case. The effectiveness of the indicator will depend on the values of the parameters involved. Thus, the more robust the model, the more accurate the indicator will be.

4. Discussion and conclusions

To the best of the author's knowledge, various works in the literature highlight the importance of an adequate irrigation technique [8, 9, 44]. The aim is to establish criteria for the quantification and timing of irrigation. These criteria are based on physiological indicators such as Ψ, g_s, and CWSI, among others. In our case, we propose a non-physiological indicator of water in the soil that depends on the parameters established in the water-soil-crop dynamics. The next-generation matrix technique was used to calculate the soil water indicator in this study. This mathematical technique was applied to a non-linear model of differential equations that describe the dynamic between water in soil and crop.

This dynamic between the water content in soil and crops is presented in two different scenarios. The first scenario describes a relationship between two variables: the amount of water in the soil $W(t)$ and trees that produce fruits biomass amount $C(t)$ at time t; this relationship is modeled in Eq (2.1). In this model the only input of water is given by the application of constant irrigation. The calculation of the soil water indicator is presented in Eq (2.3), which shows that its value depends on the established relationship between the incorporated parameters. In addition, the indicator of water in the soil was considered for the analysis of the local stability of System (2.1). However, this scenario suggests a good example for understanding the calculation of the indicator, and may not reflect the current reality when considering a constant irrigation supply in times of water scarcity and climate crisis.

The second presented scenario has considered the same relationship between the variables of System (2.1), but it differs in the application of irrigation. In this case, the water supply in the soil is described through the impulsive model given by Eq (2.4), where the irrigation is incorporated at regular intervals. In addition, the soil water indicator Eq (2.7) has been calculated and was used to determine the local stability of the impulsive system (2.4).

Biological systems present complex relationships that can be difficult to represent due to the number of variables and parameters that must be considered. In that instance, the importance of mathematical modeling in agricultural systems lies in the contribution and understanding of the relationships between climatic variables, water conditions and crops. In this sense, the construction of theoretical models allows for a better understanding of a studied phenomenon. However, the validity of these models depends on the calibration of parameters adjusted to field conditions [8]. In the model that describes the soil water-crop dynamics presented in this work and used to calculate the soil water indicator, was validated with data provided by [44], showed that the amount of water in the soil presents a trend similar to the data obtained (Figure 2(a)) from the work of [44], the errors between both works are presented in Table 3.

In addition, as a second exercise, the proposed theoretical model has been compared with data obtained from the work of Filippucci et al. [45] applied to a tomato field. In Figure 2(b), it can be seen that the model behaves similarly to the observed data. However, there is a lag in the irrigation times, which can occur because the observed data tend to present regularity over time; however, this cannot be guaranteed as required by the model assumptions. The errors for this case are presented in Table 3. Furthermore, Lin's concordance index was calculated for both works, and the results obtained suggest a poor performance of the proposed model. This result is because the proposed theoretical mathematical model is an approximate and simplified representation of reality. The proposed model is simple. Its construction is based on restrictive hypotheses and with the minimum possible parameters to describe water-soil-crop dynamics, which do not necessarily reflect actual conditions. On the other hand, the observed data correspond to an empirical model, designed to directly represent those observations [16]. Therefore, a theoretical mathematical model does not necessarily perform well compared to an empirical one because they are based on theoretical assumptions and mathematical equations, not descriptions of observable data [16, 51–53]. In both cases, the indicator is $w_d > 1$, and implies that the amount of water supplied favors the interaction between crop water in the soil. The soil water indicator was calculated in both Ooi et al. [44] and Filippucci et al. [45]. The results obtained suggest that the amount of water supplied is adequate to guarantee water-soil-crop dynamics. However, the results presented depend on the validity of the parameters involved, rather than the method used to calculate them.

One of the main contributions of this model is the extension of the next-generation matrix method, widely used to establish thresholds for the control of pests in plants. In this case, this method has been used to establish the soil water indicator. The results of this work show that it is possible to describe the crop-water interaction in the soil through mathematical models in a simple way. In addition, a non-physiological indicator of water in the soil w_d that provides information on the commitment or permanence of water over time was established. However, it is important to continue investigating the interaction between water in the soil and crop from other scenarios. For example, one should incorporate the irrigation supply at intervals. These possible studies broaden the understanding of the phenomenon and provide a different study approach than the one commonly used in agricultural systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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Supplementary

Sensitivity analysis

In the sensitivity analysis developed, we have used the technique proposed in [54, 55]. The technique proposes a formula to establish the sensitivity index of all the parameters, this is defined by

$$\Lambda_y^{w_{d1}} = \frac{\partial w_{d1}}{\partial y} x \frac{y}{w_{d1}}$$

where y represent all parameters and $w_{d1} = \frac{I}{\omega}\left(\frac{r}{\beta q} + \frac{\gamma q}{I+\beta q}\right)$. Then

$$\begin{aligned}
\frac{\partial w_{d1}}{\partial \omega} x \frac{\omega}{w_{d1}} &= -1 < 0\\
\frac{\partial w_{d1}}{\partial r} x \frac{r}{w_{d1}} &= \frac{r(I+\beta q)}{r(I+\beta q)+\beta\gamma q^2} \geq 0\\
\frac{\partial w_{d1}}{\partial \gamma} x \frac{\gamma}{w_{d1}} &= \frac{\gamma\beta q^2}{r(I+\beta q)+\beta\gamma q^2} \geq 0\\
\frac{\partial w_{d1}}{\partial \beta} x \frac{\beta}{w_{d1}} &= -\frac{r(I+\beta q)^2+\gamma\beta^2 q^3}{(I+\beta q)[r(I+\beta q)+\gamma\beta q^2]} \leq 0\\
\frac{\partial w_{d1}}{\partial q} x \frac{q}{w_{d1}} &= \frac{-r(I+\beta q)^2+\gamma\beta q^2 I}{(I+\beta q)[r(I+\beta q)+\gamma\beta q^2]}\\
\frac{\partial w_{d1}}{\partial I} x \frac{I}{w_{d1}} &= \frac{r(I+\beta q)^2+\gamma q(\beta q)^2}{(I+\beta q)[r(I+\beta q)+\gamma\beta q^2]} \geq 0
\end{aligned}$$

In our case, this sensitivity analysis provides a fundamental tool to know the level of influence of each parameter on the soil water indicator. The analysis shows that by increasing the parameters r, γ, I, and considering the other parameters constant, the soil water indicator increases. While the increase of the ω, β parameters decrease the value of the soil water indicator, which can generate compromises in the growth rates of the fruit tree. The analysis for the parameter q indicates that if $I > \frac{r\beta}{\gamma}$, then the contribution is negative for $\frac{2r\beta I - \sqrt{4r\gamma\beta I^3}}{2(\gamma\beta I - r\beta^2)} < q < \frac{2r\beta I + \sqrt{4r\gamma\beta I^3}}{2(\gamma\beta I - r\beta^2)}$.