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Research article

Fundamental boundary matrices for 36 elementary boundary value problems of finite beam deflection on elastic foundation

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Abstract: We consider the boundary value problem of finite beam deflection on elastic foundation with two point boundary conditions of the form $u^{(p)}(-l) = u^{(q)}(-l) = u^{(r)}(l) = u^{(s)}(l) = 0$, $p < q$, $r < s$, which we call elementary. We explicitly compute the fundamental boundary matrices corresponding to 7 special elementary boundary conditions called the dwarfs, and show that the fundamental boundary matrices for the whole 36 elementary boundary conditions can be derived from those for the seven dwarfs.

Keywords: beam; boundary matrix; characteristic equation; deflection; dwarf; elastic foundation; fundamental boundary matrix; integral operator; spectrum

1. Introduction

The objective of this paper is to compute explicitly the *fundamental boundary matrices* [1] corresponding to the boundary value problems consisting of the classical Euler–Bernoulli equation [2, 3]

$$EI \cdot u^{(4)}(x) + k \cdot u(x) = w(x), \qquad x \in [-l, l] \tag{1.1}$$

and two point boundary conditions of the form

$$u^{(p)}(-l) = u^{(q)}(-l) = u^{(r)}(l) = u^{(s)}(l) = 0, \qquad p, q, r, s \in \{0, 1, 2, 3\}, \quad p < q, \quad r < s \tag{1.2}$$

which we call *elementary*. (1.1), which is a most fundamental differential equation in the history of mechanical engineering [2–14], is the governing equation for small downward vertical deflection $u(x)$ at $x \in [-l, l]$ of a linear shaped beam of length $2l$ which lies horizontally on elastic foundation with spring constant density k. $w(x)$ denotes the downward loading density on the beam at x, and E, I are the Young's modulus and the mass moment of inertia of the beam respectively. For the rest of this paper, E, I and k are fixed positive constants, and we denote $\alpha = \sqrt[4]{k/(EI)}$ so that αl is a dimensionless constant.

Historically, most of the results on the boundary value problem with (1.1) have considered only a few boundary conditions, especially among those in (1.2). Our approach on boundary conditions is in a more generalized manner. Let $\mathrm{gl}(m, n, \mathbb{C})$ be the set of $m \times n$ matrices with complex entries, which is also denoted by $\mathrm{gl}(n, \mathbb{C})$ when $m = n$. In general, a two-point boundary condition for (1.1) can be represented with $\mathbf{M} \in \mathrm{gl}(4, 8, \mathbb{C})$, called a *boundary matrix*, by

$$\mathbf{M} \cdot \left(u(-l), u'(-l), u''(-l), u'''(-l), u(l), u'(l), u''(l), u'''(l)\right)^T = \mathbf{b}, \tag{1.3}$$

where $\mathbf{b} \in \mathrm{gl}(4, 1, \mathbb{C})$. It is enough to consider the case when $\mathbf{b} = \mathbf{0} = (0, 0, 0, 0)^T$, called the *homogeneous* boundary value, since *nonhomogeneous* cases can be easily settled by solving linear algebraic equations afterwards [15]. The *elementary boundary condition* (1.2) can be represented with the boundary matrix $\mathbf{E}_{pqrs} \in \mathrm{gl}(4, 8, \mathbb{C})$, called *elementary boundary matrix*, which is defined by

$$(i, j)\text{th element of } \mathbf{E}_{pqrs} = \begin{cases} 1, & \text{if } i = 1, j = p + 1, \\ 1, & \text{if } i = 2, j = q + 1, \\ 1. & \text{if } i = 3, j = r + 5, \\ 1, & \text{if } i = 4, j = s + 5, \\ 0, & \text{otherwise,} \end{cases} \qquad 1 \le i \le 4, \quad 1 \le j \le 8. \tag{1.4}$$

For example, the two-point boundary condition $u(-l) = u'(-l) = u''(l) = u'''(l) = 0$, which corresponds to the clamped end at $x = -l$ and the free end at $x = l$, and is thus called the *clamped-free* or, *cantilevered* condition, can be represented with $\mathbf{M} = \mathbf{E}_{0123}$ and $\mathbf{b} = \mathbf{0}$, where

$$\mathbf{E}_{0123} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

A boundary matrix $\mathbf{M} \in \mathrm{gl}(4, 8, \mathbb{C})$ is called *well-posed*, if it makes the boundary value problem consisting of (1.1) and (1.3) well-posed for any $\mathbf{b} \in \mathrm{gl}(4, 1, \mathbb{C})$. The set of well-posed boundary matrices is denoted by $\mathrm{wp}(4, 8, \mathbb{C})$. For $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$, the integral operator $\mathcal{K}_{\mathbf{M}} : L^2[-l, l] \to L^2[-l, l]$ on the Hilbert space of square-integrable complex-valued functions on the interval $[-l, l]$, is defined by

$$\mathcal{K}_{\mathbf{M}}[u] = \int_{-l}^{l} G_{\mathbf{M}}(x, \xi) u(\xi)\, d\xi, \qquad u \in L^2[-l, l], \tag{1.5}$$

where $G_{\mathbf{M}}(x, \xi)$ is the Green function [16, 17] corresponding to the boundary value problem (1.1) and (1.3) with $\mathbf{b} = \mathbf{0}$, so that $\mathcal{K}_{\mathbf{M}}[w]$ is the unique solution of that boundary value problem. The spectral analysis, i.e. the analysis on the eigenvalues, of the integral operator $\mathcal{K}_{\mathbf{M}}$ is important for analyzing the beam deflection problem in general, including the corresponding nonlinear problems [15, 18] and inverse problems [19].

For a well-posed boundary matrix $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$, the *spectrum* or, the set of eigenvalues, of the integral operator $\mathcal{K}_{\mathbf{M}}$ is denoted by $\mathrm{Spec}\, \mathcal{K}_{\mathbf{M}}$. Choi [20] considered the following specific well-posed boundary matrix $\mathbf{Q}$:

$$\mathbf{Q} = \left(\begin{array}{cccc|cccc} 0 & \alpha^2 & -\sqrt{2}\alpha & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\alpha^3 & -\alpha^2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^2 & \sqrt{2}\alpha & 1 \\ 0 & 0 & 0 & 0 & -\sqrt{2}\alpha^3 & -\alpha^2 & 0 & 1 \end{array}\right).$$

The corresponding Green function $G_{\mathbf{Q}}(x,\xi)$ is the one for the *infinite* version $EI\cdot u^{(4)}(x)+k\cdot u(x)=w(x)$, $x\in(-\infty,\infty)$ of (1.1) and the boundary condition $\lim_{x\to\pm\infty}u(x)=0$ [18]. By explicitly constructing and analyzing the corresponding characteristic equation, the following exceptionally detailed spectral information on the integral operator $\mathcal{K}_{\mathbf{Q}}$ was obtained.

Proposition 1.1 ([20]). *$\operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$ is of the form $\{\mu_n/k\,|\,n=1,2,3,\cdots\}\cup\{\nu_n/k\,|\,n=1,2,3,\cdots\}\subset(0,1/k)$, where $1>\mu_1>\nu_1>\mu_2>\nu_2>\cdots\searrow 0$, $\mu_n\sim\nu_n\sim n^{-4}$ and*

$$\frac{1}{1+\frac{1}{(2l\alpha)^4}\left(2\pi(n-1)-\frac{\pi}{2}\right)^4}-\mu_n\sim\frac{1}{1+\frac{1}{(2l\alpha)^4}\left(2\pi(n-1)+\frac{\pi}{2}\right)^4}-\nu_n\sim n^{-6}.$$

Here, for positive sequences a_n, b_n, we denote $a_n\sim b_n$ if there exists $N>0$ such that $m\le a_n/b_n\le M$ for every $n>N$ for some constants $0<m\le M<\infty$. See [20] for more details including numerical approximation of μ_n, ν_n with arbitrary precision.

Note that $\mathbf{M},\mathbf{N}\in\mathrm{gl}(4,8,\mathbb{C})$ represent the same boundary condition in the sense of (1.3) with $\mathbf{b}=\mathbf{0}$ if and only if there exists an invertible $\mathbf{A}\in\mathrm{gl}(4,\mathbb{C})$ such that $\mathbf{M}=\mathbf{A}\mathbf{N}$, in which case we denote $\mathbf{M}\approx\mathbf{N}$. Recently, Choi [1] explicitly constructed a transformation $\mathcal{F}:\mathrm{wp}(4,8,\mathbb{C})\to\mathrm{gl}(4,\mathbb{C})$, which satisfies Proposition 1.2 below. For $\mathbf{M}\in\mathrm{wp}(4,8,\mathbb{C})$, the 4×4 matrix $\mathcal{F}(\mathbf{M})$ is called the *fundamental boundary matrix* for $\mathbf{M}$.

Proposition 1.2 ([1], Theorem 1). *$\mathcal{F}$ is onto, and $\mathcal{F}(\mathbf{M})=\mathcal{F}(\mathbf{N})$ if and only if $\mathbf{M}\approx\mathbf{N}$. For $\mathbf{M}\in\mathrm{wp}(4,8,\mathbb{C})$ and $\lambda\in\mathbb{C}\setminus\{0\}$, $\lambda\in\operatorname{Spec}\mathcal{K}_{\mathbf{M}}$, i.e., $\mathcal{K}_{\mathbf{M}}[u]=\lambda\cdot u$ for some $0\ne u\in L^2[-l,l]$ if and only if*

$$\det\left\{\mathcal{F}(\mathbf{M})\begin{pmatrix}\mathbf{X}_\lambda^+(-l)-\mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(-l)-\mathbf{X}_\lambda^-(l)\end{pmatrix}-\begin{pmatrix}\mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(l)\end{pmatrix}\right\}=0.$$

Here, $\mathbf{X}_\lambda^\pm(x)$ are 2×2 matrices whose entries are explicitly defined holomorphic functions of $x,\lambda\in\mathbb{C}$ [1]. Let $\mathrm{wp}(4,8,\mathbb{C})/\!\approx$ be the quotient set of $\mathrm{wp}(4,8,\mathbb{C})$ by the equivalence relation $\approx$. Proposition 1.2 implies that the transformation $\mathcal{F}$ induces a *one-to-one correspondence* between $\mathrm{wp}(4,8,\mathbb{C})/\!\approx$ and the 16-dimensional algebra $\mathrm{gl}(4,\mathbb{C})$ of 4×4 matrices. Thus, the set of different well-posed boundary conditions (1.3), which is in one-to-one correspondence with the set of integral operators (1.5), forms a 16-dimensional space.

Proposition 1.2 also means that the spectrum of the integral operator $\mathcal{K}_{\mathbf{M}}$ in (1.5) corresponding to a well-posed boundary matrix $\mathbf{M}$ can be represented essentially as the zero set of one concrete holomorphic function. For example, $\mathcal{F}$ is designed so that $\mathcal{F}(\mathbf{Q})=\mathbf{O}$, hence $\operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$ is the zero set of the holomorphic function $\det\mathbf{X}_\lambda^+(l)\cdot\det\mathbf{X}_\lambda^-(l)$ in λ, from which the result in Proposition 1.1 can be recovered [1]. Note also that the information on a particular boundary condition $\mathbf{M}$ is separately encoded by the transformation $\mathcal{F}$ and can be plugged into the characteristic equation in Proposition 1.2, while the rest of the information such as $\mathbf{X}_\lambda^\pm(x)$ are already prepared and common to every well-posed boundary condition. Thus, fundamental boundary matrix is a *complete invariant* in that it contains *all* the information on *each* well-posed boundary condition in (1.3).

In contrast to the importance of fundamental boundary matrices, there have been no examples of explicit computation of them for any boundary matrix $\mathbf{M}\in\mathrm{wp}(4,8,\mathbb{C})$, except for some trivial cases such as $\mathcal{F}(\mathbf{Q})=\mathbf{O}$. This is due in part to the technical, although explicit, definition of fundamental boundary matrix. In this paper, we explicitly compute the fundamental boundary matrices for the following 7 elementary boundary matrices, called the *dwarfs*:

$$\text{01 types: } \mathbf{E}_{0101}, \mathbf{E}_{0112}, \mathbf{E}_{0123}, \qquad \text{02 types: } \mathbf{E}_{0202}, \mathbf{E}_{0213}, \qquad \text{mixed types: } \mathbf{E}_{0102}, \mathbf{E}_{0113}. \tag{1.6}$$

By exploiting various symmetries in the transformation $\mathcal{F}$, we show that computation of the fundamental boundary matrices for the above seven dwarfs is enough to explicitly generate those for the rest of all 36 elementary boundary matrices. The well-posedness of each elementary boundary matrix will be shown in the process.

Our results present the first nontrivial examples of explicit fundamental boundary matrices. Together with Proposition 1.2, it is expected to produce detailed spectral information such as those in Proposition 1.1, on the integral operators (1.5) corresponding to the elementary boundary conditions (1.2), which include important cases usually considered in practical applications.

The rest of the paper is organized as follows. Basic notations and definitions which will be used extensively in this paper are given in Section 2. In Section 3, the definition of fundamental boundary matrix is given, and our main results, Theorems 1 and 2, are stated with examples illustrating them. In Section 4, detailed analyses are done on the reduction of the 36 elementary boundary matrices to the seven dwarfs. Section 5 presents preliminary results needed for explicit computation of the fundamental boundary matrices for each of the seven dwarfs, which are actually done in Sections 6, 7 and 8. The proofs of Theorems 1 and 2 are given in Section 9. Finally, Section 10 discusses some issues related to our results and future directions.

2. Basic notations and definitions

The notations and the definitions in this section, some of which might seem nonstandard, are devised to exploit symmetries in our problem efficiently and to save space in our calculations. Readers are recommended to be sufficiently acquainted with them in advance.

We denote $\mathbb{i} = \sqrt{-1}$. Denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, the set of natural number, the set of integers, the set of real numbers, and the set of complex numbers respectively. The set of $m \times n$ matrices with entries in $\mathbb{C}$ is denoted by $\mathrm{gl}(m, n, \mathbb{C})$. When $m = n$, we also denote $\mathrm{gl}(m, n, \mathbb{C}) = \mathrm{gl}(n, \mathbb{C})$. The complex conjugate and the transpose of $\mathbf{A} \in \mathrm{gl}(m, n, \mathbb{C})$ are denoted by $\overline{\mathbf{A}}$ and $\mathbf{A}^T$ respectively. For $\mathbf{A} \in \mathrm{gl}(n, \mathbb{C})$, $\operatorname{adj} \mathbf{A}$ is the classical adjoint of $\mathbf{A}$, so that, if $\mathbf{A}$ is invertible then $\mathbf{A}^{-1} = \operatorname{adj} \mathbf{A} / \det \mathbf{A}$. The identity matrix and the zero matrix are denoted by $\mathbf{I}_n \in \mathrm{gl}(n, \mathbb{C})$ and $\mathbf{O}_{mn} \in \mathrm{gl}(m, n, \mathbb{C})$ respectively. The zero column vector is denoted by $\mathbf{0}_n \in \mathrm{gl}(n, 1, \mathbb{C})$. The subscripts in $\mathbf{I}_n$, $\mathbf{O}_{mn}$, and $\mathbf{0}_n$ can be omitted in case of no ambiguity. For $c_1, c_2, \cdots, c_n \in \mathbb{C}$, $\operatorname{diag}(c_1, c_2, \cdots, c_n)$ denotes the diagonal matrix with diagonal entries $c_1, c_2, \cdots, c_n$, and $e^{\operatorname{diag}(c_1, c_2, \cdots, c_n)} = \operatorname{diag}(e^{c_1}, e^{c_2}, e^{c_3}, e^{c_4})$.

The set of invertible matrices in $\mathrm{gl}(n, \mathbb{C})$ is denoted by $GL(n, \mathbb{C})$. For $\mathbf{A} \in GL(n, \mathbb{C})$, define the *similarity transform* by $\mathbf{A}$, $\mathrm{Sim}_{\mathbf{A}} : \mathrm{gl}(n, \mathbb{C}) \to \mathrm{gl}(n, \mathbb{C})$ by $\mathrm{Sim}_{\mathbf{A}} \mathbf{B} = \mathbf{A}^{-1} \mathbf{B} \mathbf{A}$ for $\mathbf{B} \in \mathrm{gl}(n, \mathbb{C})$, which is an isomorphism on the algebra $\mathrm{gl}(n, \mathbb{C})$. We will freely use the usual properties of Sim subsequently.

Let $\mathbf{A} \in \mathrm{gl}(m, n, \mathbb{C})$. We denote the (i, j)th entry of $\mathbf{A}$ by $\mathbf{A}_{i,j}$, $1 \le i \le m$, $1 \le j \le n$. When the (i, j)th entry of $\mathbf{A}$ is $a_{i,j}$, we write $\mathbf{A} = \left(a_{i,j}\right)_{1 \le i \le m,\, 1 \le j \le n}$. If $m = n$, then we also write $\mathbf{A} = \left(a_{i,j}\right)_{1 \le i, j \le n}$. For $1 \le p < q \le m$, $1 \le r < s \le n$, we denote by $\mathbf{A}_{p:q, r:s}$ the minor $\left(\mathbf{A}_{i,j}\right)_{p \le i \le q, r \le j \le s}$ of $\mathbf{A}$.

For $\mathbf{A}, \mathbf{B} \in \mathrm{gl}(n, \mathbb{C})$, denote $\mathbf{A} \oplus \mathbf{B} = \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \end{pmatrix} \in \mathrm{gl}(2n, \mathbb{C})$ and $\mathbf{A} \oplus' \mathbf{B} = \begin{pmatrix} \mathbf{O} & \mathbf{B} \\ \mathbf{A} & \mathbf{O} \end{pmatrix} \in \mathrm{gl}(2n, \mathbb{C})$.

The following special functions provide us compact language for our problem. See Appendix A, where we collected their properties that we need.

Definition 2.1. For $\zeta \in \mathbb{C}$, denote

$$\begin{aligned} \mathrm{c}_+(\zeta) &= \cosh\left(\sqrt{2}\zeta\right) + \cos\left(\sqrt{2}\zeta\right), & \mathrm{c}_-(\zeta) &= \cosh\left(\sqrt{2}\zeta\right) - \cos\left(\sqrt{2}\zeta\right), \\ \mathrm{s}_+(\zeta) &= \sinh\left(\sqrt{2}\zeta\right) + \sin\left(\sqrt{2}\zeta\right), & \mathrm{s}_-(\zeta) &= \sinh\left(\sqrt{2}\zeta\right) - \sin\left(\sqrt{2}\zeta\right), \\ \mathrm{ci}_+(\zeta) &= \cosh\left(\sqrt{2}\zeta\right) + \mathbb{i}\cos\left(\sqrt{2}\zeta\right), & \mathrm{ci}_-(\zeta) &= \cosh\left(\sqrt{2}\zeta\right) - \mathbb{i}\cos\left(\sqrt{2}\zeta\right), \\ \mathrm{si}_+(\zeta) &= \sinh\left(\sqrt{2}\zeta\right) + \mathbb{i}\sin\left(\sqrt{2}\zeta\right), & \mathrm{si}_-(\zeta) &= \sinh\left(\sqrt{2}\zeta\right) - \mathbb{i}\sin\left(\sqrt{2}\zeta\right). \end{aligned}$$

The following elementary fact is immediate from Definition 2.1.

Lemma 2.1. *$f(\zeta) \neq 0$ for $f \in \{\mathrm{c}_+, \mathrm{c}_-, \mathrm{s}_+, \mathrm{s}_-, \mathrm{ci}_+, \mathrm{ci}_-, \mathrm{si}_+, \mathrm{si}_-\}$ and $0 \neq \zeta \in \mathbb{R} \cup \{\mathbb{i}r \mid r \in \mathbb{R}\}$.*

Definition 2.2. Denote $\omega = e^{\mathbb{i}\frac{\pi}{4}} = (1+\mathbb{i})/\sqrt{2}$ and $\omega_n = \mathbb{i}^{n-1}\omega$, $n \in \mathbb{Z}$. Denote $\mathbf{\Omega} = \operatorname{diag}(\omega_1, \omega_2, \omega_3, \omega_4)$, $\mathbf{W}_0 = \left(\omega_j^{i-1}\right)_{1 \le i,j \le 4}$. For $x \in \mathbb{C}$, denote $\mathbf{W}(x) = \operatorname{diag}\left(1, \alpha, \alpha^2, \alpha^3\right) \cdot \mathbf{W}_0 e^{\mathbf{\Omega}\alpha x}$.

By Definition 2.2,

$$\overline{\omega} = \omega_4 = -\omega_2 = -\mathbb{i}\omega,\ \omega_3 = -\omega,\ \omega + \overline{\omega} = \sqrt{2},\ \omega - \overline{\omega} = \mathbb{i}\sqrt{2},\ \omega^2 = \mathbb{i},\ \omega\overline{\omega} = 1. \tag{2.1}$$

Definition 2.3. Denote $\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \operatorname{diag}(1, -1)$, $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{i} \end{pmatrix} = \operatorname{diag}(1, \mathbb{i})$, $\mathcal{E} = \operatorname{diag}(1, -1, -1, 1) = \mathbf{J} \oplus (-\mathbf{J})$.

By Definition 2.3,

$$\mathbf{J}^{-1} = \mathbf{J} = \mathbf{J}^T, \qquad \mathbf{K}^2 = \mathbf{J}, \qquad \mathbf{K}^{-1} = \overline{\mathbf{K}} = \mathbf{JK} = \mathbf{KJ} = \mathbf{K}^3, \qquad \mathcal{E}^{-1} = \mathcal{E} = \mathcal{E}^T. \tag{2.2}$$

Definition 2.4. Denote $\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. For $n \in \mathbb{N}$, denote $\mathbf{R}_n = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & ⋰ & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \in \mathrm{gl}(n, \mathbb{R})$.

Multiplying $\mathbf{L}$ to the left (respectively, to the right) of a matrix amounts to *lifting* up the rows (respectively, pushing to the right the columns) of that matrix cyclically by one row (respectively, by one column). In case of no ambiguity, the subscript n in $\mathbf{R}_n$ can be omitted. Multiplying $\mathbf{R}$ to the left (respectively, to the right) of a matrix amounts to *reversing* the order of the rows (respectively, the columns) of that matrix. The following are immediate from Definition 2.4.

$$\left(\mathbf{L}^2\right)^{-1} = \mathbf{L}^2 = \left(\mathbf{L}^2\right)^T, \qquad \mathbf{R}^{-1} = \mathbf{R} = \mathbf{R}^T. \tag{2.3}$$

The transform $\mathbf{A} \mapsto \mathbf{R}_m \mathbf{A} \mathbf{R}_n$ for $\mathbf{A} \in \mathrm{gl}(m, n, \mathbb{C})$ amounts to rotating the elements of $\mathbf{A}$ by 180°. For example, $\mathrm{Sim}_{\mathbf{R}}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$. By Definitions 2.3 and 2.4,

$$\mathbf{RJ} = -\mathbf{JR}, \qquad \mathbf{RK} = \begin{pmatrix} 0 & \mathbb{i} \\ 1 & 0 \end{pmatrix} = \mathbb{i}\begin{pmatrix} 0 & 1 \\ -\mathbb{i} & 0 \end{pmatrix} = \mathbb{i}\overline{\mathbf{K}}\mathbf{R}, \qquad \mathbf{R}\overline{\mathbf{K}} = -\mathbb{i}\mathbf{KR}. \tag{2.4}$$

Definition 2.5. Denote $\mathbf{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_2 & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{I}_2 \end{pmatrix}$, and

$$\mathbf{V}_{12} = \begin{pmatrix} \mathbf{R}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{V}_{34} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{V}_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multiplying the above $\mathbf{V}_{pq}$ to the left (respectively, to the right) of a matrix amounts to exchanging the pth and the qth rows (respectively, columns) of that matrix. Note from Definition 2.5 that $\mathbf{V}_{12}\mathbf{V}_{34} = \mathbf{V}_{34}\mathbf{V}_{12}$, and

$$\mathbf{V}^{-1} = \mathbf{V}^T, \qquad \mathbf{V}_{12}^{-1} = \mathbf{V}_{12} = \mathbf{V}_{12}^T, \qquad \mathbf{V}_{34}^{-1} = \mathbf{V}_{34} = \mathbf{V}_{34}^T, \qquad \mathbf{V}_{23}^{-1} = \mathbf{V}_{23} = \mathbf{V}_{23}^T. \tag{2.5}$$

Lemma 2.2. $\mathrm{Sim}_{\mathbf{V}^T}(\mathbf{A} \oplus \mathbf{A}) = \mathbf{A} \oplus \mathbf{A}$ *and* $\mathrm{Sim}_{\mathbf{V}^T}\{(-\mathbf{A}) \oplus \mathbf{A}\} = \mathbf{A} \oplus' \mathbf{A}$ *for* $\mathbf{A} \in \mathrm{gl}(2, \mathbb{C})$.

Proof. By Definition 2.5 and (2.5),

$$\mathrm{Sim}_{\mathbf{V}^T}(\mathbf{A} \oplus \mathbf{A}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{A} & \mathbf{A} \\ -\mathbf{A} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} = \mathbf{A} \oplus \mathbf{A},$$

$$\mathrm{Sim}_{\mathbf{V}^T}\{(-\mathbf{A}) \oplus \mathbf{A}\} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} -\mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\mathbf{A} & \mathbf{A} \\ \mathbf{A} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} = \mathbf{A} \oplus' \mathbf{A}. \qquad \square$$

By Definition 2.3 and Lemma 2.2,

$$\mathrm{Sim}_{\mathbf{V}^T} \mathcal{E} = \mathrm{Sim}_{\mathbf{V}^T}\{\mathbf{J} \oplus (-\mathbf{J})\} = -(\mathbf{J} \oplus' \mathbf{J}), \tag{2.6}$$

hence we have $\mathrm{Sim}_{\mathbf{V}^T} \mathrm{diag}(1,0,0,1) + \mathrm{Sim}_{\mathbf{V}^T} \mathrm{diag}(0,1,1,0) = \mathrm{Sim}_{\mathbf{V}^T} \mathbf{I} = \mathbf{I}$ and $\mathrm{Sim}_{\mathbf{V}^T} \mathrm{diag}(1,0,0,1) - \mathrm{Sim}_{\mathbf{V}^T} \mathrm{diag}(0,1,1,0) = \mathrm{Sim}_{\mathbf{V}^T} \mathcal{E} = -(\mathbf{J} \oplus' \mathbf{J})$. Thus

$$\mathrm{Sim}_{\mathbf{V}^T} \mathrm{diag}(1,0,0,1) = \frac{1}{2}\{\mathbf{I} - (\mathbf{J} \oplus' \mathbf{J})\}, \qquad \mathrm{Sim}_{\mathbf{V}^T} \mathrm{diag}(0,1,1,0) = \frac{1}{2}\{\mathbf{I} + (\mathbf{J} \oplus' \mathbf{J})\}. \tag{2.7}$$

3. Main results

Proposition 3.1 ([16], Lemma 3.1)**.** *Let* $\mathbf{M} = (\mathbf{M}^- \,|\, \mathbf{M}^+) \in \mathrm{gl}(4, 8, \mathbb{C})$ *for* $\mathbf{M}^-, \mathbf{M}^+ \in \mathrm{gl}(4, \mathbb{C})$. $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$ *if and only if* $\det\{\mathbf{M}^-\mathbf{W}(-l) + \mathbf{M}^+\mathbf{W}(l)\} \neq 0$.

Definition 3.1. Let $\mathbf{M} = (\mathbf{M}^- \,|\, \mathbf{M}^+) \in \mathrm{wp}(4, 8, \mathbb{C})$ for $\mathbf{M}^-, \mathbf{M}^+ \in \mathrm{gl}(4, \mathbb{C})$. Denote

$$\mathcal{G}(\mathbf{M}) = -\{\mathbf{M}^-\mathbf{W}(-l) + \mathbf{M}^+\mathbf{W}(l)\}^{-1} \mathbf{M}^-\mathbf{W}(-l)\mathcal{E} - \mathrm{diag}(0,1,1,0), \quad \mathcal{F}(\mathbf{M}) = \mathrm{Sim}_{\mathbf{V}^T} \mathcal{G}(\mathbf{M}).$$

Definition 3.1, which appeared in [1, 16], defines the transformation $\mathcal{F}$ in Proposition 1.2. We extend this definition in order to express the effect of the beam length l explicitly.

Definition 3.2. Let $z \in \mathbb{C}$, and let $\mathbf{M} = (\mathbf{M}^- \,|\, \mathbf{M}^+) \in \mathrm{gl}(4, 8, \mathbb{C})$ for $\mathbf{M}^-, \mathbf{M}^+ \in \mathrm{gl}(4, \mathbb{C})$. When $\det\{\mathbf{M}^-\mathbf{W}(-z/\alpha) + \mathbf{M}^+\mathbf{W}(z/\alpha)\} \neq 0$, denote

$$\mathcal{G}(\mathbf{M}, z) = -\left\{\mathbf{M}^-\mathbf{W}(-z/\alpha) + \mathbf{M}^+\mathbf{W}(z/\alpha)\right\}^{-1} \mathbf{M}^-\mathbf{W}(-z/\alpha)\mathcal{E} - \mathrm{diag}(0, 1, 1, 0), \ \mathcal{F}(\mathbf{M}, z) = \mathrm{Sim}_{\mathbf{V}^T}\, \mathcal{G}(\mathbf{M}, z).$$

Note from Definitions 3.1, 3.2, and Proposition 3.1 that

$$\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C}) \text{ if and only if } \mathcal{F}(\mathbf{M}, \alpha l) \text{ is defined}, \qquad \mathbf{M} \in \mathrm{gl}(4, 8, \mathbb{C}). \tag{3.1}$$

In particular, we have $\mathcal{G}(\mathbf{M}) = \mathcal{G}(\mathbf{M}, \alpha l)$ and $\mathcal{F}(\mathbf{M}) = \mathcal{F}(\mathbf{M}, \alpha l)$ for $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$.

Theorem 1. *For $l > 0$, every dwarf $\mathbf{E}_{pqrs}$ is well-posed, and the following hold, where $z = \alpha l$.*

$$\mathcal{F}(\mathbf{E}_{0101}, z) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{ci}_-(z)}{\mathrm{s}_+(z)} & -\frac{\sqrt{2}\omega}{\mathrm{s}_+(z)} \\ -\frac{\sqrt{2}\overline{\omega}}{\mathrm{s}_+(z)} & \frac{\mathrm{ci}_+(z)}{\mathrm{s}_+(z)} \end{pmatrix} - \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \begin{pmatrix} \frac{\mathrm{ci}_+(z)}{\mathrm{s}_-(z)} & -\frac{\sqrt{2}\omega}{\mathrm{s}_-(z)} \\ -\frac{\sqrt{2}\overline{\omega}}{\mathrm{s}_-(z)} & \frac{\mathrm{ci}_-(z)}{\mathrm{s}_-(z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

$$\mathcal{F}(\mathbf{E}_{0112}, z) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathrm{i}}{\mathrm{si}_+(z)} \\ \frac{\mathrm{i}}{\mathrm{si}_-(z)} & \frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} -\frac{4\mathrm{i}\sin^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_+(z)} \\ \frac{1}{\mathrm{c}_+(z)} & -\frac{4\mathrm{i}\sin^2(\overline{\omega}z)}{\mathrm{c}_-(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{4\mathrm{i}\cos^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_-(z)} \\ \frac{1}{\mathrm{c}_-(z)} & -\frac{4\mathrm{i}\cos^2(\overline{\omega}z)}{\mathrm{c}_-(2z)} \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathrm{i}}{\mathrm{si}_-(z)} \\ \frac{\mathrm{i}}{\mathrm{si}_+(z)} & \frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

$$\mathcal{F}(\mathbf{E}_{0123}, z) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{s}_-(z)}{\mathrm{ci}_+(z)} & 0 \\ 0 & \frac{\mathrm{s}_-(z)}{\mathrm{ci}_-(z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} 0 & -\frac{\sqrt{2}\omega}{\mathrm{ci}_+(z)} \\ \frac{\sqrt{2}\overline{\omega}}{\mathrm{ci}_-(z)} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\frac{\sqrt{2}\omega}{\mathrm{ci}_-(z)} \\ \frac{\sqrt{2}\overline{\omega}}{\mathrm{ci}_+(z)} & 0 \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{s}_+(z)}{\mathrm{ci}_-(z)} & 0 \\ 0 & \frac{\mathrm{s}_+(z)}{\mathrm{ci}_+(z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

$$\mathcal{F}(\mathbf{E}_{0202}, z) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{si}_+(z)}{\mathrm{c}_+(z)} & 0 \\ 0 & \frac{\mathrm{si}_-(z)}{\mathrm{c}_+(z)} \end{pmatrix} - \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \begin{pmatrix} \frac{\mathrm{si}_-(z)}{\mathrm{c}_-(z)} & 0 \\ 0 & \frac{\mathrm{si}_+(z)}{\mathrm{c}_-(z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

$$\mathcal{F}(\mathbf{E}_{0213}, z) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{si}_+(2z)}{\mathrm{c}_+(2z)} & 0 \\ 0 & \frac{\mathrm{si}_-(2z)}{\mathrm{c}_+(2z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} \frac{2\cos(2\omega z)}{\mathrm{c}_+(2z)} & 0 \\ 0 & -\frac{2\cos(2\overline{\omega}z)}{\mathrm{c}_+(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{2\cos(2\omega z)}{\mathrm{c}_+(2z)} & 0 \\ 0 & \frac{2\cos(2\overline{\omega}z)}{\mathrm{c}_+(2z)} \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{si}_+(2z)}{\mathrm{c}_+(2z)} & 0 \\ 0 & \frac{\mathrm{si}_-(2z)}{\mathrm{c}_+(2z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

$$\mathcal{F}(\mathbf{E}_{0102}, z) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{ci}_+(2z) - \sqrt{2}\omega\cos(2\omega z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{c}_-(z)}{\mathrm{s}_-(2z)} \\ -\frac{\sqrt{2}\overline{\omega}\,\mathrm{c}_-(z)}{\mathrm{s}_-(2z)} & \frac{\mathrm{ci}_-(2z) - \sqrt{2}\overline{\omega}\cos(2\overline{\omega}z)}{\mathrm{s}_-(2z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} \frac{\sqrt{2}\omega\sin(2\omega z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{si}_+(z)}{\mathrm{s}_-(2z)} \\ \frac{\sqrt{2}\overline{\omega}\,\mathrm{si}_-(z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\overline{\omega}\sin(2\overline{\omega}z)}{\mathrm{s}_-(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{\sqrt{2}\omega\sin(2\omega z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{si}_-(z)}{\mathrm{s}_-(2z)} \\ \frac{\sqrt{2}\overline{\omega}\,\mathrm{si}_+(z)}{\mathrm{s}_-(2z)} & \frac{\sqrt{2}\overline{\omega}\sin(2\overline{\omega}z)}{\mathrm{s}_-(2z)} \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{ci}_+(2z) + \sqrt{2}\omega\cos(2\omega z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{c}_+(z)}{\mathrm{s}_-(2z)} \\ -\frac{\sqrt{2}\overline{\omega}\,\mathrm{c}_+(z)}{\mathrm{s}_-(2z)} & \frac{\mathrm{ci}_-(2z) + \sqrt{2}\overline{\omega}\cos(2\overline{\omega}z)}{\mathrm{s}_-(2z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

$$\mathcal{F}\left(\mathbf{E}_{0113}, z\right) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{ci}_-(2z)+\sqrt{2}\overline{\omega}\cos(2\omega z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{c}_+(z)}{\mathrm{s}_+(2z)} \\ -\frac{\sqrt{2}\overline{\omega}\,\mathrm{c}_+(z)}{\mathrm{s}_+(2z)} & \frac{\mathrm{ci}_+(2z)+\sqrt{2}\omega\cos(2\overline{\omega}z)}{\mathrm{s}_+(2z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} \frac{\sqrt{2}\overline{\omega}\sin(2\omega z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{si}_-(z)}{\mathrm{s}_+(2z)} \\ \frac{\sqrt{2}\overline{\omega}\,\mathrm{si}_+(z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\sin(2\overline{\omega}z)}{\mathrm{s}_+(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{\sqrt{2}\overline{\omega}\sin(2\omega z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{si}_+(z)}{\mathrm{s}_+(2z)} \\ \frac{\sqrt{2}\overline{\omega}\,\mathrm{si}_-(z)}{\mathrm{s}_+(2z)} & \frac{\sqrt{2}\omega\sin(2\overline{\omega}z)}{\mathrm{s}_+(2z)} \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{ci}_-(2z)-\sqrt{2}\overline{\omega}\cos(2\omega z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{c}_-(z)}{\mathrm{s}_+(2z)} \\ -\frac{\sqrt{2}\overline{\omega}\,\mathrm{c}_-(z)}{\mathrm{s}_+(2z)} & \frac{\mathrm{ci}_+(2z)-\sqrt{2}\omega\cos(2\overline{\omega}z)}{\mathrm{s}_+(2z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

The proof of Theorem 1 will be given in Section 9.

Definition 3.3. Denote $\mathbb{I} = \mathbf{I}_4 \oplus' \mathbf{I}_4 = \begin{pmatrix} \mathbf{O} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{O} \end{pmatrix} \in \mathrm{gl}(8, \mathbb{C})$ and $\mathbb{L} = \begin{pmatrix} \mathbf{L} & \mathbf{O} \\ \mathbf{O} & \mathbf{L} \end{pmatrix} \in \mathrm{gl}(8, \mathbb{C})$.

Readers are advised to distinguish between $\mathbb{I}$, $\mathbb{L}$ and $\mathbf{I}$, $\mathbf{L}$. Note from (2.3) and Definition 3.3 that

$$\mathbb{I}^2 = \mathbb{L}^4 = \mathbf{I}_8, \qquad \mathbb{I}\mathbb{L} = \mathbb{L}\mathbb{I}. \tag{3.2}$$

For $\mathbf{M} = (\mathbf{M}^- \,|\, \mathbf{M}^+) \in \mathrm{gl}(4, 8, \mathbb{C})$ with $\mathbf{M}^-, \mathbf{M}^+ \in \mathrm{gl}(4, \mathbb{C})$,

$$\mathbf{M}\mathbb{I} = (\mathbf{M}^- \,|\, \mathbf{M}^+)\begin{pmatrix} \mathbf{O} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{O} \end{pmatrix} = (\mathbf{M}^+ \,|\, \mathbf{M}^-), \qquad \mathbf{M}\mathbb{L} = (\mathbf{M}^- \,|\, \mathbf{M}^+)\begin{pmatrix} \mathbf{L} & \mathbf{O} \\ \mathbf{O} & \mathbf{L} \end{pmatrix} = (\mathbf{M}^-\mathbf{L} \,|\, \mathbf{M}^+\mathbf{L}). \tag{3.3}$$

Theorem 2. *The following* (a) *and* (b) *hold for $p, q, r, s \in \{0, 1, 2, 3\}$, $p < q$, $r < s$.*

(a) *There exist $\mathbf{S} \in GL(4, \mathbb{C})$, $m \in \{0, 1\}$, $n \in \{0, 1, 2, 3\}$, and a dwarf $\mathbf{E}_{p'q'r's'}$ in* (1.6) *such that $\mathbf{E}_{pgrs} = \mathbf{S} \cdot \mathbf{E}_{p'q'r's'} \cdot \mathbb{I}^m\mathbb{L}^n \approx \mathbf{E}_{p'q'r's'} \cdot \mathbb{I}^m\mathbb{L}^n$. $\mathbf{S}$, m, n, and the corresponding dwarf $\mathbf{E}_{p'q'r's'}$ are listed explicitly for each $\mathbf{E}_{pqrs}$ in Table 1.*

(b) *For $l > 0$, $\mathbf{E}_{pqrs}$ is well-posed, and the following hold, where $z = \alpha l$.*

$$\mathcal{F}(\mathbf{E}_{pgrs} \cdot \mathbb{I}, z) = -\mathcal{F}\left(\mathbf{E}_{pgrs}, -z\right) - \mathbf{I}.$$
$$\mathcal{F}\left(\mathbf{E}_{pgrs} \cdot \mathbb{L}, z\right) = \mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}\,\mathcal{F}\left(\mathbf{E}_{pgrs}, \mathrm{i}z\right) \cdot (\mathbf{J} \oplus \mathbf{J}) - \mathrm{diag}(0, 1, 0, 1).$$
$$\mathcal{F}\left(\mathbf{E}_{pgrs} \cdot \mathbb{L}^2, z\right) = -\,\mathrm{Sim}_{\mathcal{E}}\,\mathcal{F}\left(\mathbf{E}_{pgrs}, -z\right) - \mathbf{I}.$$
$$\mathcal{F}\left(\mathbf{E}_{pgrs} \cdot \mathbb{L}^3, z\right) = -\,\mathrm{Sim}_{\mathbf{R}(\mathbf{K}\oplus\overline{\mathbf{K}})}\,\mathcal{F}\left(\mathbf{E}_{pgrs}, -\mathrm{i}z\right) \cdot (\mathbf{J} \oplus \mathbf{J}) - \mathrm{diag}(1, 0, 1, 0).$$

Theorem 2 (a) can readily be checked with Table 1. For example,

$$\begin{aligned} &\mathbf{V}_{12}\mathbf{V}_{34}\mathbf{L}^2 \cdot \mathbf{E}_{0102} \cdot \mathbb{I}^1\mathbb{L}^3 \\ &= \mathbf{V}_{12}\mathbf{V}_{34}\mathbf{L}^2 \cdot \left(\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \end{array}\right)\mathbb{I}^1 \cdot \mathbb{L}^3 = \mathbf{V}_{12}\mathbf{V}_{34} \cdot \mathbf{L}^2 \left(\begin{array}{cccc|cccc} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{array}\right) \cdot \mathbb{L}^3 \\ &= \mathbf{V}_{12}\mathbf{V}_{34} \cdot \left(\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{array}\right)\mathbb{L}^3 = \mathbf{V}_{12} \cdot \mathbf{V}_{34}\left(\begin{array}{cccc|cccc} 0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \end{array}\right) \\ &= \mathbf{V}_{12}\left(\begin{array}{cccc|cccc} 0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{array}\right) = \mathbf{E}_{1303}. \end{aligned}$$

Table 1. Reduction of 36 elementary boundary matrices to the seven dwarfs. For each elementary boundary matrix $\mathbf{E}_{pgrs}$, we have $\mathbf{E}_{pgrs} = \mathbf{S} \cdot \mathbf{E}_{p'q'r's'} \cdot \mathbb{I}^m\mathbb{L}^n \approx \mathbf{E}_{p'q'r's'} \cdot \mathbb{I}^m\mathbb{L}^n$, where $\mathbf{E}_{p'q'r's'}$ is one of the seven dwarfs in (1.6).

$\mathbf{E}_{pqrs}$	$\mathbf{S}$	corresponding dwarf $\mathbf{E}_{p'q'r's'}$	$\mathbb{I}^m\mathbb{L}^n$	$\mathbf{E}_{pqrs}$	$\mathbf{S}$	corresponding dwarf $\mathbf{E}_{p'q'r's'}$	$\mathbb{I}^m\mathbb{L}^n$
$\mathbf{E}_{0101}$	$\mathbf{I}$	$\mathbf{E}_{0101}$	$\mathbb{I}^0\mathbb{L}^0$	$\mathbf{E}_{1201}$	$\mathbf{L}^2$	$\mathbf{E}_{0112}$	$\mathbb{I}^1\mathbb{L}^0$
$\mathbf{E}_{0102}$	$\mathbf{I}$	$\mathbf{E}_{0102}$	$\mathbb{I}^0\mathbb{L}^0$	$\mathbf{E}_{1202}$	$\mathbf{V}_{34}$	$\mathbf{E}_{0113}$	$\mathbb{I}^0\mathbb{L}^1$
$\mathbf{E}_{0103}$	$\mathbf{V}_{34}$	$\mathbf{E}_{0112}$	$\mathbb{I}^1\mathbb{L}^3$	$\mathbf{E}_{1203}$	$\mathbf{V}_{34}$	$\mathbf{E}_{0123}$	$\mathbb{I}^0\mathbb{L}^1$
$\mathbf{E}_{0112}$	$\mathbf{I}$	$\mathbf{E}_{0112}$	$\mathbb{I}^0\mathbb{L}^0$	$\mathbf{E}_{1212}$	$\mathbf{I}$	$\mathbf{E}_{0101}$	$\mathbb{I}^0\mathbb{L}^1$
$\mathbf{E}_{0113}$	$\mathbf{I}$	$\mathbf{E}_{0113}$	$\mathbb{I}^0\mathbb{L}^0$	$\mathbf{E}_{1213}$	$\mathbf{I}$	$\mathbf{E}_{0102}$	$\mathbb{I}^0\mathbb{L}^1$
$\mathbf{E}_{0123}$	$\mathbf{I}$	$\mathbf{E}_{0123}$	$\mathbb{I}^0\mathbb{L}^0$	$\mathbf{E}_{1223}$	$\mathbf{I}$	$\mathbf{E}_{0112}$	$\mathbb{I}^0\mathbb{L}^1$
$\mathbf{E}_{0201}$	$\mathbf{L}^2$	$\mathbf{E}_{0102}$	$\mathbb{I}^1\mathbb{L}^0$	$\mathbf{E}_{1301}$	$\mathbf{L}^2$	$\mathbf{E}_{0113}$	$\mathbb{I}^1\mathbb{L}^0$
$\mathbf{E}_{0202}$	$\mathbf{I}$	$\mathbf{E}_{0202}$	$\mathbb{I}^0\mathbb{L}^0$	$\mathbf{E}_{1302}$	$\mathbf{L}^2$	$\mathbf{E}_{0213}$	$\mathbb{I}^1\mathbb{L}^0$
$\mathbf{E}_{0203}$	$\mathbf{V}_{34}\mathbf{L}^2$	$\mathbf{E}_{0113}$	$\mathbb{I}^1\mathbb{L}^3$	$\mathbf{E}_{1303}$	$\mathbf{V}_{12}\mathbf{V}_{34}\mathbf{L}^2$	$\mathbf{E}_{0102}$	$\mathbb{I}^1\mathbb{L}^3$
$\mathbf{E}_{0212}$	$\mathbf{V}_{12}\mathbf{L}^2$	$\mathbf{E}_{0113}$	$\mathbb{I}^1\mathbb{L}^1$	$\mathbf{E}_{1312}$	$\mathbf{L}^2$	$\mathbf{E}_{0102}$	$\mathbb{I}^1\mathbb{L}^1$
$\mathbf{E}_{0213}$	$\mathbf{I}$	$\mathbf{E}_{0213}$	$\mathbb{I}^0\mathbb{L}^0$	$\mathbf{E}_{1313}$	$\mathbf{I}$	$\mathbf{E}_{0202}$	$\mathbb{I}^0\mathbb{L}^1$
$\mathbf{E}_{0223}$	$\mathbf{V}_{12}\mathbf{L}^2$	$\mathbf{E}_{0102}$	$\mathbb{I}^1\mathbb{L}^2$	$\mathbf{E}_{1323}$	$\mathbf{V}_{12}\mathbf{L}^2$	$\mathbf{E}_{0113}$	$\mathbb{I}^1\mathbb{L}^2$
$\mathbf{E}_{0301}$	$\mathbf{V}_{12}$	$\mathbf{E}_{0112}$	$\mathbb{I}^0\mathbb{L}^3$	$\mathbf{E}_{2301}$	$\mathbf{I}$	$\mathbf{E}_{0123}$	$\mathbb{I}^0\mathbb{L}^2$
$\mathbf{E}_{0302}$	$\mathbf{V}_{12}$	$\mathbf{E}_{0113}$	$\mathbb{I}^0\mathbb{L}^3$	$\mathbf{E}_{2302}$	$\mathbf{V}_{34}$	$\mathbf{E}_{0102}$	$\mathbb{I}^0\mathbb{L}^2$
$\mathbf{E}_{0303}$	$\mathbf{V}_{12}\mathbf{V}_{34}$	$\mathbf{E}_{0101}$	$\mathbb{I}^0\mathbb{L}^3$	$\mathbf{E}_{2303}$	$\mathbf{V}_{34}$	$\mathbf{E}_{0112}$	$\mathbb{I}^0\mathbb{L}^2$
$\mathbf{E}_{0312}$	$\mathbf{V}_{12}$	$\mathbf{E}_{0123}$	$\mathbb{I}^0\mathbb{L}^3$	$\mathbf{E}_{2312}$	$\mathbf{L}^2$	$\mathbf{E}_{0112}$	$\mathbb{I}^1\mathbb{L}^1$
$\mathbf{E}_{0313}$	$\mathbf{V}_{12}\mathbf{V}_{34}$	$\mathbf{E}_{0102}$	$\mathbb{I}^0\mathbb{L}^3$	$\mathbf{E}_{2313}$	$\mathbf{V}_{34}$	$\mathbf{E}_{0113}$	$\mathbb{I}^0\mathbb{L}^2$
$\mathbf{E}_{0323}$	$\mathbf{V}_{12}\mathbf{L}^2$	$\mathbf{E}_{0112}$	$\mathbb{I}^1\mathbb{L}^2$	$\mathbf{E}_{2323}$	$\mathbf{I}$	$\mathbf{E}_{0101}$	$\mathbb{I}^0\mathbb{L}^2$

Let $z = \alpha l > 0$. By (3.2) and Theorem 2 (a) with Table 1, $\mathbf{E}_{0323} = \mathbf{V}_{12}\mathbf{L}^2 \cdot \mathbf{E}_{0112} \cdot \mathbb{L}^2\mathbb{I} \approx \mathbf{E}_{0112}\mathbb{L}^2\mathbb{I}$, hence, by Proposition 1.2 and Theorem 2 (b),

$$\begin{aligned}\mathcal{F}(\mathbf{E}_{0323}, z) &= \mathcal{F}\left(\mathbf{E}_{0112}\mathbb{L}^2 \cdot \mathbb{I}, z\right) = -\mathcal{F}\left(\mathbf{E}_{0112} \cdot \mathbb{L}^2, -z\right) - \mathbf{I} = -\left\{\mathrm{Sim}_{\mathcal{E}}\mathcal{F}(\mathbf{E}_{0112}, -(-z)) - \mathbf{I}\right\} - \mathbf{I}\\ &= -\mathrm{Sim}_{\mathcal{E}}\mathcal{F}(\mathbf{E}_{0112}, z)\,.\end{aligned}$$

So by Theorem 1, Definition 2.3, and (2.2),

$$\begin{aligned}\mathcal{F}(\mathbf{E}_{0323}, z) &= -\{\mathbf{J} \oplus (-\mathbf{J})\} \cdot \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathbb{i}}{\mathrm{si}_+(z)} \\ \frac{\mathbb{i}}{\mathrm{si}_-(z)} & \frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} -\frac{4\mathbb{i}\sin^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_+(z)} \\ \frac{1}{\mathrm{c}_+(z)} & -\frac{4\mathbb{i}\sin^2(\overline{\omega}z)}{\mathrm{c}_-(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{4\mathbb{i}\cos^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_-(z)} \\ \frac{1}{\mathrm{c}_-(z)} & -\frac{4\mathbb{i}\cos^2(\overline{\omega}z)}{\mathrm{c}_-(2z)} \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathbb{i}}{\mathrm{si}_-(z)} \\ \frac{\mathbb{i}}{\mathrm{si}_+(z)} & \frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix} - \mathbf{I} \end{pmatrix} \cdot \{\mathbf{J} \oplus (-\mathbf{J})\}\\ &= \frac{1}{2}\begin{pmatrix} -\mathbf{J}\begin{pmatrix} \frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathbb{i}}{\mathrm{si}_+(z)} \\ \frac{\mathbb{i}}{\mathrm{si}_-(z)} & \frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix}\mathbf{J} + \mathbf{JIJ} & \mathbf{J}\begin{pmatrix} -\frac{4\mathbb{i}\sin^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_+(z)} \\ \frac{1}{\mathrm{c}_+(z)} & -\frac{4\mathbb{i}\sin^2(\overline{\omega}z)}{\mathrm{c}_-(2z)} \end{pmatrix}\mathbf{J} \\ \mathbf{J}\begin{pmatrix} -\frac{4\mathbb{i}\cos^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_-(z)} \\ \frac{1}{\mathrm{c}_-(z)} & -\frac{4\mathbb{i}\cos^2(\overline{\omega}z)}{\mathrm{c}_-(2z)} \end{pmatrix}\mathbf{J} & -\mathbf{J}\begin{pmatrix} \frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathbb{i}}{\mathrm{si}_-(z)} \\ \frac{\mathbb{i}}{\mathrm{si}_+(z)} & \frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix}\mathbf{J} + \mathbf{JIJ} \end{pmatrix}\end{aligned}$$

$$= \frac{1}{2}\begin{pmatrix} \begin{pmatrix} -\frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathbb{i}}{\mathrm{si}_+(z)} \\ \frac{\mathbb{i}}{\mathrm{si}_-(z)} & -\frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix} + \mathbf{I} & \begin{pmatrix} -\frac{4\mathbb{i}\sin^2(\omega z)}{\mathrm{c}_-(2z)} & \frac{1}{\mathrm{c}_+(z)} \\ -\frac{1}{\mathrm{c}_+(z)} & -\frac{4\mathbb{i}\sin^2(\overline{\omega} z)}{\mathrm{c}_-(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{4\mathbb{i}\cos^2(\omega z)}{\mathrm{c}_-(2z)} & \frac{1}{\mathrm{c}_-(z)} \\ -\frac{1}{\mathrm{c}_-(z)} & -\frac{4\mathbb{i}\cos^2(\overline{\omega} z)}{\mathrm{c}_-(2z)} \end{pmatrix} & \begin{pmatrix} -\frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathbb{i}}{\mathrm{si}_-(z)} \\ \frac{\mathbb{i}}{\mathrm{si}_+(z)} & -\frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix} + \mathbf{I} \end{pmatrix}.$$

Note that the seven dwarfs cannot be reduced to each other in the sense of Theorem 2 (a). The proof of Theorem 2 (b) will be given in Section 9.

4. Reduction to the seven dwarfs

Definition 4.1. For $m, n \in \{0, 1, 2, 3\}$, define $\mathbf{E}_{mn} \in \mathrm{gl}(4, \mathbb{C})$ by

$$(\mathbf{E}_{mn})_{i,j} = \begin{cases} 1, & \text{if } (i = 1,\ j = m + 1) \text{ or } (i = 2,\ j = n + 1), \\ 0, & \text{otherwise,} \end{cases} \qquad 1 \le i, j \le 4.$$

By Definitions 2.4 and 4.1,

$$\left(\mathbf{L}^2\mathbf{E}_{mn}\right)_{i,j} = \begin{cases} 1, & \text{if } (i = 3,\ j = m + 1) \text{ or } (i = 4,\ j = n + 1), \\ 0, & \text{otherwise,} \end{cases} \qquad 1 \le i, j \le 4, \tag{4.1}$$

hence, by (1.4), we have the following representation of elementary boundary matrices.

$$\mathbf{E}_{pqrs} = \left(\mathbf{E}_{pq} \,|\, \mathbf{L}^2\mathbf{E}_{rs}\right), \qquad p, q, r, s \in \{0, 1, 2, 3\}, \quad p < q, \quad r < s. \tag{4.2}$$

4.1. Separable boundary conditions

Definition 4.2. $\mathbf{M} \in \mathrm{gl}(4, 8, \mathbb{C})$ is called *separable* if $\mathbf{M} \approx \mathbf{M}\cdot(\mathbf{D} \oplus \mathbf{D})$ for every diagonal $\mathbf{D} \in GL(4, \mathbb{C})$.

Let $m, n \in \{0, 1, 2, 3\}$ and $d_0, d_1, d_2, d_3 \in \mathbb{C}$. By Definition 4.1 and (4.1),

$$\begin{aligned} \{\mathbf{E}_{mn} \cdot \mathrm{diag}\,(d_0, d_1, d_2, d_3)\}_{i,j} &= \begin{cases} d_m, & \text{if } i = 1 \text{ and } j = m + 1, \\ d_n, & \text{if } i = 2 \text{ and } j = n + 1, \\ 0, & \text{otherwise} \end{cases} \\ &= \{\mathrm{diag}\,(d_m, d_n, a_3, a_4) \cdot \mathbf{E}_{mn}\}_{i,j}, \qquad 1 \le i, j \le 4, \quad a_3, a_4 \in \mathbb{C}, \\ \left\{\mathbf{L}^2\mathbf{E}_{mn} \cdot \mathrm{diag}\,(d_0, d_1, d_2, d_3)\right\}_{i,j} &= \begin{cases} d_m, & \text{if } i = 3 \text{ and } j = m + 1, \\ d_n, & \text{if } i = 4 \text{ and } j = n + 1, \\ 0, & \text{otherwise} \end{cases} \\ &= \left\{\mathrm{diag}\,(a_1, a_2, d_m, d_n) \cdot \mathbf{L}^2\mathbf{E}_{mn}\right\}_{i,j}, \qquad 1 \le i, j \le 4, \quad a_1, a_2 \in \mathbb{C}, \end{aligned}$$

hence

$$\begin{aligned} \mathbf{E}_{mn} \cdot \mathrm{diag}\,(d_0, d_1, d_2, d_3) &= \mathrm{diag}\,(d_m, d_n, a_3, a_4) \cdot \mathbf{E}_{mn}, \quad d_0, d_1, d_2, d_3, a_3, a_4 \in \mathbb{C}, \\ \mathbf{L}^2\mathbf{E}_{mn} \cdot \mathrm{diag}\,(d_0, d_1, d_2, d_3) &= \mathrm{diag}\,(a_1, a_2, d_m, d_n) \cdot \mathbf{L}^2\mathbf{E}_{mn}, \quad d_0, d_1, d_2, d_3, a_1, a_2 \in \mathbb{C}. \end{aligned} \tag{4.3}$$

Lemma 4.1. *Every elementary boundary matrix is separable.*

Proof. Let $p, q, r, s \in \{0, 1, 2, 3\}$, $p < q$, $r < s$, and let $\mathbf{D} = \operatorname{diag}(d_0, d_1, d_2, d_3) \in GL(4, \mathbb{C})$. By (4.2), $\mathbf{E}_{pqrs} \cdot (\mathbf{D} \oplus \mathbf{D}) = \left(\mathbf{E}_{pq} \,|\, \mathbf{L}^2\mathbf{E}_{rs}\right)\begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{D}\end{pmatrix} = \left(\mathbf{E}_{pq}\mathbf{D} \,|\, \mathbf{L}^2\mathbf{E}_{rs}\mathbf{D}\right)$, hence, by (4.3),

$$\begin{aligned}\mathbf{E}_{pqrs} \cdot (\mathbf{D} \oplus \mathbf{D}) &= \left(\operatorname{diag}\left(d_p, d_q, d_r, d_s\right) \cdot \mathbf{E}_{pq} \;\middle|\; \operatorname{diag}\left(d_p, d_q, d_r, d_s\right) \cdot \mathbf{L}^2\mathbf{E}_{rs} \right)\\ &= \operatorname{diag}\left(d_p, d_q, d_r, d_s\right) \cdot \left(\mathbf{E}_{pq} \,|\, \mathbf{L}^2\mathbf{E}_{rs}\right) = \operatorname{diag}\left(d_p, d_q, d_r, d_s\right) \cdot \mathbf{E}_{pqrs}.\end{aligned} \tag{4.4}$$

Since $\operatorname{diag}(d_0, d_1, d_2, d_3) \in GL(4, \mathbb{C})$, it follows that $\operatorname{diag}\left(d_p, d_q, d_r, d_s\right) \in GL(4, \mathbb{C})$. Thus $\mathbf{E}_{pqrs} \approx \mathbf{E}_{pqrs} \cdot (\mathbf{D} \oplus \mathbf{D})$ by (4.4), hence we have the proof by Definition 4.2. □

Lemma 4.2. *The following* (a), (b), (c) *hold for* $\mathbf{M}, \mathbf{N} \in \mathrm{gl}(4, 8, \mathbb{C})$.

(a) *Suppose that* $\mathbf{M} \approx \mathbf{N}$. *Then* $\mathbf{M}$ *is separable if and only if* $\mathbf{N}$ *is separable.*
(b) $\mathbf{M}$ *is separable if and only if* $\mathbf{M}\mathbb{I}$ *is separable.*
(c) $\mathbf{M}$ *is separable if and only if* $\mathbf{M}\mathbb{L}$ *is separable.*

Proof. Suppose that $\mathbf{M} \approx \mathbf{N}$ so that $\mathbf{N} = \mathbf{A}\mathbf{M}$ for some $\mathbf{A} \in GL(4, \mathbb{C})$. Suppose that $\mathbf{M}$ is separable. By Definition 4.2, there exists $\mathbf{B} \in GL(4, \mathbb{C})$ such that $\mathbf{M} \cdot (\mathbf{D} \oplus \mathbf{D}) = \mathbf{B}\mathbf{M}$ for every diagonal $\mathbf{D} \in GL(4, \mathbb{C})$. So we have $\mathbf{N} \cdot (\mathbf{D} \oplus \mathbf{D}) = \mathbf{A}\mathbf{M} \cdot (\mathbf{D} \oplus \mathbf{D}) = \mathbf{A}\mathbf{B}\mathbf{M} = \mathbf{A}\mathbf{B}\mathbf{A}^{-1} \cdot \mathbf{A}\mathbf{M} = \mathbf{A}\mathbf{B}\mathbf{A}^{-1} \cdot \mathbf{N}$, hence $\mathbf{N} \approx \mathbf{N} \cdot (\mathbf{D} \oplus \mathbf{D})$ for every diagonal $\mathbf{D} \in GL(4, \mathbb{C})$. Thus $\mathbf{N}$ is separable by Definition 4.2. Similarly, $\mathbf{M}$ is separable if $\mathbf{N}$ is separable. This shows (a).

Let $\mathbf{D} = \operatorname{diag}(d_1, d_2, d_3, d_4) \in GL(4, \mathbb{C})$. By Definition 3.3,

$$\begin{aligned}&\mathbf{M}\mathbb{I} \cdot (\mathbf{D} \oplus \mathbf{D}) = \mathbf{M} \cdot (\mathbf{D} \oplus \mathbf{D}) \cdot \mathbb{I},\\ &\mathbf{M}\mathbb{L} \cdot (\mathbf{D} \oplus \mathbf{D}) = \mathbf{M} \cdot \mathbb{L}\,(\mathbf{D} \oplus \mathbf{D})\,\mathbb{L}^{-1} \cdot \mathbb{L} = \mathbf{M} \cdot \left\{\left(\mathbf{L}\mathbf{D}\mathbf{L}^{-1}\right) \oplus \left(\mathbf{L}\mathbf{D}\mathbf{L}^{-1}\right)\right\} \cdot \mathbb{L}.\end{aligned} \tag{4.5}$$

Suppose that $\mathbf{M}$ is separable. By Definition 2.4, $\mathbf{L}\mathbf{D}\mathbf{L}^{-1} = \operatorname{diag}(d_2, d_3, d_4, d_1) \in GL(4, \mathbb{C})$, hence, by Definition 4.2, there exist $\mathbf{A}, \mathbf{B} \in GL(4, \mathbb{C})$ such that $\mathbf{M} \cdot (\mathbf{D} \oplus \mathbf{D}) = \mathbf{A}\mathbf{M}$ and $\mathbf{M} \cdot \left\{\left(\mathbf{L}\mathbf{D}\mathbf{L}^{-1}\right) \oplus \left(\mathbf{L}\mathbf{D}\mathbf{L}^{-1}\right)\right\} = \mathbf{B}\mathbf{M}$. So by (4.5), $\mathbf{M}\mathbb{I} \cdot (\mathbf{D} \oplus \mathbf{D}) = \mathbf{A} \cdot \mathbf{M}\mathbb{I} \approx \mathbf{M}\mathbb{I}$ and $\mathbf{M}\mathbb{L} \cdot (\mathbf{D} \oplus \mathbf{D}) = \mathbf{B} \cdot \mathbf{M}\mathbb{L} \approx \mathbf{M}\mathbb{L}$. It follows that $\mathbf{M}\mathbb{I}$ and $\mathbf{M}\mathbb{L}$ are separable, since $\mathbf{D}$ is arbitrary. Thus (b) and (c) follow by (3.2), and the proof is complete. □

By Definitions 2.2, 2.4, and (2.1),

$$\begin{aligned}\mathrm{Sim}_{\mathbf{L}^{-1}}\, e^{\mathbf{\Omega} z} &= \mathbf{L} \cdot \operatorname{diag}\left(e^{\omega z}, e^{\mathrm{i}\omega z}, e^{\mathrm{i}^2\omega z}, e^{\mathrm{i}^3\omega z}\right) \cdot \mathbf{L}^{-1} = \operatorname{diag}\left(e^{\mathrm{i}\omega z}, e^{\mathrm{i}^2\omega z}, e^{\mathrm{i}^3\omega z}, e^{\omega z}\right)\\ &= \operatorname{diag}\left(e^{\omega_1 \cdot \mathrm{i} z}, e^{\omega_2 \cdot \mathrm{i} z}, e^{\omega_3 \cdot \mathrm{i} z}, e^{\omega_4 \cdot \mathrm{i} z}\right) = e^{\mathbf{\Omega}\mathrm{i} z}, \qquad z \in \mathbb{C},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathbf{W}_0\mathbf{L}^{-1} &= \left(\omega_j^{i-1}\right)_{1 \le i,j \le 4} \cdot \mathbf{L}^{-1} = \left(\omega_{j+1}^{i-1}\right)_{1 \le i,j \le 4} = \left(\left(\mathrm{i}\omega_j\right)^{i-1}\right)_{1 \le i,j \le 4} = \left(\mathrm{i}^{i-1} \cdot \omega_j^{i-1}\right)_{1 \le i,j \le 4}\\ &= \operatorname{diag}\left(1, \mathrm{i}, \mathrm{i}^2, \mathrm{i}^3\right)\mathbf{W}_0,\end{aligned} \tag{4.7}$$

$$\mathbf{L}\mathbf{W}_0 = \mathbf{L} \cdot \left(\omega_j^{i-1}\right)_{1 \le i,j \le 4} = \left(\omega_j^{i}\right)_{1 \le i,j \le 4} = \left(\omega_j^{i-1} \cdot \omega_j\right)_{1 \le i,j \le 4} = \mathbf{W}_0\mathbf{\Omega}. \tag{4.8}$$

Lemma 4.3. *The following* (a) *and* (b) *hold for* $z \in \mathbb{C}$. (a) $\mathbf{W}_0 e^{\pm\mathbf{\Omega} z} = \operatorname{diag}\left(1, \mathrm{i}, \mathrm{i}^2, \mathrm{i}^3\right)\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathrm{i} z}\mathbf{L}$. (b) $\mathbf{L}\mathbf{W}_0 e^{\pm\mathbf{\Omega} z} = \operatorname{diag}\left(\mathrm{i}, \mathrm{i}^2, \mathrm{i}^3, \mathrm{i}^4\right)\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathrm{i} z}\mathbf{\Omega}\mathbf{L}$.

Proof. By (4.6) and (4.7), $\mathbf{W}_0 e^{\pm\mathbf{\Omega}z} = \mathbf{W}_0\mathbf{L}^{-1}\cdot\mathbf{L}e^{\pm\mathbf{\Omega}z}\mathbf{L}^{-1}\cdot\mathbf{L} = \operatorname{diag}\left(1, \mathbb{i}, \mathbb{i}^2, \mathbb{i}^3\right)\mathbf{W}_0\cdot e^{\pm\mathbf{\Omega}\mathbb{i}z}\cdot\mathbf{L}$, which shows (a). By (4.8), $\mathbf{L}\mathbf{W}_0 e^{\pm\mathbf{\Omega}z} = \mathbf{W}_0\mathbf{\Omega}e^{\pm\mathbf{\Omega}z} = \mathbf{W}_0 e^{\pm\mathbf{\Omega}z}\mathbf{\Omega}$, since $\mathbf{\Omega}$ and $e^{\pm\mathbf{\Omega}z}$ are diagonal, and hence commute. So by (a), $\mathbf{L}\mathbf{W}_0 e^{\pm\mathbf{\Omega}z} = \mathbf{W}_0 e^{\pm\mathbf{\Omega}z}\cdot\mathbf{\Omega} = \operatorname{diag}\left(1, \mathbb{i}, \mathbb{i}^2, \mathbb{i}^3\right)\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z}\mathbf{L}\cdot\mathbf{\Omega} = \operatorname{diag}\left(1, \mathbb{i}, \mathbb{i}^2, \mathbb{i}^3\right)\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z}\cdot\mathbf{L}\mathbf{\Omega}\mathbf{L}^{-1}\cdot\mathbf{L}$. Thus (b) follows since $\mathbf{L}\mathbf{\Omega}\mathbf{L}^{-1} = \mathbb{i}\mathbf{\Omega}$ by Definitions 2.2, 2.4, and (2.1). □

Lemma 4.4. *Let* $z \in \mathbb{C}$, *and let* $\mathbf{M} = (\mathbf{M}^- \,|\, \mathbf{M}^+) \in \mathrm{gl}(4, 8, \mathbb{C})$ *for* $\mathbf{M}^-, \mathbf{M}^+ \in \mathrm{gl}(4, \mathbb{C})$. *Suppose that* $\mathbf{M}$ *is separable. Then the following* (a), (b), *and* (c) *hold.*

(a) *There exists* $\mathbf{A} \in GL(4, \mathbb{C})$ *such that* $\mathbf{M}^\pm\mathbf{W}(\pm z/\alpha) = \mathbf{A}\cdot\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}z}$.
(b) *There exists* $\mathbf{B} \in GL(4, \mathbb{C})$ *such that* $\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}z} = \mathbf{B}\cdot\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z}\cdot\mathbf{L}$.
(c) *There exists* $\mathbf{C} \in GL(4, \mathbb{C})$ *such that* $\mathbf{M}^\pm\mathbf{L}\mathbf{W}_0 e^{\pm\mathbf{\Omega}z} = \mathbf{C}\cdot\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z}\cdot\mathbf{\Omega}\mathbf{L}$.

Proof. By Definition 4.2, there exist $\mathbf{A}, \mathbf{B} \in GL(4, \mathbb{C})$ such that

$$\mathbf{A}\mathbf{M} = \mathbf{M}\left\{\operatorname{diag}\left(1, \alpha, \alpha^2, \alpha^3\right) \oplus \operatorname{diag}\left(1, \alpha, \alpha^2, \alpha^3\right)\right\}, \quad \mathbf{B}\mathbf{M} = \mathbf{M}\left\{\operatorname{diag}\left(1, \mathbb{i}, \mathbb{i}^2, \mathbb{i}^3\right) \oplus \operatorname{diag}\left(1, \mathbb{i}, \mathbb{i}^2, \mathbb{i}^3\right)\right\},$$

hence $\mathbf{A}\mathbf{M}^\pm = \mathbf{M}^\pm \operatorname{diag}\left(1, \alpha, \alpha^2, \alpha^3\right)$, $\mathbf{B}\mathbf{M}^\pm = \mathbf{M}^\pm \operatorname{diag}\left(1, \mathbb{i}, \mathbb{i}^2, \mathbb{i}^3\right)$, $\mathbf{C}\mathbf{M}^\pm = \mathbf{M}^\pm \operatorname{diag}\left(\mathbb{i}, \mathbb{i}^2, \mathbb{i}^3, \mathbb{i}^4\right)$, where we put $\mathbf{C} = \mathbb{i}\mathbf{B} \in GL(4, \mathbb{C})$. Thus, by Definition 2.2 and Lemma 4.3 (a), (b),

$$\begin{aligned}
\mathbf{M}^\pm\mathbf{W}(\pm z/\alpha) &= \mathbf{M}^\pm \operatorname{diag}\left(1, \alpha, \alpha^2, \alpha^3\right)\mathbf{W}_0 e^{\pm\mathbf{\Omega}z} = \mathbf{A}\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}z}, \\
\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}z} &= \mathbf{M}^\pm \operatorname{diag}\left(1, \mathbb{i}, \mathbb{i}^2, \mathbb{i}^3\right)\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z}\mathbf{L} = \mathbf{B}\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z}\mathbf{L}, \\
\mathbf{M}^\pm\mathbf{L}\mathbf{W}_0 e^{\pm\mathbf{\Omega}z} &= \mathbf{M}^\pm \operatorname{diag}\left(\mathbb{i}, \mathbb{i}^2, \mathbb{i}^3, \mathbb{i}^4\right)\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z}\mathbf{\Omega}\mathbf{L} = \mathbf{C}\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z}\mathbf{\Omega}\mathbf{L},
\end{aligned}$$

which show (a), (b), and (c) respectively. □

Lemma 4.5. *Let* $z \in \mathbb{C}$, *and let* $\mathbf{M} = (\mathbf{M}^- \,|\, \mathbf{M}^+) \in \mathrm{gl}(4, 8, \mathbb{C})$ *for* $\mathbf{M}^-, \mathbf{M}^+ \in \mathrm{gl}(4, \mathbb{C})$. *Suppose that* $\mathbf{M}$ *is separable. Then the following* (a), (b), *and* (c) *hold.*

(a) $\mathcal{G}(\mathbf{M}, z)$ *is defined if and only if* $\det\left(\mathbf{M}^-\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^+\mathbf{W}_0 e^{\mathbf{\Omega}z}\right) \neq 0$.
(b) $\mathcal{G}(\mathbf{M}, z)$ *is defined if and only if* $\mathcal{G}(\mathbf{M}, \mathbb{i}z)$ *is defined.*
(c) *If* $\mathcal{G}(\mathbf{M}, z)$ *is defined, then*

$$\begin{aligned}
\mathcal{G}(\mathbf{M}, z) &= -\left(\mathbf{M}^-\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^+\mathbf{W}_0 e^{\mathbf{\Omega}z}\right)^{-1}\mathbf{M}^-\mathbf{W}_0 e^{-\mathbf{\Omega}z}\mathcal{E} - \operatorname{diag}(0, 1, 1, 0) \\
&= \left(\mathbf{M}^-\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^+\mathbf{W}_0 e^{\mathbf{\Omega}z}\right)^{-1}\mathbf{M}^+\mathbf{W}_0 e^{\mathbf{\Omega}z}\mathcal{E} - \operatorname{diag}(1, 0, 0, 1).
\end{aligned}$$

Proof. By Lemma 4.4 (a), there exists $\mathbf{A} \in GL(4, \mathbb{C})$ such that

$$\begin{aligned}
\mathbf{M}^\pm\mathbf{W}(\pm z/\alpha) &= \mathbf{A}\cdot\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}z}, \\
\mathbf{M}^-\mathbf{W}(-z/\alpha) + \mathbf{M}^+\mathbf{W}(z/\alpha) &= \mathbf{A}\mathbf{M}^-\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{A}\mathbf{M}^+\mathbf{W}_0 e^{\mathbf{\Omega}z} = \mathbf{A}\left\{\mathbf{M}^-\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^+\mathbf{W}_0 e^{\mathbf{\Omega}z}\right\}.
\end{aligned} \tag{4.9}$$

Thus (a) follows from Definition 3.2 since $\mathbf{A}$ is invertible. By Lemma 4.4 (b), there exists $\mathbf{B} \in GL(4, \mathbb{C})$ such that $\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}\mathbb{i}z} = \mathbf{B}^{-1}\cdot\mathbf{M}^\pm\mathbf{W}_0 e^{\pm\mathbf{\Omega}z}\cdot\mathbf{L}^{-1}$, hence

$$\mathbf{M}^-\mathbf{W}_0 e^{-\mathbf{\Omega}\cdot\mathbb{i}z} + \mathbf{M}^+\mathbf{W}_0 e^{\mathbf{\Omega}\cdot\mathbb{i}z} = \mathbf{B}^{-1}\left(\mathbf{M}^-\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^+\mathbf{W}_0 e^{\mathbf{\Omega}z}\right)\mathbf{L}^{-1}.$$

Thus (b) follows by (a) and Definition 3.2, since $\mathbf{B}$, $\mathbf{L}$ are invertible.

Suppose that $\mathcal{G}(\mathbf{M}, z)$ is defined. By Definition 3.2 and (4.9),

$$\mathcal{G}(\mathbf{M}, z) = -\left\{\mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}z}\right\}^{-1} \mathbf{A}^{-1} \cdot \mathbf{A}\mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z}\mathcal{E} - \operatorname{diag}(0, 1, 1, 0),$$

from which follows the first equality in (c). By Definition 2.3,

$$\begin{aligned}
&\left[\left(\mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}z}\right)^{-1} \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}z}\mathcal{E} - \operatorname{diag}(1, 0, 0, 1)\right] \\
&-\left[-\left(\mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}z}\right)^{-1} \mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z}\mathcal{E} - \operatorname{diag}(0, 1, 1, 0)\right] \\
&= \left(\mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}z}\right)^{-1} \left(\mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}z}\right)\mathcal{E} - \mathcal{E} = \mathbf{O},
\end{aligned}$$

from which follows the second equality in (c), and the proof is complete. □

4.2. Transformation with $\mathbb{I}$

Lemma 4.6. *Let* $z \in \mathbb{C}$*, and let* $\mathbf{M} \in \mathrm{gl}(4, 8, \mathbb{C})$ *be separable. Suppose that* $\mathcal{F}(\mathbf{M}, z)$ *is defined. Then* $\mathcal{G}(\mathbf{M}\mathbb{I}, z)$ *and* $\mathcal{F}(\mathbf{M}\mathbb{I}, z)$ *are defined, and* $\mathcal{G}(\mathbf{M}\mathbb{I}, z) = -\mathcal{G}(\mathbf{M}, -z) - \mathbf{I}$*,* $\mathcal{F}(\mathbf{M}\mathbb{I}, z) = -\mathcal{F}(\mathbf{M}, -z) - \mathbf{I}$.

Proof. Let $\mathbf{M} = (\mathbf{M}^{-} \,|\, \mathbf{M}^{+})$ for $\mathbf{M}^{-}, \mathbf{M}^{+} \in \mathrm{gl}(4, \mathbb{C})$. Denote $\mathbf{N} = \mathbf{M}\mathbb{I}$ and let $\mathbf{N} = (\mathbf{N}^{-} \,|\, \mathbf{N}^{+})$ for $\mathbf{N}^{-}, \mathbf{N}^{+} \in \mathrm{gl}(4, \mathbb{C})$. Then by (3.3), $(\mathbf{N}^{-} \,|\, \mathbf{N}^{+}) = (\mathbf{M}^{+} \,|\, \mathbf{M}^{-})$, hence

$$\mathbf{N}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z} = \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}(-z)}, \qquad \mathbf{N}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}z} = \mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}(-z)}. \tag{4.10}$$

Suppose that $\mathcal{F}(\mathbf{M}, z)$ is defined, so that $\mathcal{G}(\mathbf{M}, z)$ is also defined by Definition 3.2. By Lemma 4.5 (b), $\mathcal{G}(\mathbf{M}, -z) = \mathcal{G}(\mathbf{M}, \mathbb{i} \cdot \mathbb{i}z)$ is defined, hence, by Lemma 4.5 (a) and (4.10), $\mathcal{G}(\mathbf{N}, z) = \mathcal{G}(\mathbf{M}\mathbb{I}, z)$ is defined. By Lemma 4.2 (b), $\mathbf{N}$ is separable, hence, by Lemma 4.5 (c) and (4.10),

$$\begin{aligned}
\mathcal{G}(\mathbf{M}\mathbb{I}, z) = \mathcal{G}(\mathbf{N}, z) &= -\left\{\mathbf{N}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{N}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}z}\right\}^{-1} \mathbf{N}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}z}\mathcal{E} - \operatorname{diag}(0, 1, 1, 0) \\
&= -\left\{\mathbf{M}^{-}\mathbf{W}_0 e^{-\mathbf{\Omega}\cdot(-z)} + \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}\cdot(-z)}\right\}^{-1} \mathbf{M}^{+}\mathbf{W}_0 e^{\mathbf{\Omega}\cdot(-z)}\mathcal{E} - \operatorname{diag}(0, 1, 1, 0) \\
&= -\left\{\mathcal{G}(\mathbf{M}, -z) + \operatorname{diag}(1, 0, 0, 1)\right\} - \operatorname{diag}(0, 1, 1, 0) = -\mathcal{G}(\mathbf{M}, -z) - \mathbf{I}.
\end{aligned}$$

Thus, by Definition 3.2, $\mathcal{F}(\mathbf{M}\mathbb{I}, z)$ is also defined, and $\mathcal{F}(\mathbf{M}\mathbb{I}, z) = \operatorname{Sim}_{\mathbf{V}^T}\{-\mathcal{G}(\mathbf{M}, -z) - \mathbf{I}\} = -\operatorname{Sim}_{\mathbf{V}^T}\mathcal{G}(\mathbf{M}, -z) - \operatorname{Sim}_{\mathbf{V}^T}\mathbf{I} = -\mathcal{F}(\mathbf{M}, -z) - \mathbf{I}$, which completes the proof. □

4.3. Transformation with $\mathbb{L}$

By Definitions 2.2, 2.4, and (2.1), $\mathbf{\Omega}\mathbf{L} = \begin{pmatrix} 0 & \omega & 0 & 0 \\ 0 & 0 & -\overline{\omega} & 0 \\ 0 & 0 & 0 & -\omega \\ \overline{\omega} & 0 & 0 & 0 \end{pmatrix}$, hence, by Definitions 2.3, 2.4, 2.5, and (2.1), (2.5),

$$\operatorname{Sim}_{\mathcal{E}}(\mathbf{\Omega}\mathbf{L}) = \mathcal{E}^{-1}\begin{pmatrix} 0 & \omega & 0 & 0 \\ 0 & 0 & -\overline{\omega} & 0 \\ 0 & 0 & 0 & -\omega \\ \overline{\omega} & 0 & 0 & 0 \end{pmatrix}\operatorname{diag}(1, -1, -1, 1) = \operatorname{diag}(1, -1, -1, 1)\begin{pmatrix} 0 & -\omega & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & 0 & 0 & -\omega \\ \overline{\omega} & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\omega & 0 & 0 \\ 0 & 0 & -\overline{\omega} & 0 \\ 0 & 0 & 0 & \omega \\ \overline{\omega} & 0 & 0 & 0 \end{pmatrix} = \mathbf{\Omega L}\,\mathrm{diag}(1,-1,1,-1) = \mathbf{\Omega L}(\mathbf{J}\oplus\mathbf{J}), \tag{4.11}$$

$$\mathrm{Sim}_{\mathbf{V}^T}(\mathbf{\Omega L}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & \omega & 0 & 0 \\ 0 & 0 & -\overline{\omega} & 0 \\ 0 & 0 & 0 & -\omega \\ \overline{\omega} & 0 & 0 & 0 \end{pmatrix}\mathbf{V}^T = \frac{1}{2}\begin{pmatrix} 0 & \omega & 0 & -\omega \\ \overline{\omega} & 0 & -\overline{\omega} & 0 \\ 0 & -\omega & 0 & -\omega \\ \overline{\omega} & 0 & \overline{\omega} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\omega \\ 0 & 0 & -\overline{\omega} & 0 \\ 0 & -\omega & 0 & 0 \\ \overline{\omega} & 0 & 0 & 0 \end{pmatrix} = -\omega\cdot\mathrm{diag}(1,-\mathbb{i},1,\mathbb{i})\cdot\mathbf{R} = -\omega\left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\mathbf{R}. \tag{4.12}$$

Lemma 4.7. *Let $z\in\mathbb{C}$, and let $\mathbf{M}\in\mathrm{gl}(4,8,\mathbb{C})$ be separable. Suppose that $\mathcal{F}(\mathbf{M},z)$ is defined. Then $\mathcal{G}(\mathbf{M}\mathbb{L},z)$ and $\mathcal{F}(\mathbf{M}\mathbb{L},z)$ are defined, and $\mathcal{G}(\mathbf{M}\mathbb{L},z) = \mathrm{Sim}_{\mathbf{\Omega L}}\,\mathcal{G}(\mathbf{M},\mathbb{i}z)\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1)$, $\mathcal{F}(\mathbf{M}\mathbb{L},z) = \mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}\,\mathcal{F}(\mathbf{M},\mathbb{i}z)\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1)$.*

Proof. Let $\mathbf{M} = (\mathbf{M}^-\,|\,\mathbf{M}^+)$ for $\mathbf{M}^-,\mathbf{M}^+\in\mathrm{gl}(4,\mathbb{C})$. Denote $\mathbf{N} = \mathbf{M}\mathbb{L}$ and let $\mathbf{N} = (\mathbf{N}^-\,|\,\mathbf{N}^+)$ for $\mathbf{N}^-,\mathbf{N}^+\in\mathrm{gl}(4,\mathbb{C})$. Then by (3.3), $(\mathbf{N}^-\,|\,\mathbf{N}^+) = (\mathbf{M}^-\mathbf{L}\,|\,\mathbf{M}^+\mathbf{L})$, hence $\mathbf{N}^- = \mathbf{M}^-\mathbf{L}$, $\mathbf{N}^+ = \mathbf{M}^+\mathbf{L}$. So by Lemma 4.4 (c), there exists $\mathbf{C}\in GL(4,\mathbb{C})$ such that

$$\begin{gathered}\mathbf{N}^\pm\mathbf{W}_0e^{\pm\mathbf{\Omega}z} = \mathbf{M}^\pm\mathbf{L}\mathbf{W}_0e^{\pm\mathbf{\Omega}z} = \mathbf{C}\cdot\mathbf{M}^\pm\mathbf{W}_0e^{\pm\mathbf{\Omega}\cdot\mathbb{i}z}\cdot\mathbf{\Omega L},\\ \mathbf{N}^-\mathbf{W}_0e^{-\mathbf{\Omega}z} + \mathbf{N}^+\mathbf{W}_0e^{\mathbf{\Omega}z} = \mathbf{C}\left\{\mathbf{M}^-\mathbf{W}_0e^{-\mathbf{\Omega}\cdot\mathbb{i}z} + \mathbf{M}^+\mathbf{W}_0e^{\mathbf{\Omega}\cdot\mathbb{i}z}\right\}\mathbf{\Omega L}.\end{gathered} \tag{4.13}$$

Suppose that $\mathcal{F}(\mathbf{M},z)$ is defined, so that $\mathcal{G}(\mathbf{M},z)$ is also defined by Definition 3.2. By Lemma 4.5 (b), $\mathcal{G}(\mathbf{M},\mathbb{i}z)$ is defined, hence, by Lemma 4.5 (a) and (4.13), $\mathcal{G}(\mathbf{N},z) = \mathcal{G}(\mathbf{M}\mathbb{L},z)$ is defined, since $\mathbf{C}$ and $\mathbf{\Omega L}$ are invertible. By Lemma 4.2 (c), $\mathbf{N}$ is separable, hence, by Lemma 4.5 (c), and (4.11), (4.13),

$$\begin{aligned}\mathcal{G}(\mathbf{M}\mathbb{L},z) = \mathcal{G}(\mathbf{N},z) &= \left\{\mathbf{N}^-\mathbf{W}_0e^{-\mathbf{\Omega}z} + \mathbf{N}^+\mathbf{W}_0e^{\mathbf{\Omega}z}\right\}^{-1}\mathbf{N}^+\mathbf{W}_0e^{\mathbf{\Omega}z}\mathcal{E} - \mathrm{diag}(1,0,0,1)\\ &= (\mathbf{\Omega L})^{-1}\left\{\mathbf{M}^-\mathbf{W}_0e^{-\mathbf{\Omega}\cdot\mathbb{i}z} + \mathbf{M}^+\mathbf{W}_0e^{\mathbf{\Omega}\cdot\mathbb{i}z}\right\}^{-1}\mathbf{C}^{-1}\cdot\mathbf{C}\cdot\mathbf{M}^+\mathbf{W}_0e^{\mathbf{\Omega}\cdot\mathbb{i}z}\cdot\mathbf{\Omega L}\mathcal{E} - \mathrm{diag}(1,0,0,1)\\ &= (\mathbf{\Omega L})^{-1}\left[\left\{\mathbf{M}^-\mathbf{W}_0e^{-\mathbf{\Omega}\cdot\mathbb{i}z} + \mathbf{M}^+\mathbf{W}_0e^{\mathbf{\Omega}\cdot\mathbb{i}z}\right\}^{-1}\mathbf{M}^+\mathbf{W}_0e^{\mathbf{\Omega}\cdot\mathbb{i}z}\mathcal{E}\right]\mathcal{E}^{-1}\mathbf{\Omega L}\mathcal{E} - \mathrm{diag}(1,0,0,1)\\ &= (\mathbf{\Omega L})^{-1}\left\{\mathcal{G}(\mathbf{M},\mathbb{i}z) + \mathrm{diag}(1,0,0,1)\right\}\mathbf{\Omega L}(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(1,0,0,1).\end{aligned} \tag{4.14}$$

By Definitions 2.2, 2.3, and 2.4,

$$\begin{aligned}&(\mathbf{\Omega L})^{-1}\,\mathrm{diag}(1,0,0,1)\mathbf{\Omega L}(\mathbf{J}\oplus\mathbf{J}) = \mathbf{L}^{-1}\cdot\mathbf{\Omega}^{-1}\,\mathrm{diag}(1,0,0,1)\mathbf{\Omega}\cdot\mathbf{L}(\mathbf{J}\oplus\mathbf{J})\\ &\quad= \mathbf{L}^{-1}\,\mathrm{diag}(1,0,0,1)\mathbf{L}\cdot(\mathbf{J}\oplus\mathbf{J}) = \mathrm{diag}(1,1,0,0)(\mathbf{J}\oplus\mathbf{J}) = \mathrm{diag}(1,-1,0,0),\end{aligned}$$

hence, by (4.14), $\mathcal{G}(\mathbf{M}\mathbb{L},z) = (\mathbf{\Omega L})^{-1}\cdot\mathcal{G}(\mathbf{M},\mathbb{i}z)\cdot\mathbf{\Omega L}\cdot(\mathbf{J}\oplus\mathbf{J}) + \mathrm{diag}(1,-1,0,0) - \mathrm{diag}(1,0,0,1)$, from which follows the first equality. By Definition 3.2 and the first equality,

$$\begin{aligned}\mathcal{F}(\mathbf{M}\mathbb{L},z) &= \mathrm{Sim}_{\mathbf{V}^T}\,\mathcal{G}(\mathbf{M}\mathbb{L},z) = \mathrm{Sim}_{\mathbf{V}^T}\left\{\mathrm{Sim}_{\mathbf{\Omega L}}\,\mathcal{G}(\mathbf{M},\mathbb{i}z)\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1)\right\}\\ &= \mathrm{Sim}_{\mathbf{V}^T}\left\{\mathrm{Sim}_{\mathbf{\Omega L}}\,\mathcal{G}(\mathbf{M},\mathbb{i}z)\right\}\cdot\mathrm{Sim}_{\mathbf{V}^T}(\mathbf{J}\oplus\mathbf{J}) - \mathrm{Sim}_{\mathbf{V}^T}\,\mathrm{diag}(0,1,0,1)\end{aligned}$$

$$= \mathrm{Sim}_{\mathbf{V}^T}\left[\mathrm{Sim}_{\mathbf{\Omega L}}\left\{\mathrm{Sim}_{\mathbf{V}}\,\mathcal{F}(\mathbf{M}, \mathbb{i}z)\right\}\right]\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1),$$

since $\mathrm{Sim}_{\mathbf{V}^T}(\mathbf{J}\oplus\mathbf{J}) = \mathbf{J}\oplus\mathbf{J}$ and $\mathrm{Sim}_{\mathbf{V}^T}\,\mathrm{diag}(0,1,0,1) = \mathrm{diag}(0,1,0,1)$ by Lemma 2.2. Thus the proof is complete, since $\mathrm{Sim}_{\mathbf{V}^T}\,\mathrm{Sim}_{\mathbf{\Omega L}}\,\mathrm{Sim}_{\mathbf{V}} = \mathrm{Sim}_{\mathbf{V}\cdot\mathbf{\Omega L}\cdot\mathbf{V}^T} = \mathrm{Sim}_{-\omega(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}} = \mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}$ by (4.12). □

By Definitions 2.3, 2.4, and (2.2),

$$\begin{aligned}
\mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}(\mathbf{J}\oplus\mathbf{J}) &= \left\{\left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\mathbf{R}\right\}^{-1}(\mathbf{J}\oplus\mathbf{J})\left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\mathbf{R} = \mathbf{R}\cdot\left(\mathbf{K}\oplus\overline{\mathbf{K}}\right)(\mathbf{J}\oplus\mathbf{J})\left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\cdot\mathbf{R} \\
&= \mathbf{R}(\mathbf{J}\oplus\mathbf{J})\mathbf{R} = -(\mathbf{J}\oplus\mathbf{J}), \qquad (4.15)\\
\mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}\,\mathrm{diag}(0,1,0,1) &= \mathbf{R}\cdot\left(\mathbf{K}\oplus\overline{\mathbf{K}}\right)\mathrm{diag}(0,1,0,1)\left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\cdot\mathbf{R} = \mathbf{R}\,\mathrm{diag}(0,1,0,1)\mathbf{R}\\
&= \mathrm{diag}(1,0,1,0), \qquad (4.16)\\
\mathrm{Sim}_{\mathbf{R}(\mathbf{K}\oplus\overline{\mathbf{K}})}\,\mathrm{diag}(0,1,0,1) &= \left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\cdot\mathbf{R}\,\mathrm{diag}(0,1,0,1)\mathbf{R}\cdot\left(\mathbf{K}\oplus\overline{\mathbf{K}}\right) = \left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\mathrm{diag}(1,0,1,0)\left(\mathbf{K}\oplus\overline{\mathbf{K}}\right)\\
&= \mathrm{diag}(1,0,1,0). \qquad (4.17)
\end{aligned}$$

Lemma 4.8. *Let $z \in \mathbb{C}$, and let $\mathbf{M} \in \mathrm{gl}(4,8,\mathbb{C})$ be separable. Suppose that $\mathcal{F}(\mathbf{M},z)$ is defined. Then $\mathcal{F}\left(\mathbf{M}\mathbb{L}^2, z\right)$ and $\mathcal{F}\left(\mathbf{M}\mathbb{L}^3, z\right)$ are defined, and $\mathcal{F}\left(\mathbf{M}\mathbb{L}^2, z\right) = -\mathrm{Sim}_{\mathcal{E}}\mathcal{F}(\mathbf{M},-z) - \mathbf{I}$, $\mathcal{F}\left(\mathbf{M}\mathbb{L}^3, z\right) = -\mathrm{Sim}_{\mathbf{R}(\mathbf{K}\oplus\overline{\mathbf{K}})}\mathcal{F}(\mathbf{M},-\mathbb{i}z)\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(1,0,1,0)$.*

Proof. Suppose that $\mathcal{F}(\mathbf{M},z)$ is defined. By Lemma 4.7, $\mathcal{F}\left(\mathbf{M}\mathbb{L}^2, z\right) = \mathcal{F}(\mathbf{M}\mathbb{L}\cdot\mathbb{L}, z)$ and $\mathcal{F}\left(\mathbf{M}\mathbb{L}^2\cdot\mathbb{L}, z\right)$ are defined, and

$$\begin{aligned}
\mathcal{F}\left(\mathbf{M}\mathbb{L}^2, z\right) &= \mathcal{F}(\mathbf{M}\mathbb{L}\cdot\mathbb{L}, z) = \mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}\mathcal{F}(\mathbf{M}\mathbb{L}, \mathbb{i}z)\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1)\\
&= \mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}\left[\mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}\mathcal{F}(\mathbf{M},\mathbb{i}\cdot\mathbb{i}z)\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1)\right]\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1)\\
&= \mathrm{Sim}_{\{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}\}^2}\mathcal{F}(\mathbf{M},-z)\cdot\mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}(\mathbf{J}\oplus\mathbf{J})\cdot(\mathbf{J}\oplus\mathbf{J})\\
&\quad - \mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}\,\mathrm{diag}(0,1,0,1)\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1).
\end{aligned} \qquad (4.18)$$

By Definition 2.4,

$$\left\{\left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\mathbf{R}\right\}^2 = \left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\cdot\mathbf{R}\left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\mathbf{R} = \left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\left\{(\mathbf{RKR})\oplus\left(\mathbf{R}\overline{\mathbf{K}}\mathbf{R}\right)\right\} = \left(\overline{\mathbf{K}}\mathbf{R}\cdot\mathbf{KR}\right)\oplus\left(\mathbf{KR}\cdot\overline{\mathbf{K}}\mathbf{R}\right),$$

hence, by (2.2), (2.3), (2.4), and Definition 2.3,

$$\left\{\left(\overline{\mathbf{K}}\oplus\mathbf{K}\right)\mathbf{R}\right\}^2 = (-\mathbb{i}\mathbf{RK}\cdot\mathbf{KR})\oplus(-\mathbf{KR}\cdot\mathbb{i}\mathbf{RK}) = -\mathbb{i}\,(\mathbf{RJR}\oplus\mathbf{J}) = -\mathbb{i}\,\{(-\mathbf{J})\oplus\mathbf{J}\} = \mathbb{i}\mathcal{E}.$$

Thus, by (4.15), (4.16), (4.18),

$$\mathcal{F}\left(\mathbf{M}\mathbb{L}^2, z\right) = \mathrm{Sim}_{\mathcal{E}}\mathcal{F}(\mathbf{M},-z)\cdot\{-(\mathbf{J}\oplus\mathbf{J})\}\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(1,0,1,0)\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1),$$

from which follows the first equality by Definition 2.3.

Let $\mathbf{N} = \mathbf{M}\mathbb{L}^3$, so that $\mathbf{M} = \mathbf{N}\mathbb{L}$ by (3.2). By Definitions 3.2 and Lemma 4.5 (b), $\mathcal{F}(\mathbf{M},-\mathbb{i}z)$ is defined, hence, by Lemma 4.7,

$$\mathcal{F}(\mathbf{M},-\mathbb{i}z) = \mathcal{F}(\mathbf{N}\mathbb{L},-\mathbb{i}z) = \mathrm{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}}\mathcal{F}(\mathbf{N},\mathbb{i}\cdot(-\mathbb{i}z))\cdot(\mathbf{J}\oplus\mathbf{J}) - \mathrm{diag}(0,1,0,1)$$

$$= \operatorname{Sim}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}} \mathcal{F}\left(\mathbf{M}\mathbb{L}^3, z\right) \cdot (\mathbf{J}\oplus\mathbf{J}) - \operatorname{diag}(0,1,0,1),$$

hence, by (2.2), (2.3),

$$\begin{aligned}
\mathcal{F}\left(\mathbf{M}\mathbb{L}^3, z\right) &= \operatorname{Sim}_{\{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}\}^{-1}} \left[\{\mathcal{F}(\mathbf{M}, -\mathbb{i}z) + \operatorname{diag}(0,1,0,1)\} \cdot (\mathbf{J}\oplus\mathbf{J})\right] \\
&= \left[\operatorname{Sim}_{\{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}\}^{-1}} \mathcal{F}(\mathbf{M}, -\mathbb{i}z) + \operatorname{Sim}_{\{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}\}^{-1}} \operatorname{diag}(0,1,0,1)\right] \cdot \operatorname{Sim}_{\{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}\}^{-1}}(\mathbf{J}\oplus\mathbf{J}) \\
&= \left\{\operatorname{Sim}_{\mathbf{R}(\mathbf{K}\oplus\overline{\mathbf{K}})} \mathcal{F}(\mathbf{M}, -\mathbb{i}z) + \operatorname{Sim}_{\mathbf{R}(\mathbf{K}\oplus\overline{\mathbf{K}})} \operatorname{diag}(0,1,0,1)\right\} \cdot \left[-\operatorname{Sim}^{-1}_{(\overline{\mathbf{K}}\oplus\mathbf{K})\mathbf{R}} \{-(\mathbf{J}\oplus\mathbf{J})\}\right].
\end{aligned}$$

Thus the second equality follows by (4.15) and (4.17), and the proof is complete. □

5. Preparation for the seven dwarfs

Definition 5.1. For $z \in \mathbb{C}$ and $m, n \in \{0,1,2,3\}$, denote $\mathbf{D}^-_{mn}(z) = \operatorname{Sim}_{\mathbf{V}^T}\left(\mathbf{E}_{mn}\mathbf{W}_0 e^{-\mathbf{\Omega}z}\right)$ and $\mathbf{D}^+_{mn}(z) = \operatorname{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{mn}\mathbf{W}_0 e^{\mathbf{\Omega}z}\right)$. Denote $\mathbf{D}_{pqrs}(z) = \left\{\mathbf{D}^-_{pq}(z) + \mathbf{D}^+_{rs}(z)\right\}^{-1}\mathbf{D}^-_{pq}(z)$ for $p, q, r, s \in \{0,1,2,3\}$, $p < q$, $r < s$, and $z \in \mathbb{C}$ such that $\det\left\{\mathbf{D}^-_{pq}(z) + \mathbf{D}^+_{rs}(z)\right\} \neq 0$.

Lemma 5.1. *Let $z \in \mathbb{C}$, and $p, q, r, s \in \{0,1,2,3\}$, $p < q$, $r < s$. $\mathcal{F}\left(\mathbf{E}_{pqrs}, z\right)$ is defined if and only if $\det\left\{\mathbf{D}^-_{pq}(z) + \mathbf{D}^+_{rs}(z)\right\} \neq 0$. If $\mathcal{F}\left(\mathbf{E}_{pqrs}, z\right)$ is defined, then*

$$\mathcal{F}\left(\mathbf{E}_{pqrs}, z\right) = \left\{\mathbf{D}_{pqrs}(z) - \frac{1}{2}\mathbf{I}\right\}(\mathbf{J}\oplus'\mathbf{J}) - \frac{1}{2}\mathbf{I} = \begin{pmatrix} \left\{\mathbf{D}_{pqrs}(z)\right\}_{1:2,3:4} \cdot \mathbf{J} - \frac{1}{2}\mathbf{I} & \left[\left\{\mathbf{D}_{pqrs}(z)\right\}_{1:2,1:2} - \frac{1}{2}\mathbf{I}\right]\mathbf{J} \\ \left[\left\{\mathbf{D}_{pqrs}(z)\right\}_{3:4,3:4} - \frac{1}{2}\mathbf{I}\right]\mathbf{J} & \left\{\mathbf{D}_{pqrs}(z)\right\}_{3:4,1:2} \cdot \mathbf{J} - \frac{1}{2}\mathbf{I} \end{pmatrix}.$$

Proof. Let $\mathbf{E}_{pqrs} = \left(\mathbf{E}^-_{pqrs} \,|\, \mathbf{E}^+_{pqrs}\right)$, where $\mathbf{E}^\pm_{pqrs} \in \operatorname{gl}(4,\mathbb{C})$. By (2.5), (4.2), and Definition 5.1,

$$\mathbf{E}^-_{pqrs}\mathbf{W}_0 e^{-\mathbf{\Omega}z} = \mathbf{E}_{pq}\mathbf{W}_0 e^{-\mathbf{\Omega}z} = \operatorname{Sim}_{\mathbf{V}} \mathbf{D}^-_{pq}(z), \qquad \mathbf{E}^+_{pqrs}\mathbf{W}_0 e^{\mathbf{\Omega}z} = \mathbf{L}^2\mathbf{E}_{rs}\mathbf{W}_0 e^{\mathbf{\Omega}z} = \operatorname{Sim}_{\mathbf{V}} \mathbf{D}^+_{rs}(z), \tag{5.1}$$

$$\mathbf{E}^-_{pqrs}\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{E}^+_{pqrs}\mathbf{W}_0 e^{\mathbf{\Omega}z} = \operatorname{Sim}_{\mathbf{V}}\left\{\mathbf{D}^-_{pq}(z) + \mathbf{D}^+_{rs}(z)\right\}. \tag{5.2}$$

Note from Definition 3.2 that $\mathcal{F}\left(\mathbf{E}_{pqrs}, z\right)$ is defined if and only if $\mathcal{G}\left(\mathbf{E}_{pqrs}, z\right)$ is defined. So (a) follows by Lemma 4.1, Lemma 4.5 (a), and (5.2), since $\mathbf{V}$ is invertible.

Suppose that $\mathcal{F}\left(\mathbf{E}_{pqrs}, z\right)$ is defined. By Definition 5.1, Lemma 4.1, Lemma 4.5 (c), and (5.1), (5.2),

$$\begin{aligned}
\mathcal{G}\left(\mathbf{E}_{pqrs}, z\right) &= -\left\{\mathbf{E}^-_{pqrs}\mathbf{W}_0 e^{-\mathbf{\Omega}z} + \mathbf{E}^+_{pqrs}\mathbf{W}_0 e^{\mathbf{\Omega}z}\right\}^{-1} \mathbf{E}^-_{pqrs}\mathbf{W}_0 e^{-\mathbf{\Omega}z}\mathcal{E} - \operatorname{diag}(0,1,1,0) \\
&= -\left[\operatorname{Sim}_{\mathbf{V}}\left\{\mathbf{D}^-_{pq}(z) + \mathbf{D}^+_{rs}(z)\right\}\right]^{-1} \operatorname{Sim}_{\mathbf{V}} \mathbf{D}^-_{pq}(z) \cdot \mathcal{E} - \operatorname{diag}(0,1,1,0) \\
&= -\operatorname{Sim}_{\mathbf{V}}\left[\left\{\mathbf{D}^-_{pq}(z) + \mathbf{D}^-_{rs}(z)\right\}^{-1}\right] \cdot \operatorname{Sim}_{\mathbf{V}} \mathbf{D}^-_{pq}(z) \cdot \mathcal{E} - \operatorname{diag}(0,1,1,0) \\
&= -\operatorname{Sim}_{\mathbf{V}} \mathbf{D}_{pqrs}(z) \cdot \mathcal{E} - \operatorname{diag}(0,1,1,0),
\end{aligned}$$

hence, by Definition 3.2, and (2.6), (2.7),

$$\mathcal{F}\left(\mathbf{E}_{pqrs}, z\right) = \operatorname{Sim}_{\mathbf{V}^T}\left\{\mathcal{G}\left(\mathbf{E}_{pqrs}, z\right)\right\} = -\mathbf{D}_{pqrs}(z) \cdot \operatorname{Sim}_{\mathbf{V}^T} \mathcal{E} - \operatorname{Sim}_{\mathbf{V}^T} \operatorname{diag}(0,1,1,0)$$

$$= \mathbf{D}_{pqrs}(z) \cdot (\mathbf{J} \oplus' \mathbf{J}) - \frac{1}{2}\{\mathbf{I} + (\mathbf{J} \oplus' \mathbf{J})\} = \left\{\mathbf{D}_{pqrs}(z) - \frac{1}{2}\mathbf{I}\right\}(\mathbf{J} \oplus' \mathbf{J}) - \frac{1}{2}\mathbf{I}$$

$$= \begin{pmatrix} \left\{\mathbf{D}_{pqrs}(z)\right\}_{1:2,1:2} - \frac{1}{2}\mathbf{I} & \left\{\mathbf{D}_{pqrs}(z)\right\}_{1:2,3:4} \\ \left\{\mathbf{D}_{pqrs}(z)\right\}_{3:4,1:2} & \left\{\mathbf{D}_{pqrs}(z)\right\}_{3:4,3:4} - \frac{1}{2}\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{J} \\ \mathbf{J} & \mathbf{O} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix},$$

from which follows the result. □

The following 2×2 matrices will play special roles in further computations. See Appendix B for their properties.

Definition 5.2. For $z \in \mathbb{C}$, denote

$$\mathbf{A}(z) = \begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega} z) \\ \omega \sinh(\omega z) & \overline{\omega} \sinh(\overline{\omega} z) \end{pmatrix}, \qquad \mathbf{B}(z) = \begin{pmatrix} \sinh(\omega z) & -\sinh(\overline{\omega} z) \\ \omega \cosh(\omega z) & -\overline{\omega} \cosh(\overline{\omega} z) \end{pmatrix},$$

$$\mathbf{C}(z) = \begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega} z) \\ \mathbb{i} \cosh(\omega z) & -\mathbb{i} \cosh(\overline{\omega} z) \end{pmatrix}, \qquad \mathbf{D}(z) = \begin{pmatrix} \sinh(\omega z) & -\sinh(\overline{\omega} z) \\ \mathbb{i} \sinh(\omega z) & \mathbb{i} \sinh(\overline{\omega} z) \end{pmatrix}.$$

Lemma 5.2. *The following hold for* $z \in \mathbb{C}$.

(a) $\mathbf{D}^-_{01}(z) = \begin{pmatrix} \mathbf{JA}(z) & \mathbf{JB}(z) \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix}$.

(b) $\mathbf{D}^-_{02}(z) = \begin{pmatrix} \mathbf{C}(z) & \mathbf{D}(z) \\ -\mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix}$.

(c) $\mathbf{D}^+_{01}(z) = \begin{pmatrix} \mathbf{A}(z) & -\mathbf{B}(z) \\ \mathbf{A}(z) & -\mathbf{B}(z) \end{pmatrix}$.

(d) $\mathbf{D}^+_{02}(z) = \begin{pmatrix} \mathbf{C}(z) & -\mathbf{D}(z) \\ \mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix}$.

(e) $\mathbf{D}^+_{12}(z) = \omega \begin{pmatrix} \mathbf{B}(z)\mathbf{K} & -\mathbf{A}(z)\mathbf{K} \\ \mathbf{B}(z)\mathbf{K} & -\mathbf{A}(z)\mathbf{K} \end{pmatrix}$.

(f) $\mathbf{D}^+_{13}(z) = \omega \begin{pmatrix} \mathbf{D}(z)\mathbf{K} & -\mathbf{C}(z)\mathbf{K} \\ \mathbf{D}(z)\mathbf{K} & -\mathbf{C}(z)\mathbf{K} \end{pmatrix}$.

(g) $\mathbf{D}^+_{23}(z) = \mathbb{i} \begin{pmatrix} \mathbf{A}(z)\mathbf{J} & -\mathbf{B}(z)\mathbf{J} \\ \mathbf{A}(z)\mathbf{J} & -\mathbf{B}(z)\mathbf{J} \end{pmatrix}$.

The proof of Lemma 5.2 will be given at the end of this section.

Lemma 5.3. $\mathrm{Sim}_{\mathbf{V}^T} \mathbf{W}_0 = \sqrt{2}\mathbf{V}_{23}\left[\omega(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J} \oplus \{-\mathbb{i}(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{JK}\}\right]$.

Proof. By Definitions 2.2, 2.5, and (2.1),

$$\mathrm{Sim}_{\mathbf{V}^T} \mathbf{W}_0 = \mathrm{Sim}_{\mathbf{V}^T} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_4^2 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 & \omega_4^3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ \omega & -\overline{\omega} & -\omega & \overline{\omega} \\ \mathbb{i} & -\mathbb{i} & \mathbb{i} & -\mathbb{i} \\ -\overline{\omega} & \omega & \overline{\omega} & -\omega \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix}$$

$$= \frac{1}{2} \cdot \sqrt{2} \begin{pmatrix} \omega & \overline{\omega} & \omega & \overline{\omega} \\ \mathbb{i} & \mathbb{i} & -\mathbb{i} & -\mathbb{i} \\ -\overline{\omega} & -\omega & -\overline{\omega} & -\omega \\ -1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot 2 \begin{pmatrix} \omega & \overline{\omega} & 0 & 0 \\ 0 & 0 & -\mathbb{i} & -\mathbb{i} \\ -\overline{\omega} & -\omega & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \sqrt{2}\mathbf{V}_{23} \begin{pmatrix} \omega & \overline{\omega} & 0 & 0 \\ -\overline{\omega} & -\omega & 0 & 0 \\ 0 & 0 & -\mathbb{i} & -\mathbb{i} \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$= \sqrt{2}\mathbf{V}_{23}(\mathbf{A} \oplus \mathbf{B}), \tag{5.3}$$

where we put $\mathbf{A} = \begin{pmatrix} \omega & \overline{\omega} \\ -\overline{\omega} & -\omega \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -\mathbb{i} & -\mathbb{i} \\ 1 & -1 \end{pmatrix}$. By Definitions 2.3, 2.4, and (2.1), (2.2),

$$\mathbf{A} = \begin{pmatrix} \omega & \overline{\omega} \\ -\overline{\omega} & -\omega \end{pmatrix} = \omega \begin{pmatrix} 1 & -\mathbb{i} \\ \mathbb{i} & -1 \end{pmatrix} = \omega(\mathbf{J} + \mathbb{i}\mathbf{RJ}) = \omega(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J},$$
$$\mathbf{B} = \begin{pmatrix} -\mathbb{i} & -\mathbb{i} \\ 1 & -1 \end{pmatrix} = -\mathbb{i} \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{i} \end{pmatrix} + \begin{pmatrix} 0 & -\mathbb{i} \\ 1 & 0 \end{pmatrix} = -\mathbb{i}\mathbf{JK} + \mathbf{RJK} = -\mathbb{i}(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{JK}.$$

Thus the result follows from (5.3). □

Lemma 5.4. *For* $z \in \mathbb{C}$, $\mathrm{Sim}_{\mathbf{V}^T}\, e^{\pm\mathbf{\Omega}z} = \begin{pmatrix} \cosh(\omega z, \overline{\omega}z) & \pm\sinh(-\omega z, \overline{\omega}z) \\ \pm\sinh(-\omega z, \overline{\omega}z) & \cosh(\omega z, \overline{\omega}z) \end{pmatrix}$, *where we denote* $\cosh(\zeta_1, \zeta_2) = \mathrm{diag}(\cosh\zeta_1, \cosh\zeta_2)$, $\sinh(\zeta_1, \zeta_2) = \mathrm{diag}(\sinh\zeta_1, \sinh\zeta_2)$ *for* $\zeta_1, \zeta_2 \in \mathbb{C}$.

Proof. By Definitions 2.2, 2.5, and (2.1),

$$\begin{aligned}
\mathrm{Sim}_{\mathbf{V}^T}\, e^{\pm\mathbf{\Omega}z} &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{diag}\left(e^{\pm\omega z}, e^{\mp\overline{\omega}z}\right) & \mathbf{O} \\ \mathbf{O} & \mathrm{diag}\left(e^{\mp\omega z}, e^{\pm\overline{\omega}z}\right) \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \mathrm{diag}\left(e^{\pm\omega z}, e^{\mp\overline{\omega}z}\right) & \mathrm{diag}\left(e^{\mp\omega z}, e^{\pm\overline{\omega}z}\right) \\ -\mathrm{diag}\left(e^{\pm\omega z}, e^{\mp\overline{\omega}z}\right) & \mathrm{diag}\left(e^{\mp\omega z}, e^{\pm\overline{\omega}z}\right) \end{pmatrix} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \mathrm{diag}\left(e^{\pm\omega z}, e^{\mp\overline{\omega}z}\right) + \mathrm{diag}\left(e^{\mp\omega z}, e^{\pm\overline{\omega}z}\right) & -\mathrm{diag}\left(e^{\pm\omega z}, e^{\mp\overline{\omega}z}\right) + \mathrm{diag}\left(e^{\mp\omega z}, e^{\pm\overline{\omega}z}\right) \\ -\mathrm{diag}\left(e^{\pm\omega z}, e^{\mp\overline{\omega}z}\right) + \mathrm{diag}\left(e^{\mp\omega z}, e^{\pm\overline{\omega}z}\right) & \mathrm{diag}\left(e^{\pm\omega z}, e^{\mp\overline{\omega}z}\right) + \mathrm{diag}\left(e^{\mp\omega z}, e^{\pm\overline{\omega}z}\right) \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{diag}(\cosh(\omega z), \cosh(\overline{\omega}z)) & \pm\mathrm{diag}(-\sinh(\omega z), \sinh(\overline{\omega}z)) \\ \pm\mathrm{diag}(-\sinh(\omega z), \sinh(\overline{\omega}z)) & \mathrm{diag}(\cosh(\omega z), \cosh(\overline{\omega}z)) \end{pmatrix},
\end{aligned}$$

from which the result follows. □

By Lemma 5.4,

$$\mathrm{Sim}_{\mathbf{I}\oplus(-\mathbf{I})}\left(\mathrm{Sim}_{\mathbf{V}^T}\, e^{\pm\mathbf{\Omega}z}\right) = \mathrm{Sim}_{\mathbf{V}^T}\, e^{\mp\mathbf{\Omega}z}, \qquad z \in \mathbb{C}, \tag{5.4}$$

since

$$\mathrm{Sim}_{\mathbf{I}\oplus(-\mathbf{I})} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ -\mathbf{C} & \mathbf{D} \end{pmatrix}, \quad \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathrm{gl}(2, \mathbb{C}). \tag{5.5}$$

By Definitions 2.3, 2.4, 2.5, and (2.5),

$$\mathrm{Sim}_{\mathbf{V}_{23}}\{\mathbf{I} \oplus (-\mathbf{I})\} = \mathbf{V}_{23} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \mathbf{V}_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \mathbf{V}_{23} = \mathbf{J} \oplus \mathbf{J}, \tag{5.6}$$

$$\mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mathbf{V}^T = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \mathbf{R} \oplus (\mathbf{J}\mathbf{R}), \tag{5.7}$$

$$\mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{L} \cdot \mathbf{V}_{23} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \mathbf{V}_{23} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \mathbf{V}_{23} \cdot (\mathbf{J} \oplus' \mathbf{I}). \tag{5.8}$$

Lemma 5.5. *The following* (a) *and* (b) *hold for* $z \in \mathbb{C}$.

(a) $\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0 e^{\pm\mathbf{\Omega}z}\right) \cdot \{\mathbf{I} \oplus (-\mathbf{I})\} = (\mathbf{J} \oplus \mathbf{J}) \cdot \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0 e^{\mp\mathbf{\Omega}z}\right)$.

(b) $\mathbf{V}_{23}\, \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}\mathbf{W}_0 e^{\pm\mathbf{\Omega}z}\right) \cdot (\mathbf{K} \oplus' \mathbf{K})^{-1} = -\omega \{\mathbf{I} \oplus (-\mathbf{R})\} \cdot \mathbf{V}_{23}\, \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0 e^{\pm\mathbf{\Omega}z}\right)$.

Proof. By Lemma 5.3 and (5.5),

$$\begin{aligned} \frac{1}{\sqrt{2}}\mathbf{V}_{23}\, \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0 \cdot \{\mathbf{I} \oplus (-\mathbf{I})\} &= \{\mathbf{I} \oplus (-\mathbf{I})\} \cdot \mathrm{Sim}_{\mathbf{I}\oplus(-\mathbf{I})}\left[\frac{1}{\sqrt{2}}\mathbf{V}_{23}\, \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0\right] \\ &= \{\mathbf{I} \oplus (-\mathbf{I})\} \cdot \frac{1}{\sqrt{2}}\mathbf{V}_{23}\, \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0, \end{aligned}$$

hence, by (2.5) and (5.6), $\mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0 \cdot \{\mathbf{I} \oplus (-\mathbf{I})\} = \mathrm{Sim}_{\mathbf{V}_{23}}\, \{\mathbf{I} \oplus (-\mathbf{I})\} \cdot \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0 = (\mathbf{J} \oplus \mathbf{J}) \cdot \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0$. Thus, by (5.4),

$$\begin{aligned} &\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0 e^{\pm\mathbf{\Omega}z}\right) \cdot \{\mathbf{I} \oplus (-\mathbf{I})\} = \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0 \cdot \mathrm{Sim}_{\mathbf{V}^T}\, e^{\pm\mathbf{\Omega}z} \cdot \{\mathbf{I} \oplus (-\mathbf{I})\} \\ &= \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0 \cdot \{\mathbf{I} \oplus (-\mathbf{I})\} \cdot \mathrm{Sim}_{\mathbf{I}\oplus(-\mathbf{I})}\left(\mathrm{Sim}_{\mathbf{V}^T}\, e^{\pm\mathbf{\Omega}z}\right) = (\mathbf{J} \oplus \mathbf{J}) \cdot \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0 \cdot \mathrm{Sim}_{\mathbf{V}^T}\, e^{\mp\mathbf{\Omega}z}, \end{aligned}$$

from which follows (a). By Lemma 2.2, $\mathrm{Sim}_{\mathbf{V}^T}\left\{\left(-\overline{\mathbf{K}}\right) \oplus \overline{\mathbf{K}}\right\} = \overline{\mathbf{K}} \oplus' \overline{\mathbf{K}}$. So we have

$$\begin{aligned} \mathrm{Sim}_{\mathbf{V}^T}\, e^{\pm\mathbf{\Omega}z} \cdot \left(\overline{\mathbf{K}} \oplus' \overline{\mathbf{K}}\right) &= \mathrm{Sim}_{\mathbf{V}^T}\left[e^{\pm\mathbf{\Omega}z}\left\{\left(-\overline{\mathbf{K}}\right) \oplus \overline{\mathbf{K}}\right\}\right] = \mathrm{Sim}_{\mathbf{V}^T}\left[\left\{\left(-\overline{\mathbf{K}}\right) \oplus \overline{\mathbf{K}}\right\} e^{\pm\mathbf{\Omega}z}\right] \\ &= \mathrm{Sim}_{\mathbf{V}^T}\left\{\left(-\overline{\mathbf{K}}\right) \oplus \overline{\mathbf{K}}\right\} \cdot \mathrm{Sim}_{\mathbf{V}^T}\, e^{\pm\mathbf{\Omega}z} = \left(\overline{\mathbf{K}} \oplus' \overline{\mathbf{K}}\right) \cdot \mathrm{Sim}_{\mathbf{V}^T}\, e^{\pm\mathbf{\Omega}z}, \end{aligned} \tag{5.9}$$

where we used the fact that $e^{\pm\mathbf{\Omega}z}$ and $\left(-\overline{\mathbf{K}}\right) \oplus \overline{\mathbf{K}}$ commute since they are diagonal.

Since $(\mathbf{I} + \mathbb{i}\mathbf{R})(\mathbf{I} - \mathbb{i}\mathbf{R}) = 2\mathbf{I}$ by (2.3), we have $(\mathbf{I} \pm \mathbb{i}\mathbf{R})^{-1} = \frac{1}{2}(\mathbf{I} \mp \mathbb{i}\mathbf{R})$, hence, by Lemma 5.3, and (2.1), (2.2),

$$\begin{aligned} &\left(\frac{1}{\sqrt{2}}\mathbf{V}_{23}\, \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0\right) \cdot \left(\overline{\mathbf{K}} \oplus' \overline{\mathbf{K}}\right) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{V}_{23}\, \mathrm{Sim}_{\mathbf{V}^T}\, \mathbf{W}_0\right)^{-1} \\ &= \begin{pmatrix} \omega(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J} & \mathbf{O} \\ \mathbf{O} & -\mathbb{i}(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J}\mathbf{K} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \overline{\mathbf{K}} \\ \overline{\mathbf{K}} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \omega(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J} & \mathbf{O} \\ \mathbf{O} & -\mathbb{i}(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J}\mathbf{K} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathbf{O} & \omega(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J}\overline{\mathbf{K}} \\ -\mathbb{i}(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\overline{\omega}\mathbf{J}(\mathbf{I} - \mathbb{i}\mathbf{R}) & \mathbf{O} \\ \mathbf{O} & \frac{1}{2}\mathbb{i}\overline{\mathbf{K}}\mathbf{J}(\mathbf{I} - \mathbb{i}\mathbf{R}) \end{pmatrix} = \begin{pmatrix} \mathbf{O} & -\frac{1}{2}\overline{\omega}(\mathbf{I} + \mathbb{i}\mathbf{R})\mathbf{J}(\mathbf{I} - \mathbb{i}\mathbf{R}) \\ -\omega\mathbf{I} & \mathbf{O} \end{pmatrix}, \end{aligned}$$

hence

$$\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{W}_0\cdot\left(\overline{\mathbf{K}}\oplus'\overline{\mathbf{K}}\right)=-\omega(\mathbf{I}\oplus'\mathbf{RJ})\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{W}_0,\tag{5.10}$$

since $\overline{\omega}(\mathbf{I}+\mathbb{i}\mathbf{R})\cdot\mathbf{J}(\mathbf{I}-\mathbb{i}\mathbf{R})=\overline{\omega}(\mathbf{I}+\mathbb{i}\mathbf{R})\cdot(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J}=\overline{\omega}\cdot 2\mathbb{i}\mathbf{RJ}=2\omega\mathbf{RJ}$ by (2.1), (2.3) and (2.4). By (2.2), (2.4), and (5.8),

$$\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}\cdot\mathbf{V}_{23}\cdot(\mathbf{I}\oplus'\mathbf{RJ})=\mathbf{V}_{23}(\mathbf{J}\oplus'\mathbf{I})(\mathbf{I}\oplus'\mathbf{RJ})=\mathbf{V}_{23}\begin{pmatrix}\mathbf{O}&\mathbf{I}\\\mathbf{J}&\mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{O}&\mathbf{RJ}\\\mathbf{I}&\mathbf{O}\end{pmatrix}=\mathbf{V}_{23}\{\mathbf{I}\oplus(-\mathbf{R})\}.\tag{5.11}$$

By (5.9), (5.10), and (5.11),

$$\begin{aligned}&\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{LW}_0e^{\pm\mathbf{\Omega}z}\right)\cdot\left(\overline{\mathbf{K}}\oplus'\overline{\mathbf{K}}\right)=\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}(\mathbf{LW}_0)\cdot\mathrm{Sim}_{\mathbf{V}^T}\,e^{\pm\mathbf{\Omega}z}\cdot\left(\overline{\mathbf{K}}\oplus'\overline{\mathbf{K}}\right)\\&=\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}\cdot\mathbf{V}_{23}\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{W}_0\cdot\left(\overline{\mathbf{K}}\oplus'\overline{\mathbf{K}}\right)\cdot\mathrm{Sim}_{\mathbf{V}^T}\,e^{\pm\mathbf{\Omega}z}\\&=\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}\cdot\mathbf{V}_{23}\cdot(-\omega)(\mathbf{I}\oplus'\mathbf{RJ})\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{W}_0\cdot\mathrm{Sim}_{\mathbf{V}^T}\,e^{\pm\mathbf{\Omega}z}\\&=-\omega\mathbf{V}_{23}\cdot\mathbf{V}_{23}\{\mathbf{I}\oplus(-\mathbf{R})\}\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\pm\mathbf{\Omega}z}\right)=-\omega\{\mathbf{I}\oplus(-\mathbf{R})\}\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\pm\mathbf{\Omega}z}\right).\end{aligned}$$

Thus (b) follows since $\left(\overline{\mathbf{K}}\oplus'\overline{\mathbf{K}}\right)(\mathbf{K}\oplus'\mathbf{K})=\mathbf{I}$ by (2.2) so that $\overline{\mathbf{K}}\oplus'\overline{\mathbf{K}}=(\mathbf{K}\oplus'\mathbf{K})^{-1}$, and the proof is complete. □

By (2.3), (2.4), and (5.7),

$$\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}^2=\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}\cdot\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}=\{\mathbf{R}\oplus(\mathbf{JR})\}\{\mathbf{R}\oplus(\mathbf{JR})\}=\mathbf{I}\oplus(-\mathbf{I}).\tag{5.12}$$

By Definitions 2.3, 2.4, 2.5, and 4.1,

$$\begin{aligned}&\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{E}_{01}=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{I}&\mathbf{I}\\-\mathbf{I}&\mathbf{I}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{I}&\mathbf{O}\\\mathbf{O}&\mathbf{O}\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{I}&-\mathbf{I}\\\mathbf{I}&\mathbf{I}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\mathbf{I}&\mathbf{O}\\-\mathbf{I}&\mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{I}&-\mathbf{I}\\\mathbf{I}&\mathbf{I}\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&-1&0&0\\0&0&1&-1\\-1&1&0&0\\0&0&-1&1\end{pmatrix}\mathbf{V}_{23}=\frac{1}{2}\mathbf{V}_{23}\{(\mathbf{I}-\mathbf{R})\oplus(\mathbf{I}-\mathbf{R})\}\mathbf{V}_{23},\end{aligned}\tag{5.13}$$

$$\begin{aligned}&\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{E}_{02}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\mathbf{V}^T=\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&1&0\\-1&0&0&0\\0&0&-1&0\end{pmatrix}\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\1&0&1&0\\0&1&0&1\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}1&0&-1&0\\1&0&1&0\\-1&0&1&0\\-1&0&-1&0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&-1&0&0\\1&1&0&0\\-1&1&0&0\\-1&-1&0&0\end{pmatrix}\mathbf{V}_{23}=\frac{1}{2}\mathbf{V}_{23}\begin{pmatrix}\mathbf{I}-\mathbf{R}&\mathbf{O}\\\mathbf{J}(\mathbf{I}+\mathbf{R})&\mathbf{O}\end{pmatrix}\mathbf{V}_{23}.\end{aligned}\tag{5.14}$$

Lemma 5.6. *The following* (a), (b), (c) *hold for* $z\in\mathbb{C}$.

(a) $\mathbf{D}^+_{01}(z)=\{\mathbf{J}\oplus(-\mathbf{J})\}\mathbf{D}^-_{01}(z)\{\mathbf{I}\oplus(-\mathbf{I})\}$ *and* $\mathbf{D}^+_{02}(z)=\{\mathbf{I}\oplus(-\mathbf{I})\}\mathbf{D}^-_{02}(z)\{\mathbf{I}\oplus(-\mathbf{I})\}$.
(b) $\mathbf{D}^+_{1\,n+1}(z)=-\omega\mathbf{D}^+_{0n}(z)(\mathbf{K}\oplus'\mathbf{K})$ *for* $n=1,2$.

(c) $\mathbf{D}^+_{23}(z) = \mathbb{i}\mathbf{D}^+_{01}(z)\,(\mathbf{J}\oplus\mathbf{J})$.

Proof. By (5.5), (5.6), (5.12), (5.13), (5.14),

$$\begin{aligned}
&\{\mathbf{J}\oplus(-\mathbf{J})\}\cdot \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{01}\right) = \{\mathbf{J}\oplus(-\mathbf{J})\}\cdot \mathrm{Sim}_{\mathbf{V}^T}\mathbf{L}^2\cdot \mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{01}\\
&= (\mathbf{J}\oplus\mathbf{J})\cdot\frac{1}{2}\mathbf{V}_{23}\{(\mathbf{I}-\mathbf{R})\oplus(\mathbf{I}-\mathbf{R})\}\mathbf{V}_{23} = \frac{1}{2}\mathbf{V}_{23}\{\mathbf{I}\oplus(-\mathbf{I})\}\cdot\{(\mathbf{I}-\mathbf{R})\oplus(\mathbf{I}-\mathbf{R})\}\mathbf{V}_{23}\\
&= \frac{1}{2}\mathbf{V}_{23}\cdot\{(\mathbf{I}-\mathbf{R})\oplus(\mathbf{I}-\mathbf{R})\}\{\mathbf{I}\oplus(-\mathbf{I})\}\cdot\mathbf{V}_{23} = \frac{1}{2}\mathbf{V}_{23}\{(\mathbf{I}-\mathbf{R})\oplus(\mathbf{I}-\mathbf{R})\}\cdot\mathbf{V}_{23}\,(\mathbf{J}\oplus\mathbf{J})\\
&= \mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{01}\cdot(\mathbf{J}\oplus\mathbf{J})\,,
\end{aligned}\tag{5.15}$$

$$\begin{aligned}
&\{\mathbf{I}\oplus(-\mathbf{I})\}\cdot \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{02}\right) = \{\mathbf{I}\oplus(-\mathbf{I})\}\cdot \mathrm{Sim}_{\mathbf{V}^T}\mathbf{L}^2\cdot \mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{02}\\
&= \frac{1}{2}\mathbf{V}_{23}\begin{pmatrix}\mathbf{I}-\mathbf{R} & \mathbf{O}\\ \mathbf{J}(\mathbf{I}+\mathbf{R}) & \mathbf{O}\end{pmatrix}\mathbf{V}_{23} = \frac{1}{2}\mathbf{V}_{23}\begin{pmatrix}\mathbf{I}-\mathbf{R} & \mathbf{O}\\ \mathbf{J}(\mathbf{I}+\mathbf{R}) & \mathbf{O}\end{pmatrix}\{\mathbf{I}\oplus(-\mathbf{I})\}\,\mathbf{V}_{23} = \frac{1}{2}\mathbf{V}_{23}\begin{pmatrix}\mathbf{I}-\mathbf{R} & \mathbf{O}\\ \mathbf{J}(\mathbf{I}+\mathbf{R}) & \mathbf{O}\end{pmatrix}\cdot\mathbf{V}_{23}\,(\mathbf{J}\oplus\mathbf{J})\\
&= \mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{02}\cdot(\mathbf{J}\oplus\mathbf{J})\,.
\end{aligned}\tag{5.16}$$

By Definition 5.1, Lemma 5.5 (a), and (2.2), (5.15), (5.16),

$$\begin{aligned}
&\{\mathbf{J}\oplus(-\mathbf{J})\}\cdot\mathbf{D}^+_{01}(z)\cdot\{\mathbf{I}\oplus(-\mathbf{I})\} = \left[\{\mathbf{J}\oplus(-\mathbf{J})\}\cdot\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{01}\right)\right]\cdot\left[\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot\{\mathbf{I}\oplus(-\mathbf{I})\}\right]\\
&= \mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{01}\cdot(\mathbf{J}\oplus\mathbf{J})\cdot(\mathbf{J}\oplus\mathbf{J})\cdot\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{-\mathbf{\Omega}z}\right) = \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{E}_{01}\mathbf{W}_0e^{-\mathbf{\Omega}z}\right) = \mathbf{D}^-_{01}(z),\\
&\{\mathbf{I}\oplus(-\mathbf{I})\}\cdot\mathbf{D}^+_{02}(z)\cdot\{\mathbf{I}\oplus(-\mathbf{I})\} = \left[\{\mathbf{I}\oplus(-\mathbf{I})\}\cdot\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{02}\right)\right]\cdot\left[\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot\{\mathbf{I}\oplus(-\mathbf{I})\}\right]\\
&= \mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{02}\cdot(\mathbf{J}\oplus\mathbf{J})\cdot(\mathbf{J}\oplus\mathbf{J})\cdot\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{-\mathbf{\Omega}z}\right) = \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{E}_{02}\mathbf{W}_0e^{-\mathbf{\Omega}z}\right) = \mathbf{D}^-_{02}(z).
\end{aligned}$$

Thus (a) follows, since $\{\mathbf{I}\oplus(-\mathbf{I})\}^{-1} = \mathbf{I}\oplus(-\mathbf{I})$ and $\{\mathbf{J}\oplus(-\mathbf{J})\}^{-1} = \mathbf{J}\oplus(-\mathbf{J})$.

Let n be 1 or 2. By Definitions 2.4 and 4.1, $\mathbf{E}_{1\,n+1} = \mathbf{E}_{0n}\mathbf{L}$, hence, by Definition 5.1, $\mathbf{D}^+_{1\,n+1}(z) = \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{0n}\mathbf{L}\mathbf{W}_0e^{\mathbf{\Omega}z}\right) = \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{0n}\right)\cdot\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}\mathbf{W}_0e^{\mathbf{\Omega}z}\right)$. So by Lemma 5.5 (b) and (2.5),

$$\begin{aligned}
\mathbf{D}^+_{1\,n+1}(z)\cdot(\mathbf{K}\oplus'\mathbf{K})^{-1} &= \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{0n}\right)\cdot\mathbf{V}_{23}\cdot\left\{\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot(\mathbf{K}\oplus'\mathbf{K})^{-1}\right\}\\
&= -\omega\,\mathrm{Sim}_{\mathbf{V}^T}\mathbf{L}^2\cdot[\mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{0n}\cdot\mathbf{V}_{23}\cdot\{\mathbf{I}\oplus(-\mathbf{R})\}]\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right).
\end{aligned}\tag{5.17}$$

By (5.13) and (5.14),

$$\begin{aligned}
&\mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{01}\cdot\mathbf{V}_{23}\cdot\{\mathbf{I}\oplus(-\mathbf{R})\} = \frac{1}{2}\mathbf{V}_{23}\{(\mathbf{I}-\mathbf{R})\oplus(\mathbf{I}-\mathbf{R})\}\mathbf{V}_{23}\cdot\mathbf{V}_{23}\cdot\{\mathbf{I}\oplus(-\mathbf{R})\} = \mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{01}\cdot\mathbf{V}_{23},\\
&\mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{02}\cdot\mathbf{V}_{23}\cdot\{\mathbf{I}\oplus(-\mathbf{R})\} = \frac{1}{2}\mathbf{V}_{23}\begin{pmatrix}\mathbf{I}-\mathbf{R} & \mathbf{O}\\ \mathbf{J}(\mathbf{I}+\mathbf{R}) & \mathbf{O}\end{pmatrix}\mathbf{V}_{23}\cdot\mathbf{V}_{23}\cdot\{\mathbf{I}\oplus(-\mathbf{R})\} = \mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{02}\cdot\mathbf{V}_{23},
\end{aligned}$$

hence, by (5.17) and Definition 5.1,

$$\mathbf{D}^+_{1\,n+1}(z)\cdot(\mathbf{K}\oplus'\mathbf{K})^{-1} = -\omega\,\mathrm{Sim}_{\mathbf{V}^T}\mathbf{L}^2\cdot(\mathrm{Sim}_{\mathbf{V}^T}\mathbf{E}_{0n}\cdot\mathbf{V}_{23})\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right) = -\omega\mathbf{D}^+_{0n}(z),$$

which shows (b).

By (2.2), $\left\{(\mathbf{K}\oplus'\mathbf{K})^{-1}\right\}^2 = \left(\overline{\mathbf{K}}\oplus'\overline{\mathbf{K}}\right)^2 = \mathbf{J}\oplus\mathbf{J}$, hence, by Lemma 5.5 (b) and (5.8),

$$\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot(\mathbf{J}\oplus\mathbf{J})$$

$$= \{\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}\cdot\mathbf{V}_{23}\}\cdot\left\{\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot(\mathbf{K}\oplus'\mathbf{K})^{-1}\right\}\cdot(\mathbf{K}\oplus'\mathbf{K})^{-1}$$
$$= (\mathbf{J}\oplus'\mathbf{I})\cdot(-\omega)\{\mathbf{I}\oplus(-\mathbf{R})\}\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot(\mathbf{K}\oplus'\mathbf{K})^{-1}$$
$$= -\omega(\mathbf{J}\oplus'\mathbf{I})\{\mathbf{I}\oplus(-\mathbf{R})\}\cdot\left\{\mathbf{V}_{23}\,(\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L})^{-1}\,\mathbf{V}_{23}\right\}\cdot\left\{\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot(\mathbf{K}\oplus'\mathbf{K})^{-1}\right\}$$
$$= -\omega(\mathbf{J}\oplus'\mathbf{I})\{\mathbf{I}\oplus(-\mathbf{R})\}\cdot(\mathbf{J}\oplus'\mathbf{I})^{-1}\cdot(-\omega)\{\mathbf{I}\oplus(-\mathbf{R})\}\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right)$$
$$= \mathbb{i}\,\{(-\mathbf{R})\oplus(-\mathbf{R})\}\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right), \tag{5.18}$$

since $(\mathbf{J}\oplus'\mathbf{I})^{-1} = \mathbf{I}\oplus'\mathbf{J}$, and hence

$$(\mathbf{J}\oplus'\mathbf{I})\{\mathbf{I}\oplus(-\mathbf{R})\}(\mathbf{J}\oplus'\mathbf{I})^{-1}\{\mathbf{I}\oplus(-\mathbf{R})\} = \begin{pmatrix}\mathbf{O} & \mathbf{I}\\ \mathbf{J} & \mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & -\mathbf{R}\end{pmatrix}\begin{pmatrix}\mathbf{O} & \mathbf{J}\\ \mathbf{I} & \mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & -\mathbf{R}\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{O} & -\mathbf{R}\\ \mathbf{J} & \mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{O} & \mathbf{J}\\ \mathbf{I} & \mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & -\mathbf{R}\end{pmatrix} = \begin{pmatrix}-\mathbf{R} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & -\mathbf{R}\end{pmatrix} = (-\mathbf{R})\oplus(-\mathbf{R})$$

by (2.2). By (2.3) and (5.13),

$$\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{E}_{01}\cdot\mathbf{V}_{23}\{(-\mathbf{R})\oplus(-\mathbf{R})\} = \frac{1}{2}\mathbf{V}_{23}\{(\mathbf{I}-\mathbf{R})\oplus(\mathbf{I}-\mathbf{R})\}\mathbf{V}_{23}\cdot\mathbf{V}_{23}\{(-\mathbf{R})\oplus(-\mathbf{R})\} = \mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{E}_{01}\cdot\mathbf{V}_{23},$$

hence, by Definitions 2.4, 5.1, and (5.18),

$$\mathbf{D}_{23}^{+}(z)\cdot(\mathbf{J}\oplus\mathbf{J}) = \mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{01}\mathbf{L}^2\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot(\mathbf{J}\oplus\mathbf{J})$$
$$= \mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}^2\cdot\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{E}_{01}\cdot\mathbf{V}_{23}\cdot\left\{\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{W}_0e^{\mathbf{\Omega}z}\right)\cdot(\mathbf{J}\oplus\mathbf{J})\right\}$$
$$= \mathbb{i}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}^2\cdot[\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{E}_{01}\cdot\mathbf{V}_{23}\cdot\{(-\mathbf{R})\oplus(-\mathbf{R})\}]\cdot\mathbf{V}_{23}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right)$$
$$= \mathbb{i}\,\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{L}^2\cdot\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{E}_{01}\cdot\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{W}_0e^{\mathbf{\Omega}z}\right) = \mathbb{i}\,\mathrm{Sim}_{\mathbf{V}^T}\left(\mathbf{L}^2\mathbf{E}_{01}\mathbf{W}_0e^{\mathbf{\Omega}z}\right) = \mathbb{i}\mathbf{D}_{01}^{+}(z),$$

from which (c) follows, since $(\mathbf{J}\oplus\mathbf{J})^{-1} = \mathbf{J}\oplus\mathbf{J}$, and the proof is complete. □

By Definition 2.2 and (2.4),

$$(\mathbf{I}\pm\mathbf{R})(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J} = \{(1\pm\mathbb{i})\mathbf{I}\pm(1\pm\mathbb{i})\mathbf{R}\}\mathbf{J} = \sqrt{2}\omega^{\pm1}(\mathbf{I}\pm\mathbf{R})\mathbf{J} = \sqrt{2}\omega^{\pm1}\mathbf{J}(\mathbf{I}\mp\mathbf{R}). \tag{5.19}$$

Proof of Lemma 5.2. Denote $\cosh(\zeta_1,\zeta_2) = \mathrm{diag}(\cosh\zeta_1,\cosh\zeta_2)$, $\sinh(\zeta_1,\zeta_2) = \mathrm{diag}(\sinh\zeta_1,\sinh\zeta_2)$ for $\zeta_1,\zeta_2\in\mathbb{C}$. By Lemma 5.3 and (2.1), (5.13), (5.19),

$$\mathrm{Sim}_{\mathbf{V}^T}(\mathbf{E}_{01}\mathbf{W}_0) = \mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{E}_{01}\cdot\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{W}_0$$
$$= \frac{1}{2}\mathbf{V}_{23}\{(\mathbf{I}-\mathbf{R})\oplus(\mathbf{I}-\mathbf{R})\}\mathbf{V}_{23}\cdot\sqrt{2}\mathbf{V}_{23}[\{\omega(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J}\}\oplus\{-\mathbb{i}(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J}\mathbf{K}\}]$$
$$= \frac{1}{\sqrt{2}}\mathbf{V}_{23}[\{\omega(\mathbf{I}-\mathbf{R})(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J}\}\oplus\{-\mathbb{i}(\mathbf{I}-\mathbf{R})(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J}\mathbf{K}\}] = \mathbf{V}_{23}[\{\mathbf{J}(\mathbf{I}+\mathbf{R})\}\oplus\{-\omega\mathbf{J}(\mathbf{I}+\mathbf{R})\mathbf{K}\}],$$

hence, by Definitions 2.5, 5.1, and Lemma 5.4,

$$\mathbf{D}_{01}^{-}(z) = \mathrm{Sim}_{\mathbf{V}^T}(\mathbf{E}_{01}\mathbf{W}_0)\cdot\mathrm{Sim}_{\mathbf{V}^T}\,e^{-\mathbf{\Omega}z} = \mathbf{V}_{23}\begin{pmatrix}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\cosh(\omega z,\overline{\omega}z) & \begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\sinh(-\omega z,\overline{\omega}z)\\ \begin{pmatrix}\omega & -\overline{\omega}\\ -\omega & \overline{\omega}\end{pmatrix}\sinh(-\omega z,\overline{\omega}z) & \begin{pmatrix}-\omega & \overline{\omega}\\ \omega & -\overline{\omega}\end{pmatrix}\cosh(\omega z,\overline{\omega}z)\end{pmatrix}$$

$$= \mathbf{V}_{23}\begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega}z) & \sinh(\omega z) & -\sinh(\overline{\omega}z) \\ -\cosh(\omega z) & -\cosh(\overline{\omega}z) & -\sinh(\omega z) & \sinh(\overline{\omega}z) \\ -\omega\sinh(\omega z) & -\overline{\omega}\sinh(\overline{\omega}z) & -\omega\cosh(\omega z) & \overline{\omega}\cosh(\overline{\omega}z) \\ \omega\sinh(\omega z) & \overline{\omega}\sinh(\overline{\omega}z) & \omega\cosh(\omega z) & -\overline{\omega}\cosh(\overline{\omega}z) \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega}z) & \sinh(\omega z) & -\sinh(\overline{\omega}z) \\ -\omega\sinh(\omega z) & -\overline{\omega}\sinh(\overline{\omega}z) & -\omega\cosh(\omega z) & \overline{\omega}\cosh(\overline{\omega}z) \\ -\cosh(\omega z) & -\cosh(\overline{\omega}z) & -\sinh(\omega z) & \sinh(\overline{\omega}z) \\ \omega\sinh(\omega z) & \overline{\omega}\sinh(\overline{\omega}z) & \omega\cosh(\omega z) & -\overline{\omega}\cosh(\overline{\omega}z) \end{pmatrix},$$

from which (a) follows by Definition 5.2. By Lemma 5.3 and (2.1), (2.2), (5.14), (5.19),

$$\begin{aligned} \operatorname{Sim}_{\mathbf{V}^T}(\mathbf{E}_{02}\mathbf{W}_0) &= \operatorname{Sim}_{\mathbf{V}^T}\mathbf{E}_{02}\cdot\operatorname{Sim}_{\mathbf{V}^T}\mathbf{W}_0 \\ &= \frac{1}{2}\mathbf{V}_{23}\begin{pmatrix} \mathbf{I}-\mathbf{R} & \mathbf{O} \\ \mathbf{J}(\mathbf{I}+\mathbf{R}) & \mathbf{O} \end{pmatrix}\mathbf{V}_{23}\cdot\ \sqrt{2}\mathbf{V}_{23}\left[\{\omega(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J}\}\oplus\{-\mathbb{i}(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{JK}\}\right] \\ &= \frac{1}{\sqrt{2}}\mathbf{V}_{23}\begin{pmatrix} \omega(\mathbf{I}-\mathbf{R})(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J} & \mathbf{O} \\ \omega\mathbf{J}\cdot(\mathbf{I}+\mathbf{R})(\mathbf{I}+\mathbb{i}\mathbf{R})\mathbf{J} & \mathbf{O} \end{pmatrix} = \mathbf{V}_{23}\begin{pmatrix} \mathbf{J}(\mathbf{I}+\mathbf{R}) & \mathbf{O} \\ \mathbb{i}(\mathbf{I}-\mathbf{R}) & \mathbf{O} \end{pmatrix}, \end{aligned}$$

hence, by Definitions 2.5, 5.1 and Lemma 5.4,

$$\mathbf{D}_{02}^{-}(z) = \operatorname{Sim}_{\mathbf{V}^T}(\mathbf{E}_{02}\mathbf{W}_0)\cdot\operatorname{Sim}_{\mathbf{V}^T}e^{-\mathbf{\Omega}z} = \mathbf{V}_{23}\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\cosh(\omega z,\overline{\omega}z) & \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\sinh(-\omega z,\overline{\omega}z) \\ \begin{pmatrix} \mathbb{i} & -\mathbb{i} \\ -\mathbb{i} & \mathbb{i} \end{pmatrix}\cosh(\omega z,\overline{\omega}z) & \begin{pmatrix} -\mathbb{i} & \mathbb{i} \\ \mathbb{i} & -\mathbb{i} \end{pmatrix}\sinh(-\omega z,\overline{\omega}z) \end{pmatrix}$$

$$= \mathbf{V}_{23}\begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega}z) & \sinh(\omega z) & -\sinh(\overline{\omega}z) \\ -\cosh(\omega z) & -\cosh(\overline{\omega}z) & -\sinh(\omega z) & \sinh(\overline{\omega}z) \\ \mathbb{i}\cosh(\omega z) & -\mathbb{i}\cosh(\overline{\omega}z) & \mathbb{i}\sinh(\omega z) & \mathbb{i}\sinh(\overline{\omega}z) \\ -\mathbb{i}\cosh(\omega z) & \mathbb{i}\cosh(\overline{\omega}z) & -\mathbb{i}\sinh(\omega z) & -\mathbb{i}\sinh(\overline{\omega}z) \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega}z) & \sinh(\omega z) & -\sinh(\overline{\omega}z) \\ \mathbb{i}\cosh(\omega z) & -\mathbb{i}\cosh(\overline{\omega}z) & \mathbb{i}\sinh(\omega z) & \mathbb{i}\sinh(\overline{\omega}z) \\ -\cosh(\omega z) & -\cosh(\overline{\omega}z) & -\sinh(\omega z) & \sinh(\overline{\omega}z) \\ -\mathbb{i}\cosh(\omega z) & \mathbb{i}\cosh(\overline{\omega}z) & -\mathbb{i}\sinh(\omega z) & -\mathbb{i}\sinh(\overline{\omega}z) \end{pmatrix},$$

from which (b) follows by Definition 5.2.

(c), (d) follow from (a), (b), and Lemma 5.6 (a). (e), (f) follow from (c), (d), and Lemma 5.6 (b). (g) follows from (c) and Lemma 5.6 (c), and the proof is complete. □

6. Fundamental boundary matrices for 01 types

6.1. $\mathcal{F}(\mathbf{E}_{0101}, z)$

By Lemma 5.2 (a), (c),

$$\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{01}^{+}(z) = \begin{pmatrix} \mathbf{JA}(z) & \mathbf{JB}(z) \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix} + \begin{pmatrix} \mathbf{A}(z) & -\mathbf{B}(z) \\ \mathbf{A}(z) & -\mathbf{B}(z) \end{pmatrix} = \begin{pmatrix} \mathbf{JA}(z)+\mathbf{A}(z) & \mathbf{JB}(z)-\mathbf{B}(z) \\ -\mathbf{JA}(z)+\mathbf{A}(z) & -\mathbf{JB}(z)-\mathbf{B}(z) \end{pmatrix}.$$

So by Definition 2.5,

$$\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{01}^{+}(z)\right\}=\frac{1}{2}\begin{pmatrix}\mathbf{I} & \mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{JA}(z)+\mathbf{A}(z) & \mathbf{JB}(z)-\mathbf{B}(z)\\ -\mathbf{JA}(z)+\mathbf{A}(z) & -\mathbf{JB}(z)-\mathbf{B}(z)\end{pmatrix}=\begin{pmatrix}\mathbf{A}(z) & -\mathbf{B}(z)\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix},$$

hence, by (2.2),

$$\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{01}^{+}(z)\right\}\cdot\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbf{A}(z)^{-1}\mathbf{J}\\ -\mathbf{B}(z)^{-1} & -\mathbf{B}(z)^{-1}\mathbf{J}\end{pmatrix}=\begin{pmatrix}\mathbf{A}(z) & -\mathbf{B}(z)\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix}\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbf{A}(z)^{-1}\mathbf{J}\\ -\mathbf{B}(z)^{-1} & -\mathbf{B}(z)^{-1}\mathbf{J}\end{pmatrix}=2\mathbf{I}.$$

Thus we have

$$\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{01}^{+}(z)\right\}^{-1}=\frac{1}{2\sqrt{2}}\left[\frac{1}{2\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{01}^{+}(z)\right\}\right]^{-1}\mathbf{V}=\frac{1}{2\sqrt{2}}\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbf{A}(z)^{-1}\mathbf{J}\\ -\mathbf{B}(z)^{-1} & -\mathbf{B}(z)^{-1}\mathbf{J}\end{pmatrix}\mathbf{V}. \tag{6.1}$$

By Definition 2.5 and Lemma 5.2 (a),

$$\mathbf{V}\cdot\mathbf{D}_{01}^{-}(z)=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{I} & \mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{JA}(z) & \mathbf{JB}(z)\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix}=\sqrt{2}\begin{pmatrix}\mathbf{O} & \mathbf{O}\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix}, \tag{6.2}$$

hence, by Definition 5.1, and (2.2), (6.1),

$$\begin{aligned}\mathbf{D}_{0101}(z)&=\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{01}^{+}(z)\right\}^{-1}\mathbf{D}_{01}^{-}(z)\\ &=\frac{1}{2\sqrt{2}}\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbf{A}(z)^{-1}\mathbf{J}\\ -\mathbf{B}(z)^{-1} & -\mathbf{B}(z)^{-1}\mathbf{J}\end{pmatrix}\cdot\sqrt{2}\begin{pmatrix}\mathbf{O} & \mathbf{O}\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\mathbf{I} & \mathbf{A}(z)^{-1}\mathbf{B}(z)\\ \mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathbf{I}\end{pmatrix}.\end{aligned} \tag{6.3}$$

Lemma 6.1. *For* $z\in\mathbb{C}$ *such that* $\mathrm{s}_+(z)\,\mathrm{s}_-(z)\neq 0$,

$$\mathcal{F}(\mathbf{E}_{0101},z)=\frac{1}{2}\begin{pmatrix}\begin{pmatrix}\frac{\mathrm{ci}_-(z)}{\mathrm{s}_+(z)} & -\frac{\sqrt{2}\omega}{\mathrm{s}_+(z)}\\ -\frac{\sqrt{2}\overline{\omega}}{\mathrm{s}_+(z)} & \frac{\mathrm{ci}_+(z)}{\mathrm{s}_+(z)}\end{pmatrix}-\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \begin{pmatrix}\frac{\mathrm{ci}_+(z)}{\mathrm{s}_-(z)} & -\frac{\sqrt{2}\omega}{\mathrm{s}_-(z)}\\ -\frac{\sqrt{2}\overline{\omega}}{\mathrm{s}_-(z)} & \frac{\mathrm{ci}_-(z)}{\mathrm{s}_-(z)}\end{pmatrix}-\mathbf{I}\end{pmatrix}.$$

Proof. By Lemma 5.1 and (6.3),

$$\begin{aligned}\mathcal{F}(\mathbf{E}_{0101},z)&=\left\{\frac{1}{2}\begin{pmatrix}\mathbf{I} & \mathbf{A}(z)^{-1}\mathbf{B}(z)\\ \mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathbf{I}\end{pmatrix}-\frac{1}{2}\mathbf{I}\right\}(\mathbf{J}\oplus'\mathbf{J})-\frac{1}{2}\mathbf{I}\\ &=\frac{1}{2}\begin{pmatrix}\mathbf{O} & \mathbf{A}(z)^{-1}\mathbf{B}(z)\\ \mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{O} & \mathbf{J}\\ \mathbf{J} & \mathbf{O}\end{pmatrix}-\frac{1}{2}\mathbf{I}=\frac{1}{2}\begin{pmatrix}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{J}-\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{J}-\mathbf{I}\end{pmatrix},\end{aligned}$$

from which the result follows by Definition 2.3 and Lemma B.2 (a), (b). □

6.2. $\mathcal{F}(\mathbf{E}_{0123},z)$

By Lemma 5.2 (a), (g),

$$\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{23}^{+}(z)=\begin{pmatrix}\mathbf{JA}(z) & \mathbf{JB}(z)\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix}+\mathbb{i}\begin{pmatrix}\mathbf{A}(z)\mathbf{J} & -\mathbf{B}(z)\mathbf{J}\\ \mathbf{A}(z)\mathbf{J} & -\mathbf{B}(z)\mathbf{J}\end{pmatrix}=\begin{pmatrix}\mathbf{JA}(z)+\mathbb{i}\mathbf{A}(z)\mathbf{J} & \mathbf{JB}(z)-\mathbb{i}\mathbf{B}(z)\mathbf{J}\\ -\mathbf{JA}(z)+\mathbb{i}\mathbf{A}(z)\mathbf{J} & -\mathbf{JB}(z)-\mathbb{i}\mathbf{B}(z)\mathbf{J}\end{pmatrix}.$$

So by Definition 2.5,

$$\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{23}^{+}(z)\right\}=\frac{1}{2}\begin{pmatrix}\mathbf{I} & \mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{J}\mathbf{A}(z)+\mathbb{i}\mathbf{A}(z)\mathbf{J} & \mathbf{J}\mathbf{B}(z)-\mathbb{i}\mathbf{B}(z)\mathbf{J}\\ -\mathbf{J}\mathbf{A}(z)+\mathbb{i}\mathbf{A}(z)\mathbf{J} & -\mathbf{J}\mathbf{B}(z)-\mathbb{i}\mathbf{B}(z)\mathbf{J}\end{pmatrix}=\begin{pmatrix}\mathbb{i}\mathbf{A}(z)\mathbf{J} & -\mathbb{i}\mathbf{B}(z)\mathbf{J}\\ -\mathbf{J}\mathbf{A}(z) & -\mathbf{J}\mathbf{B}(z)\end{pmatrix},$$

hence, by (2.2),

$$\begin{aligned}&\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{23}^{+}(z)\right\}\cdot\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbb{i}\mathbf{J}\mathbf{A}(z)^{-1}\\ -\mathbf{B}(z)^{-1} & -\mathbb{i}\mathbf{J}\mathbf{B}(z)^{-1}\end{pmatrix}=\begin{pmatrix}\mathbb{i}\mathbf{A}(z)\mathbf{J} & -\mathbb{i}\mathbf{B}(z)\mathbf{J}\\ -\mathbf{J}\mathbf{A}(z) & -\mathbf{J}\mathbf{B}(z)\end{pmatrix}\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbb{i}\mathbf{J}\mathbf{A}(z)^{-1}\\ -\mathbf{B}(z)^{-1} & -\mathbb{i}\mathbf{J}\mathbf{B}(z)^{-1}\end{pmatrix}\\&=\mathbb{i}\begin{pmatrix}\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}+\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1} & \mathbf{O}\\ \mathbf{O} & \mathbf{J}\left\{\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}+\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\right\}\end{pmatrix}.\end{aligned}$$

So by Lemma B.4 (a) and (2.2),

$$\begin{aligned}&\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{23}^{+}(z)\right\}^{-1}=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{23}^{+}(z)\right\}\right]^{-1}\mathbf{V}\\&=\frac{-\mathbb{i}}{\sqrt{2}}\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbb{i}\mathbf{J}\mathbf{A}(z)^{-1}\\ -\mathbf{B}(z)^{-1} & -\mathbb{i}\mathbf{J}\mathbf{B}(z)^{-1}\end{pmatrix}\cdot\begin{pmatrix}\left\{\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}+\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\right\}^{-1} & \mathbf{O}\\ \mathbf{O} & \left\{\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}+\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\right\}^{-1}\mathbf{J}\end{pmatrix}\mathbf{V}\\&=\frac{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}{4\sqrt{2}\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)}\begin{pmatrix}-\mathbb{i}\mathbf{A}(z)^{-1} & -\mathbf{J}\mathbf{A}(z)^{-1}\\ \mathbb{i}\mathbf{B}(z)^{-1} & -\mathbf{J}\mathbf{B}(z)^{-1}\end{pmatrix}\mathbf{H}(z)\mathbf{V},\end{aligned}$$

where we put $\mathbf{H}(z)=\left\{\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}+\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\right\}\oplus\left[\left\{\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}+\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\right\}\mathbf{J}\right]$. Thus, by Definition 5.1 and (2.2), (6.2),

$$\begin{aligned}&\mathbf{D}_{0123}(z)=\left\{\mathbf{D}_{01}^{-}(z)+\mathbf{D}_{23}^{+}(z)\right\}^{-1}\mathbf{D}_{01}^{-}(z)\\&=\frac{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}{4\sqrt{2}\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)}\begin{pmatrix}-\mathbb{i}\mathbf{A}(z)^{-1} & -\mathbf{J}\mathbf{A}(z)^{-1}\\ \mathbb{i}\mathbf{B}(z)^{-1} & -\mathbf{J}\mathbf{B}(z)^{-1}\end{pmatrix}\cdot\mathbf{H}(z)\cdot\sqrt{2}\begin{pmatrix}\mathbf{O} & \mathbf{O}\\ -\mathbf{J}\mathbf{A}(z) & -\mathbf{J}\mathbf{B}(z)\end{pmatrix}\\&=\frac{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}{4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)}\begin{pmatrix}\mathbb{i}\mathbf{A}(z)^{-1} & \mathbf{J}\mathbf{A}(z)^{-1}\\ -\mathbb{i}\mathbf{B}(z)^{-1} & \mathbf{J}\mathbf{B}(z)^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{O} & \mathbf{O}\\ \mathbf{A}(z)\mathbf{J}+\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}\mathbf{B}(z)+\mathbf{B}(z)\mathbf{J}\end{pmatrix}\\&=\frac{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}{4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)}\begin{pmatrix}\mathbf{I}+\mathbf{J}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathbf{A}(z)^{-1}\mathbf{B}(z)+\mathbf{J}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{J}\\ \mathbf{J}\mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{J}+\mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathbf{J}\mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}\mathbf{B}(z)+\mathbf{I}\end{pmatrix}.\end{aligned}\tag{6.4}$$

Lemma 6.2. *For $z\in\mathbb{C}$ such that* $\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\neq 0$,

$$\mathscr{F}(\mathbf{E}_{0123},z)=\frac{1}{2}\begin{pmatrix}\begin{pmatrix}\frac{\mathrm{s}_-(z)}{\mathrm{ci}_+(z)} & 0\\ 0 & \frac{\mathrm{s}_-(z)}{\mathrm{ci}_-(z)}\end{pmatrix}-\mathbf{I} & \begin{pmatrix}0 & -\frac{\sqrt{2}\omega}{\mathrm{ci}_+(z)}\\ \frac{\sqrt{2}\overline{\omega}}{\mathrm{ci}_-(z)} & 0\end{pmatrix}\\ \begin{pmatrix}0 & -\frac{\sqrt{2}\omega}{\mathrm{ci}_-(z)}\\ \frac{\sqrt{2}\overline{\omega}}{\mathrm{ci}_+(z)} & 0\end{pmatrix} & \begin{pmatrix}\frac{\mathrm{s}_+(z)}{\mathrm{ci}_-(z)} & 0\\ 0 & \frac{\mathrm{s}_+(z)}{\mathrm{ci}_+(z)}\end{pmatrix}-\mathbf{I}\end{pmatrix}.$$

Proof. By Lemma B.2 (a), (b), and (2.2), (6.4),

$$4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\left[\left\{\mathbf{D}_{0123}(z)\right\}_{1:2,1:2}-\frac{1}{2}\mathbf{I}\right]=\mathrm{s}_+(z)\,\mathrm{s}_-(z)\left\{\mathbf{I}+\mathbf{J}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\mathbf{A}(z)\right\}-2\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\mathbf{I}$$

$$= \{\mathrm{s}_+(z)\,\mathrm{s}_-(z) - 2\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\}\,\mathbf{I} + \mathrm{s}_+(z)\,\mathrm{s}_-(z)\mathbf{J}\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix}\cdot\mathbf{J}\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}$$

$$= \{\mathrm{s}_+(z)\,\mathrm{s}_-(z) - 2\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\}\,\mathbf{I} + \begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ \sqrt{2}\overline{\omega} & \mathrm{ci}_+(z)\end{pmatrix}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ \sqrt{2}\overline{\omega} & \mathrm{ci}_-(z)\end{pmatrix}$$

$$= \begin{pmatrix}\mathrm{s}_+(z)\,\mathrm{s}_-(z) - \mathrm{ci}_+(z)\,\mathrm{ci}_-(z) + 2 & 2\sqrt{2}\omega\,\mathrm{ci}_-(z)\\ 2\sqrt{2}\overline{\omega}\,\mathrm{ci}_+(z) & \mathrm{s}_+(z)\,\mathrm{s}_-(z) - \mathrm{ci}_+(z)\,\mathrm{ci}_-(z) + 2\end{pmatrix}, \tag{6.5}$$

$$4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\left[\{\mathbf{D}_{0123}(z)\}_{3:4,3:4} - \frac{1}{2}\mathbf{I}\right] = \mathrm{s}_+(z)\,\mathrm{s}_-(z)\left\{\mathbf{J}\mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathbf{I}\right\} - 2\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\mathbf{I}$$

$$= \{\mathrm{s}_+(z)\,\mathrm{s}_-(z) - 2\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\}\,\mathbf{I} + \mathrm{s}_+(z)\,\mathrm{s}_-(z)\mathbf{J}\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}\cdot\mathbf{J}\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix}$$

$$= \{\mathrm{s}_+(z)\,\mathrm{s}_-(z) - 2\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\}\,\mathbf{I} + \begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ \sqrt{2}\overline{\omega} & \mathrm{ci}_-(z)\end{pmatrix}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ \sqrt{2}\overline{\omega} & \mathrm{ci}_+(z)\end{pmatrix}$$

$$= \begin{pmatrix}\mathrm{s}_+(z)\,\mathrm{s}_-(z) - \mathrm{ci}_+(z)\,\mathrm{ci}_-(z) + 2 & 2\sqrt{2}\omega\,\mathrm{ci}_+(z)\\ 2\sqrt{2}\overline{\omega}\,\mathrm{ci}_-(z) & \mathrm{s}_+(z)\,\mathrm{s}_-(z) - \mathrm{ci}_+(z)\,\mathrm{ci}_-(z) + 2\end{pmatrix}, \tag{6.6}$$

$$4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\cdot\{\mathbf{D}_{0123}(z)\}_{1:2,3:4}\cdot\mathbf{J} = \mathrm{s}_+(z)\,\mathrm{s}_-(z)\left\{\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathbf{J}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{J}\right\}\mathbf{J}$$

$$= \mathrm{s}_+(z)\,\mathrm{s}_-(z)\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix}\mathbf{J} + \mathrm{s}_+(z)\,\mathrm{s}_-(z)\cdot\mathbf{J}\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix}$$

$$= \mathrm{s}_-(z)\begin{pmatrix}\mathrm{ci}_-(z) & -\sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & \mathrm{ci}_+(z)\end{pmatrix} + \mathrm{s}_-(z)\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ \sqrt{2}\overline{\omega} & \mathrm{ci}_+(z)\end{pmatrix} = 2\,\mathrm{s}_-(z)\begin{pmatrix}\mathrm{ci}_-(z) & 0\\ 0 & \mathrm{ci}_+(z)\end{pmatrix}, \tag{6.7}$$

$$4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)\cdot\{\mathbf{D}_{0123}(z)\}_{3:4,1:2}\cdot\mathbf{J} = \mathrm{s}_+(z)\,\mathrm{s}_-(z)\left\{\mathbf{J}\mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{J} + \mathbf{B}(z)^{-1}\mathbf{A}(z)\right\}\mathbf{J}$$

$$= \mathrm{s}_+(z)\,\mathrm{s}_-(z)\cdot\mathbf{J}\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix} + \mathrm{s}_+(z)\,\mathrm{s}_-(z)\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}\mathbf{J}$$

$$= \mathrm{s}_+(z)\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ \sqrt{2}\overline{\omega} & \mathrm{ci}_-(z)\end{pmatrix} + \mathrm{s}_+(z)\begin{pmatrix}\mathrm{ci}_+(z) & -\sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & \mathrm{ci}_-(z)\end{pmatrix} = 2\,\mathrm{s}_+(z)\begin{pmatrix}\mathrm{ci}_+(z) & 0\\ 0 & \mathrm{ci}_-(z)\end{pmatrix}. \tag{6.8}$$

By (A.4), $\mathrm{s}_+(z)\,\mathrm{s}_-(z) - \mathrm{ci}_+(z)\,\mathrm{ci}_-(z) + 2 = 0$, hence, by Definition 2.3 and (6.5), (6.6),

$$\left[\{\mathbf{D}_{0123}(z)\}_{1:2,1:2} - \frac{1}{2}\mathbf{I}\right]\mathbf{J} = \frac{1}{4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)}\begin{pmatrix}0 & -2\sqrt{2}\omega\,\mathrm{ci}_-(z)\\ 2\sqrt{2}\overline{\omega}\,\mathrm{ci}_+(z) & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & -\frac{\sqrt{2}\omega}{\mathrm{ci}_+(z)}\\ \frac{\sqrt{2}\overline{\omega}}{\mathrm{ci}_-(z)} & 0\end{pmatrix}, \tag{6.9}$$

$$\left[\{\mathbf{D}_{0123}(z)\}_{3:4.3:4} - \frac{1}{2}\mathbf{I}\right]\mathbf{J} = \frac{1}{4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)}\begin{pmatrix}0 & -2\sqrt{2}\omega\,\mathrm{ci}_+(z)\\ 2\sqrt{2}\overline{\omega}\,\mathrm{ci}_-(z) & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & -\frac{\sqrt{2}\omega}{\mathrm{ci}_-(z)}\\ \frac{\sqrt{2}\overline{\omega}}{\mathrm{ci}_+(z)} & 0\end{pmatrix}. \tag{6.10}$$

Thus the result follows by (6.7), (6.8), (6.9), (6.10), and Lemma 5.1. □

6.3. $\mathcal{F}(\mathbf{E}_{0112}, z)$

By Lemma 5.2 (a), (e),

$$\mathbf{D}_{01}^{-}(z) + \mathbf{D}_{12}^{+}(z)$$

$$= \begin{pmatrix} \mathbf{JA}(z) & \mathbf{JB}(z) \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix} + \omega \begin{pmatrix} \mathbf{B}(z)\mathbf{K} & -\mathbf{A}(z)\mathbf{K} \\ \mathbf{B}(z)\mathbf{K} & -\mathbf{A}(z)\mathbf{K} \end{pmatrix} = \begin{pmatrix} \mathbf{JA}(z) + \omega\mathbf{B}(z)\mathbf{K} & \mathbf{JB}(z) - \omega\mathbf{A}(z)\mathbf{K} \\ -\mathbf{JA}(z) + \omega\mathbf{B}(z)\mathbf{K} & -\mathbf{JB}(z) - \omega\mathbf{A}(z)\mathbf{K} \end{pmatrix}.$$

So by Definition 2.5,

$$\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z) + \mathbf{D}_{12}^{+}(z)\right\} = \frac{1}{2}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{JA}(z) + \omega\mathbf{B}(z)\mathbf{K} & \mathbf{JB}(z) - \omega\mathbf{A}(z)\mathbf{K} \\ -\mathbf{JA}(z) + \omega\mathbf{B}(z)\mathbf{K} & -\mathbf{JB}(z) - \omega\mathbf{A}(z)\mathbf{K} \end{pmatrix} = \begin{pmatrix} \omega\mathbf{B}(z)\mathbf{K} & -\omega\mathbf{A}(z)\mathbf{K} \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix},$$

hence, by (2.2),

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z) + \mathbf{D}_{12}^{+}(z)\right\} \cdot \begin{pmatrix} \mathbf{A}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{B}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{A}(z)^{-1} \end{pmatrix} = \begin{pmatrix} \omega\mathbf{B}(z)\mathbf{K} & -\omega\mathbf{A}(z)\mathbf{K} \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix}\begin{pmatrix} \mathbf{A}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{B}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{A}(z)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \omega\left\{\mathbf{A}(z)\mathbf{KB}(z)^{-1} + \mathbf{B}(z)\mathbf{KA}(z)^{-1}\right\} & \mathbf{O} \\ \mathbf{O} & -\overline{\omega}\mathbf{J}\left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\right\} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}(z)\mathbf{KB}(z)^{-1} + \mathbf{B}(z)\mathbf{KA}(z)^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{J}\left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\right\} \end{pmatrix}\begin{pmatrix} \omega\mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\overline{\omega}\mathbf{I} \end{pmatrix}.
\end{aligned}$$

So by Lemma B.4 (b) and (2.1), (2.2),

$$\begin{aligned}
&\left\{\mathbf{D}_{01}^{-}(z) + \mathbf{D}_{12}^{+}(z)\right\}^{-1} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^{-}(z) + \mathbf{D}_{12}^{+}(z)\right\}\right]^{-1}\mathbf{V} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{A}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{B}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{A}(z)^{-1} \end{pmatrix} \cdot \begin{pmatrix} \overline{\omega}\mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\omega\mathbf{I} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \left\{\mathbf{A}(z)\mathbf{KB}(z)^{-1} + \mathbf{B}(z)\mathbf{KA}(z)^{-1}\right\}^{-1} & \mathbf{O} \\ \mathbf{O} & \left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\right\}^{-1}\mathbf{J} \end{pmatrix}\mathbf{V} \\
&= \frac{1}{2\sqrt{2}\,\mathrm{c}_{-}(2z)}\begin{pmatrix} \overline{\omega}\mathbf{A}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{B}(z)^{-1} \\ -\overline{\omega}\mathbf{B}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{A}(z)^{-1} \end{pmatrix}\mathbf{H}(z)\mathbf{V},
\end{aligned}$$

where we put

$$\mathbf{H}(z) = \left\{\mathrm{s}_{-}^{2}(z)\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} + \mathrm{s}_{+}^{2}(z)\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\right\} \oplus \left[\left\{\mathrm{s}_{-}^{2}(z)\mathbf{A}(z)\mathbf{KB}(z)^{-1} + \mathrm{s}_{+}^{2}(z)\mathbf{B}(z)\mathbf{KA}(z)^{-1}\right\}\mathbf{J}\right].$$

Thus, by Definition 5.1, and (2.2), (6.2),

$$\begin{aligned}
&\mathbf{D}_{0112}(z) = \left\{\mathbf{D}_{01}^{-}(z) + \mathbf{D}_{12}^{+}(z)\right\}^{-1}\mathbf{D}_{01}^{-}(z) \\
&= \frac{1}{2\sqrt{2}\,\mathrm{c}_{-}(2z)}\begin{pmatrix} \overline{\omega}\mathbf{A}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{B}(z)^{-1} \\ -\overline{\omega}\mathbf{B}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{A}(z)^{-1} \end{pmatrix} \cdot \mathbf{H}(z) \cdot \sqrt{2}\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix} \\
&= \frac{1}{2\,\mathrm{c}_{-}(2z)}\begin{pmatrix} -\overline{\omega}\mathbf{A}(z)^{-1} & \mathbf{K}^{-1}\mathbf{B}(z)^{-1} \\ \overline{\omega}\mathbf{B}(z)^{-1} & \mathbf{K}^{-1}\mathbf{A}(z)^{-1} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathrm{s}_{-}^{2}(z)\mathbf{A}(z)\mathbf{KB}(z)^{-1}\mathbf{A}(z) + \mathrm{s}_{+}^{2}(z)\mathbf{B}(z)\mathbf{K} & \mathrm{s}_{-}^{2}(z)\mathbf{A}(z)\mathbf{K} + \mathrm{s}_{+}^{2}(z)\mathbf{B}(z)\mathbf{KA}(z)^{-1}\mathbf{B}(z) \end{pmatrix} \\
&= \frac{1}{2\,\mathrm{c}_{-}(2z)}\begin{pmatrix} \mathrm{s}_{-}^{2}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{KB}(z)^{-1}\mathbf{A}(z) + \mathrm{s}_{+}^{2}(z)\mathbf{I} & \mathrm{s}_{-}^{2}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{K} + \mathrm{s}_{+}^{2}(z)\mathbf{A}(z)^{-1}\mathbf{B}(z) \\ \mathrm{s}_{-}^{2}(z)\mathbf{B}(z)^{-1}\mathbf{A}(z) + \mathrm{s}_{+}^{2}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{K} & \mathrm{s}_{-}^{2}(z)\mathbf{I} + \mathrm{s}_{+}^{2}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{KA}(z)^{-1}\mathbf{B}(z) \end{pmatrix}.
\end{aligned} \tag{6.11}$$

Lemma 6.3. *For $z \in \mathbb{C}$ such that* $\mathrm{c}_-(2z) \neq 0$*,*

$$\mathcal{F}(\mathbf{E}_{0112}, z) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathbb{i}}{\mathrm{si}_+(z)} \\ \frac{\mathbb{i}}{\mathrm{si}_-(z)} & \frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} -\frac{4\mathbb{i}\sin^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_+(z)} \\ \frac{1}{\mathrm{c}_+(z)} & -\frac{4\mathbb{i}\sin^2(\overline{\omega} z)}{\mathrm{c}_-(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{4\mathbb{i}\cos^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_-(z)} \\ \frac{1}{\mathrm{c}_-(z)} & -\frac{4\mathbb{i}\cos^2(\overline{\omega} z)}{\mathrm{c}_-(2z)} \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{si}_-(2z)}{\mathrm{c}_-(2z)} & -\frac{\mathbb{i}}{\mathrm{si}_-(z)} \\ \frac{\mathbb{i}}{\mathrm{si}_+(z)} & \frac{\mathrm{si}_+(2z)}{\mathrm{c}_-(2z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

Proof. By Definition 2.1 and (2.1),

$$\begin{aligned}
&\sqrt{2}\{\omega\,\mathrm{ci}_\pm(z) + \overline{\omega}\,\mathrm{ci}_\mp(z)\} = \sqrt{2}\omega\left[\left\{\cosh\left(\sqrt{2}z\right) \pm \mathbb{i}\cos\left(\sqrt{2}z\right)\right\} - \mathbb{i}\left\{\cosh\left(\sqrt{2}z\right) \mp \mathbb{i}\cos\left(\sqrt{2}z\right)\right\}\right] \\
&= \sqrt{2}\omega\left\{\sqrt{2}\overline{\omega}\cosh\left(\sqrt{2}z\right) \mp \sqrt{2}\overline{\omega}\cos\left(\sqrt{2}z\right)\right\} = 2\,\mathrm{c}_\mp(z), \\
&\sqrt{2}\{\omega\,\mathrm{s}_\pm(z) - \overline{\omega}\,\mathrm{s}_\mp(z)\} = \sqrt{2}\omega\left[\left\{\sinh\left(\sqrt{2}z\right) \pm \sin\left(\sqrt{2}z\right)\right\} + \mathbb{i}\left\{\sinh\left(\sqrt{2}z\right) \mp \sin\left(\sqrt{2}z\right)\right\}\right] \\
&= \sqrt{2}\omega\left\{\sqrt{2}\omega\sinh\left(\sqrt{2}z\right) \pm \sqrt{2}\overline{\omega}\sin\left(\sqrt{2}z\right)\right\} = 2\mathbb{i}\left\{\sinh\left(\sqrt{2}z\right) \mp \mathbb{i}\sin\left(\sqrt{2}z\right)\right\} = 2\mathbb{i}\,\mathrm{si}_\mp(z),
\end{aligned}$$

hence, by Definition 2.3, Lemma B.2 (a), (b), and (6.11), (A.6), (A.7),

$$\begin{aligned}
&2\,\mathrm{c}_-(2z)\cdot\{\mathbf{D}_{0112}(z)\}_{1:2,1:2} = \mathrm{s}_-^2(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1}\mathbf{A}(z) + \mathrm{s}_+^2(z)\mathbf{I} \\
&= \mathrm{s}_-^2(z)\begin{pmatrix}1 & 0 \\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}\cdot\begin{pmatrix}1 & 0 \\ 0 & \mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix} + \mathrm{s}_+^2(z)\mathbf{I} \\
&= \begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega \\ \sqrt{2}\omega & \mathbb{i}\,\mathrm{ci}_-(z)\end{pmatrix}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega \\ -\sqrt{2}\omega & -\mathbb{i}\,\mathrm{ci}_-(z)\end{pmatrix} + \mathrm{s}_+^2(z)\mathbf{I} = \begin{pmatrix}\mathrm{s}_+^2(z) + \mathrm{ci}_+^2(z) - 2\mathbb{i} & 2\,\mathrm{c}_-(z) \\ 2\,\mathrm{c}_-(z) & \mathrm{s}_+^2(z) + \mathrm{ci}_-^2(z) + 2\mathbb{i}\end{pmatrix}, \quad (6.12) \\
&2\,\mathrm{c}_-(2z)\cdot\{\mathbf{D}_{0112}(z)\}_{3:4,3:4} = \mathrm{s}_-^2(z)\mathbf{I} + \mathrm{s}_+^2(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1}\mathbf{B}(z) \\
&= \mathrm{s}_-^2(z)\mathbf{I} + \mathrm{s}_+^2(z)\begin{pmatrix}1 & 0 \\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix}\cdot\begin{pmatrix}1 & 0 \\ 0 & \mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix} \\
&= \mathrm{s}_-^2(z)\mathbf{I} + \begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega \\ \sqrt{2}\omega & \mathbb{i}\,\mathrm{ci}_+(z)\end{pmatrix}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\omega & -\mathbb{i}\,\mathrm{ci}_+(z)\end{pmatrix} = \begin{pmatrix}\mathrm{s}_-^2(z) + \mathrm{ci}_-^2(z) - 2\mathbb{i} & 2\,\mathrm{c}_+(z) \\ 2\,\mathrm{c}_+(z) & \mathrm{s}_-^2(z) + \mathrm{ci}_+^2(z) + 2\mathbb{i}\end{pmatrix}, \quad (6.13) \\
&2\,\mathrm{c}_-(2z)\cdot\{\mathbf{D}_{0112}(z)\}_{1:2,3:4}\cdot\mathbf{J} = \left\{\mathrm{s}_-^2(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1}\mathbf{A}(z)\mathbf{K} + \mathrm{s}_+^2(z)\mathbf{A}(z)^{-1}\mathbf{B}(z)\right\}\mathbf{J} \\
&= \mathrm{s}_-^2(z)\begin{pmatrix}1 & 0 \\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & \mathbb{i}\end{pmatrix}\mathbf{J} + \mathrm{s}_+^2(z)\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix}\mathbf{J} \\
&= \mathrm{s}_-(z)\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\overline{\omega} \\ \sqrt{2}\omega & \mathrm{ci}_-(z)\end{pmatrix} + \mathrm{s}_+(z)\begin{pmatrix}\mathrm{ci}_-(z) & -\sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & \mathrm{ci}_+(z)\end{pmatrix} = \begin{pmatrix}\mathrm{si}_-(2z) & -2\mathbb{i}\,\mathrm{si}_-(z) \\ 2\mathbb{i}\,\mathrm{si}_+(z) & \mathrm{si}_+(2z)\end{pmatrix}, \quad (6.14) \\
&2\,\mathrm{c}_-(2z)\cdot\{\mathbf{D}_{0112}(z)\}_{3:4.1:2}\cdot\mathbf{J} = \left\{\mathrm{s}_-^2(z)\mathbf{B}(z)^{-1}\mathbf{A}(z) + \mathrm{s}_+^2(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\mathbf{B}(z)\mathbf{K}\right\}\mathbf{J} \\
&= \mathrm{s}_-^2(z)\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}\mathbf{J} + \mathrm{s}_+^2(z)\begin{pmatrix}1 & 0 \\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & \mathbb{i}\end{pmatrix}\mathbf{J} \\
&= \mathrm{s}_-(z)\begin{pmatrix}\mathrm{ci}_+(z) & -\sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & \mathrm{ci}_-(z)\end{pmatrix} + \mathrm{s}_+(z)\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\overline{\omega} \\ \sqrt{2}\omega & \mathrm{ci}_+(z)\end{pmatrix} = \begin{pmatrix}\mathrm{si}_-(2z) & -2\mathbb{i}\,\mathrm{si}_+(z) \\ 2\mathbb{i}\,\mathrm{si}_-(z) & \mathrm{si}_+(2z)\end{pmatrix}. \quad (6.15)
\end{aligned}$$

By Definition 2.1, Lemma A.1 (a), (h), and (2.1),

$$\mathrm{s}_\pm^2(z) + \mathrm{ci}_\pm^2(z) - 2\mathbb{i} - \mathrm{c}_-(2z) = \left\{\frac{1}{2}\,\mathrm{c}_-(2z) \mp \mathbb{i}\,\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{c}_-(2z) \pm \mathbb{i}\,\mathrm{c}_+\left(\sqrt{2}\omega z\right)\right\} - 2\mathbb{i} - \mathrm{c}_-(2z)$$

$$= \pm \mathrm{i}\left\{\mathrm{c}_+\left(\sqrt{2}\omega z\right) - \mathrm{c}_-\left(\sqrt{2}\omega z\right) \mp 2\right\} = \pm 2\mathrm{i}\left\{\cos(2\omega z) \mp 1\right\} = -4\mathrm{i}\cdot\frac{1 \mp \cos(2\omega z)}{2},$$

$$\mathrm{s}_\pm^2(z) + \mathrm{ci}_\mp^2(z) + 2\mathrm{i} - \mathrm{c}_-(2z) = \left\{\frac{1}{2}\mathrm{c}_-(2z) \mp \mathrm{i}\,\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\mathrm{c}_-(2z) \mp \mathrm{i}\,\mathrm{c}_+\left(\sqrt{2}\omega z\right)\right\} + 2\mathrm{i} - \mathrm{c}_-(2z)$$

$$= \mp \mathrm{i}\left\{\mathrm{c}_+\left(\sqrt{2}\omega z\right) + \mathrm{c}_-\left(\sqrt{2}\omega z\right) \mp 2\right\} = \mp 2\mathrm{i}\left\{\cosh(2\omega z) \mp 1\right\} = 4\mathrm{i}\cdot\frac{1 \mp \cos(2\overline{\omega} z)}{2},$$

hence, by Definition 2.3, (6.12), (6.13), and Lemma A.1 (d),

$$\left[\{\mathbf{D}_{0112}(z)\}_{1:2,1:2} - \frac{1}{2}\mathbf{I}\right]\mathbf{J} = \frac{1}{2\,\mathrm{c}_-(2z)}\begin{pmatrix} \mathrm{s}_+^2(z) + \mathrm{ci}_+^2(z) - 2\mathrm{i} - \mathrm{c}_-(2z) & 2\,\mathrm{c}_-(z) \\ 2\,\mathrm{c}_-(z) & \mathrm{s}_+^2(z) + \mathrm{ci}_-^2(z) + 2\mathrm{i} - \mathrm{c}_-(2z) \end{pmatrix}\mathbf{J}$$

$$= \frac{1}{2\,\mathrm{c}_-(2z)}\begin{pmatrix} -4\mathrm{i}\sin^2(\omega z) & -2\,\mathrm{c}_-(z) \\ 2\,\mathrm{c}_-(z) & -4\mathrm{i}\sin^2(\overline{\omega} z) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\frac{4\mathrm{i}\sin^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_+(z)} \\ \frac{1}{\mathrm{c}_+(z)} & -\frac{4\mathrm{i}\sin^2(\overline{\omega} z)}{\mathrm{c}_-(2z)} \end{pmatrix}, \tag{6.16}$$

$$\left[\{\mathbf{D}_{0112}(z)\}_{3:4,3:4} - \frac{1}{2}\mathbf{I}\right]\mathbf{J} = \frac{1}{2\,\mathrm{c}_-(2z)}\begin{pmatrix} \mathrm{s}_-^2(z) + \mathrm{ci}_-^2(z) - 2\mathrm{i} - \mathrm{c}_-(2z) & 2\,\mathrm{c}_+(z) \\ 2\,\mathrm{c}_+(z) & \mathrm{s}_-^2(z) + \mathrm{ci}_+^2(z) + 2\mathrm{i} - \mathrm{c}_-(2z) \end{pmatrix}\mathbf{J}$$

$$= \frac{1}{2\,\mathrm{c}_-(2z)}\begin{pmatrix} -4\mathrm{i}\cos^2(\omega z) & -2\,\mathrm{c}_+(z) \\ 2\,\mathrm{c}_+(z) & -4\mathrm{i}\cos^2(\overline{\omega} z) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\frac{4\mathrm{i}\cos^2(\omega z)}{\mathrm{c}_-(2z)} & -\frac{1}{\mathrm{c}_-(z)} \\ \frac{1}{\mathrm{c}_-(z)} & -\frac{4\mathrm{i}\cos^2(\overline{\omega} z)}{\mathrm{c}_-(2z)} \end{pmatrix}. \tag{6.17}$$

Thus the result follows by Lemma 5.1, (6.14), (6.15), (6.16), (6.17), and Lemma A.1 (i). □

7. Fundamental boundary matrices for 02 types

7.1. $\mathcal{F}(\mathbf{E}_{0202}, z)$

By Lemma 5.2 (b), (d),

$$\mathbf{D}_{02}^-(z) + \mathbf{D}_{02}^+(z) = \begin{pmatrix} \mathbf{C}(z) & \mathbf{D}(z) \\ -\mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix} + \begin{pmatrix} \mathbf{C}(z) & -\mathbf{D}(z) \\ \mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix} = 2\begin{pmatrix} \mathbf{C}(z) & \mathbf{O} \\ \mathbf{O} & -\mathbf{D}(z) \end{pmatrix},$$

hence

$$\{\mathbf{D}_{02}^-(z) + \mathbf{D}_{02}^+(z)\}^{-1} = \frac{1}{2}\begin{pmatrix} \mathbf{C}(z) & \mathbf{O} \\ \mathbf{O} & -\mathbf{D}(z) \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} \mathbf{C}(z)^{-1} & \mathbf{O} \\ \mathbf{O} & -\mathbf{D}(z)^{-1} \end{pmatrix}.$$

Thus, by Definition 5.1 and Lemma 5.2 (b),

$$\begin{aligned} \mathbf{D}_{0202}(z) &= \{\mathbf{D}_{02}^-(z) + \mathbf{D}_{02}^+(z)\}^{-1}\,\mathbf{D}_{02}^-(z) \\ &= \frac{1}{2}\begin{pmatrix} \mathbf{C}(z)^{-1} & \mathbf{O} \\ \mathbf{O} & -\mathbf{D}(z)^{-1} \end{pmatrix}\cdot\begin{pmatrix} \mathbf{C}(z) & \mathbf{D}(z) \\ -\mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{I} & \mathbf{C}(z)^{-1}\mathbf{D}(z) \\ \mathbf{D}(z)^{-1}\mathbf{C}(z) & \mathbf{I} \end{pmatrix}. \end{aligned} \tag{7.1}$$

Lemma 7.1. *For $z \in \mathbb{C}$ such that $\mathrm{c}_+(z)\,\mathrm{c}_-(z) \neq 0$,*

$$\mathcal{F}(\mathbf{E}_{0202}, z) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} \frac{\mathrm{si}_+(z)}{\mathrm{c}_+(z)} & 0 \\ 0 & \frac{\mathrm{si}_-(z)}{\mathrm{c}_+(z)} \end{pmatrix} - \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \begin{pmatrix} \frac{\mathrm{si}_-(z)}{\mathrm{c}_-(z)} & 0 \\ 0 & \frac{\mathrm{si}_+(z)}{\mathrm{c}_-(z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

Proof. By Lemma 5.1 and (7.1),

$$\mathscr{F}(\mathbf{E}_{0202}, z) = \frac{1}{2}\begin{pmatrix} \mathbf{O} & \mathbf{C}(z)^{-1}\mathbf{D}(z) \\ \mathbf{D}(z)^{-1}\mathbf{C}(z) & \mathbf{O} \end{pmatrix}(\mathbf{J} \oplus' \mathbf{J}) - \frac{1}{2}\mathbf{I} = \frac{1}{2}\begin{pmatrix} \mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{J} - \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{J} - \mathbf{I} \end{pmatrix}.$$

Thus the result follows by Definition 2.3 and Lemma B.2 (c), (d). □

7.2. $\mathscr{F}(\mathbf{E}_{0213}, z)$

By Lemma 5.2 (b), (f),

$$\mathbf{D}^-_{02}(z) + \mathbf{D}^+_{13}(z) = \begin{pmatrix} \mathbf{C}(z) & \mathbf{D}(z) \\ -\mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix} + \omega\begin{pmatrix} \mathbf{D}(z)\mathbf{K} & -\mathbf{C}(z)\mathbf{K} \\ \mathbf{D}(z)\mathbf{K} & -\mathbf{C}(z)\mathbf{K} \end{pmatrix} = \begin{pmatrix} \mathbf{C}(z) + \omega\mathbf{D}(z)\mathbf{K} & \mathbf{D}(z) - \omega\mathbf{C}(z)\mathbf{K} \\ -\mathbf{C}(z) + \omega\mathbf{D}(z)\mathbf{K} & -\mathbf{D}(z) - \omega\mathbf{C}(z)\mathbf{K} \end{pmatrix}.$$

So by Definition 2.5,

$$\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}^-_{02}(z) + \mathbf{D}^+_{13}(z)\right\} = \frac{1}{2}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{C}(z) + \omega\mathbf{D}(z)\mathbf{K} & \mathbf{D}(z) - \omega\mathbf{C}(z)\mathbf{K} \\ -\mathbf{C}(z) + \omega\mathbf{D}(z)\mathbf{K} & -\mathbf{D}(z) - \omega\mathbf{C}(z)\mathbf{K} \end{pmatrix} = \begin{pmatrix} \omega\mathbf{D}(z)\mathbf{K} & -\omega\mathbf{C}(z)\mathbf{K} \\ -\mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix},$$

hence, by (2.1),

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}^-_{02}(z) + \mathbf{D}^+_{13}(z)\right\}\cdot\begin{pmatrix} \mathbf{C}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\mathbf{D}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix} = \begin{pmatrix} \omega\mathbf{D}(z)\mathbf{K} & -\omega\mathbf{C}(z)\mathbf{K} \\ -\mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix}\begin{pmatrix} \mathbf{C}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\mathbf{D}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \omega\left\{\mathbf{C}(z)\mathbf{K}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}\mathbf{C}(z)^{-1}\right\} & \mathbf{O} \\ \mathbf{O} & -\overline{\omega}\left\{\mathbf{C}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{C}(z)\mathbf{K}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}\mathbf{C}(z)^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{C}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix}\begin{pmatrix} \omega\mathbf{I} & 0 \\ 0 & -\overline{\omega}\mathbf{I} \end{pmatrix}.
\end{aligned}$$

So by Lemma B.4 (c) and (2.1),

$$\begin{aligned}
\left\{\mathbf{D}^-_{02}(z) + \mathbf{D}^+_{13}(z)\right\}^{-1} &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}^-_{02}(z) + \mathbf{D}^+_{13}(z)\right\}\right]^{-1}\mathbf{V} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{C}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\mathbf{D}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix}\cdot\begin{pmatrix} \overline{\omega}\mathbf{I} & 0 \\ 0 & -\omega\mathbf{I} \end{pmatrix} \\
&\quad\cdot\begin{pmatrix} \left\{\mathbf{C}(z)\mathbf{K}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}\mathbf{C}(z)^{-1}\right\}^{-1} & \mathbf{O} \\ \mathbf{O} & \left\{\mathbf{C}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\}^{-1} \end{pmatrix}\mathbf{V} \\
&= \frac{1}{2\sqrt{2}\,\mathrm{c}_+(2z)}\begin{pmatrix} \overline{\omega}\mathbf{C}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\overline{\omega}\mathbf{D}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix}\mathbf{H}(z)\mathbf{V}, \qquad (7.2)
\end{aligned}$$

where we put

$$\mathbf{H}(z) = \left\{\mathrm{c}^2_-(z)\mathbf{C}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathrm{c}^2_+(z)\mathbf{D}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\} \oplus \left\{\mathrm{c}^2_-(z)\mathbf{C}(z)\mathbf{K}\mathbf{D}(z)^{-1} + \mathrm{c}^2_+(z)\mathbf{D}(z)\mathbf{K}\mathbf{C}(z)^{-1}\right\}.$$

By Definition 2.5 and Lemma 5.2 (b),

$$\mathbf{V}\mathbf{D}^-_{02}(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}\cdot\begin{pmatrix} \mathbf{C}(z) & \mathbf{D}(z) \\ -\mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix} = \sqrt{2}\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ -\mathbf{C}(z) & -\mathbf{D}(z) \end{pmatrix},$$

hence, by Definition 5.1 and (7.2),

$$\mathbf{D}_{0213}(z) = \{\mathbf{D}^-_{02}(z) + \mathbf{D}^+_{13}(z)\}^{-1}\,\mathbf{D}^-_{02}(z) = \frac{1}{2\,\mathrm{c}_+(2z)}\begin{pmatrix}\overline{\omega}\mathbf{C}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\\ -\overline{\omega}\mathbf{D}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\end{pmatrix}\cdot\mathbf{H}(z)\begin{pmatrix}\mathbf{O} & \mathbf{O}\\ -\mathbf{C}(z) & -\mathbf{D}(z)\end{pmatrix}$$
$$= \frac{1}{2\,\mathrm{c}_+(2z)}\begin{pmatrix}-\overline{\omega}\mathbf{C}(z)^{-1} & \mathbf{K}^{-1}\mathbf{D}(z)^{-1}\\ \overline{\omega}\mathbf{D}(z)^{-1} & \mathbf{K}^{-1}\mathbf{C}(z)^{-1}\end{pmatrix}$$
$$\cdot\begin{pmatrix}\mathbf{O} & \mathbf{O}\\ \mathrm{c}_-^2(z)\mathbf{C}(z)\mathbf{K}\mathbf{D}(z)^{-1}\mathbf{C}(z) + \mathrm{c}_+^2(z)\mathbf{D}(z)\mathbf{K} & \mathrm{c}_-^2(z)\mathbf{C}(z)\mathbf{K} + \mathrm{c}_+^2(z)\mathbf{D}(z)\mathbf{K}\mathbf{C}(z)^{-1}\mathbf{D}(z)\end{pmatrix}$$
$$= \frac{1}{2\,\mathrm{c}_+(2z)}\begin{pmatrix}\mathrm{c}_-^2(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{K}\mathbf{D}(z)^{-1}\mathbf{C}(z) + \mathrm{c}_+^2(z)\mathbf{I} & \mathrm{c}_-^2(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{K} + \mathrm{c}_+^2(z)\mathbf{C}(z)^{-1}\mathbf{D}(z)\\ \mathrm{c}_-^2(z)\mathbf{D}(z)^{-1}\mathbf{C}(z) + \mathrm{c}_+^2(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{K} & \mathrm{c}_-^2(z)\mathbf{I} + \mathrm{c}_+^2(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{K}\mathbf{C}(z)^{-1}\mathbf{D}(z)\end{pmatrix}. \tag{7.3}$$

Lemma 7.2. *For $z \in \mathbb{C}$ such that* $\mathrm{c}_+(2z) \neq 0$,

$$\mathcal{F}(\mathbf{E}_{0213}, z) = \frac{1}{2}\begin{pmatrix}\begin{pmatrix}\frac{\mathrm{si}_+(2z)}{\mathrm{c}_+(2z)} & 0\\ 0 & \frac{\mathrm{si}_-(2z)}{\mathrm{c}_+(2z)}\end{pmatrix} - \mathbf{I} & \begin{pmatrix}\frac{2\cos(2\omega z)}{\mathrm{c}_+(2z)} & 0\\ 0 & -\frac{2\cos(2\overline{\omega}z)}{\mathrm{c}_+(2z)}\end{pmatrix}\\ \begin{pmatrix}-\frac{2\cos(2\omega z)}{\mathrm{c}_+(2z)} & 0\\ 0 & \frac{2\cos(2\overline{\omega}z)}{\mathrm{c}_+(2z)}\end{pmatrix} & \begin{pmatrix}\frac{\mathrm{si}_+(2z)}{\mathrm{c}_+(2z)} & 0\\ 0 & \frac{\mathrm{si}_-(2z)}{\mathrm{c}_+(2z)}\end{pmatrix} - \mathbf{I}\end{pmatrix}.$$

Proof. By Definition 2.3, Lemma B.2 (c), (d), and (7.3), (A.6), (A.7),

$$2\,\mathrm{c}_+(2z)\cdot\{\mathbf{D}_{0213}(z)\}_{1:2,1:2} = \mathrm{c}_-^2(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{K}\mathbf{D}(z)^{-1}\mathbf{C}(z) + \mathrm{c}_+^2(z)\mathbf{I}$$
$$= \mathrm{c}_-^2(z)\begin{pmatrix}1 & 0\\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{c}_-(z)}\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & -\mathrm{si}_+(z)\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{c}_-(z)}\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & -\mathrm{si}_+(z)\end{pmatrix} + \mathrm{c}_+^2(z)\mathbf{I}$$
$$= \begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & \mathbb{i}\,\mathrm{si}_+(z)\end{pmatrix}\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & -\mathbb{i}\,\mathrm{si}_+(z)\end{pmatrix} + \mathrm{c}_+^2(z)\mathbf{I} = \begin{pmatrix}\mathrm{c}_+^2(z) + \mathrm{si}_-^2(z) & 0\\ 0 & \mathrm{c}_+^2(z) + \mathrm{si}_+^2(z)\end{pmatrix}, \tag{7.4}$$

$$2\,\mathrm{c}_+(2z)\cdot\{\mathbf{D}_{0213}(z)\}_{3:4,3:4} = \mathrm{c}_-^2(z)\mathbf{I} + \mathrm{c}_+^2(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{K}\mathbf{C}(z)^{-1}\mathbf{D}(z)$$
$$= \mathrm{c}_-^2(z)\mathbf{I} + \mathrm{c}_+^2(z)\begin{pmatrix}1 & 0\\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathrm{si}_-(z)\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathrm{si}_-(z)\end{pmatrix}$$
$$= \mathrm{c}_-^2(z)\mathbf{I} + \begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & \mathbb{i}\,\mathrm{si}_-(z)\end{pmatrix}\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathbb{i}\,\mathrm{si}_-(z)\end{pmatrix} = \begin{pmatrix}\mathrm{c}_-^2(z) + \mathrm{si}_+^2(z) & 0\\ 0 & \mathrm{c}_-^2(z) + \mathrm{si}_-^2(z)\end{pmatrix}, \tag{7.5}$$

$$2\,\mathrm{c}_+(2z)\cdot\{\mathbf{D}_{0213}\}_{1:2,3:4}\cdot\mathbf{J} = \left\{\mathrm{c}_-^2(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{K} + \mathrm{c}_+^2(z)\mathbf{C}(z)^{-1}\mathbf{D}(z)\right\}\mathbf{J}$$
$$= \mathrm{c}_-^2(z)\begin{pmatrix}1 & 0\\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{c}_-(z)}\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & -\mathrm{si}_+(z)\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathbb{i}\end{pmatrix}\mathbf{J} + \mathrm{c}_+^2(z)\cdot\frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathrm{si}_-(z)\end{pmatrix}\mathbf{J}$$
$$= \mathrm{c}_-(z)\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & \mathrm{si}_+(z)\end{pmatrix} + \mathrm{c}_+(z)\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & \mathrm{si}_-(z)\end{pmatrix} = \begin{pmatrix}\mathrm{si}_+(2z) & 0\\ 0 & \mathrm{si}_-(2z)\end{pmatrix}, \tag{7.6}$$

$$2\,\mathrm{c}_+(2z)\cdot\{\mathbf{D}_{0213}\}_{3:4,1:2}\cdot\mathbf{J} = \left\{\mathrm{c}_-^2(z)\mathbf{D}(z)^{-1}\mathbf{C}(z) + \mathrm{c}_+^2(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{K}\right\}\mathbf{J}$$
$$= \mathrm{c}_-^2(z)\cdot\frac{1}{\mathrm{c}_-(z)}\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & -\mathrm{si}_+(z)\end{pmatrix}\mathbf{J} + \mathrm{c}_+^2(z)\begin{pmatrix}1 & 0\\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathrm{si}_-(z)\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & \mathbb{i}\end{pmatrix}\mathbf{J}$$
$$= \mathrm{c}_-(z)\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & \mathrm{si}_+(z)\end{pmatrix} + \mathrm{c}_+(z)\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & \mathrm{si}_-(z)\end{pmatrix} = \begin{pmatrix}\mathrm{si}_+(2z) & 0\\ 0 & \mathrm{si}_-(2z)\end{pmatrix}. \tag{7.7}$$

By Lemma A.1 (b), (g) and (2.1),

$$\begin{aligned}
\mathrm{c}_{\pm}^2(z)+\mathrm{si}_{\mp}^2(z)-\mathrm{c}_+(2z) &= \left\{\frac{1}{2}\,\mathrm{c}_+(2z)+1\pm\mathrm{c}_+\left(\sqrt{2}\omega z\right)\right\}+\left\{\frac{1}{2}\,\mathrm{c}_+(2z)-1\mp\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\}-\mathrm{c}_+(2z)\\
&= \pm\left\{\mathrm{c}_+\left(\sqrt{2}\omega z\right)-\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\}=\pm 2\cos(2\omega z),\\
\mathrm{c}_{\pm}^2(z)+\mathrm{si}_{\pm}^2(z)-\mathrm{c}_+(2z) &= \left\{\frac{1}{2}\,\mathrm{c}_+(2z)+1\pm\mathrm{c}_+\left(\sqrt{2}\omega z\right)\right\}+\left\{\frac{1}{2}\,\mathrm{c}_+(2z)-1\pm\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\}-\mathrm{c}_+(2z)\\
&= \pm\left\{\mathrm{c}_+\left(\sqrt{2}\omega z\right)+\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\}=\pm 2\cosh(2\omega z)=\pm 2\cos(2\overline{\omega}z),
\end{aligned}$$

hence, by Definition 2.3 and (7.4), (7.5),

$$\begin{aligned}
\left[\{\mathbf{D}_{0213}\}_{1:2,1:2}-\frac{1}{2}\mathbf{I}\right]\mathbf{J} &= \frac{1}{2\,\mathrm{c}_+(2z)}\begin{pmatrix}\mathrm{c}_+^2(z)+\mathrm{si}_-^2(z)-\mathrm{c}_+(2z) & 0\\ 0 & \mathrm{c}_+^2(z)+\mathrm{si}_+^2(z)-\mathrm{c}_+(2z)\end{pmatrix}\mathbf{J}\\
&= \frac{1}{2\,\mathrm{c}_+(2z)}\begin{pmatrix}2\cos(2\omega z) & 0\\ 0 & -2\cos(2\overline{\omega}z)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\frac{2\cos(2\omega z)}{\mathrm{c}_+(2z)} & 0\\ 0 & -\frac{2\cos(2\overline{\omega}z)}{\mathrm{c}_+(2z)}\end{pmatrix}, \qquad (7.8)\\
\left[\{\mathbf{D}_{0213}\}_{3:4,3:4}-\frac{1}{2}\mathbf{I}\right]\mathbf{J} &= \frac{1}{2\,\mathrm{c}_+(2z)}\begin{pmatrix}\mathrm{c}_-^2(z)+\mathrm{si}_+^2(z)-\mathrm{c}_+(2z) & 0\\ 0 & \mathrm{c}_-^2(z)+\mathrm{si}_-^2(z)-\mathrm{c}_+(2z)\end{pmatrix}\mathbf{J}\\
&= \frac{1}{2\,\mathrm{c}_+(2z)}\begin{pmatrix}-2\cos(2\omega z) & 0\\ 0 & 2\cos(2\overline{\omega}z)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-\frac{2\cos(2\omega z)}{\mathrm{c}_+(2z)} & 0\\ 0 & \frac{2\cos(2\overline{\omega}z)}{\mathrm{c}_+(2z)}\end{pmatrix}. \qquad (7.9)
\end{aligned}$$

Thus the result follows by (7.6), (7.7), (7.8), (7.9), and Lemma 5.1. □

8. Fundamental boundary matrices for mixed types

8.1. $\mathcal{F}(\mathbf{E}_{0102}, z)$

By Lemma 5.2 (a), (d),

$$\mathbf{D}_{01}^-(z)+\mathbf{D}_{02}^+(z)=\begin{pmatrix}\mathbf{JA}(z) & \mathbf{JB}(z)\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix}+\begin{pmatrix}\mathbf{C}(z) & -\mathbf{D}(z)\\ \mathbf{C}(z) & -\mathbf{D}(z)\end{pmatrix}=\begin{pmatrix}\mathbf{JA}(z)+\mathbf{C}(z) & \mathbf{JB}(z)-\mathbf{D}(z)\\ -\mathbf{JA}(z)+\mathbf{C}(z) & -\mathbf{JB}(z)-\mathbf{D}(z)\end{pmatrix}.$$

So by Definition 2.5,

$$\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^-(z)+\mathbf{D}_{02}^+(z)\right\}=\frac{1}{2}\begin{pmatrix}\mathbf{I} & \mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{JA}(z)+\mathbf{C}(z) & \mathbf{JB}(z)-\mathbf{D}(z)\\ -\mathbf{JA}(z)+\mathbf{C}(z) & -\mathbf{JB}(z)-\mathbf{D}(z)\end{pmatrix}=\begin{pmatrix}\mathbf{C}(z) & -\mathbf{D}(z)\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix},$$

hence

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^-(z)+\mathbf{D}_{02}^+(z)\right\}\cdot\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbf{C}(z)^{-1}\\ -\mathbf{B}(z)^{-1} & -\mathbf{D}(z)^{-1}\end{pmatrix}=\begin{pmatrix}\mathbf{C}(z) & -\mathbf{D}(z)\\ -\mathbf{JA}(z) & -\mathbf{JB}(z)\end{pmatrix}\begin{pmatrix}\mathbf{A}(z)^{-1} & -\mathbf{C}(z)^{-1}\\ -\mathbf{B}(z)^{-1} & -\mathbf{D}(z)^{-1}\end{pmatrix}\\
&=\begin{pmatrix}\mathbf{C}(z)\mathbf{A}(z)^{-1}+\mathbf{D}(z)\mathbf{B}(z)^{-1} & \mathbf{O}\\ \mathbf{O} & \mathbf{J}\left\{\mathbf{A}(z)\mathbf{C}(z)^{-1}+\mathbf{B}(z)\mathbf{D}(z)^{-1}\right\}\end{pmatrix}.
\end{aligned}$$

So by Lemma B.4 (d), (e), and (2.2),

$$\left\{\mathbf{D}_{01}^-(z)+\mathbf{D}_{02}^+(z)\right\}^{-1}=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\mathbf{V}\left\{\mathbf{D}_{01}^-(z)+\mathbf{D}_{02}^+(z)\right\}\right]^{-1}\mathbf{V}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{A}(z)^{-1} & -\mathbf{C}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & -\mathbf{D}(z)^{-1} \end{pmatrix} \begin{pmatrix} \left\{\mathbf{C}(z)\mathbf{A}(z)^{-1} + \mathbf{D}(z)\mathbf{B}(z)^{-1}\right\}^{-1} & \mathbf{O} \\ \mathbf{O} & \left\{\mathbf{A}(z)\mathbf{C}(z)^{-1} + \mathbf{B}(z)\mathbf{D}(z)^{-1}\right\}^{-1} \mathbf{J} \end{pmatrix} \mathbf{V}$$

$$= \frac{1}{2\sqrt{2}\,\mathrm{s}_-(2z)} \begin{pmatrix} \mathbf{A}(z)^{-1} & -\mathbf{C}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & -\mathbf{D}(z)^{-1} \end{pmatrix} \mathbf{H}(z)\mathbf{V},$$

where we put

$$\mathbf{H}(z) = \left\{\mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{A}(z)\mathbf{C}(z)^{-1} + \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{B}(z)\mathbf{D}(z)^{-1}\right\}$$
$$\oplus \left[\left\{\mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{A}(z)^{-1} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{B}(z)^{-1}\right\} \mathbf{J}\right].$$

Thus, by Definition 5.1, and (2.2), (6.2),

$$\mathbf{D}_{0102}(z) = \{\mathbf{D}_{01}^-(z) + \mathbf{D}_{02}^+(z)\}^{-1}\,\mathbf{D}_{01}^-(z) = \frac{1}{2\,\mathrm{s}_-(2z)} \begin{pmatrix} \mathbf{A}(z)^{-1} & -\mathbf{C}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & -\mathbf{D}(z)^{-1} \end{pmatrix} \mathbf{H}(z) \cdot \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ -\mathbf{J}\mathbf{A}(z) & -\mathbf{J}\mathbf{B}(z) \end{pmatrix}$$

$$= \frac{1}{2\,\mathrm{s}_-(2z)} \begin{pmatrix} -\mathbf{A}(z)^{-1} & \mathbf{C}(z)^{-1} \\ \mathbf{B}(z)^{-1} & \mathbf{D}(z)^{-1} \end{pmatrix}$$
$$\cdot \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{C}(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{D}(z) \end{pmatrix}$$
$$= \frac{1}{2\,\mathrm{s}_-(2z)}$$
$$\cdot \begin{pmatrix} \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{I} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{C}(z)^{-1}\mathbf{D}(z) \\ \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{D}(z)^{-1}\mathbf{C}(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{I} \end{pmatrix}. \tag{8.1}$$

Lemma 8.1. *For $z \in \mathbb{C}$ such that* $\mathrm{s}_-(2z) \neq 0$,

$$\mathcal{F}(\mathbf{E}_{0102}, z) = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} \frac{\mathrm{ci}_+(2z) - \sqrt{2}\omega\cos(2\omega z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{c}_-(z)}{\mathrm{s}_-(2z)} \\ -\frac{\sqrt{2}\overline{\omega}\,\mathrm{c}_-(z)}{\mathrm{s}_-(2z)} & \frac{\mathrm{ci}_-(2z) - \sqrt{2}\overline{\omega}\cos(2\overline{\omega}z)}{\mathrm{s}_-(2z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} \frac{\sqrt{2}\omega\sin(2\omega z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{si}_+(z)}{\mathrm{s}_-(2z)} \\ \frac{\sqrt{2}\overline{\omega}\,\mathrm{si}_-(z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\overline{\omega}\sin(2\overline{\omega}z)}{\mathrm{s}_-(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{\sqrt{2}\omega\sin(2\omega z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{si}_-(z)}{\mathrm{s}_-(2z)} \\ \frac{\sqrt{2}\overline{\omega}\,\mathrm{si}_+(z)}{\mathrm{s}_-(2z)} & \frac{\sqrt{2}\overline{\omega}\sin(2\overline{\omega}z)}{\mathrm{s}_-(2z)} \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{ci}_+(2z) + \sqrt{2}\omega\cos(2\omega z)}{\mathrm{s}_-(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{c}_+(z)}{\mathrm{s}_-(2z)} \\ -\frac{\sqrt{2}\overline{\omega}\,\mathrm{c}_+(z)}{\mathrm{s}_-(2z)} & \frac{\mathrm{ci}_-(2z) + \sqrt{2}\overline{\omega}\cos(2\overline{\omega}z)}{\mathrm{s}_-(2z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

Proof. By Lemma B.2 (a), (b), (c), (d), and (8.1),

$$2\,\mathrm{s}_-(2z) \cdot \{\mathbf{D}_{0102}(z)\}_{1:2,1:2} = \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{I} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{B}(z)^{-1}\mathbf{A}(z)$$
$$= \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{I} + \mathrm{s}_-(z)\,\mathrm{c}_+(z) \cdot \frac{1}{\mathrm{c}_+(z)} \begin{pmatrix} \mathrm{si}_+(z) & 0 \\ 0 & -\mathrm{si}_-(z) \end{pmatrix} \cdot \frac{1}{\mathrm{s}_-(z)} \begin{pmatrix} \mathrm{ci}_+(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z) \end{pmatrix}, \tag{8.2}$$

$$2\,\mathrm{s}_-(2z) \cdot \{\mathbf{D}_{0102}(z)\}_{3:4,3:4} = \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{I}$$
$$= \mathrm{s}_+(z)\,\mathrm{c}_-(z) \cdot \frac{1}{\mathrm{c}_-(z)} \begin{pmatrix} \mathrm{si}_-(z) & 0 \\ 0 & -\mathrm{si}_+(z) \end{pmatrix} \cdot \frac{1}{\mathrm{s}_+(z)} \begin{pmatrix} \mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z) \end{pmatrix} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{I}, \tag{8.3}$$

$$2\,\mathrm{s}_-(2z) \cdot \{\mathbf{D}_{0102}(z)\}_{1:2,3:4} = \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{C}(z)^{-1}\mathbf{D}(z)$$
$$= \mathrm{s}_+(z)\,\mathrm{c}_-(z) \cdot \frac{1}{\mathrm{s}_+(z)} \begin{pmatrix} \mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z) \end{pmatrix} + \mathrm{s}_-(z)\,\mathrm{c}_+(z) \cdot \frac{1}{\mathrm{c}_+(z)} \begin{pmatrix} \mathrm{si}_+(z) & 0 \\ 0 & -\mathrm{si}_-(z) \end{pmatrix}, \tag{8.4}$$

$$2\,\mathrm{s}_-(2z)\cdot\{\mathbf{D}_{0102}(z)\}_{3:4,1:2} = \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{D}(z)^{-1}\mathbf{C}(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{B}(z)^{-1}\mathbf{A}(z)$$
$$= \mathrm{s}_+(z)\,\mathrm{c}_-(z)\cdot\frac{1}{\mathrm{c}_-(z)}\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & -\mathrm{si}_+(z)\end{pmatrix} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}. \tag{8.5}$$

By Lemma A.1 (f), (k), and (2.1),

$$\mathrm{s}_\pm(z)\,\mathrm{c}_\mp(z) + \mathrm{si}_\pm(z)\,\mathrm{ci}_\pm(z) = \left\{\frac{1}{2}\,\mathrm{s}_-(2z) \mp \frac{\omega}{\sqrt{2}}\,\mathrm{s}_-\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{s}_-(2z) \pm \frac{\omega}{\sqrt{2}}\,\mathrm{s}_+\left(\sqrt{2}\omega z\right)\right\}$$
$$= \mathrm{s}_-(2z) \pm \frac{\omega}{\sqrt{2}}\left\{\mathrm{s}_+\left(\sqrt{2}\omega z\right) - \mathrm{s}_-\left(\sqrt{2}\omega z\right)\right\} = \mathrm{s}_-(2z) \pm \sqrt{2}\omega\sin(2\omega z),$$
$$\mathrm{s}_\pm(z)\,\mathrm{c}_\mp(z) + \mathrm{si}_\mp(z)\,\mathrm{ci}_\mp(z) = \left\{\frac{1}{2}\,\mathrm{s}_-(2z) \mp \frac{\omega}{\sqrt{2}}\,\mathrm{s}_-\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{s}_-(2z) \mp \frac{\omega}{\sqrt{2}}\,\mathrm{s}_+\left(\sqrt{2}\omega z\right)\right\}$$
$$= \mathrm{s}_-(2z) \mp \frac{\omega}{\sqrt{2}}\left\{\mathrm{s}_+\left(\sqrt{2}\omega z\right) + \mathrm{s}_-\left(\sqrt{2}\omega z\right)\right\} = \mathrm{s}_-(2z) \mp \sqrt{2}\omega\sinh(2\omega z) = \mathrm{s}_-(2z) \pm \sqrt{2}\overline{\omega}\sin(2\overline{\omega}z),$$

hence, by Definition 2.3, and (8.2), (8.3),

$$\left[\{\mathbf{D}_{0102}(z)\}_{1:2,1:2} - \frac{1}{2}\mathbf{I}\right]\mathbf{J}$$
$$= \frac{1}{2\,\mathrm{s}_-(2z)}\begin{pmatrix}\mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{si}_+(z)\,\mathrm{ci}_+(z) - \mathrm{s}_-(2z) & \sqrt{2}\omega\,\mathrm{si}_+(z)\\ \sqrt{2}\overline{\omega}\,\mathrm{si}_-(z) & \mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{si}_-(z)\,\mathrm{ci}_-(z) - \mathrm{s}_-(2z)\end{pmatrix}\mathbf{J}$$
$$= \frac{1}{2\,\mathrm{s}_-(2z)}\begin{pmatrix}\sqrt{2}\omega\sin(2\omega z) & -\sqrt{2}\omega\,\mathrm{si}_+(z)\\ \sqrt{2}\overline{\omega}\,\mathrm{si}_-(z) & -\sqrt{2}\overline{\omega}\sin(2\overline{\omega}z)\end{pmatrix}, \tag{8.6}$$
$$\left[\{\mathbf{D}_{0102}(z)\}_{3:4,3:4} - \frac{1}{2}\mathbf{I}\right]\mathbf{J}$$
$$= \frac{1}{2\,\mathrm{s}_-(2z)}\begin{pmatrix}\mathrm{s}_-(z)\,\mathrm{c}_+(z) + \mathrm{si}_-(z)\,\mathrm{ci}_-(z) - \mathrm{s}_-(2z) & \sqrt{2}\omega\,\mathrm{si}_-(z)\\ \sqrt{2}\overline{\omega}\,\mathrm{si}_+(z) & \mathrm{s}_-(z)\,\mathrm{c}_+(z) + \mathrm{si}_+(z)\,\mathrm{ci}_+(z) - \mathrm{s}_-(2z)\end{pmatrix}\mathbf{J}$$
$$= \frac{1}{2\,\mathrm{s}_-(2z)}\begin{pmatrix}-\sqrt{2}\omega\sin(2\omega z) & -\sqrt{2}\omega\,\mathrm{si}_-(z)\\ \sqrt{2}\overline{\omega}\,\mathrm{si}_+(z) & \sqrt{2}\overline{\omega}\sin(2\overline{\omega}z)\end{pmatrix}. \tag{8.7}$$

By Lemma A.1 (m), (n), (o), (p), and (2.1),

$$\mathrm{s}_\mp(z)\,\mathrm{si}_\pm(z) + \mathrm{c}_\mp(z)\,\mathrm{ci}_\mp(z) = \left\{\frac{1}{2}\,\mathrm{ci}_+(2z) - \frac{\omega}{\sqrt{2}} \pm \frac{\omega}{\sqrt{2}}\,\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{ci}_+(2z) + \frac{\omega}{\sqrt{2}} \mp \frac{\omega}{\sqrt{2}}\,\mathrm{c}_+\left(\sqrt{2}\omega z\right)\right\}$$
$$= \mathrm{ci}_+(2z) \mp \frac{\omega}{\sqrt{2}}\left\{\mathrm{c}_+\left(\sqrt{2}\omega z\right) - \mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} = \mathrm{ci}_+(2z) \mp \sqrt{2}\omega\cos(2\omega z),$$
$$\mathrm{s}_\mp(z)\,\mathrm{si}_\mp(z) + \mathrm{c}_\mp(z)\,\mathrm{ci}_\pm(z) = \left\{\frac{1}{2}\,\mathrm{ci}_-(2z) - \frac{\overline{\omega}}{\sqrt{2}} \mp \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{ci}_-(2z) + \frac{\overline{\omega}}{\sqrt{2}} \mp \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{c}_+\left(\sqrt{2}\omega z\right)\right\}$$
$$= \mathrm{ci}_-(2z) \mp \frac{\overline{\omega}}{\sqrt{2}}\left\{\mathrm{c}_+\left(\sqrt{2}\omega z\right) + \mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} = \mathrm{ci}_-(2z) \mp \sqrt{2}\overline{\omega}\cosh(2\omega z) = \mathrm{ci}_-(2z) \mp \sqrt{2}\overline{\omega}\cos\left(2\overline{\omega}z\right),$$

hence, by Definition 2.3, and (8.4), (8.5),

$$\{\mathbf{D}_{0102}(z)\}_{1:2,3:4}\cdot\mathbf{J} = \frac{1}{2\,\mathrm{s}_-(2z)}\begin{pmatrix}\mathrm{s}_-(z)\,\mathrm{si}_+(z) + \mathrm{c}_-(z)\,\mathrm{ci}_-(z) & \sqrt{2}\omega\,\mathrm{c}_-(z)\\ -\sqrt{2}\overline{\omega}\,\mathrm{c}_-(z) & -\{\mathrm{s}_-(z)\,\mathrm{si}_-(z) + \mathrm{c}_-(z)\,\mathrm{ci}_+(z)\}\end{pmatrix}\mathbf{J}$$

$$= \frac{1}{2\,\mathrm{s}_-(2z)} \begin{pmatrix} \mathrm{ci}_+(2z) - \sqrt{2}\omega\cos(2\omega z) & -\sqrt{2}\omega\,\mathrm{c}_-(z) \\ -\sqrt{2}\overline{\omega}\,\mathrm{c}_-(z) & \mathrm{ci}_-(2z) - \sqrt{2}\overline{\omega}\cos(2\overline{\omega}z) \end{pmatrix}, \tag{8.8}$$

$$\{\mathbf{D}_{0102}(z)\}_{3:4,1:2}\cdot\mathbf{J} = \frac{1}{2\,\mathrm{s}_-(2z)} \begin{pmatrix} \mathrm{s}_+(z)\,\mathrm{si}_-(z) + \mathrm{c}_+(z)\,\mathrm{ci}_+(z) & \sqrt{2}\omega\,\mathrm{c}_+(z) \\ -\sqrt{2}\overline{\omega}\,\mathrm{c}_+(z) & -\{\mathrm{s}_+(z)\,\mathrm{si}_+(z) + \mathrm{c}_+(z)\,\mathrm{ci}_-(z)\} \end{pmatrix}\mathbf{J}$$

$$= \frac{1}{2\,\mathrm{s}_-(2z)} \begin{pmatrix} \mathrm{ci}_+(2z) + \sqrt{2}\omega\cos(2\omega z) & -\sqrt{2}\omega\,\mathrm{c}_+(z) \\ -\sqrt{2}\overline{\omega}\,\mathrm{c}_+(z) & \mathrm{ci}_-(2z) + \sqrt{2}\overline{\omega}\cos(2\overline{\omega}z) \end{pmatrix}. \tag{8.9}$$

Thus the result follows by (8.6), (8.7), (8.8), (8.9), and Lemma 5.1. □

8.2. $\mathcal{F}(\mathbf{E}_{0113}, z)$

By Lemma 5.2 (a), (f),

$$\mathbf{D}_{01}^-(z) + \mathbf{D}_{13}^+(z)$$
$$= \begin{pmatrix} \mathbf{JA}(z) & \mathbf{JB}(z) \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix} + \omega \begin{pmatrix} \mathbf{D}(z)\mathbf{K} & -\mathbf{C}(z)\mathbf{K} \\ \mathbf{D}(z)\mathbf{K} & -\mathbf{C}(z)\mathbf{K} \end{pmatrix} = \begin{pmatrix} \mathbf{JA}(z) + \omega\mathbf{D}(z)\mathbf{K} & \mathbf{JB}(z) - \omega\mathbf{C}(z)\mathbf{K} \\ -\mathbf{JA}(z) + \omega\mathbf{D}(z)\mathbf{K} & -\mathbf{JB}(z) - \omega\mathbf{C}(z)\mathbf{K} \end{pmatrix}.$$

So by Definition 2.5,

$$\frac{1}{\sqrt{2}}\mathbf{V}\{\mathbf{D}_{01}^-(z) + \mathbf{D}_{13}^+(z)\} = \frac{1}{2}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{JA}(z) + \omega\mathbf{D}(z)\mathbf{K} & \mathbf{JB}(z) - \omega\mathbf{C}(z)\mathbf{K} \\ -\mathbf{JA}(z) + \omega\mathbf{D}(z)\mathbf{K} & -\mathbf{JB}(z) - \omega\mathbf{C}(z)\mathbf{K} \end{pmatrix} = \begin{pmatrix} \omega\mathbf{D}(z)\mathbf{K} & -\omega\mathbf{C}(z)\mathbf{K} \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix},$$

hence, by (2.1),

$$\frac{1}{\sqrt{2}}\mathbf{V}\{\mathbf{D}_{01}^-(z) + \mathbf{D}_{13}^+(z)\}\cdot\begin{pmatrix} \mathbf{A}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix} = \begin{pmatrix} \omega\mathbf{D}(z)\mathbf{K} & -\omega\mathbf{C}(z)\mathbf{K} \\ -\mathbf{JA}(z) & -\mathbf{JB}(z) \end{pmatrix}\begin{pmatrix} \mathbf{A}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} \omega\left\{\mathbf{D}(z)\mathbf{KA}(z)^{-1} + \mathbf{C}(z)\mathbf{KB}(z)^{-1}\right\} & \mathbf{O} \\ \mathbf{O} & -\overline{\omega}\mathbf{J}\left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{D}(z)\mathbf{KA}(z)^{-1} + \mathbf{C}(z)\mathbf{KB}(z)^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{J}\left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\} \end{pmatrix}\begin{pmatrix} \omega\mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\overline{\omega}\mathbf{I} \end{pmatrix}.$$

So by Lemma B.4 (f), (g), and (2.1), (2.2),

$$\{\mathbf{D}_{01}^-(z) + \mathbf{D}_{13}^+(z)\}^{-1} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\mathbf{V}\{\mathbf{D}_{01}^-(z) + \mathbf{D}_{13}^+(z)\}\right]^{-1}\mathbf{V}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{A}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\mathbf{B}(z)^{-1} & \overline{\omega}\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix}\cdot\begin{pmatrix} \overline{\omega}\mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\omega\mathbf{I} \end{pmatrix}$$
$$\cdot\begin{pmatrix} \left\{\mathbf{D}(z)\mathbf{KA}(z)^{-1} + \mathbf{C}(z)\mathbf{KB}(z)^{-1}\right\}^{-1} & \mathbf{O} \\ \mathbf{O} & \left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\}^{-1}\mathbf{J} \end{pmatrix}\mathbf{V}$$
$$= \frac{1}{2\sqrt{2}\,\mathrm{s}_+(2z)}\begin{pmatrix} \overline{\omega}\mathbf{A}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\overline{\omega}\mathbf{B}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix}\mathbf{H}(z)\mathbf{V},$$

where we put

$$\mathbf{H}(z) = \left\{\mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\}$$

$$\oplus \left[\left\{ \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1} \right\} \mathbf{J} \right].$$

Thus, by Definition 5.1 and (6.2),

$$\mathbf{D}_{0113}(z) = \{\mathbf{D}_{01}^-(z) + \mathbf{D}_{13}^+(z)\}^{-1}\,\mathbf{D}_{01}^-(z) = \frac{1}{2\,\mathrm{s}_+(2z)} \begin{pmatrix} \overline{\omega}\mathbf{A}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ -\overline{\omega}\mathbf{B}(z)^{-1} & -\mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix} \mathbf{H}(z) \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ -\mathbf{J}\mathbf{A}(z) & -\mathbf{J}\mathbf{B}(z) \end{pmatrix}$$

$$= \frac{1}{2\,\mathrm{s}_+(2z)} \begin{pmatrix} -\overline{\omega}\mathbf{A}(z)^{-1} & \mathbf{K}^{-1}\mathbf{D}(z)^{-1} \\ \overline{\omega}\mathbf{B}(z)^{-1} & \mathbf{K}^{-1}\mathbf{C}(z)^{-1} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{K} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}\mathbf{A}(z) & \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{K} \end{pmatrix}$$

$$= \frac{1}{2\,\mathrm{s}_+(2z)} \cdot \left(\begin{matrix} \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{I} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}\mathbf{A}(z) \\ \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{K} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{B}(z)^{-1}\mathbf{A}(z) \end{matrix} \right| \tag{8.10}$$

$$\left. \begin{matrix} \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{K} \\ \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{I} \end{matrix} \right).$$

Lemma 8.2. *For $z \in \mathbb{C}$ such that $\mathrm{s}_+(2z) \neq 0$,*

$$\mathcal{F}(\mathbf{E}_{0113}, z) = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} \frac{\mathrm{ci}_-(2z) + \sqrt{2}\overline{\omega}\cos(2\omega z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{c}_+(z)}{\mathrm{s}_+(2z)} \\ -\frac{\sqrt{2}\overline{\omega}\,\mathrm{c}_+(z)}{\mathrm{s}_+(2z)} & \frac{\mathrm{ci}_+(2z) + \sqrt{2}\omega\cos(2\overline{\omega}z)}{\mathrm{s}_+(2z)} \end{pmatrix} - \mathbf{I} & \begin{pmatrix} \frac{\sqrt{2}\overline{\omega}\sin(2\omega z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{si}_-(z)}{\mathrm{s}_+(2z)} \\ \frac{\sqrt{2}\overline{\omega}\,\mathrm{si}_+(z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\sin(2\overline{\omega}z)}{\mathrm{s}_+(2z)} \end{pmatrix} \\ \begin{pmatrix} -\frac{\sqrt{2}\overline{\omega}\sin(2\omega z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{si}_+(z)}{\mathrm{s}_+(2z)} \\ \frac{\sqrt{2}\overline{\omega}\,\mathrm{si}_-(z)}{\mathrm{s}_+(2z)} & \frac{\sqrt{2}\omega\sin(2\overline{\omega}z)}{\mathrm{s}_+(2z)} \end{pmatrix} & \begin{pmatrix} \frac{\mathrm{ci}_-(2z) - \sqrt{2}\overline{\omega}\cos(2\omega z)}{\mathrm{s}_+(2z)} & -\frac{\sqrt{2}\omega\,\mathrm{c}_-(z)}{\mathrm{s}_+(2z)} \\ -\frac{\sqrt{2}\overline{\omega}\,\mathrm{c}_-(z)}{\mathrm{s}_+(2z)} & \frac{\mathrm{ci}_+(2z) - \sqrt{2}\omega\cos(2\overline{\omega}z)}{\mathrm{s}_+(2z)} \end{pmatrix} - \mathbf{I} \end{pmatrix}.$$

Proof. By Definition 2.3, Lemma B.2 (a), (b), (c), (d), and (8.10),

$$2\,\mathrm{s}_+(2z) \cdot \{\mathbf{D}_{0113}(z)\}_{1:2,1:2} = \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{I} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}\mathbf{A}(z)$$

$$= \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{I} + \mathrm{s}_-(z)\,\mathrm{c}_-(z) \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{i} \end{pmatrix} \cdot \frac{1}{\mathrm{c}_-(z)} \begin{pmatrix} \mathrm{si}_-(z) & 0 \\ 0 & -\mathrm{si}_+(z) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{i} \end{pmatrix} \cdot \frac{1}{\mathrm{s}_-(z)} \begin{pmatrix} \mathrm{ci}_+(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z) \end{pmatrix}$$

$$= \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{I} + \begin{pmatrix} \mathrm{si}_-(z) & 0 \\ 0 & -\mathrm{si}_+(z) \end{pmatrix} \begin{pmatrix} \mathrm{ci}_+(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z) \end{pmatrix}, \tag{8.11}$$

$$2\,\mathrm{s}_+(2z) \cdot \{\mathbf{D}_{0113}(z)\}_{3:4,3:4} = \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{I}$$

$$= \mathrm{s}_+(z)\,\mathrm{c}_+(z) \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{i} \end{pmatrix} \cdot \frac{1}{\mathrm{c}_+(z)} \begin{pmatrix} \mathrm{si}_+(z) & 0 \\ 0 & -\mathrm{si}_-(z) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{i} \end{pmatrix} \cdot \frac{1}{\mathrm{s}_+(z)} \begin{pmatrix} \mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z) \end{pmatrix} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{I}$$

$$= \begin{pmatrix} \mathrm{si}_+(z) & 0 \\ 0 & -\mathrm{si}_-(z) \end{pmatrix} \begin{pmatrix} \mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z) \end{pmatrix} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{I}, \tag{8.12}$$

$$2\,\mathrm{s}_+(2z) \cdot \{\mathbf{D}_{0113}(z)\}_{1:2,3:4} = \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{A}(z)^{-1}\mathbf{B}(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1}\mathbf{C}(z)\mathbf{K}$$

$$= \mathrm{s}_+(z)\,\mathrm{c}_+(z) \cdot \frac{1}{\mathrm{s}_+(z)} \begin{pmatrix} \mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z) \end{pmatrix} + \mathrm{s}_-(z)\,\mathrm{c}_-(z) \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{i} \end{pmatrix} \cdot \frac{1}{\mathrm{c}_-(z)} \begin{pmatrix} \mathrm{si}_-(z) & 0 \\ 0 & -\mathrm{si}_+(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{i} \end{pmatrix}$$

$$= \mathrm{c}_+(z) \begin{pmatrix} \mathrm{ci}_-(z) & \sqrt{2}\omega \\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z) \end{pmatrix} + \mathrm{s}_-(z) \begin{pmatrix} \mathrm{si}_-(z) & 0 \\ 0 & -\mathrm{si}_+(z) \end{pmatrix}, \tag{8.13}$$

$$2\,\mathrm{s}_+(2z) \cdot \{\mathbf{D}_{0113}(z)\}_{3:4,1:2} = \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\mathbf{D}(z)\mathbf{K} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{B}(z)^{-1}\mathbf{A}(z)$$

$$= \mathrm{s}_+(z)\,\mathrm{c}_+(z)\cdot\begin{pmatrix}1 & 0\\ 0 & -\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathrm{si}_-(z)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \mathbb{i}\end{pmatrix} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}$$

$$= \mathrm{s}_+(z)\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathrm{si}_-(z)\end{pmatrix} + \mathrm{c}_-(z)\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}. \tag{8.14}$$

By Definition 2.1, Lemma A.1 (e), (l), and (2.1),

$$\mathrm{s}_\pm(z)\,\mathrm{c}_\pm(z) + \mathrm{si}_\mp(z)\,\mathrm{ci}_\pm(z) = \left\{\frac{1}{2}\,\mathrm{s}_+(2z) \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{s}_+\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{s}_+(2z) \mp \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{s}_-\left(\sqrt{2}\omega z\right)\right\}$$

$$= \mathrm{s}_+(2z) \pm \frac{\overline{\omega}}{\sqrt{2}}\left\{\mathrm{s}_+\left(\sqrt{2}\omega z\right) - \mathrm{s}_-\left(\sqrt{2}\omega z\right)\right\} = \mathrm{s}_+(2z) \pm \sqrt{2}\overline{\omega}\sin(2\omega z),$$

$$\mathrm{s}_\pm(z)\,\mathrm{c}_\pm(z) + \mathrm{si}_\pm(z)\,\mathrm{ci}_\mp(z) = \left\{\frac{1}{2}\,\mathrm{s}_+(2z) \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{s}_+\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{s}_+(2z) \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{s}_-\left(\sqrt{2}\omega z\right)\right\}$$

$$= \mathrm{s}_+(2z) \pm \frac{\overline{\omega}}{\sqrt{2}}\left\{\mathrm{s}_+\left(\sqrt{2}\omega z\right) + \mathrm{s}_-\left(\sqrt{2}\omega z\right)\right\} = \mathrm{s}_+(2z) \pm \sqrt{2}\overline{\omega}\sinh(2\omega z) = \mathrm{s}_+(2z) \pm \sqrt{2}\omega\sin\left(2\overline{\omega}z\right),$$

hence, by Definition 2.3, and (8.11), (8.12),

$$\left[\{\mathbf{D}_{0113}(z)\}_{1:2,1:2} - \frac{1}{2}\mathbf{I}\right]\mathbf{J}$$

$$= \frac{1}{2\,\mathrm{s}_+(2z)}\begin{pmatrix}\mathrm{s}_+(z)\,\mathrm{c}_+(z) + \mathrm{si}_-(z)\,\mathrm{ci}_+(z) - \mathrm{s}_+(2z) & \sqrt{2}\omega\,\mathrm{si}_-(z)\\ \sqrt{2}\overline{\omega}\,\mathrm{si}_+(z) & \mathrm{s}_+(z)\,\mathrm{c}_+(z) + \mathrm{si}_+(z)\,\mathrm{ci}_-(z) - \mathrm{s}_+(2z)\end{pmatrix}\mathbf{J}$$

$$= \frac{1}{2\,\mathrm{s}_+(2z)}\begin{pmatrix}\sqrt{2}\overline{\omega}\sin(2\omega z) & -\sqrt{2}\omega\,\mathrm{si}_-(z)\\ \sqrt{2}\overline{\omega}\,\mathrm{si}_+(z) & -\sqrt{2}\omega\sin\left(2\overline{\omega}z\right)\end{pmatrix}, \tag{8.15}$$

$$\left[\{\mathbf{D}_{0113}(z)\}_{3:4,3:4} - \frac{1}{2}\mathbf{I}\right]\mathbf{J}$$

$$= \frac{1}{2\,\mathrm{s}_+(2z)}\begin{pmatrix}\mathrm{s}_-(z)\,\mathrm{c}_-(z) + \mathrm{si}_+(z)\,\mathrm{ci}_-(z) - \mathrm{s}_+(2z) & \sqrt{2}\omega\,\mathrm{si}_+(z)\\ \sqrt{2}\overline{\omega}\,\mathrm{si}_-(z) & \mathrm{s}_-(z)\,\mathrm{c}_-(z) + \mathrm{si}_-(z)\,\mathrm{ci}_+(z) - \mathrm{s}_+(2z)\end{pmatrix}\mathbf{J}$$

$$= \frac{1}{2\,\mathrm{s}_+(2z)}\begin{pmatrix}-\sqrt{2}\overline{\omega}\sin(2\omega z) & -\sqrt{2}\omega\,\mathrm{si}_+(z)\\ \sqrt{2}\overline{\omega}\,\mathrm{si}_-(z) & \sqrt{2}\omega\sin\left(2\overline{\omega}z\right)\end{pmatrix}. \tag{8.16}$$

By Lemma A.1 (m), (n), (o), (p), and (2.1),

$$\mathrm{s}_\mp(z)\,\mathrm{si}_\mp(z) + \mathrm{c}_\pm(z)\,\mathrm{ci}_\mp(z) = \left\{\frac{1}{2}\,\mathrm{ci}_-(2z) - \frac{\overline{\omega}}{\sqrt{2}} \mp \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{ci}_-(2z) + \frac{\overline{\omega}}{\sqrt{2}} \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{c}_+\left(\sqrt{2}\omega z\right)\right\}$$

$$= \mathrm{ci}_-(2z) \pm \frac{\overline{\omega}}{\sqrt{2}}\left\{\mathrm{c}_+\left(\sqrt{2}\omega z\right) - \mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} = \mathrm{ci}_-(2z) \pm \sqrt{2}\overline{\omega}\cos(2\omega z),$$

$$\mathrm{s}_\mp(z)\,\mathrm{si}_\pm(z) + \mathrm{c}_\pm(z)\,\mathrm{ci}_\pm(z) = \left\{\frac{1}{2}\,\mathrm{ci}_+(2z) - \frac{\omega}{\sqrt{2}} \pm \frac{\omega}{\sqrt{2}}\,\mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} + \left\{\frac{1}{2}\,\mathrm{ci}_+(2z) + \frac{\omega}{\sqrt{2}} \pm \frac{\omega}{\sqrt{2}}\,\mathrm{c}_+\left(\sqrt{2}\omega z\right)\right\}$$

$$= \mathrm{ci}_+(2z) \pm \frac{\omega}{\sqrt{2}}\left\{\mathrm{c}_+\left(\sqrt{2}\omega z\right) + \mathrm{c}_-\left(\sqrt{2}\omega z\right)\right\} = \mathrm{ci}_+(2z) \pm \sqrt{2}\omega\cosh(2\omega z) = \mathrm{ci}_+(2z) \pm \sqrt{2}\omega\cos\left(2\overline{\omega}z\right),$$

hence, by Definition 2.3, and (8.13), (8.14),

$$\{\mathbf{D}_{0113}(z)\}_{1:2,3:4}\cdot\mathbf{J} = \frac{1}{2\,\mathrm{s}_+(2z)}\begin{pmatrix}\mathrm{s}_-(z)\,\mathrm{si}_-(z)+\mathrm{c}_+(z)\,\mathrm{ci}_-(z) & \sqrt{2}\omega\,\mathrm{c}_+(z)\\ -\sqrt{2}\overline{\omega}\,\mathrm{c}_+(z) & -\{\mathrm{s}_-(z)\,\mathrm{si}_+(z)+\mathrm{c}_+(z)\,\mathrm{ci}_+(z)\}\end{pmatrix}\mathbf{J}$$
$$= \frac{1}{2\,\mathrm{s}_+(2z)}\begin{pmatrix}\mathrm{ci}_-(2z)+\sqrt{2}\overline{\omega}\cos(2\omega z) & -\sqrt{2}\omega\,\mathrm{c}_+(z)\\ -\sqrt{2}\overline{\omega}\,\mathrm{c}_+(z) & \mathrm{ci}_+(2z)+\sqrt{2}\omega\cos(2\overline{\omega}z)\end{pmatrix}, \tag{8.17}$$
$$\{\mathbf{D}_{0113}(z)\}_{3:4,1:2}\cdot\mathbf{J} = \frac{1}{2\,\mathrm{s}_+(2z)}\begin{pmatrix}\mathrm{s}_+(z)\,\mathrm{si}_+(z)+\mathrm{c}_-(z)\,\mathrm{ci}_+(z) & \sqrt{2}\omega\,\mathrm{c}_-(z)\\ -\sqrt{2}\overline{\omega}\,\mathrm{c}_-(z) & -\{\mathrm{s}_+(z)\,\mathrm{si}_-(z)+\mathrm{c}_-(z)\,\mathrm{ci}_-(z)\}\end{pmatrix}\mathbf{J}$$
$$= \frac{1}{2\,\mathrm{s}_+(2z)}\begin{pmatrix}\mathrm{ci}_-(2z)-\sqrt{2}\overline{\omega}\cos(2\omega z) & -\sqrt{2}\omega\,\mathrm{c}_-(z)\\ -\sqrt{2}\overline{\omega}\,\mathrm{c}_-(z) & \mathrm{ci}_+(2z)-\sqrt{2}\omega\cos(2\overline{\omega}z)\end{pmatrix}. \tag{8.18}$$

Thus the result follows by (8.15), (8.16), (8.17), (8.18), and Lemma 5.1. □

9. Proofs of Theorems 1 and 2

Theorem 1 follows from (3.1), Lemma 2.1, and Lemmas 6.1, 6.2, 6.3, 7.1, 7.2, 8.1, 8.2. Theorem 2 (a) can be checked with Table 1. Theorem 2 (b) follows from Theorem 1, Theorem 2 (a), Lemma 4.2 (a), and Lemmas 2.1, 4.6, 4.7, 4.8.

10. Discussion

The properties of the transformation $\mathcal{F}$ in Proposition 1.2 imply in particular that the set of all well-posed two-point boundary conditions forms a 16-dimensional space $\mathrm{gl}(4,\mathbb{C})$. Although elementary boundary conditions consist of only 36 conditions inside this huge space, they still include important boundary conditions useful in practical engineering situations. Following are a few examples:

- $\mathbf{E}_{0101}$: Clamped at both ends or, *bi-clamped*.
- $\mathbf{E}_{0202}$: Hinged at both ends or, *bi-hinged*.
- $\mathbf{E}_{0123}$: Clamped-free or, *cantilevered*.
- $\mathbf{E}_{0102}$: Clamped-hinged.

The matrix $\mathbf{X}^{\pm}_{\lambda}(x)$ in Proposition 1.2 has a compact representation with concretely defined holomorphic functions $\delta^{\pm}(z,\kappa)$ [1]. So Theorems 1 and 2, together with Proposition 1.2, provide us 36 concrete instances of holomorphic equations which completely describe the spectral information of the 36 different integral operators $\mathcal{K}_{\mathbf{M}}$ in (1.5) corresponding to elementary boundary conditions. Detailed analysis of these equations is expected to produce complete spectral information such as Proposition 1.1 for the 36 elementary boundary value problems, which will be valuable data for further understanding of the whole 16-dimensional boundary value problems of beam deflection.

Use of AI tools declaration

The authors have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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Appendix

A. Elementary calculus with the functions $\mathrm{s}_\pm$, $\mathrm{c}_\pm$, $\mathrm{si}_\pm$, $\mathrm{ci}_\pm$

Let $\zeta \in \mathbb{C}$. By Definition 2.1 and (2.1),

$$
\begin{aligned}
&\sinh\zeta\cos\zeta = \sinh\zeta\cosh(\mathbb{i}\zeta) = \frac{1}{2}\left\{\sinh\left(\sqrt{2}\omega\zeta\right) + \sinh\left(\sqrt{2}\overline{\omega}\zeta\right)\right\}\\
&= \frac{1}{2}\left\{\sinh\left(\sqrt{2}\omega\zeta\right) + \sinh\left(-\mathbb{i}\cdot\sqrt{2}\omega\zeta\right)\right\} = \frac{1}{2}\left\{\sinh\left(\sqrt{2}\omega\zeta\right) - \mathbb{i}\sin\left(\sqrt{2}\omega\zeta\right)\right\} = \frac{1}{2}\mathrm{si}_-(\omega\zeta),\\
&\cosh\zeta\sin\zeta = -\mathbb{i}\cosh\zeta\sinh(\mathbb{i}\zeta) = -\frac{\mathbb{i}}{2}\left\{\sinh\left(\sqrt{2}\omega\zeta\right) - \sinh\left(\sqrt{2}\overline{\omega}\zeta\right)\right\}\\
&= -\frac{\mathbb{i}}{2}\left\{\sinh\left(\sqrt{2}\omega\zeta\right) - \sinh\left(-\mathbb{i}\cdot\sqrt{2}\omega\zeta\right)\right\} = -\frac{\mathbb{i}}{2}\left\{\sinh\left(\sqrt{2}\omega\zeta\right) + \mathbb{i}\sin\left(\sqrt{2}\omega\zeta\right)\right\} = -\frac{\mathbb{i}}{2}\mathrm{si}_+(\omega\zeta),\\
&\cosh\zeta\cos\zeta = \cosh\zeta\cosh(\mathbb{i}\zeta) = \frac{1}{2}\left\{\cosh\left(\sqrt{2}\omega\zeta\right) + \cosh\left(\sqrt{2}\overline{\omega}\zeta\right)\right\}\\
&= \frac{1}{2}\left\{\cosh\left(\sqrt{2}\omega\zeta\right) + \cosh\left(-\mathbb{i}\cdot\sqrt{2}\omega\zeta\right)\right\} = \frac{1}{2}\left\{\cosh\left(\sqrt{2}\omega\zeta\right) + \cos\left(\sqrt{2}\omega\zeta\right)\right\} = \frac{1}{2}\mathrm{c}_+(\omega\zeta),\\
&\sinh\zeta\sin\zeta = -\mathbb{i}\sinh\zeta\sinh(\mathbb{i}\zeta) = -\frac{\mathbb{i}}{2}\left\{\cosh\left(\sqrt{2}\omega\zeta\right) - \cosh\left(\sqrt{2}\overline{\omega}\zeta\right)\right\}\\
&= -\frac{\mathbb{i}}{2}\left\{\cosh\left(\sqrt{2}\omega\zeta\right) - \cosh\left(-\mathbb{i}\cdot\sqrt{2}\omega\zeta\right)\right\} = -\frac{\mathbb{i}}{2}\left\{\cosh\left(\sqrt{2}\omega\zeta\right) - \cos\left(\sqrt{2}\omega\zeta\right)\right\} = -\frac{\mathbb{i}}{2}\mathrm{c}_-(\omega\zeta).
\end{aligned}
\tag{A.1}
$$

Lemma A.1. *The following hold for $\zeta \in \mathbb{C}$.*

(a) $\mathrm{s}_\pm^2(\zeta) = \frac{1}{2}\,\mathrm{c}_-(2\zeta) \mp \mathbb{i}\,\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right)$.

(b) $\mathrm{c}_\pm^2(\zeta) = \frac{1}{2}\,\mathrm{c}_+(2\zeta) + 1 \pm \mathrm{c}_+\left(\sqrt{2}\omega\zeta\right)$.

(c) $\mathrm{s}_+(\zeta)\,\mathrm{s}_-(\zeta) = \frac{1}{2}\,\mathrm{c}_+(2\zeta) - 1$.

(d) $\mathrm{c}_+(\zeta)\,\mathrm{c}_-(\zeta) = \frac{1}{2}\,\mathrm{c}_-(2\zeta)$.

(e) $\mathrm{s}_\pm(\zeta)\,\mathrm{c}_\pm(\zeta) = \frac{1}{2}\,\mathrm{s}_+(2\zeta) \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{s}_+\left(\sqrt{2}\omega\zeta\right)$.

(f) $\mathrm{s}_\pm(\zeta)\,\mathrm{c}_\mp(\zeta) = \frac{1}{2}\,\mathrm{s}_-(2\zeta) \mp \frac{\omega}{\sqrt{2}}\,\mathrm{s}_-\left(\sqrt{2}\omega\zeta\right)$.

(g) $\mathrm{si}_\pm^2(\zeta) = \frac{1}{2}\,\mathrm{c}_+(2\zeta) - 1 \pm \mathrm{c}_-\left(\sqrt{2}\omega\zeta\right)$.

(h) $\mathrm{ci}_\pm^2(\zeta) = \frac{1}{2}\,\mathrm{c}_-(2\zeta) \pm \mathbb{i}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right)$.

(i) $\mathrm{si}_+(\zeta)\,\mathrm{si}_-(\zeta) = \frac{1}{2}\,\mathrm{c}_-(2\zeta)$.

(j) $\mathrm{ci}_+(\zeta)\,\mathrm{ci}_-(\zeta) = \frac{1}{2}\,\mathrm{c}_+(2\zeta) + 1$.

(k) $\mathrm{si}_\pm(\zeta)\,\mathrm{ci}_\pm(\zeta) = \frac{1}{2}\,\mathrm{s}_-(2\zeta) \pm \frac{\omega}{\sqrt{2}}\,\mathrm{s}_+\left(\sqrt{2}\omega\zeta\right)$.

(l) $\mathrm{si}_\pm(\zeta)\,\mathrm{ci}_\mp(\zeta) = \frac{1}{2}\,\mathrm{s}_+(2\zeta) \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{s}_-\left(\sqrt{2}\omega\zeta\right)$.

(m) $\mathrm{s}_\pm(\zeta)\,\mathrm{si}_\pm(\zeta) = \frac{1}{2}\,\mathrm{ci}_-(2\zeta) - \frac{\overline{\omega}}{\sqrt{2}} \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right)$.

(n) $\mathrm{s}_\pm(\zeta)\,\mathrm{si}_\mp(\zeta) = \frac{1}{2}\,\mathrm{ci}_+(2\zeta) - \frac{\omega}{\sqrt{2}} \mp \frac{\omega}{\sqrt{2}}\,\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right)$.

(o) $\mathrm{c}_\pm(\zeta)\,\mathrm{ci}_\pm(\zeta) = \frac{1}{2}\,\mathrm{ci}_+(2\zeta) + \frac{\omega}{\sqrt{2}} \pm \frac{\omega}{\sqrt{2}}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right)$.

(p) $\mathrm{c}_\pm(\zeta)\,\mathrm{ci}_\mp(\zeta) = \frac{1}{2}\,\mathrm{ci}_-(2\zeta) + \frac{\overline{\omega}}{\sqrt{2}} \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right)$.

(q) $\mathrm{s}_\pm(\zeta)\,\mathrm{ci}_\pm(\zeta) = \frac{1}{2}\,\mathrm{si}_+(2\zeta) \pm \sin(2\omega\zeta)$.

(r) $\mathrm{s}_\pm(\zeta)\,\mathrm{ci}_\mp(\zeta) = \frac{1}{2}\,\mathrm{si}_-(2\zeta) \mp \mathbb{i}\sinh(2\omega\zeta)$.

(s) $\mathrm{c}_\pm(\zeta)\,\mathrm{si}_\pm(\zeta) = \frac{1}{2}\,\mathrm{si}_+(2\zeta) \pm \sinh(2\omega\zeta)$.

(t) $\mathrm{c}_\pm(\zeta)\,\mathrm{si}_\mp(\zeta) = \frac{1}{2}\,\mathrm{si}_-(2\zeta) \mp \mathbb{i}\sin(2\omega\zeta)$.

Proof. By Definition 2.1, and (2.1), (A.1),

$$\begin{aligned}
\mathrm{s}_\pm^2(\zeta) &= \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)\right\}^2 = \sinh^2\left(\sqrt{2}\zeta\right) + \sin^2\left(\sqrt{2}\zeta\right) \pm 2\sinh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) - 1\right\} - \frac{1}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) - 1\right\} \pm 2\cdot\frac{-\mathbb{i}}{2}\,\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right) = \frac{1}{2}\,\mathrm{c}_-(2\zeta) \mp \mathbb{i}\,\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right),\\
\mathrm{c}_\pm^2(\zeta) &= \left\{\cosh\left(\sqrt{2}\zeta\right) \pm \cos\left(\sqrt{2}\zeta\right)\right\}^2 = \cosh^2\left(\sqrt{2}\zeta\right) + \cos^2\left(\sqrt{2}\zeta\right) \pm 2\cosh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) + 1\right\} + \frac{1}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) + 1\right\} \pm 2\cdot\frac{1}{2}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right) = \frac{1}{2}\,\mathrm{c}_+(2\zeta) + 1 \pm \mathrm{c}_+\left(\sqrt{2}\omega\zeta\right),\\
\mathrm{s}_+(\zeta)\,\mathrm{s}_-(\zeta) &= \left\{\sinh\left(\sqrt{2}\zeta\right) + \sin\left(\sqrt{2}\zeta\right)\right\}\left\{\sinh\left(\sqrt{2}\zeta\right) - \sin\left(\sqrt{2}\zeta\right)\right\} = \sinh^2\left(\sqrt{2}\zeta\right) - \sin^2\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) - 1\right\} + \frac{1}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) - 1\right\} = \frac{1}{2}\,\mathrm{c}_+(2\zeta) - 1,\\
\mathrm{c}_+(\zeta)\,\mathrm{c}_-(\zeta) &= \left\{\cosh\left(\sqrt{2}\zeta\right) + \cos\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) - \cos\left(\sqrt{2}\zeta\right)\right\} = \cosh^2\left(\sqrt{2}\zeta\right) - \cos^2\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) + 1\right\} - \frac{1}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) + 1\right\} = \frac{1}{2}\,\mathrm{c}_-(2\zeta),\\
\mathrm{s}_\pm(\zeta)\,\mathrm{c}_\pm(\zeta) &= \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) \pm \cos\left(\sqrt{2}\zeta\right)\right\}\\
&= \sinh\left(\sqrt{2}\zeta\right)\cosh\left(\sqrt{2}\zeta\right) + \sin\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \sinh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \cosh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\sinh\left(2\sqrt{2}\zeta\right) + \frac{1}{2}\sin\left(2\sqrt{2}\zeta\right) \pm \frac{1}{2}\,\mathrm{si}_-\left(\sqrt{2}\omega\zeta\right) \pm \frac{-\mathbb{i}}{2}\,\mathrm{si}_+\left(\sqrt{2}\omega\zeta\right)\\
&= \frac{1}{2}\,\mathrm{s}_+(2\zeta) \pm \frac{1}{2}\left\{\sqrt{2}\overline{\omega}\sinh(2\omega\zeta) + \sqrt{2}\overline{\omega}\sin(2\omega\zeta)\right\} = \frac{1}{2}\,\mathrm{s}_+(2\zeta) \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{s}_+\left(\sqrt{2}\omega\zeta\right),\\
\mathrm{s}_\pm(\zeta)\,\mathrm{c}_\mp(\zeta) &= \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) \mp \cos\left(\sqrt{2}\zeta\right)\right\}\\
&= \sinh\left(\sqrt{2}\zeta\right)\cosh\left(\sqrt{2}\zeta\right) - \sin\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \mp \sinh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \cosh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\sinh\left(2\sqrt{2}\zeta\right) - \frac{1}{2}\sin\left(2\sqrt{2}\zeta\right) \mp \frac{1}{2}\,\mathrm{si}_-\left(\sqrt{2}\omega\zeta\right) \pm \frac{-\mathbb{i}}{2}\,\mathrm{si}_+\left(\sqrt{2}\omega\zeta\right)\\
&= \frac{1}{2}\,\mathrm{s}_-(2\zeta) \mp \frac{1}{2}\left\{\sqrt{2}\omega\sinh(2\omega\zeta) - \sqrt{2}\omega\sin(2\omega\zeta)\right\} = \frac{1}{2}\,\mathrm{s}_-(2\zeta) \mp \frac{\omega}{\sqrt{2}}\,\mathrm{s}_-\left(\sqrt{2}\omega\zeta\right),
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{si}_\pm^2(\zeta) = \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\sin\left(\sqrt{2}\zeta\right)\right\}^2 = \sinh^2\left(\sqrt{2}\zeta\right) - \sin^2\left(\sqrt{2}\zeta\right) \pm 2\mathrm{i}\sinh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) - 1\right\} + \frac{1}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) - 1\right\} \pm 2\mathrm{i}\cdot\frac{-\mathrm{i}}{2}\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right) = \frac{1}{2}\mathrm{c}_+(2\zeta) - 1 \pm \mathrm{c}_-\left(\sqrt{2}\omega\zeta\right),\\
&\mathrm{ci}_\pm^2(\zeta) = \left\{\cosh\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\cos\left(\sqrt{2}\zeta\right)\right\}^2 = \cosh^2\left(\sqrt{2}\zeta\right) - \cos^2\left(\sqrt{2}\zeta\right) \pm 2\mathrm{i}\cosh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) + 1\right\} - \frac{1}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) + 1\right\} \pm 2\mathrm{i}\cdot\frac{1}{2}\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right) = \frac{1}{2}\mathrm{c}_-(2\zeta) \pm \mathrm{i}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right),\\
&\mathrm{si}_+(\zeta)\,\mathrm{si}_-(\zeta) = \left\{\sinh\left(\sqrt{2}\zeta\right) + \mathrm{i}\sin\left(\sqrt{2}\zeta\right)\right\}\left\{\sinh\left(\sqrt{2}\zeta\right) - \mathrm{i}\sin\left(\sqrt{2}\zeta\right)\right\}\\
&= \sinh^2\left(\sqrt{2}\zeta\right) + \sin^2\left(\sqrt{2}\zeta\right) = \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) - 1\right\} - \frac{1}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) - 1\right\} = \frac{1}{2}\mathrm{c}_-(2\zeta),\\
&\mathrm{ci}_+(\zeta)\,\mathrm{ci}_-(\zeta) = \left\{\cosh\left(\sqrt{2}\zeta\right) + \mathrm{i}\cos\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) - \mathrm{i}\cos\left(\sqrt{2}\zeta\right)\right\}\\
&= \cosh^2\left(\sqrt{2}\zeta\right) + \cos^2\left(\sqrt{2}\zeta\right) = \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) + 1\right\} + \frac{1}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) + 1\right\} = \frac{1}{2}\mathrm{c}_+(2\zeta) + 1,\\
&\mathrm{si}_\pm(\zeta)\,\mathrm{ci}_\pm(\zeta) = \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\sin\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\cos\left(\sqrt{2}\zeta\right)\right\}\\
&= \sinh\left(\sqrt{2}\zeta\right)\cosh\left(\sqrt{2}\zeta\right) - \sin\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\sinh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\cosh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\sinh\left(2\sqrt{2}\zeta\right) - \frac{1}{2}\sin\left(2\sqrt{2}\zeta\right) \pm \mathrm{i}\cdot\frac{1}{2}\mathrm{si}_-\left(\sqrt{2}\omega\zeta\right) \pm \mathrm{i}\cdot\frac{-\mathrm{i}}{2}\mathrm{si}_+\left(\sqrt{2}\omega\zeta\right)\\
&= \frac{1}{2}\mathrm{s}_-(2\zeta) \pm \frac{1}{2}\left\{\sqrt{2}\omega\sinh(2\omega\zeta) + \sqrt{2}\omega\sin(2\omega\zeta)\right\} = \frac{1}{2}\mathrm{s}_-(2\zeta) \pm \frac{\omega}{\sqrt{2}}\mathrm{s}_+\left(\sqrt{2}\omega\zeta\right),\\
&\mathrm{si}_\pm(\zeta)\,\mathrm{ci}_\mp(\zeta) = \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\sin\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) \mp \mathrm{i}\cos\left(\sqrt{2}\zeta\right)\right\}\\
&= \sinh\left(\sqrt{2}\zeta\right)\cosh\left(\sqrt{2}\zeta\right) + \sin\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \mp \mathrm{i}\sinh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\cosh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\sinh\left(2\sqrt{2}\zeta\right) + \frac{1}{2}\sin\left(2\sqrt{2}\zeta\right) \mp \mathrm{i}\cdot\frac{1}{2}\mathrm{si}_-\left(\sqrt{2}\omega\zeta\right) \pm \mathrm{i}\cdot\frac{-\mathrm{i}}{2}\mathrm{si}_+\left(\sqrt{2}\omega\zeta\right)\\
&= \frac{1}{2}\mathrm{s}_+(2\zeta) \pm \frac{1}{2}\left\{\sqrt{2}\overline{\omega}\sinh(2\omega\zeta) - \sqrt{2}\overline{\omega}\sin(2\omega\zeta)\right\} = \frac{1}{2}\mathrm{s}_+(2\zeta) \pm \frac{\overline{\omega}}{\sqrt{2}}\mathrm{s}_-\left(\sqrt{2}\omega\zeta\right),\\
&\mathrm{s}_\pm(\zeta)\,\mathrm{si}_\pm(\zeta) = \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)\right\}\left\{\sinh\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\sin\left(\sqrt{2}\zeta\right)\right\}\\
&= \sinh^2\left(\sqrt{2}\zeta\right) + \mathrm{i}\sin^2\left(\sqrt{2}\zeta\right) \pm \sqrt{2}\omega\sinh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) - 1\right\} - \frac{\mathrm{i}}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) - 1\right\} \pm \sqrt{2}\omega\cdot\frac{-\mathrm{i}}{2}\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right)\\
&= \frac{1}{2}\mathrm{ci}_-(2\zeta) - \frac{\overline{\omega}}{\sqrt{2}} \pm \frac{\overline{\omega}}{\sqrt{2}}\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right),\\
&\mathrm{s}_\pm(\zeta)\,\mathrm{si}_\mp(\zeta) = \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)\right\}\left\{\sinh\left(\sqrt{2}\zeta\right) \mp \mathrm{i}\sin\left(\sqrt{2}\zeta\right)\right\}\\
&= \sinh^2\left(\sqrt{2}\zeta\right) - \mathrm{i}\sin^2\left(\sqrt{2}\zeta\right) \pm \sqrt{2}\overline{\omega}\sinh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)\\
&= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right) - 1\right\} + \frac{\mathrm{i}}{2}\left\{\cos\left(2\sqrt{2}\zeta\right) - 1\right\} \pm \sqrt{2}\overline{\omega}\cdot\frac{-\mathrm{i}}{2}\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right)\\
&= \frac{1}{2}\mathrm{ci}_+(2\zeta) - \frac{\omega}{\sqrt{2}} \mp \frac{\omega}{\sqrt{2}}\mathrm{c}_-\left(\sqrt{2}\omega\zeta\right),\\
&\mathrm{c}_\pm(\zeta)\,\mathrm{ci}_\pm(\zeta) = \left\{\cosh\left(\sqrt{2}\zeta\right) \pm \cos\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) \pm \mathrm{i}\cos\left(\sqrt{2}\zeta\right)\right\}
\end{aligned}
$$

$$= \cosh^2\left(\sqrt{2}\zeta\right) + \mathbb{i}\cos^2\left(\sqrt{2}\zeta\right) \pm \sqrt{2}\omega\cosh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right)$$
$$= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right)+1\right\} + \frac{\mathbb{i}}{2}\left\{\cos\left(2\sqrt{2}\zeta\right)+1\right\} \pm \sqrt{2}\omega\cdot\frac{1}{2}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right)$$
$$= \frac{1}{2}\,\mathrm{ci}_+(2\zeta) + \frac{\omega}{\sqrt{2}} \pm \frac{\omega}{\sqrt{2}}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right),$$
$$\mathrm{c}_\pm(\zeta)\,\mathrm{ci}_\mp(\zeta) = \left\{\cosh\left(\sqrt{2}\zeta\right) \pm \cos\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) \mp \mathbb{i}\cos\left(\sqrt{2}\zeta\right)\right\}$$
$$= \cosh^2\left(\sqrt{2}\zeta\right) - \mathbb{i}\cos^2\left(\sqrt{2}\zeta\right) \pm \sqrt{2}\overline{\omega}\cosh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right)$$
$$= \frac{1}{2}\left\{\cosh\left(2\sqrt{2}\zeta\right)+1\right\} - \frac{\mathbb{i}}{2}\left\{\cos\left(2\sqrt{2}\zeta\right)+1\right\} \pm \sqrt{2}\overline{\omega}\cdot\frac{1}{2}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right)$$
$$= \frac{1}{2}\,\mathrm{ci}_-(2\zeta) + \frac{\overline{\omega}}{\sqrt{2}} \pm \frac{\overline{\omega}}{\sqrt{2}}\,\mathrm{c}_+\left(\sqrt{2}\omega\zeta\right),$$
$$\mathrm{s}_\pm(\zeta)\,\mathrm{ci}_\pm(\zeta) = \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) \pm \mathbb{i}\cos\left(\sqrt{2}\zeta\right)\right\}$$
$$= \sinh\left(\sqrt{2}\zeta\right)\cosh\left(\sqrt{2}\zeta\right) + \mathbb{i}\sin\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \mathbb{i}\sinh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \cosh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)$$
$$= \frac{1}{2}\sinh\left(2\sqrt{2}\zeta\right) + \frac{\mathbb{i}}{2}\sin\left(2\sqrt{2}\zeta\right) \pm \mathbb{i}\cdot\frac{1}{2}\,\mathrm{si}_-\left(\sqrt{2}\omega\zeta\right) \pm \frac{-\mathbb{i}}{2}\,\mathrm{si}_+\left(\sqrt{2}\omega\zeta\right) = \frac{1}{2}\,\mathrm{si}_+(2\zeta) \pm \sin(2\omega\zeta),$$
$$\mathrm{s}_\pm(\zeta)\,\mathrm{ci}_\mp(\zeta) = \left\{\sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)\right\}\left\{\cosh\left(\sqrt{2}\zeta\right) \mp \mathbb{i}\cos\left(\sqrt{2}\zeta\right)\right\}$$
$$= \sinh\left(\sqrt{2}\zeta\right)\cosh\left(\sqrt{2}\zeta\right) - \mathbb{i}\sin\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \mp \mathbb{i}\sinh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \cosh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)$$
$$= \frac{1}{2}\sinh\left(2\sqrt{2}\zeta\right) - \frac{\mathbb{i}}{2}\sin\left(2\sqrt{2}\zeta\right) \mp \mathbb{i}\cdot\frac{1}{2}\,\mathrm{si}_-\left(\sqrt{2}\omega\zeta\right) \pm \frac{-\mathbb{i}}{2}\,\mathrm{si}_+\left(\sqrt{2}\omega\zeta\right) = \frac{1}{2}\,\mathrm{si}_-(2\zeta) \mp \mathbb{i}\sinh(2\omega\zeta),$$
$$\mathrm{c}_\pm(\zeta)\,\mathrm{si}_\pm(\zeta) = \left\{\cosh\left(\sqrt{2}\zeta\right) \pm \cos\left(\sqrt{2}\zeta\right)\right\}\left\{\sinh\left(\sqrt{2}\zeta\right) \pm \mathbb{i}\sin\left(\sqrt{2}\zeta\right)\right\}$$
$$= \sinh\left(\sqrt{2}\zeta\right)\cosh\left(\sqrt{2}\zeta\right) + \mathbb{i}\sin\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \sinh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \mathbb{i}\cosh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)$$
$$= \frac{1}{2}\sinh\left(2\sqrt{2}\zeta\right) + \frac{\mathbb{i}}{2}\sin\left(2\sqrt{2}\zeta\right) \pm \frac{1}{2}\,\mathrm{si}_-\left(\sqrt{2}\omega\zeta\right) \pm \mathbb{i}\cdot\frac{-\mathbb{i}}{2}\,\mathrm{si}_+\left(\sqrt{2}\omega\zeta\right) = \frac{1}{2}\,\mathrm{si}_+(2\zeta) \pm \sinh(2\omega\zeta),$$
$$\mathrm{c}_\pm(\zeta)\,\mathrm{si}_\mp(\zeta) = \left\{\cosh\left(\sqrt{2}\zeta\right) \pm \cos\left(\sqrt{2}\zeta\right)\right\}\left\{\sinh\left(\sqrt{2}\zeta\right) \mp \mathbb{i}\sin\left(\sqrt{2}\zeta\right)\right\}$$
$$= \sinh\left(\sqrt{2}\zeta\right)\cosh\left(\sqrt{2}\zeta\right) - \mathbb{i}\sin\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \pm \sinh\left(\sqrt{2}\zeta\right)\cos\left(\sqrt{2}\zeta\right) \mp \mathbb{i}\cosh\left(\sqrt{2}\zeta\right)\sin\left(\sqrt{2}\zeta\right)$$
$$= \frac{1}{2}\sinh\left(2\sqrt{2}\zeta\right) - \frac{\mathbb{i}}{2}\sin\left(2\sqrt{2}\zeta\right) \pm \frac{1}{2}\,\mathrm{si}_-\left(\sqrt{2}\omega\zeta\right) \mp \mathbb{i}\cdot\frac{-\mathbb{i}}{2}\,\mathrm{si}_+\left(\sqrt{2}\omega\zeta\right) = \frac{1}{2}\,\mathrm{si}_-(2\zeta) \mp \mathbb{i}\sin(2\omega\zeta). \qquad \square$$

By Lemma A.1, we have the following equalities for $\zeta \in \mathbb{C}$.

$$\mathrm{s}_+(2\zeta) = \mathrm{s}_+(\zeta)\,\mathrm{c}_+(\zeta) + \mathrm{s}_-(\zeta)\,\mathrm{c}_-(\zeta) = \mathrm{si}_+(\zeta)\,\mathrm{ci}_-(\zeta) + \mathrm{si}_-(\zeta)\,\mathrm{ci}_+(\zeta), \tag{A.2}$$
$$\mathrm{s}_-(2\zeta) = \mathrm{s}_+(\zeta)\,\mathrm{c}_-(\zeta) + \mathrm{s}_-(\zeta)\,\mathrm{c}_+(\zeta) = \mathrm{si}_+(\zeta)\,\mathrm{ci}_+(\zeta) + \mathrm{si}_-(\zeta)\,\mathrm{ci}_-(\zeta), \tag{A.3}$$
$$\mathrm{c}_+(2\zeta) = \mathrm{c}_+^2(\zeta) + \mathrm{c}_-^2(\zeta) - 2 = 2\,\mathrm{s}_+(\zeta)\,\mathrm{s}_-(\zeta) + 2 = \mathrm{si}_+^2(\zeta) + \mathrm{si}_-^2(\zeta) + 2 = 2\,\mathrm{ci}_+(\zeta)\,\mathrm{ci}_-(\zeta) - 2, \tag{A.4}$$
$$\mathrm{c}_-(2\zeta) = \mathrm{s}_+^2(\zeta) + \mathrm{s}_-^2(\zeta) = 2\,\mathrm{c}_+(\zeta)\,\mathrm{c}_-(\zeta) = 2\,\mathrm{si}_+(\zeta)\,\mathrm{si}_-(\zeta) = \mathrm{ci}_+^2(\zeta) + \mathrm{ci}_-^2(\zeta), \tag{A.5}$$
$$\mathrm{si}_+(2\zeta) = \mathrm{s}_+(\zeta)\,\mathrm{ci}_+(\zeta) + \mathrm{s}_-(\zeta)\,\mathrm{ci}_-(\zeta) = \mathrm{c}_+(\zeta)\,\mathrm{si}_+(\zeta) + \mathrm{c}_-(\zeta)\,\mathrm{si}_-(\zeta), \tag{A.6}$$
$$\mathrm{si}_-(2\zeta) = \mathrm{s}_+(\zeta)\,\mathrm{ci}_-(\zeta) + \mathrm{s}_-(\zeta)\,\mathrm{ci}_+(\zeta) = \mathrm{c}_+(\zeta)\,\mathrm{si}_-(\zeta) + \mathrm{c}_-(\zeta)\,\mathrm{si}_+(\zeta). \tag{A.7}$$

B. The matrices $\mathbf{A}(z)$, $\mathbf{B}(z)$, $\mathbf{C}(z)$, $\mathbf{D}(z)$

By Definition 2.1 and (2.1),

$$\begin{aligned} \sinh(\omega z)\cosh(\overline{\omega}z) &= \frac{1}{2}\left\{\sinh\left(\sqrt{2}z\right) + \sinh\left(\mathbb{i}\sqrt{2}z\right)\right\} = \frac{1}{2}\,\mathrm{si}_+(z), \\ \cosh(\omega z)\sinh(\overline{\omega}z) &= \frac{1}{2}\left\{\sinh\left(\sqrt{2}z\right) - \sinh\left(\mathbb{i}\sqrt{2}z\right)\right\} = \frac{1}{2}\,\mathrm{si}_-(z), \\ \cosh(\omega z)\cosh(\overline{\omega}z) &= \frac{1}{2}\left\{\cosh\left(\sqrt{2}z\right) + \cosh\left(\mathbb{i}\sqrt{2}z\right)\right\} = \frac{1}{2}\,\mathrm{c}_+(z), \\ \sinh(\omega z)\sinh(\overline{\omega}z) &= \frac{1}{2}\left\{\cosh\left(\sqrt{2}z\right) - \cosh\left(\mathbb{i}\sqrt{2}z\right)\right\} = \frac{1}{2}\,\mathrm{c}_-(z), \end{aligned} \tag{B.1}$$

hence,

$$\begin{aligned} &\sinh(\omega z)\cosh(\overline{\omega}z) \pm \mathbb{i}\cosh(\omega z)\sinh(\overline{\omega}z) \\ &= \frac{1}{2}\,\mathrm{si}_+(z) \pm \frac{\mathbb{i}}{2}\,\mathrm{si}_-(z) = \frac{\omega^{\pm 1}}{\sqrt{2}}\sinh\left(\sqrt{2}z\right) \pm \frac{\omega^{\pm 1}}{\sqrt{2}}\sin\left(\sqrt{2}z\right) = \frac{\omega^{\pm 1}}{\sqrt{2}}\,\mathrm{s}_\pm(z), \end{aligned} \tag{B.2}$$

$$\begin{aligned} &\cosh(\omega z)\cosh(\overline{\omega}z) \pm \mathbb{i}\sinh(\omega z)\sinh(\overline{\omega}z) = \frac{1}{2}\,\mathrm{c}_+(z) \pm \frac{\mathbb{i}}{2}\,\mathrm{c}_-(z) \\ &= \frac{\omega^{\pm 1}}{\sqrt{2}}\cosh\left(\sqrt{2}z\right) + \frac{\omega^{\mp 1}}{\sqrt{2}}\cos\left(\sqrt{2}z\right) = \frac{\omega^{\pm 1}}{\sqrt{2}}\left\{\cosh\left(\sqrt{2}z\right) \mp \mathbb{i}\cos\left(\sqrt{2}z\right)\right\} = \frac{\omega^{\pm 1}}{\sqrt{2}}\,\mathrm{ci}_\mp(z). \end{aligned} \tag{B.3}$$

Lemma B.1. *For $z \in \mathbb{C}$, $\det\mathbf{A}(z) = -\frac{\mathbb{i}}{\sqrt{2}}\,\mathrm{s}_+(z)$, $\det\mathbf{B}(z) = \frac{\mathbb{i}}{\sqrt{2}}\,\mathrm{s}_-(z)$, $\det\mathbf{C}(z) = -\mathbb{i}\,\mathrm{c}_+(z)$, $\det\mathbf{D}(z) = \mathbb{i}\,\mathrm{c}_-(z)$. The following hold for $z \in \mathbb{C}$ such that the respective right hand sides are defined.*

$$\mathbf{A}(z)^{-1} = \frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix} \omega\sinh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z) \\ \overline{\omega}\sinh(\omega z) & \mathbb{i}\cosh(\omega z) \end{pmatrix}, \qquad \mathbf{B}(z)^{-1} = \frac{\sqrt{2}}{\mathrm{s}_-(z)}\begin{pmatrix} \omega\cosh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z) \\ -\overline{\omega}\cosh(\omega z) & -\mathbb{i}\sinh(\omega z) \end{pmatrix},$$

$$\mathbf{C}(z)^{-1} = \frac{1}{\mathrm{c}_+(z)}\begin{pmatrix} \cosh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z) \\ \cosh(\omega z) & \mathbb{i}\cosh(\omega z) \end{pmatrix}, \qquad \mathbf{D}(z)^{-1} = \frac{1}{\mathrm{c}_-(z)}\begin{pmatrix} \sinh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z) \\ -\sinh(\omega z) & -\mathbb{i}\sinh(\omega z) \end{pmatrix}.$$

Proof. By Definition 5.2, and (2.1), (B.1), (B.2),

$$\begin{aligned} \det\mathbf{A}(z) &= \overline{\omega}\cosh(\omega z)\sinh(\overline{\omega}z) - \omega\sinh(\omega z)\cosh(\overline{\omega}z) \\ &= -\omega\{\sinh(\omega z)\cosh(\overline{\omega}z) + \mathbb{i}\cosh(\omega z)\sinh(\overline{\omega}z)\} = -\omega\cdot\frac{\omega}{\sqrt{2}}\,\mathrm{s}_+(z) = -\frac{\mathbb{i}}{\sqrt{2}}\,\mathrm{s}_+(z), \\ \det\mathbf{B}(z) &= -\overline{\omega}\sinh(\omega z)\cosh(\overline{\omega}z) + \omega\cosh(\omega z)\sinh(\overline{\omega}z) \\ &= -\overline{\omega}\{\sinh(\omega z)\cosh(\overline{\omega}z) - \mathbb{i}\cosh(\omega z)\sinh(\overline{\omega}z)\} = -\overline{\omega}\cdot\frac{\omega^{-1}}{\sqrt{2}}\,\mathrm{s}_-(z) = \frac{\mathbb{i}}{\sqrt{2}}\,\mathrm{s}_-(z), \\ \det\mathbf{C}(z) &= -2\mathbb{i}\cosh(\omega z)\cosh(\overline{\omega}z) = -2\mathbb{i}\cdot\frac{1}{2}\,\mathrm{c}_+(z) = -\mathbb{i}\,\mathrm{c}_+(z), \\ \det\mathbf{D}(z) &= 2\mathbb{i}\sinh(\omega z)\sinh(\overline{\omega}z) = 2\mathbb{i}\cdot\frac{1}{2}\,\mathrm{c}_-(z) = \mathbb{i}\,\mathrm{c}_-(z), \end{aligned}$$

hence

$$\mathbf{A}(z)^{-1} = \frac{\mathrm{adj}\,\mathbf{A}(z)}{\det\mathbf{A}(z)} = \frac{\mathbb{i}\sqrt{2}}{\mathrm{s}_+(z)}\cdot\begin{pmatrix} \overline{\omega}\sinh(\overline{\omega}z) & -\cosh(\overline{\omega}z) \\ -\omega\sinh(\omega z) & \cosh(\omega z) \end{pmatrix} = \frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix} \omega\sinh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z) \\ \overline{\omega}\sinh(\omega z) & \mathbb{i}\cosh(\omega z) \end{pmatrix},$$

$$\mathbf{B}(z)^{-1} = \frac{\operatorname{adj}\mathbf{B}(z)}{\det\mathbf{B}(z)} = -\frac{\mathbb{i}\sqrt{2}}{\mathrm{s}_-(z)}\cdot\begin{pmatrix}-\overline{\omega}\cosh(\overline{\omega}z) & \sinh(\overline{\omega}z)\\ -\omega\cosh(\omega z) & \sinh(\omega z)\end{pmatrix} = \frac{\sqrt{2}}{\mathrm{s}_-(z)}\begin{pmatrix}\omega\cosh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z)\\ -\overline{\omega}\cosh(\omega z) & -\mathbb{i}\sinh(\omega z)\end{pmatrix},$$

$$\mathbf{C}(z)^{-1} = \frac{\operatorname{adj}\mathbf{C}(z)}{\det\mathbf{C}(z)} = \frac{\mathbb{i}}{\mathrm{c}_+(z)}\cdot\begin{pmatrix}-\mathbb{i}\cosh(\overline{\omega}z) & -\cosh(\overline{\omega}z)\\ -\mathbb{i}\cosh(\omega z) & \cosh(\omega z)\end{pmatrix} = \frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\cosh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z)\\ \cosh(\omega z) & \mathbb{i}\cosh(\omega z)\end{pmatrix},$$

$$\mathbf{D}(z)^{-1} = \frac{\operatorname{adj}\mathbf{D}(z)}{\det\mathbf{D}(z)} = -\frac{\mathbb{i}}{\mathrm{c}_-(z)}\cdot\begin{pmatrix}\mathbb{i}\sinh(\overline{\omega}z) & \sinh(\overline{\omega}z)\\ -\mathbb{i}\sinh(\omega z) & \sinh(\omega z)\end{pmatrix} = \frac{1}{\mathrm{c}_-(z)}\begin{pmatrix}\sinh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z)\\ -\sinh(\omega z) & -\mathbb{i}\sinh(\omega z)\end{pmatrix}. \quad \square$$

Lemma B.2. *The following hold for $z \in \mathbb{C}$ such that the respective right hand sides are defined.*

(a) $\mathbf{A}(z)^{-1}\mathbf{B}(z) = \frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix}$.

(b) $\mathbf{B}(z)^{-1}\mathbf{A}(z) = \frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{ci}_+(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_-(z)\end{pmatrix}$.

(c) $\mathbf{C}(z)^{-1}\mathbf{D}(z) = \frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathrm{si}_-(z)\end{pmatrix}$.

(d) $\mathbf{D}(z)^{-1}\mathbf{C}(z) = \frac{1}{\mathrm{c}_-(z)}\begin{pmatrix}\mathrm{si}_-(z) & 0\\ 0 & -\mathrm{si}_+(z)\end{pmatrix}$.

(e) $\mathbf{C}(z)\mathbf{A}(z)^{-1} = \frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{s}_+(z) & 0\\ -\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z)\end{pmatrix}$.

(f) $\mathbf{A}(z)\mathbf{C}(z)^{-1} = \frac{1}{\sqrt{2}\,\mathrm{c}_+(z)}\begin{pmatrix}\sqrt{2}\,\mathrm{c}_+(z) & 0\\ \mathrm{s}_-(z) & \mathrm{s}_+(z)\end{pmatrix}$.

(g) $\mathbf{D}(z)\mathbf{B}(z)^{-1} = \frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{s}_-(z) & 0\\ -\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z)\end{pmatrix}$.

(h) $\mathbf{B}(z)\mathbf{D}(z)^{-1} = \frac{1}{\sqrt{2}\,\mathrm{c}_-(z)}\begin{pmatrix}\sqrt{2}\,\mathrm{c}_-(z) & 0\\ \mathrm{s}_+(z) & \mathrm{s}_-(z)\end{pmatrix}$.

Proof. By Definition 5.2, Lemma B.1, and (2.1), (B.1), (B.2), (B.3),

$$\begin{aligned}
\mathbf{A}(z)^{-1}\mathbf{B}(z) &= \frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix}\omega\sinh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z)\\ \overline{\omega}\sinh(\omega z) & \mathbb{i}\cosh(\omega z)\end{pmatrix}\cdot\begin{pmatrix}\sinh(\omega z) & -\sinh(\overline{\omega}z)\\ \omega\cosh(\omega z) & -\overline{\omega}\cosh(\overline{\omega}z)\end{pmatrix}\\
&= \frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix}\overline{\omega}\cdot\frac{\omega}{\sqrt{2}}\mathrm{ci}_-(z) & \omega\\ -\overline{\omega} & -\omega\cdot\frac{\overline{\omega}}{\sqrt{2}}\mathrm{ci}_+(z)\end{pmatrix} = \frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{ci}_-(z) & \sqrt{2}\omega\\ -\sqrt{2}\overline{\omega} & -\mathrm{ci}_+(z)\end{pmatrix},\\
\mathbf{C}(z)^{-1}\mathbf{D}(z) &= \frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\cosh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z)\\ \cosh(\omega z) & \mathbb{i}\cosh(\omega z)\end{pmatrix}\cdot\begin{pmatrix}\sinh(\omega z) & -\sinh(\overline{\omega}z)\\ \mathbb{i}\sinh(\omega z) & \mathbb{i}\sinh(\overline{\omega}z)\end{pmatrix} = \frac{1}{\mathrm{c}_+(z)}\begin{pmatrix}\mathrm{si}_+(z) & 0\\ 0 & -\mathrm{si}_-(z)\end{pmatrix},\\
\mathbf{C}(z)\mathbf{A}(z)^{-1} &= \begin{pmatrix}\cosh(\omega z) & \cosh(\overline{\omega}z)\\ \mathbb{i}\cosh(\omega z) & -\mathbb{i}\cosh(\overline{\omega}z)\end{pmatrix}\cdot\frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix}\omega\sinh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z)\\ \overline{\omega}\sinh(\omega z) & \mathbb{i}\cosh(\omega z)\end{pmatrix}\\
&= \frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix}\overline{\omega}\cdot\frac{\omega}{\sqrt{2}}\mathrm{s}_+(z) & 0\\ -\omega\cdot\frac{\overline{\omega}}{\sqrt{2}}\mathrm{s}_-(z) & \mathrm{c}_+(z)\end{pmatrix} = \frac{1}{\mathrm{s}_+(z)}\begin{pmatrix}\mathrm{s}_+(z) & 0\\ -\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z)\end{pmatrix},\\
\mathbf{D}(z)\mathbf{B}(z)^{-1} &= \begin{pmatrix}\sinh(\omega z) & -\sinh(\overline{\omega}z)\\ \mathbb{i}\sinh(\omega z) & \mathbb{i}\sinh(\overline{\omega}z)\end{pmatrix}\cdot\frac{\sqrt{2}}{\mathrm{s}_-(z)}\begin{pmatrix}\omega\cosh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z)\\ -\overline{\omega}\cosh(\omega z) & -\mathbb{i}\sinh(\omega z)\end{pmatrix}\\
&= \frac{\sqrt{2}}{\mathrm{s}_-(z)}\begin{pmatrix}\omega\cdot\frac{\overline{\omega}}{\sqrt{2}}\mathrm{s}_-(z) & 0\\ -\overline{\omega}\cdot\frac{\omega}{\sqrt{2}}\mathrm{s}_+(z) & \mathrm{c}_-(z)\end{pmatrix} = \frac{1}{\mathrm{s}_-(z)}\begin{pmatrix}\mathrm{s}_-(z) & 0\\ -\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z)\end{pmatrix},
\end{aligned}$$

which shows (a), (c), (e), (g).

(b) follows from (a) and (A.4), since $\mathbf{B}(z)^{-1}\mathbf{A}(z) = \left\{\mathbf{A}(z)^{-1}\mathbf{B}(z)\right\}^{-1}$. (d) follows from (c) and (A.5), since $\mathbf{D}(z)^{-1}\mathbf{C}(z) = \left\{\mathbf{C}(z)^{-1}\mathbf{D}(z)\right\}^{-1}$. Since $\mathbf{A}(z)\mathbf{C}(z)^{-1} = \left\{\mathbf{C}(z)\mathbf{A}(z)^{-1}\right\}^{-1}$ and $\mathbf{B}(z)\mathbf{D}(z)^{-1} = \left\{\mathbf{D}(z)\mathbf{B}(z)^{-1}\right\}^{-1}$, (f) and (h) follow respectively from (e) and (g), and the proof is complete. $\square$

Lemma B.3. *The following hold for $z \in \mathbb{C}$ such that the respective right hand sides are defined.*

(a) $\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1} = \frac{\mathbb{i}}{\mathrm{s}_+(z)}\begin{pmatrix} \mathrm{s}_-(z) & -\sqrt{2}\,\mathrm{c}_+(z) \\ \sqrt{2}\,\mathrm{c}_-(z) & -\mathrm{s}_-(z)\end{pmatrix}$.
(b) $\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1} = \frac{\mathbb{i}}{\mathrm{s}_-(z)}\begin{pmatrix} \mathrm{s}_+(z) & -\sqrt{2}\,\mathrm{c}_-(z) \\ \sqrt{2}\,\mathrm{c}_+(z) & -\mathrm{s}_+(z)\end{pmatrix}$.

(c) $\mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1} = \frac{\overline{\omega}}{\mathrm{s}_-(z)}\begin{pmatrix} 0 & \mathrm{s}_-(z) \\ -\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z)\end{pmatrix}$.
(d) $\mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1} = \frac{\overline{\omega}}{\mathrm{s}_+(z)}\begin{pmatrix} 0 & \mathrm{s}_+(z) \\ -\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z)\end{pmatrix}$.

(e) $\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} = \frac{\omega}{\mathrm{s}_-(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_+(z) & -\mathrm{s}_+(z) \\ \mathrm{s}_-(z) & 0\end{pmatrix}$.
(f) $\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1} = \frac{\omega}{\mathrm{s}_+(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_-(z) & -\mathrm{s}_-(z) \\ \mathrm{s}_+(z) & 0\end{pmatrix}$.

(g) $\mathbf{C}(z)\mathbf{K}^{\pm1}\mathbf{D}(z)^{-1} = \frac{\omega^{\mp1}}{\sqrt{2}\,\mathrm{c}_-(z)}\begin{pmatrix} \mathrm{s}_\pm(z) & \pm\,\mathrm{s}_\mp(z) \\ \mp\,\mathrm{s}_\mp(z) & \mathrm{s}_\pm(z)\end{pmatrix}$.
(h) $\mathbf{D}(z)\mathbf{K}^{\pm1}\mathbf{C}(z)^{-1} = \frac{\omega^{\mp1}}{\sqrt{2}\,\mathrm{c}_+(z)}\begin{pmatrix} \mathrm{s}_\mp(z) & \pm\,\mathrm{s}_\pm(z) \\ \mp\,\mathrm{s}_\pm(z) & \mathrm{s}_\mp(z)\end{pmatrix}$.

(i) $\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} = \frac{\overline{\omega}}{\mathrm{s}_+(z)}\begin{pmatrix} 0 & \mathrm{s}_+(z) \\ -\sqrt{2}\,\mathrm{c}_-(z) & \mathrm{s}_-(z)\end{pmatrix}$.
(j) $\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} = \frac{\omega}{\sqrt{2}\,\mathrm{c}_-(z)}\begin{pmatrix} \mathrm{s}_-(z) & -\mathrm{s}_+(z) \\ \sqrt{2}\,\mathrm{c}_-(z) & 0\end{pmatrix}$.

(k) $\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1} = \frac{\overline{\omega}}{\mathrm{s}_-(z)}\begin{pmatrix} 0 & \mathrm{s}_-(z) \\ -\sqrt{2}\,\mathrm{c}_+(z) & \mathrm{s}_+(z)\end{pmatrix}$.
(l) $\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1} = \frac{\omega}{\sqrt{2}\,\mathrm{c}_+(z)}\begin{pmatrix} \mathrm{s}_+(z) & -\mathrm{s}_-(z) \\ \sqrt{2}\,\mathrm{c}_+(z) & 0\end{pmatrix}$.

Proof. By Definitions 2.3, 5.2, Lemma B.1, and (2.1), (B.1), (B.2),

$$\begin{aligned}
\mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1} &= \begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega}z) \\ \omega\sinh(\omega z) & \overline{\omega}\sinh(\overline{\omega}z)\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathbb{i}\end{pmatrix}\cdot\frac{\sqrt{2}}{\mathrm{s}_-(z)}\begin{pmatrix} \omega\cosh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z) \\ -\overline{\omega}\cosh(\omega z) & -\mathbb{i}\sinh(\omega z)\end{pmatrix} \\
&= \frac{\sqrt{2}}{\mathrm{s}_-(z)}\begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega}z) \\ \omega\sinh(\omega z) & \overline{\omega}\sinh(\overline{\omega}z)\end{pmatrix}\begin{pmatrix} \omega\cosh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z) \\ -\omega\cosh(\omega z) & \sinh(\omega z)\end{pmatrix} \\
&= \frac{\sqrt{2}}{\mathrm{s}_-(z)}\begin{pmatrix} 0 & \frac{\overline{\omega}}{\sqrt{2}}\mathrm{s}_-(z) \\ \mathbb{i}\cdot\frac{\omega}{\sqrt{2}}\mathrm{s}_+(z) & \overline{\omega}\cdot\mathrm{c}_-(z)\end{pmatrix} = \frac{\overline{\omega}}{\mathrm{s}_-(z)}\begin{pmatrix} 0 & \mathrm{s}_-(z) \\ -\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z)\end{pmatrix}, \\
\mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1} &= \begin{pmatrix} \sinh(\omega z) & -\sinh(\overline{\omega}z) \\ \omega\cosh(\omega z) & -\overline{\omega}\cosh(\overline{\omega}z)\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathbb{i}\end{pmatrix}\cdot\frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix} \omega\sinh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z) \\ \overline{\omega}\sinh(\omega z) & \mathbb{i}\cosh(\omega z)\end{pmatrix} \\
&= \frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix} \sinh(\omega z) & -\sinh(\overline{\omega}z) \\ \omega\cosh(\omega z) & -\overline{\omega}\cosh(\overline{\omega}z)\end{pmatrix}\begin{pmatrix} \omega\sinh(\overline{\omega}z) & -\mathbb{i}\cosh(\overline{\omega}z) \\ \omega\sinh(\omega z) & -\cosh(\omega z)\end{pmatrix} \\
&= \frac{\sqrt{2}}{\mathrm{s}_+(z)}\begin{pmatrix} 0 & -\mathbb{i}\cdot\frac{\omega}{\sqrt{2}}\mathrm{s}_+(z) \\ -\frac{\overline{\omega}}{\sqrt{2}}\mathrm{s}_-(z) & \overline{\omega}\cdot\mathrm{c}_+(z)\end{pmatrix} = \frac{\overline{\omega}}{\mathrm{s}_+(z)}\begin{pmatrix} 0 & \mathrm{s}_+(z) \\ -\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z)\end{pmatrix}, \\
\mathbf{C}(z)\mathbf{K}^{\pm1}\mathbf{D}(z)^{-1} &= \begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega}z) \\ \mathbb{i}\cosh(\omega z) & -\mathbb{i}\cosh(\overline{\omega}z)\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \pm\mathbb{i}\end{pmatrix}\cdot\frac{1}{\mathrm{c}_-(z)}\begin{pmatrix} \sinh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z) \\ -\sinh(\omega z) & -\mathbb{i}\sinh(\omega z)\end{pmatrix} \\
&= \frac{1}{\mathrm{c}_-(z)}\begin{pmatrix} \cosh(\omega z) & \cosh(\overline{\omega}z) \\ \mathbb{i}\cosh(\omega z) & -\mathbb{i}\cosh(\overline{\omega}z)\end{pmatrix}\begin{pmatrix} \sinh(\overline{\omega}z) & -\mathbb{i}\sinh(\overline{\omega}z) \\ \mp\mathbb{i}\sinh(\omega z) & \pm\sinh(\omega z)\end{pmatrix} \\
&= \frac{1}{\mathrm{c}_-(z)}\begin{pmatrix} \mp\mathbb{i}\cdot\frac{\omega^{\pm1}}{\sqrt{2}}\mathrm{s}_\pm(z) & \pm\frac{\omega^{\mp1}}{\sqrt{2}}\mathrm{s}_\mp(z) \\ \mp\frac{\omega^{\mp1}}{\sqrt{2}}\mathrm{s}_\mp(z) & \mp\mathbb{i}\cdot\frac{\omega^{\pm1}}{\sqrt{2}}\mathrm{s}_\pm(z)\end{pmatrix} = \frac{\omega^{\mp1}}{\sqrt{2}\,\mathrm{c}_-(z)}\begin{pmatrix} \mathrm{s}_\pm(z) & \pm\,\mathrm{s}_\mp(z) \\ \mp\,\mathrm{s}_\mp(z) & \mathrm{s}_\pm(z)\end{pmatrix},
\end{aligned}$$

which shows (c), (d), and (g). (e), (f) follow from (d), (c) respectively, since $\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} = \left\{\mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1}\right\}^{-1}$, $\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1} = \left\{\mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1}\right\}^{-1}$. (h) follows from (g) and (A.5), since $\mathbf{D}(z)\mathbf{K}^{\pm1}\mathbf{C}(z)^{-1} = \left\{\mathbf{C}(z)\mathbf{K}^{\mp1}\mathbf{D}(z)^{-1}\right\}^{-1}$. By (h), (g), and Lemma B.2 (e), (g),

$$\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} = \mathbf{D}(z)\mathbf{K}\mathbf{C}(z)^{-1}\cdot\mathbf{C}(z)\mathbf{A}(z)^{-1} = \frac{\overline{\omega}}{\sqrt{2}\,\mathrm{c}_+(z)}\begin{pmatrix} \mathrm{s}_-(z) & \mathrm{s}_+(z) \\ -\mathrm{s}_+(z) & \mathrm{s}_-(z)\end{pmatrix}\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix} \mathrm{s}_+(z) & 0 \\ -\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z)\end{pmatrix},$$

$$\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1} = \mathbf{C}(z)\mathbf{K}\mathbf{D}(z)^{-1} \cdot \mathbf{D}(z)\mathbf{B}(z)^{-1} = \frac{\overline{\omega}}{\sqrt{2}\,\mathrm{c}_-(z)} \begin{pmatrix} \mathrm{s}_+(z) & \mathrm{s}_-(z) \\ -\,\mathrm{s}_-(z) & \mathrm{s}_+(z) \end{pmatrix} \cdot \frac{1}{\mathrm{s}_-(z)} \begin{pmatrix} \mathrm{s}_-(z) & 0 \\ -\,\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z) \end{pmatrix},$$

hence (i) and (k) follow by (A.5). (j), (l) follow from (i), (k) respectively, since $\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} = \left\{\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1}\right\}^{-1}$ and $\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1} = \left\{\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}\right\}^{-1}$. By (c), (d), and (2.2),

$$\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1} = \mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1} \cdot \mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1} = \frac{\overline{\omega}}{\mathrm{s}_-(z)} \begin{pmatrix} 0 & \mathrm{s}_-(z) \\ -\,\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z) \end{pmatrix} \cdot \frac{\overline{\omega}}{\mathrm{s}_+(z)} \begin{pmatrix} 0 & \mathrm{s}_+(z) \\ -\,\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z) \end{pmatrix},$$

$$\mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1} = \mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1} \cdot \mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1} = \frac{\overline{\omega}}{\mathrm{s}_+(z)} \begin{pmatrix} 0 & \mathrm{s}_+(z) \\ -\,\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z) \end{pmatrix} \cdot \frac{\overline{\omega}}{\mathrm{s}_-(z)} \begin{pmatrix} 0 & \mathrm{s}_-(z) \\ -\,\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z) \end{pmatrix},$$

hence (a) and (b) follow by (A.5). Thus the proof is complete. □

Lemma B.4. *The following hold for $z \in \mathbb{C}$ such that the respective right hand sides are defined.*

(a) $\left\{\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1} + \mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\right\}^{-1} = \frac{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}{4\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z)} \left\{\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1} + \mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\right\}$.

(b) $\left\{\mathbf{A}(z)\mathbf{K}^{\pm1}\mathbf{B}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{\pm1}\mathbf{A}(z)^{-1}\right\}^{-1} = \frac{1}{2\,\mathrm{c}_-(2z)} \left\{\mathrm{s}_-^2(z)\mathbf{A}(z)\mathbf{K}^{\mp1}\mathbf{B}(z)^{-1} + \mathrm{s}_+^2(z)\mathbf{B}(z)\mathbf{K}^{\mp1}\mathbf{A}(z)^{-1}\right\}$.

(c) $\left\{\mathbf{C}(z)\mathbf{K}^{\pm1}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}^{\pm1}\mathbf{C}(z)^{-1}\right\}^{-1} = \frac{1}{2\,\mathrm{c}_+(2z)} \left\{\mathrm{c}_-^2(z)\mathbf{C}(z)\mathbf{K}^{\mp1}\mathbf{D}(z)^{-1} + \mathrm{c}_+^2(z)\mathbf{D}(z)\mathbf{K}^{\mp1}\mathbf{C}(z)^{-1}\right\}$.

(d) $\left\{\mathbf{C}(z)\mathbf{A}(z)^{-1} + \mathbf{D}(z)\mathbf{B}(z)^{-1}\right\}^{-1} = \frac{1}{2\,\mathrm{s}_-(2z)} \left\{\mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{A}(z)\mathbf{C}(z)^{-1} + \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{B}(z)\mathbf{D}(z)^{-1}\right\}$.

(e) $\left\{\mathbf{A}(z)\mathbf{C}(z)^{-1} + \mathbf{B}(z)\mathbf{D}(z)^{-1}\right\}^{-1} = \frac{1}{2\,\mathrm{s}_-(2z)} \left\{\mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{A}(z)^{-1} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{B}(z)^{-1}\right\}$.

(f) $\left\{\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} + \mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}\right\}^{-1} = \frac{1}{2\,\mathrm{s}_+(2z)} \left\{\mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\}$.

(g) $\left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\}^{-1} = \frac{1}{2\,\mathrm{s}_+(2z)} \left\{\mathrm{s}_+(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}\right\}$.

Proof. By Lemma B.3 (a), (b), and (A.2), (A.3), (A.5),

$$\begin{aligned}
\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1} + \mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1} &= \frac{\mathbb{i}}{\mathrm{s}_+(z)} \begin{pmatrix} \mathrm{s}_-(z) & -\sqrt{2}\,\mathrm{c}_+(z) \\ \sqrt{2}\,\mathrm{c}_-(z) & -\,\mathrm{s}_-(z) \end{pmatrix} + \frac{\mathbb{i}}{\mathrm{s}_-(z)} \begin{pmatrix} \mathrm{s}_+(z) & -\sqrt{2}\,\mathrm{c}_-(z) \\ \sqrt{2}\,\mathrm{c}_+(z) & -\,\mathrm{s}_+(z) \end{pmatrix} \\
&= \frac{\mathbb{i}}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)} \begin{pmatrix} \mathrm{s}_+^2(z) + \mathrm{s}_-^2(z) & -\sqrt{2}\,\{\mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\} \\ \sqrt{2}\,\{\mathrm{s}_+(z)\,\mathrm{c}_+(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\} & -\left\{\mathrm{s}_+^2(z) + \mathrm{s}_-^2(z)\right\} \end{pmatrix} \\
&= \frac{\mathbb{i}}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)} \begin{pmatrix} \mathrm{c}_-(2z) & -\sqrt{2}\,\mathrm{s}_-(2z) \\ \sqrt{2}\,\mathrm{s}_+(2z) & -\,\mathrm{c}_-(2z) \end{pmatrix},
\end{aligned}$$

hence

$$\begin{aligned}
&\left\{\mathbf{A}(z)\mathbf{J}\mathbf{A}(z)^{-1} + \mathbf{B}(z)\mathbf{J}\mathbf{B}(z)^{-1}\right\}^2 \\
&= -\frac{1}{\mathrm{s}_+^2(z)\,\mathrm{s}_-^2(z)} \begin{pmatrix} \mathrm{c}_-(2z) & -\sqrt{2}\,\mathrm{s}_-(2z) \\ \sqrt{2}\,\mathrm{s}_+(2z) & -\,\mathrm{c}_-(2z) \end{pmatrix} \begin{pmatrix} \mathrm{c}_-(2z) & -\sqrt{2}\,\mathrm{s}_-(2z) \\ \sqrt{2}\,\mathrm{s}_+(2z) & -\,\mathrm{c}_-(2z) \end{pmatrix} = -\frac{\mathrm{c}_-^2(2z) - 2\,\mathrm{s}_+(2z)\,\mathrm{s}_-(2z)}{\mathrm{s}_+^2(z)\,\mathrm{s}_-^2(z)} \cdot \mathbf{I}.
\end{aligned}$$

Thus follows (a), since $-\,\mathrm{c}_-^2(2z) + 2\,\mathrm{s}_+(2z)\,\mathrm{s}_-(2z) = \mathrm{c}_+^2(2z) - 4 = \{\mathrm{c}_+(2z) + 2\}\{\mathrm{c}_+(2z) - 2\} = 2\,\mathrm{ci}_+(z)\,\mathrm{ci}_-(z) \cdot 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z)$ by (A.4). By Lemma B.3 (c), (d), (e), (f), and (A.2), (A.3), (A.5),

$$\mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1} + \mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1} = \frac{\overline{\omega}}{\mathrm{s}_-(z)} \begin{pmatrix} 0 & \mathrm{s}_-(z) \\ -\,\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z) \end{pmatrix} + \frac{\overline{\omega}}{\mathrm{s}_+(z)} \begin{pmatrix} 0 & \mathrm{s}_+(z) \\ -\,\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z) \end{pmatrix}$$

$$= \frac{\overline{\omega}}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)} \begin{pmatrix} 0 & 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) \\ -\left\{\mathrm{s}_+(z)^2 + \mathrm{s}_-(z)^2\right\} & \sqrt{2}\,\{\mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\} \end{pmatrix} = \frac{\overline{\omega}}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)} \begin{pmatrix} 0 & 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) \\ -\,\mathrm{c}_-(2z) & \sqrt{2}\,\mathrm{s}_-(2z) \end{pmatrix},$$

$$\mathrm{s}_-^2(z)\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} + \mathrm{s}_+^2(z)\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}$$

$$= \mathrm{s}_-^2(z)\cdot\frac{\omega}{\mathrm{s}_-(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_+(z) & -\,\mathrm{s}_+(z) \\ \mathrm{s}_-(z) & 0 \end{pmatrix} + \mathrm{s}_+^2(z)\cdot\frac{\omega}{\mathrm{s}_+(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_-(z) & -\,\mathrm{s}_-(z) \\ \mathrm{s}_+(z) & 0 \end{pmatrix}$$

$$= \omega\begin{pmatrix} \sqrt{2}\,\{\mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\} & -2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) \\ \mathrm{s}_+(z)^2 + \mathrm{s}_-(z)^2 & 0 \end{pmatrix} = \omega\begin{pmatrix} \sqrt{2}\,\mathrm{s}_-(2z) & -2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) \\ \mathrm{c}_-(2z) & 0 \end{pmatrix},$$

$$\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1} = \frac{\omega}{\mathrm{s}_-(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_+(z) & -\,\mathrm{s}_+(z) \\ \mathrm{s}_-(z) & 0 \end{pmatrix} + \frac{\omega}{\mathrm{s}_+(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_-(z) & -\,\mathrm{s}_-(z) \\ \mathrm{s}_+(z) & 0 \end{pmatrix}$$

$$= \frac{\omega}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}\begin{pmatrix} \sqrt{2}\,\{\mathrm{s}_+(z)\,\mathrm{c}_+(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\} & -\left\{\mathrm{s}_+(z)^2 + \mathrm{s}_-(z)^2\right\} \\ 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & 0 \end{pmatrix} = \frac{\omega}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{s}_+(2z) & -\,\mathrm{c}_-(2z) \\ 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & 0 \end{pmatrix},$$

$$\mathrm{s}_-^2(z)\mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1} + \mathrm{s}_+^2(z)\mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1}$$

$$= \mathrm{s}_-^2(z)\cdot\frac{\overline{\omega}}{\mathrm{s}_-(z)}\begin{pmatrix} 0 & \mathrm{s}_-(z) \\ -\,\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z) \end{pmatrix} + \mathrm{s}_+^2(z)\cdot\frac{\overline{\omega}}{\mathrm{s}_+(z)}\begin{pmatrix} 0 & \mathrm{s}_+(z) \\ -\,\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z) \end{pmatrix}$$

$$= \overline{\omega}\begin{pmatrix} 0 & \mathrm{s}_+(z)^2 + \mathrm{s}_-(z)^2 \\ -2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & \sqrt{2}\,\{\mathrm{s}_+(z)\,\mathrm{c}_+(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\} \end{pmatrix} = \overline{\omega}\begin{pmatrix} 0 & \mathrm{c}_-(2z) \\ -2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{s}_+(2z) \end{pmatrix},$$

hence

$$\left\{\mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1} + \mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1}\right\}\left\{\mathrm{s}_-^2(z)\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} + \mathrm{s}_+^2(z)\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\right\}$$

$$= \frac{\overline{\omega}}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}\begin{pmatrix} 0 & 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) \\ -\,\mathrm{c}_-(2z) & \sqrt{2}\,\mathrm{s}_-(2z) \end{pmatrix}\cdot\omega\begin{pmatrix} \sqrt{2}\,\mathrm{s}_-(2z) & -2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) \\ \mathrm{c}_-(2z) & 0 \end{pmatrix} = 2\,\mathrm{c}_-(2z)\mathbf{I},$$

$$\left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{B}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{A}(z)^{-1}\right\}\left\{\mathrm{s}_-^2(z)\mathbf{A}(z)\mathbf{K}\mathbf{B}(z)^{-1} + \mathrm{s}_+^2(z)\mathbf{B}(z)\mathbf{K}\mathbf{A}(z)^{-1}\right\}$$

$$= \frac{\omega}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{s}_+(2z) & -\,\mathrm{c}_-(2z) \\ 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & 0 \end{pmatrix}\cdot\overline{\omega}\begin{pmatrix} 0 & \mathrm{c}_-(2z) \\ -2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{s}_+(2z) \end{pmatrix} = 2\,\mathrm{c}_-(2z)\mathbf{I},$$

from which follows (b). By Lemma B.3 (g), (h), and (A.2), (A.3), (A.5),

$$\mathbf{C}(z)\mathbf{K}^{\pm 1}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}^{\pm 1}\mathbf{C}(z)^{-1} = \frac{\omega^{\mp 1}}{\sqrt{2}\,\mathrm{c}_-(z)}\begin{pmatrix} \mathrm{s}_\pm(z) & \pm\,\mathrm{s}_\mp(z) \\ \mp\,\mathrm{s}_\mp(z) & \mathrm{s}_\pm(z) \end{pmatrix} + \frac{\omega^{\mp 1}}{\sqrt{2}\,\mathrm{c}_+(z)}\begin{pmatrix} \mathrm{s}_\mp(z) & \pm\,\mathrm{s}_\pm(z) \\ \mp\,\mathrm{s}_\pm(z) & \mathrm{s}_\mp(z) \end{pmatrix}$$

$$= \frac{\omega^{\mp 1}}{\sqrt{2}\,\mathrm{c}_+(z)\,\mathrm{c}_-(z)}\begin{pmatrix} \mathrm{s}_\pm(z)\,\mathrm{c}_+(z) + \mathrm{s}_\mp(z)\,\mathrm{c}_-(z) & \pm\,\{\mathrm{s}_\pm(z)\,\mathrm{c}_-(z) + \mathrm{s}_\mp(z)\,\mathrm{c}_+(z)\} \\ \mp\,\{\mathrm{s}_\pm(z)\,\mathrm{c}_-(z) + \mathrm{s}_\mp(z)\,\mathrm{c}_+(z)\} & \mathrm{s}_\pm(z)\,\mathrm{c}_+(z) + \mathrm{s}_\mp(z)\,\mathrm{c}_-(z) \end{pmatrix}$$

$$= \frac{\sqrt{2}\omega^{\mp 1}}{\mathrm{c}_-(2z)}\begin{pmatrix} \mathrm{s}_\pm(2z) & \pm\,\mathrm{s}_\mp(2z) \\ \mp\,\mathrm{s}_\mp(2z) & \mathrm{s}_\pm(2z) \end{pmatrix},$$

$$\mathrm{c}_-^2(z)\mathbf{C}(z)\mathbf{K}^{\mp 1}\mathbf{D}(z)^{-1} + \mathrm{c}_+^2(z)\mathbf{D}(z)\mathbf{K}^{\mp 1}\mathbf{C}(z)^{-1}$$

$$= \mathrm{c}_-^2(z)\cdot\frac{\omega^{\pm 1}}{\sqrt{2}\,\mathrm{c}_-(z)}\begin{pmatrix} \mathrm{s}_\mp(z) & \mp\,\mathrm{s}_\pm(z) \\ \pm\,\mathrm{s}_\pm(z) & \mathrm{s}_\mp(z) \end{pmatrix} + \mathrm{c}_+^2(z)\cdot\frac{\omega^{\pm 1}}{\sqrt{2}\,\mathrm{c}_+(z)}\begin{pmatrix} \mathrm{s}_\pm(z) & \mp\,\mathrm{s}_\mp(z) \\ \pm\,\mathrm{s}_\mp(z) & \mathrm{s}_\pm(z) \end{pmatrix}$$

$$= \frac{\omega^{\pm 1}}{\sqrt{2}}\begin{pmatrix} \mathrm{s}_\pm(z)\,\mathrm{c}_+(z) + \mathrm{s}_\mp(z)\,\mathrm{c}_-(z) & \mp\,\{\mathrm{s}_\pm(z)\,\mathrm{c}_-(z) + \mathrm{s}_\mp(z)\,\mathrm{c}_+(z)\} \\ \pm\,\{\mathrm{s}_\pm(z)\,\mathrm{c}_-(z) + \mathrm{s}_\mp(z)\,\mathrm{c}_+(z)\} & \mathrm{s}_\pm(z)\,\mathrm{c}_+(z) + \mathrm{s}_\mp(z)\,\mathrm{c}_-(z) \end{pmatrix} = \frac{\omega^{\pm 1}}{\sqrt{2}}\begin{pmatrix} \mathrm{s}_\pm(2z) & \mp\,\mathrm{s}_\mp(2z) \\ \pm\,\mathrm{s}_\mp(2z) & \mathrm{s}_\pm(2z) \end{pmatrix},$$

hence, by (A.5),

$$\left\{\mathbf{C}(z)\mathbf{K}^{\pm1}\mathbf{D}(z)^{-1} + \mathbf{D}(z)\mathbf{K}^{\pm1}\mathbf{C}(z)^{-1}\right\}\left\{\mathrm{c}_-^2(z)\mathbf{C}(z)\mathbf{K}^{\mp1}\mathbf{D}(z)^{-1} + \mathrm{c}_+^2(z)\mathbf{D}(z)\mathbf{K}^{\mp1}\mathbf{C}(z)^{-1}\right\}$$
$$= \frac{\sqrt{2}\omega^{\mp1}}{\mathrm{c}_-(2z)}\begin{pmatrix} \mathrm{s}_\pm(2z) & \pm\,\mathrm{s}_\mp(2z) \\ \mp\,\mathrm{s}_\mp(2z) & \mathrm{s}_\pm(2z)\end{pmatrix} \cdot \frac{\omega^{\pm1}}{\sqrt{2}}\begin{pmatrix} \mathrm{s}_\pm(2z) & \mp\,\mathrm{s}_\mp(2z) \\ \pm\,\mathrm{s}_\mp(2z) & \mathrm{s}_\pm(2z)\end{pmatrix} = \frac{\mathrm{s}_+^2(2z) + \mathrm{s}_-^2(2z)}{\mathrm{c}_-(2z)} \cdot \mathbf{I} = 2\,\mathrm{c}_+(2z)\cdot\mathbf{I},$$

from which follows (c). By Lemma B.2 (e), (f), (g), (h), and (A.2), (A.3), (A.5),

$$\mathbf{C}(z)\mathbf{A}(z)^{-1} + \mathbf{D}(z)\mathbf{B}(z)^{-1} = \frac{1}{\mathrm{s}_+(z)}\begin{pmatrix} \mathrm{s}_+(z) & 0 \\ -\,\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z)\end{pmatrix} + \frac{1}{\mathrm{s}_-(z)}\begin{pmatrix} \mathrm{s}_-(z) & 0 \\ -\,\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z)\end{pmatrix}$$
$$= \frac{1}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}\begin{pmatrix} 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & 0 \\ -\left\{\mathrm{s}_+^2(z) + \mathrm{s}_-^2(z)\right\} & \sqrt{2}\,\{\mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\}\end{pmatrix} = \frac{1}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}\begin{pmatrix} 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & 0 \\ -\,\mathrm{c}_-(2z) & \sqrt{2}\,\mathrm{s}_-(2z)\end{pmatrix},$$

$$\mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{A}(z)\mathbf{C}(z)^{-1} + \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{B}(z)\mathbf{D}(z)^{-1}$$
$$= \mathrm{s}_-(z)\,\mathrm{c}_+(z)\cdot\frac{1}{\sqrt{2}\,\mathrm{c}_+(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_+(z) & 0 \\ \mathrm{s}_-(z) & \mathrm{s}_+(z)\end{pmatrix} + \mathrm{s}_+(z)\,\mathrm{c}_-(z)\cdot\frac{1}{\sqrt{2}\,\mathrm{c}_-(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_-(z) & 0 \\ \mathrm{s}_+(z) & \mathrm{s}_-(z)\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\,\{\mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\} & 0 \\ \mathrm{s}_+^2(z) + \mathrm{s}_-^2(z) & 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\,\mathrm{s}_-(2z) & 0 \\ \mathrm{c}_-(2z) & 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z)\end{pmatrix},$$

$$\mathbf{A}(z)\mathbf{C}(z)^{-1} + \mathbf{B}(z)\mathbf{D}(z)^{-1} = \frac{1}{\sqrt{2}\,\mathrm{c}_+(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_+(z) & 0 \\ \mathrm{s}_-(z) & \mathrm{s}_+(z)\end{pmatrix} + \frac{1}{\sqrt{2}\,\mathrm{c}_-(z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_-(z) & 0 \\ \mathrm{s}_+(z) & \mathrm{s}_-(z)\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}\,\mathrm{c}_+(z)\,\mathrm{c}_-(z)}\begin{pmatrix} 2\sqrt{2}\,\mathrm{c}_+(z)\,\mathrm{c}_-(z) & 0 \\ \mathrm{s}_+(z)\,\mathrm{c}_+(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z) & \mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\end{pmatrix} = \frac{\sqrt{2}}{\mathrm{c}_-(2z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_-(2z) & 0 \\ \mathrm{s}_+(2z) & \mathrm{s}_-(2z)\end{pmatrix},$$

$$\mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{A}(z)^{-1} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{B}(z)^{-1}$$
$$= \mathrm{s}_+(z)\,\mathrm{c}_-(z)\cdot\frac{1}{\mathrm{s}_+(z)}\begin{pmatrix} \mathrm{s}_+(z) & 0 \\ -\,\mathrm{s}_-(z) & \sqrt{2}\,\mathrm{c}_+(z)\end{pmatrix} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\cdot\frac{1}{\mathrm{s}_-(z)}\begin{pmatrix} \mathrm{s}_-(z) & 0 \\ -\,\mathrm{s}_+(z) & \sqrt{2}\,\mathrm{c}_-(z)\end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{s}_+(z)\,\mathrm{c}_-(z) + \mathrm{s}_-(z)\,\mathrm{c}_+(z) & 0 \\ -\,\{\mathrm{s}_+(z)\,\mathrm{c}_+(z) + \mathrm{s}_-(z)\,\mathrm{c}_-(z)\} & 2\sqrt{2}\,\mathrm{c}_+(z)\,\mathrm{c}_-(z)\end{pmatrix} = \begin{pmatrix} \mathrm{s}_-(2z) & 0 \\ -\,\mathrm{s}_+(2z) & \sqrt{2}\,\mathrm{c}_-(2z)\end{pmatrix},$$

hence

$$\left\{\mathbf{C}(z)\mathbf{A}(z)^{-1} + \mathbf{D}(z)\mathbf{B}(z)^{-1}\right\}\left\{\mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{A}(z)\mathbf{C}(z)^{-1} + \mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{B}(z)\mathbf{D}(z)^{-1}\right\}$$
$$= \frac{1}{\mathrm{s}_+(z)\,\mathrm{s}_-(z)}\begin{pmatrix} 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z) & 0 \\ -\,\mathrm{c}_-(2z) & \sqrt{2}\,\mathrm{s}_-(2z)\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\,\mathrm{s}_-(2z) & 0 \\ \mathrm{c}_-(2z) & 2\,\mathrm{s}_+(z)\,\mathrm{s}_-(z)\end{pmatrix} = 2\,\mathrm{s}_-(2z)\cdot\mathbf{I},$$
$$\left\{\mathbf{A}(z)\mathbf{C}(z)^{-1} + \mathbf{B}(z)\mathbf{D}(z)^{-1}\right\}\left\{\mathrm{s}_+(z)\,\mathrm{c}_-(z)\mathbf{C}(z)\mathbf{A}(z)^{-1} + \mathrm{s}_-(z)\,\mathrm{c}_+(z)\mathbf{D}(z)\mathbf{B}(z)^{-1}\right\}$$
$$= \frac{\sqrt{2}}{\mathrm{c}_-(2z)}\begin{pmatrix} \sqrt{2}\,\mathrm{c}_-(2z) & 0 \\ \mathrm{s}_+(2z) & \mathrm{s}_-(2z)\end{pmatrix} \cdot \begin{pmatrix} \mathrm{s}_-(2z) & 0 \\ -\,\mathrm{s}_+(2z) & \sqrt{2}\,\mathrm{c}_-(2z)\end{pmatrix} = 2\,\mathrm{s}_-(2z)\cdot\mathbf{I},$$

from which follow (d) and (e). By Lemma B.3 (i), (j), (k), (l), and (A.2), (A.3), (A.5),

$$\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} + \mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1} = \frac{\overline{\omega}}{\mathrm{s}_+(z)}\begin{pmatrix} 0 & \mathrm{s}_+(z) \\ -\sqrt{2}\,\mathrm{c}_-(z) & \mathrm{s}_-(z)\end{pmatrix} + \frac{\overline{\omega}}{\mathrm{s}_-(z)}\begin{pmatrix} 0 & \mathrm{s}_-(z) \\ -\sqrt{2}\,\mathrm{c}_+(z) & \mathrm{s}_+(z)\end{pmatrix}$$

$$= \frac{\overline{\omega}}{s_+(z)\,s_-(z)} \begin{pmatrix} 0 & 2\,s_+(z)\,s_-(z) \\ -\sqrt{2}\,\{s_+(z)\,c_+(z) + s_-(z)\,c_-(z)\} & s_+^2(z) + s_-^2(z) \end{pmatrix}$$

$$= \frac{\overline{\omega}}{s_+(z)\,s_-(z)} \begin{pmatrix} 0 & 2\,s_+(z)\,s_-(z) \\ -\sqrt{2}\,s_+(2z) & c_-(2z) \end{pmatrix},$$

$$s_-(z)\,c_-(z)\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + s_+(z)\,c_+(z)\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}$$

$$= s_-(z)\,c_-(z) \cdot \frac{\omega}{\sqrt{2}\,c_-(z)} \begin{pmatrix} s_-(z) & -s_+(z) \\ \sqrt{2}\,c_-(z) & 0 \end{pmatrix} + s_+(z)\,c_+(z) \cdot \frac{\omega}{\sqrt{2}\,c_+(z)} \begin{pmatrix} s_+(z) & -s_-(z) \\ \sqrt{2}\,c_+(z) & 0 \end{pmatrix}$$

$$= \frac{\omega}{\sqrt{2}} \begin{pmatrix} s_+^2(z) + s_-^2(z) & -2\,s_+(z)\,s_-(z) \\ \sqrt{2}\,\{s_+(z)\,c_+(z) + s_-(z)\,c_-(z)\} & 0 \end{pmatrix} = \frac{\omega}{\sqrt{2}} \begin{pmatrix} c_-(2z) & -2\,s_+(z)\,s_-(z) \\ \sqrt{2}\,s_+(2z) & 0 \end{pmatrix},$$

$$\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1} = \frac{\omega}{\sqrt{2}\,c_-(z)} \begin{pmatrix} s_-(z) & -s_+(z) \\ \sqrt{2}\,c_-(z) & 0 \end{pmatrix} + \frac{\omega}{\sqrt{2}\,c_+(z)} \begin{pmatrix} s_+(z) & -s_-(z) \\ \sqrt{2}\,c_+(z) & 0 \end{pmatrix}$$

$$= \frac{\omega}{\sqrt{2}\,c_+(z)\,c_-(z)} \begin{pmatrix} s_+(z)\,c_-(z) + s_-(z)\,c_+(z) & -\{s_+(z)\,c_+(z) + s_-(z)\,c_-(z)\} \\ 2\sqrt{2}\,c_+(z)\,c_-(z) & 0 \end{pmatrix}$$

$$= \frac{\sqrt{2}\omega}{c_-(2z)} \begin{pmatrix} s_-(2z) & -s_+(2z) \\ \sqrt{2}\,c_-(2z) & 0 \end{pmatrix},$$

$$s_+(z)\,c_+(z)\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} + s_-(z)\,c_-(z)\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}$$

$$= s_+(z)\,c_+(z) \cdot \frac{\overline{\omega}}{s_+(z)} \begin{pmatrix} 0 & s_+(z) \\ -\sqrt{2}\,c_-(z) & s_-(z) \end{pmatrix} + s_-(z)\,c_-(z) \cdot \frac{\overline{\omega}}{s_-(z)} \begin{pmatrix} 0 & s_-(z) \\ -\sqrt{2}\,c_+(z) & s_+(z) \end{pmatrix}$$

$$= \overline{\omega} \begin{pmatrix} 0 & s_+(z)\,c_+(z) + s_-(z)\,c_-(z) \\ -2\sqrt{2}\,c_+(z)\,c_-(z) & s_+(z)\,c_-(z) + s_-(z)\,c_+(z) \end{pmatrix} = \overline{\omega} \begin{pmatrix} 0 & s_+(2z) \\ -\sqrt{2}\,c_-(2z) & s_-(2z) \end{pmatrix},$$

hence

$$\left\{\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} + \mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}\right\}\left\{s_-(z)\,c_-(z)\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + s_+(z)\,c_+(z)\mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\}$$

$$= \frac{\overline{\omega}}{s_+(z)\,s_-(z)} \begin{pmatrix} 0 & 2\,s_+(z)\,s_-(z) \\ -\sqrt{2}\,s_+(2z) & c_-(2z) \end{pmatrix} \cdot \frac{\omega}{\sqrt{2}} \begin{pmatrix} c_-(2z) & -2\,s_+(z)\,s_-(z) \\ \sqrt{2}\,s_+(2z) & 0 \end{pmatrix} = 2\,s_+(2z) \cdot \mathbf{I},$$

$$\left\{\mathbf{A}(z)\mathbf{K}^{-1}\mathbf{D}(z)^{-1} + \mathbf{B}(z)\mathbf{K}^{-1}\mathbf{C}(z)^{-1}\right\}\left\{s_+(z)\,c_+(z)\mathbf{D}(z)\mathbf{K}\mathbf{A}(z)^{-1} + s_-(z)\,c_-(z)\mathbf{C}(z)\mathbf{K}\mathbf{B}(z)^{-1}\right\}$$

$$= \frac{\sqrt{2}\omega}{c_-(2z)} \begin{pmatrix} s_-(2z) & -s_+(2z) \\ \sqrt{2}\,c_-(2z) & 0 \end{pmatrix} \cdot \overline{\omega} \begin{pmatrix} 0 & s_+(2z) \\ -\sqrt{2}\,c_-(2z) & s_-(2z) \end{pmatrix} = 2\,s_+(2z) \cdot \mathbf{I},$$

from which follow (f) and (g), and the proof is complete. □