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Research article

Predictive modeling of reliability engineering data using a new version of the flexible Weibull model

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Abstract: The combined-unified hybrid sampling approach was introduced as a general model that combines the unified hybrid censoring sampling approach and the combined hybrid censoring approach into a unified approach. In this paper, we apply this censoring sampling approach to improve the estimation of the parameter via a novel five-parameter expansion distribution, which we call the generalized Weibull-modified Weibull model. The new distribution contains five parameters and is therefore very flexible in terms of accommodating different types of data. The new distribution provides graphs of the probability density function, e.g., symmetric or right skewed. The graph of the risk function can have a shape similar to a monomer of the increasing or decreasing model. Using the Monte Carlo method, the maximum likelihood approach is used in the estimation procedure. The Copula model was used to discuss the two marginal univariate distributions. The asymptotic confidence intervals of the parameters were developed. We present some simulation results to validate the theoretical results. Finally, a data set with failure times for 50 electronic components was analyzed to illustrate the applicability and potential of the proposed model.

Keywords: modified Weibull; generalized Weibull; maximum likelihood estimation; statistical modeling

1. Introduction

There are many phenomena in this world that need statistical description to be more understandable to the reader, but there is no specific statistical distribution that describes all of them. Therefore, many researchers have recently tried to develop new families by adding one, two or three parameters, e.g., [1–5]. By adding two additional parameters β and γ, Cordeiro et al. [6] introduced the generalized

Weibull distribution family. We continue this line of research by proposing a novel family, namely, a generalized Weibull-modified Weibull model. A model was developed that can fit real data because the model has great flexibility in representing nonlinear dynamics. The model is useful for studying data science through statistical modeling and it has been applied to engineering data. Its application in engineering shows that the developed model is adaptable and flexible in terms of ability to represent complex data. For any cumulative function (CDF) $W(x)$ and probability density function (PDF) $w(x)$, the CDF and the PDF of the proposed family are respectively given by

$$F(x;\beta,\gamma) = 1 - \exp\left[-\beta(-\log[1 - W(x)])^{\gamma}\right], \quad x \in \mathbb{R};\beta,\gamma > 0, \tag{1.1}$$

and

$$f(x;\beta,\gamma) = \frac{\beta\gamma w(x)}{1 - W(x)}(-\log[1 - W(x)])^{\gamma-1} \exp\left(-\beta\left(-\log\left(1 - W(x)\right)\right)^{\gamma}\right), \; x \in \mathbb{R};\beta,\gamma > 0. \tag{1.2}$$

Type-I and Type- II -censorship schemes are the two most common and popular censorship schemes. Type-I and type- II censorship schemes were merged by Epstein [7] in the hybrid censorship scheme. For a more brief review of censoring schemes, we refer the reader to [8–14]. Balakrishnan et al. [15] have proposed a unified hybrid censoring method. Huang and Yang [16] have considered a combined hybrid censoring sample. Emam and Sultan [17] combined the unified hybrid censoring sampling method and the combined hybrid censoring sampling method into a unified approach known as C-UHCS($m, r; T_1, T_2$), which refers to the combined-unified hybrid censoring method. The likelihood function of C-UHCS($k, r; T_1, T_2$) is

$$L(\Omega|\mathbf{x}_k) = \frac{n!}{(n-k)!}[1 - F(T)]^{n-k} \prod_{i=1}^{k} f(x_i), \tag{1.3}$$

where k and T can be chosen as:

Cases	$L^{(C)}(\Omega\|\mathbf{x})$		$L^{(U)}(\Omega\|\mathbf{x})$	
	k	T	k	T
$1: \; 0 < T_1 < X_{k:n} < T_2 < X_{r:n}$	m	$X_{m:n}$	D_2	T_2
$2: \; 0 < T_1 < X_{k:n} < X_{r:n} < T_2$	m	$X_{m:n}$	r	$X_{r:n}$
$3: \; 0 < T_1 < T_2 < X_{k:n} < X_{r:n}$	D_2	T_2	m	$X_{m:n}$
$4: \; 0 < X_{k:n} < X_{r:n} < T_1 < T_2$	r	$X_{r:n}$	D_1	T_1
$5: \; 0 < X_{k:n} < T_1 < X_{r:n} < T_2$	D_1	T_1	r	$X_{r:n}$
$6: \; 0 < X_{k:n} < T_1 < T_2 < X_{r:n}$	D_1	T_1	D_2	T_2

Then, for a parameter space Ω, the likelihood function of C-UHCS($m, r; T_1, T_2$), which represents all possible likelihood functions under different values of k, T, and $\mathbf{x}_k = (x_1, x_2, ..., x_k)$, can be written as

$$L(\Omega|\mathbf{x}_k) = \frac{n!}{(n-k)!}\left(\prod_{i=1}^{k} f(x_i)\right)(1 - F(T))^{n-k}. \tag{1.4}$$

The authors believe that this problem deserves investigation. The main motivations for using the GMW-X family in practice are as follows: 1) It is an excellent way to enter additional parameters to create an extended version of the basic model. 2) It can improve the properties of the traditional distributions. 3) It can create symmetric, right-skewed and left-skewed distributions. 4) It can provide a consistently better fit than other models. This was a good incentive to study the problem, and this was supported by the numerical results, which confirmed the superiority of the new model over many of the basic and competing models.

The rest of this work is presented here as follows. The generalized Weibull-modified Weibull distribution (GWMWD) is presented in Section 2. The bivariate extension of the generalized Weibull-modified Weibull model is discussed in Section 3. Based on C-UHCS$(m, r; T_1, T_2)$, Section 4 is devoted to applying the maximum likelihood approach to the GWMWD. Section 5 presents the Monte Carlo procedure. Section 6 applies the GWMWD to a data set of 50 electronic component failures. Section 7 shows some conclusions.

2. Generalized Weibull-modified Weibull model

Let X be a random variable (R.V.) with the modified Weibull distribution (ω, θ, ν) distribution suggested by Sarhan and Zaindin [18]; then, its CDF is

$$W(x; \omega, \delta, \nu) = 1 - \exp\left(-\omega x - \delta x^{\nu}\right), \quad x > 0, \tag{2.1}$$

and its PDF is given by

$$w(x; \omega, \delta, \nu) = \left(\omega + \delta \nu x^{\nu-1}\right) \exp\left(-\omega x - \delta x^{\nu}\right), \quad x > 0, \tag{2.2}$$

where $\omega \geq 0$ is a scale parameter, while $\delta \geq 0$ and $\nu > 0$ are shape parameters such that $\nu + \delta > 0$.

The generalized Weibull distribution family generalizes the generalized Weibull normal distribution when $\beta = \gamma = 1$, the generalized Weibull Gumbel distribution when the generalized Weibull family $\beta\gamma = 1$, and the generalized Weibull logistic distribution. The GWMWD is defined from (1.1) by replacing $W(x)$ and $w(x)$ with $W(x; \omega, \delta, \nu)$ and $w(x; \omega, \delta, \nu)$, respectively. The CDF and PDF of GWMWD, respectively, are

$$F(x; \beta, \gamma, \omega, \delta, \nu) = 1 - e^{-\beta(x\omega + x^{\nu}\delta)^{\gamma}}, \qquad x > 0; \beta, \gamma, \omega, \delta, \nu > 0, \tag{2.3}$$

and

$$f(x; \beta, \gamma, \omega, \delta, \nu) = e^{-\beta(x\omega + x^{\nu}\delta)^{\gamma}} \beta\gamma \left(\omega + x^{\nu-1}\delta\nu\right) (x\omega + x^{\nu}\delta)^{\gamma-1}, x > 0; \beta, \gamma, \omega, \delta, \nu > 0. \tag{2.4}$$

The survival function (SF) and hazard rate function (HRF) via the GWMWD of time t, respectively, are

$$S(t; \beta, \gamma, \omega, \delta, \nu) = e^{-\beta(t\omega + t^{\nu}\delta)^{\gamma}}, \tag{2.5}$$

and

$$H(t; \beta, \gamma, \omega, \delta, \nu) = \beta\gamma (t\omega + t^{\nu}\delta)^{\gamma-1} \left(\omega + t^{\nu-1}\delta\nu\right). \tag{2.6}$$

In particular, the GWMWD generalizes the generalized Weibull-Weibull distribution (when $\omega = 0$), the generalized Weibull-Rayleigh distribution (when $\omega = 0$ and $\nu = 2$), the generalized linear Weibull exponential distribution (when $\nu = 2$ and $\delta = \omega/2$, $\omega > 0$), and the generalized Weibull exponential distribution (for $\nu = 0$). In what follows, an R.V. X with the GWMWD PDF (2.4) is written as $X \sim GWMWD(\beta, \gamma, \omega, \delta, \nu)$.

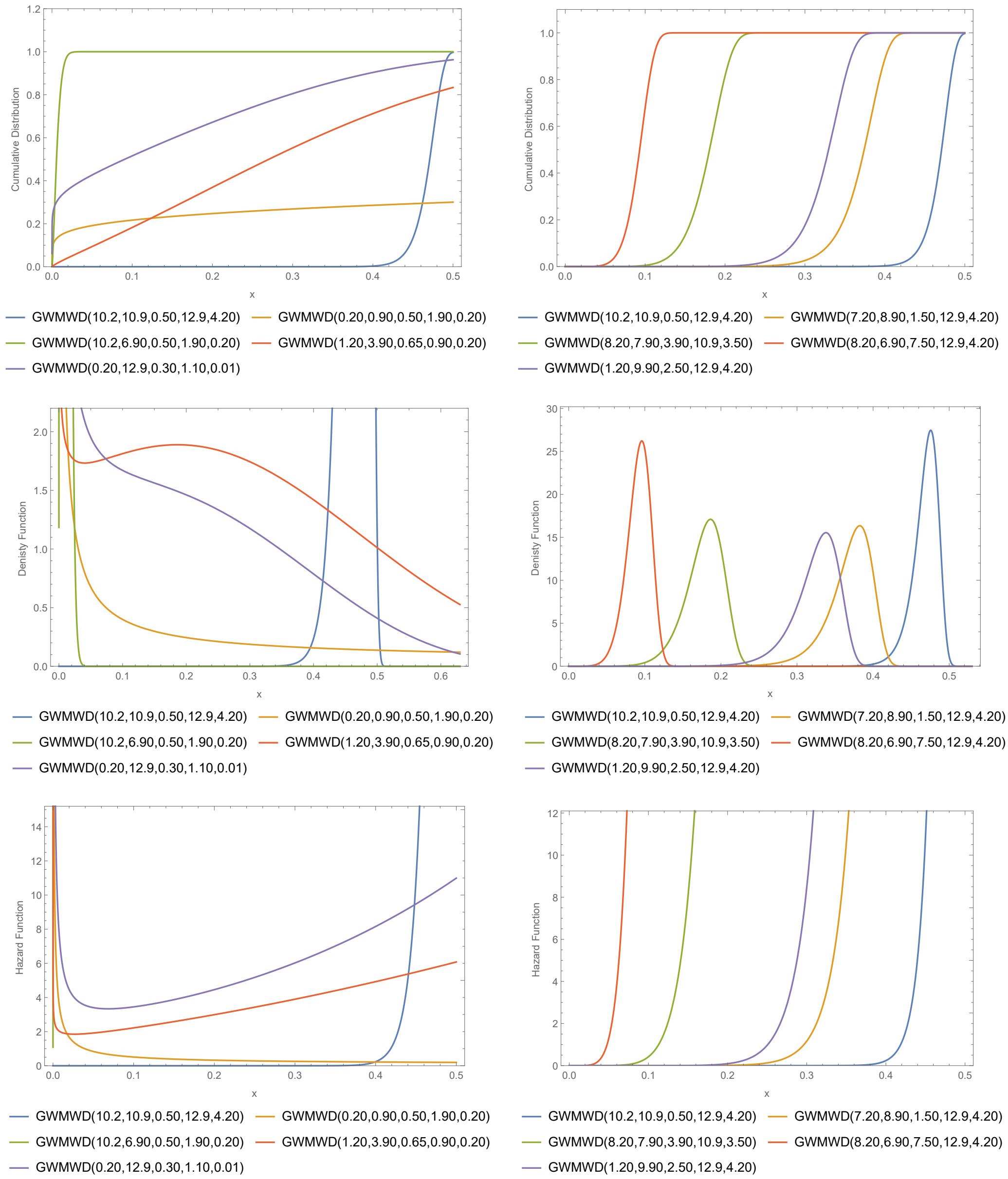

Figure 1. Plots for different CDFs, PDFs and HRFs for GWMWD $(\beta, \gamma, \omega, \delta, \nu)$.

Some possible behaviors of the CDF, PDF, and HRF for GWMWD $(\beta, \gamma, \omega, \delta, \nu)$ are shown in Figure 1. The left panel shows GWMWD (10.2,10.9, 0.50,12.9, 4.20), GWMWD(0.20, 0.90, 0.50,1.90, 0.20), GWMWD (10.20, 6.90, 0.50,1.90, 0.20), GWMWD (1.20, 3.90, 0.65, 0.90, 0.20), and GWMWD (0.20,12.9, 0.30,1.10, 0.01), while the right panel shows GWMWD (10.2, 10.9, 0.50, 12.9, 4.20) ,GWMWD (7.20, 8.90, 1.50,12.9, 4.20), GWMWD (8.20, 7.90, 3.90, 10.9,3.50), GWMWD (8.20, 6.90, 7.50, 12.9, 4.20), and GWMWD (1.20,9.90, 2.50, 12.9, 4.20). From Figure 1, it can be seen that the CDF increases faster with increasing x for the parameters β and $\delta > 1$; then, it is constant and the graph grows exponentially, the PDF increases faster with increasing x for the parameters β and $\delta > 1$, and the proposed GWMWD has strong spurs; the HRF is constant and then increases faster with increasing x when the parameters β, γ, δ and $\alpha > 1$. The PDF shape is at times very flexible. It appears to approximate a bell curve with some twist. At other times it appears to have strong tails. Because of the divergent behavior of the proposed model, it could be a good candidate for modeling semi-normal and strong-tailed data in various industrial, financial, and medical applications.

3. Bivariate GWMWD

The copula model was introduced by Morgenstern [19] to represent the joint CDF of the two marginal univariate distributions. Let $F\left(x_j\right)$ be the CDF of $X_j, j = 1, 2$. Conway [20] introduced the joint CDF and PDF of the copula model, respectively, as

$$F(x_1, x_2) = F(x_1)F(x_2)\left[1 + \rho(1 - F(x_1))(1 - F(x_2))\right], \qquad -1 < \rho < 1, \tag{3.1}$$

and

$$f(x_1, x_2) = f(x_1)f(x_2)\left[1 + \rho(1 - 2F(x_1))(1 - 2F(x_2))\right], \tag{3.2}$$

where ρ is the dependence measure between X_1 and X_2. Let the R.V.s $X_1 \sim$ GWMWD $(\beta_1, \gamma_1, \omega_1, \delta_1, \nu_1)$ and $X_2 \sim$ GWMWD $(\beta_2, \gamma_2, \omega_2, \delta_2, \nu_2)$; then, the corresponding joint CDF and PDF are, respectively, given by

$$\begin{aligned} F(x_1, x_2) = &\left(1 - e^{-\beta_1\left(x_1\omega_1 + x_1^{\nu_1}\delta_1\right)^{\gamma_1}}\right)\left(1 - e^{-\beta_2\left(x_2\omega_2 + x_2^{\nu_2}\delta_2\right)^{\gamma_2}}\right) \\ &\times\left(1 + \rho e^{-\beta_1\left(x_1\omega_1 + x_1^{\nu_1}\delta_1\right)^{\gamma_1} - \beta_2\left(x_2\omega_2 + x_2^{\nu_2}\delta_2\right)^{\gamma_2}}\right), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} f(x_1, x_2) = &\beta_1\beta_2\gamma_1\gamma_2 e^{-\beta_1\left(x_1\omega_1 + x_1^{\nu_1}\delta_1\right)^{\gamma_1} - \beta_2\left(x_2\omega_2 + x_2^{\nu_2}\delta_2\right)^{\gamma_2}}\left(x_1\omega_1 + x_1^{\nu_1}\delta_1\right)^{\gamma_1 - 1} \\ &\times\left(x_2\omega_2 + x_2^{\nu_2}\delta_2\right)^{\omega_1 - 1}\left(\omega_1 + x_1^{\nu - 1_1}\delta_1\nu_1\right)\left(\omega_2 + x_2^{\nu - 1_2}\delta_2\nu_2\right) \\ &\times\left(1 + \rho\left(1 - 2e^{-\beta_1\left(x_1\omega_1 + x_1^{\nu_1}\delta_1\right)^{\gamma_1}}\beta_1\gamma_1\left(x_1\omega_1 + x_1^{\nu_1}\delta_1\right)^{\gamma - 1_2}\left(\omega_1 + x_1^{\nu - 1_1}\delta_1\nu_1\right)\right)\right. \\ &\times\left.\left(1 - 2e^{-\beta_2\left(x_2\omega_2 + x_2^{\nu_2}\delta_2\right)^{\gamma_2}}\beta_2\gamma_2\left(x_2\omega_2 + x_2^{\nu_2}\delta_2\right)^{\gamma - 1_2}\left(\omega_2 + x_2^{\nu - 1_2}\delta_2\nu_2\right)\right)\right). \end{aligned} \tag{3.4}$$

Figure 2 presents the CDF for the bivariate GWMWD (0.1, 0.8, 4.4,6.1,3.20) and GWMWD (0.1, 0.8, 4.4,6.1,0.20) when the parameter ν increases, and for $\rho = 0.2$.

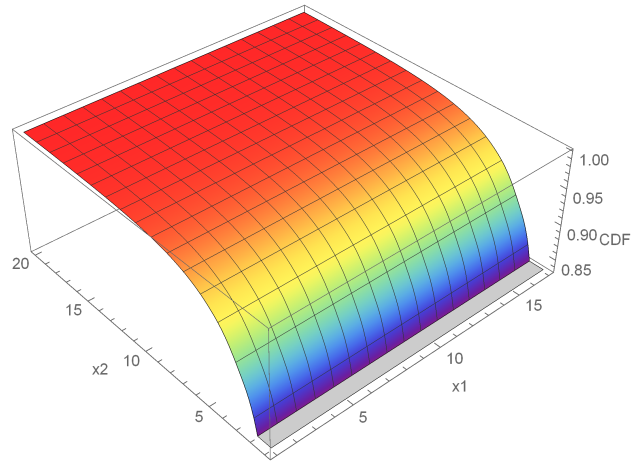

Figure 2. CDF for the bivariate GWMWD (0.1,0.8,4.4,6.1,3.20) and GWMWD (0.1,0.8,4.4,6.1,0.20).

The copula function is a way to construct bivariate distributions. Other methods can be reviewed and may be helpful to introduce some new bivariate distributions (see, Xu et al. [21] and Luo et al. [22].)

4. Likelihood function under C-UHCS

Suppose that $\{x_1, x_2, ..., x_k\}$ is an observed sample from $X \sim GWMWD(\beta, \gamma, \omega, \delta, \nu)$. The likelihood function of $\beta, \gamma, \omega, \delta$, and ν becomes

$$l = \frac{n!}{(n-k)!} e^{-\beta(n-k)(T\omega+T^{\nu}\delta)^{\gamma}} (\beta\gamma)^k \, e^{-\beta\sum_{i=1}^{k}\left(x_i\omega+x_i^{\nu}\delta\right)^{\gamma}} \prod_{i=1}^{k}\left(\omega + x_i^{\nu-1}\delta\nu\right)\left(x_i\omega + x_i^{\nu}\delta\right)^{\gamma-1}, \tag{4.1}$$

and the log-likelihood function (L) is

$$L = \log\left[\frac{n!}{(n-k)!}\right] - \beta(n-k)\left(T\omega + T^{\nu}\delta\right)^{\gamma} + k\log[\beta\gamma] - \beta\sum_{i=1}^{k}\left(x_i\omega + x_i^{\nu}\delta\right)^{\gamma}$$
$$+ \sum_{i=1}^{k}\log\left[\omega + x_i^{\nu-1}\delta\nu\right] + (\gamma-1)\sum_{i=1}^{k}\log\left[x_i\omega + x_i^{\nu}\delta\right]. \tag{4.2}$$

Let $Q(x) = \omega x + \delta x^{\nu}$. The first partial derivatives of (4.2) with respect to $\beta, \gamma, \omega, \delta$ and ν are given by

$$\frac{\partial L}{\partial \beta} = \frac{k}{\beta} - (n-k)Q(T)^{\gamma} - \sum_{i=1}^{k} Q(x_i)^{\gamma}, \tag{4.3}$$

$$\frac{\partial L}{\partial \gamma} = \frac{k}{\gamma} - (n-k)\beta Q(T)^{\gamma}\log Q(T) + \sum_{i=1}^{k}\log Q(x_i)$$
$$-\beta\sum_{i=1}^{k} Q(x_i)^{\gamma}\log Q(x_i), \tag{4.4}$$

$$\frac{\partial L}{\partial \omega} = -(n-k)T\beta\gamma Q(T)^{\gamma-1} + \sum_{i=1}^{k} \frac{1}{\omega + \delta\nu x_i^{\nu-1}} + (\gamma-1)\sum_{i=1}^{k} \frac{x_i}{Q(x_i)} -\beta\gamma \sum_{i=1}^{k} x_i Q(x_i)^{\gamma-1}, \tag{4.5}$$

$$\frac{\partial L}{\partial \delta} = -(n-k)T^{\nu}\beta\gamma Q(T)^{\gamma-1} + \sum_{i=1}^{k} \frac{\nu x_i^{\nu-1}}{\omega + \delta\nu x_i^{\nu-1}} + (\gamma-1)\sum_{i=1}^{k} \frac{x_i^{\nu}}{Q(x_i)} -\beta\gamma \sum_{i=1}^{k} x_i^{\nu} Q(x_i)^{\gamma-1}, \tag{4.6}$$

$$\frac{\partial L}{\partial \nu} = -(n-k)T^{\nu}\beta\gamma\delta Q(T)^{\gamma-1}\log[T] + \delta \sum_{i=1}^{k} \frac{x_i^{\nu-1} + \nu \log[x_i]\, x_i^{\nu-1}}{\omega + \delta\nu x_i^{\nu-1}} +(\gamma-1)\delta \sum_{i=1}^{k} \frac{\log[x_i]\, x_i^{\nu}}{Q(x_i)} - \beta\gamma\delta \sum_{i=1}^{k} \log[x_i]\, x_i^{\nu} Q(x_i)^{\gamma-1}. \tag{4.7}$$

The maximum likelihood estimators $\hat{\beta}_{ML}, \hat{\gamma}_{ML}, \hat{\omega}_{ML}, \hat{\delta}_{ML}$, and $\hat{\nu}_{ML}$ of the GWMWD $(\beta, \gamma, \omega, \delta, \nu)$ parameters are the solutions of (4.3)–(4.7). The asymptotic confidence intervals of the parameters $\beta, \gamma, \omega, \delta$ and ν can be calculated. $\hat{V} = V(\hat{\beta}_{ML}, \hat{\gamma}_{ML}, \hat{\omega}_{ML}, \hat{\delta}_{ML}, \hat{\nu}_{ML})$ is the observed variance covariance matrix, such that

$$V(\beta,\gamma,\omega,\delta,\nu) = -\begin{bmatrix} \frac{\partial^2 L}{\partial \beta^2} & \frac{\partial^2 L}{\partial \beta \partial \gamma} & \frac{\partial^2 L}{\partial \beta \partial \omega} & \frac{\partial^2 L}{\partial \beta \partial \delta} & \frac{\partial^2 L}{\partial \beta \partial \nu} \\ \frac{\partial^2 L}{\partial \gamma \partial \beta} & \frac{\partial^2 L}{\partial \gamma^2} & \frac{\partial^2 L}{\partial \gamma \partial \omega} & \frac{\partial^2 L}{\partial \gamma \partial \delta} & \frac{\partial^2 L}{\partial \gamma \partial \nu} \\ \frac{\partial^2 L}{\partial \omega \partial \beta} & \frac{\partial^2 L}{\partial \omega \partial \gamma} & \frac{\partial^2 L}{\partial \omega^2} & \frac{\partial^2 L}{\partial \omega \partial \delta} & \frac{\partial^2 L}{\partial \omega \partial \nu} \\ \frac{\partial^2 L}{\partial \delta \partial \beta} & \frac{\partial^2 L}{\partial \delta \partial \gamma} & \frac{\partial^2 L}{\partial \delta \partial \omega} & \frac{\partial^2 L}{\partial \delta^2} & \frac{\partial^2 L}{\partial \delta \partial \nu} \\ \frac{\partial^2 L}{\partial \nu \partial \beta} & \frac{\partial^2 L}{\partial \nu \partial \gamma} & \frac{\partial^2 L}{\partial \nu \partial \omega} & \frac{\partial^2 L}{\partial \nu \partial \delta} & \frac{\partial^2 L}{\partial \nu^2} \end{bmatrix}^{-1}, \tag{4.8}$$

where

$$\frac{\partial^2 L}{\partial \beta^2} = -\frac{k}{\beta^2}, \tag{4.9}$$

$$\frac{\partial^2 L}{\partial \beta \partial \gamma} = (k-n)Q(T)^{\gamma}\log\delta^2 - \sum_{i=1}^{k} Q(x_i)^{\gamma}\log Q(x_i), \tag{4.10}$$

$$\frac{\partial^2 L}{\partial\beta\partial\omega} =(k-n)T\gamma Q(T)^{\gamma-1} - \gamma\sum_{i=1}^{k} x_i Q(x_i)^{\gamma-1}, \tag{4.11}$$

$$\frac{\partial^2 L}{\partial\beta\partial\delta} =(k-n)T^{\nu}\gamma Q(T)^{\gamma-1} - \gamma\sum_{i=1}^{k} x_i^{\nu} Q(x_i)^{\gamma-1}, \tag{4.12}$$

$$\frac{\partial^2 L}{\partial\beta\partial\nu} =(k-n)T^{\nu}\gamma\delta Q(T)^{\gamma-1}\log[T] - \gamma\delta\sum_{i=1}^{k} x_i^{\nu} Q(x_i)^{\gamma-1}\log[x_i], \tag{4.13}$$

$$\frac{\partial^2 L}{\partial\gamma^2} =-\frac{k}{\gamma^2}(k-n)\beta Q(T)^{\gamma}\log Q(T)^2 - \beta\sum_{i=1}^{k} Q(x_i)^{\gamma}\log[Q(x_i)]^2, \tag{4.14}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\gamma\partial\omega} =&(k-n)^2T^2\beta^2\gamma Q(T)^{2\gamma-2}\log Q(T)\\ &+\sum_{i=1}^{k}\frac{x_i}{Q(x_i)} - \beta\sum_{i=1}^{k} x_i Q(x_i)^{\gamma-1}\left(1+\gamma\log Q(x_i)\right),\end{aligned} \tag{4.15}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\gamma\partial\delta} =&(k-n)^2T^{2\nu}\beta^2\gamma Q(T)^{2\gamma-2}\log Q(T)\\ &+\sum_{i=1}^{k}\frac{x_i^{\nu}}{Q(x_i)} - \beta\sum_{i=1}^{k} x_i^{\nu} Q(x_i)^{\gamma-1}\left(1+\gamma\log Q(x_i)\right),\end{aligned} \tag{4.16}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\gamma\partial\nu} =&(k-n)T^{\nu}\beta\delta Q(T)^{\gamma-1}\log[T]\left(1+\gamma\log Q(T)\right) + \sum_{i=1}^{k}\frac{\delta\log[x_i]\,x_i^{\nu}}{Q(x_i)}\\ &-\beta\delta\sum_{i=1}^{k} x_i^{\nu} Q(x_i)^{\gamma-1}\log[x_i]\left(1+\gamma\log Q(x_i)\right),\end{aligned} \tag{4.17}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\omega^2} =&(k-n)T^2\beta(\gamma-1)\gamma Q(T)^{\gamma-2} + \sum_{i=1}^{k} -\frac{1}{Q(x_i)^2}\\ &-(\gamma-1)\sum_{i=1}^{k}\frac{x_i^2}{Q(x_i)^2} + \beta\gamma(1-\gamma)\sum_{i=1}^{k} {x_i}^2 Q(x_i)^{\gamma-2},\end{aligned} \tag{4.18}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\omega\partial\delta} =&(k-n)T^{1+\nu}\beta(\gamma-1)\gamma Q(T)^{\gamma-2} + \sum_{i=1}^{k} -\frac{\nu x_i^{\nu-1}}{\left(\omega+\delta\nu x_i^{\nu-1}\right)^2}\\ &-(\gamma-1)\sum_{i=1}^{k}\frac{x_i^{1+\nu}}{Q(x_i)^2} + \beta\gamma\sum_{i=1}^{k} {x_i}^{\nu+1} Q(x_i)^{\gamma-2}(1-\gamma),\end{aligned} \tag{4.19}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial\omega\partial\nu} =&(k-n)T^{1+\nu}\beta(\gamma-1)\gamma\delta Q(T)^{\gamma-2}\log[T]\\ &-(\gamma-1)\sum_{i=1}^{k}\frac{\delta\log[x_i]\,x_i^{1+\nu}}{Q(x_i)^2} - \delta\sum_{i=1}^{k}\left(\frac{x_i^{\nu-1}}{\left(\omega+\delta\nu x_i^{\nu-1}\right)^2} + \frac{\nu\log[x_i]\,x_i^{\nu-1}}{\left(\omega+\delta\nu x_i^{\nu-1}\right)^2}\right)\end{aligned}$$

$$+\beta\gamma\delta\sum_{i=1}^{k} x_i^{\nu+1} Q(x_i)^{\gamma-2}\log[x_i](1-\gamma), \tag{4.20}$$

$$\frac{\partial^2 L}{\partial\delta^2} = (k-n)T^{2\nu}\beta(\gamma-1)\gamma Q(T)^{\gamma-2} - \sum_{i=1}^{k}\frac{\nu^2 x_i^{-2+2\nu}}{\left(\omega+\delta\nu x_i^{\nu-1}\right)^2}$$
$$-\sum_{i=1}^{k}\frac{(\gamma-1)x_i^{2\nu}}{Q(x_i)^2} + \beta\gamma\sum_{i=1}^{k} x_i^{2\nu} Q(x_i)^{\gamma-2}(1+\gamma), \tag{4.21}$$

$$\frac{\partial^2 L}{\partial\delta\partial\nu} = (k-n)T^{\nu}\beta\gamma Q(T)^{\gamma-1}\log[T]\left(T^{\nu}(\gamma-1)\delta(\omega T+\delta T^{\nu})^{-1}-1\right)$$
$$+(\gamma-1)\sum_{i=1}^{k}\frac{\log[x_i]\,x_i^{\nu}}{Q(x_i)}\left(1-\frac{\delta\,x_i^{\nu}}{Q(x_i)}\right)$$
$$+\sum_{i=1}^{k}\frac{x_i^{\nu-1}}{\omega+\delta\nu x_i^{\nu-1}}\left(1+\nu\log[x_i]-\nu x_i^{\nu-1}\left(\frac{\delta(1-\nu\log[x_i])}{\omega+\delta\nu x_i^{\nu-1}}\right)\right)$$
$$-\beta\gamma\sum_{i=1}^{k} x_i^{\nu}\log[x_i]\left(Q(x_i)^{\gamma-1}+(\gamma-1)\delta Q(x-i)^{\gamma-2}\right), \tag{4.22}$$

$$\frac{\partial^2 L}{\partial\nu^2} = (k-n)^2 T^{3\nu}\beta^2(\gamma-1)\gamma^2\delta^3 Q(T)^{2\gamma-3}\left(\log[T]^2\right)^2$$
$$+(\gamma-1)\sum_{i=1}^{k}\left(-\frac{\delta^2\log[x_i]^2\,x_i^{2\nu}}{Q(x_i)^2}+\frac{\delta\log[x_i]^2\,x_i^{\nu}}{Q(x_i)}\right)+\frac{2\delta\log[x_i]\,x_i^{\nu-1}+\delta\nu\log[x_i]^2\,x_i^{\nu-1}}{\omega+\delta\nu x_i^{\nu-1}}$$
$$-\delta^2\sum_{i=1}^{k} x_i^{2\nu-2}(1+\nu\log[x_i])\left(\frac{1}{\left(\omega+\nu x_i^{\nu-1}\right)^2}+\frac{\nu\log[x_i]}{\left(\omega+\delta\nu x_i^{\nu-1}\right)^2}\right)$$
$$-\beta\gamma\delta\sum_{i=1}^{k} x_i^{\nu} Q(x_i)^{\gamma-1}\log[x_i]\left(\log[x_i]+(x_i^{\nu})^{-1}(\gamma-1)\right). \tag{4.23}$$

An approximate $100(1-\epsilon)\%$ two-sided C.Is for the parameters $\beta, \gamma, \omega, \delta$ and ν are

$$\widehat{\beta} \pm z_{\epsilon/2}\sqrt{V(\widehat{\beta})}, \tag{4.24}$$

$$\widehat{\gamma} \pm z_{\epsilon/2}\sqrt{V(\widehat{\gamma})}, \tag{4.25}$$

$$\widehat{\omega} \pm z_{\epsilon/2}\sqrt{V(\widehat{\omega})}, \tag{4.26}$$

$$\widehat{\delta} \pm z_{\epsilon/2}\sqrt{V(\widehat{\delta})}, \tag{4.27}$$

and

$$\widehat{\nu} \pm z_{\epsilon/2}\sqrt{V(\widehat{\nu})}, \tag{4.28}$$

respectively, where the diagonal elements of $\hat{V}$ $V(\hat{\beta})$, $V(\hat{\gamma})$, $V(\hat{\omega})$, $V(\hat{\delta})$, and $V(\hat{\nu})$ are the estimated variances of $\hat{\beta}_{ML}$,$\hat{\gamma}_{ML}$,$\hat{\omega}_{ML}$,$\hat{\delta}_{ML}$, and $\hat{\nu}_{ML}$, and $z_{\epsilon/2}$ is the upper $\left(\frac{\epsilon}{2}\right)$ percentile of the normal(0,1) distribution.

5. Monte Carlo simulation study

Let U have a uniform (0,1) distribution. The GWMWD can be simulated by using the solution of the nonlinear equation

$$0 = 1 - u - exp^{-\beta(x\omega + x^{\nu}\delta)^{\gamma}}. \tag{5.1}$$

We simulate the GWMWD for two sets of the parameters: Set 1: $\beta = 1.4, \gamma = 3.0, \omega = 0.7, \delta = 1.3, \nu = 0.4$, and Set 2: $\beta = 0.4, \gamma = 1.0, \omega = 1.7, \delta = 1.5, \nu = 1.8$. The empirical results of the Monte Carlo simulation study are given in Table 1 for Set 1. The empirical results of the Monte Carlo simulation study are given in Table 2 for Set 2. Suppose that the data were observed for the GWMWD under the censoring scheme C-UHCS$(m, r; T_1, T_2)$ and set the arbitrary values for termination as $T = X_k$ and $k = \frac{4}{5}n$. The simulation study is carried out as follows

1) Random samples of size $n = 25, 50, ..., 1000$ were simulated from the GWMWD.

2) The model parameters were estimated via the maximum likelihood method.

3) 1000 iterations were made to obtain the MLEs, biases and MSEs of these estimators.

4) Let $\hat{\vartheta}$ be the MLE of $\vartheta = (\beta, \gamma, \omega, \delta, \nu)$. The MLEs, biases and MSEs are given, respectively, by

$$\hat{\vartheta} = \frac{1}{1000} \sum_{i=1}^{M} \hat{\vartheta}_i, \tag{5.2}$$

$$Bias\left(\hat{\vartheta}\right) = \frac{1}{1000} \sum_{i=1}^{M} \left(\hat{\vartheta}_i - \vartheta\right), \tag{5.3}$$

and

$$MSE\left(\hat{\vartheta}\right) = \frac{1}{1000} \sum_{i=1}^{1000} \left(\hat{\vartheta}_i - \vartheta\right)^2, \tag{5.4}$$

5) The 90% and 95% approximate confidence intervals with their width were calculated.

Table 1. Point and interval estimation of the parameters for $\beta = 1.4, \gamma = 3.0, \omega = 0.7, \delta = 1.3$, and $\nu = 0.4$ with different values of n.

Par.	n	Bias	MSEs	90%L	90%U	90%W	95%L	95%U	95%W
$\hat{\beta}$	25	1.284	1.955	0.001	5.89	5.89	0.001	6.516	6.516
	50	1.163	1.479	0.137	4.989	4.852	0.001	5.462	5.462
	100	1.124	1.325	0.35	4.698	4.347	0.001	5.122	5.122
	200	1.099	1.241	0.464	4.533	4.069	0.067	4.93	4.863
	400	1.08	1.18	0.545	4.415	3.87	0.167	4.793	4.625
	600	1.076	1.169	0.559	4.394	3.835	0.185	4.768	4.583
	800	1.079	1.171	0.559	4.399	3.841	0.184	4.774	4.59
	1000	1.078	1.169	0.561	4.395	3.834	0.187	4.77	4.582
$\hat{\gamma}$	25	-0.307	0.403	2.032	3.354	1.322	1.903	3.483	1.58
	50	-0.428	0.311	2.061	3.082	1.021	1.962	3.182	1.22
	100	-0.475	0.288	2.053	2.997	0.944	1.961	3.089	1.129
	200	-0.502	0.281	2.038	2.959	0.921	1.948	3.049	1.101
	400	-0.51	0.276	2.038	2.941	0.904	1.949	3.03	1.08
	600	-0.52	0.28	2.02	2.939	0.919	1.931	3.029	1.099
	800	-0.521	0.279	2.021	2.937	0.915	1.932	3.026	1.094
	1000	-0.524	0.28	2.018	2.935	0.918	1.928	3.025	1.097
$\hat{\omega}$	25	0.99	1.265	0.616	4.764	4.148	0.212	5.169	4.957
	50	0.886	0.921	1.076	4.096	3.02	0.781	4.39	3.609
	100	0.827	0.743	1.308	3.746	2.438	1.07	3.984	2.914
	200	0.798	0.668	1.403	3.593	2.19	1.189	3.807	2.617
	400	0.783	0.629	1.452	3.515	2.062	1.251	3.716	2.465
	600	0.78	0.619	1.466	3.495	2.029	1.268	3.693	2.425
	800	0.777	0.61	1.476	3.477	2.002	1.281	3.673	2.392
	1000	0.772	0.601	1.486	3.457	1.971	1.294	3.649	2.355
$\hat{\delta}$	25	1.102	1.355	0.379	4.824	4.445	-0.054	5.258	5.312
	50	1.057	1.212	0.57	4.544	3.974	0.182	4.932	4.75
	100	1.017	1.086	0.735	4.298	3.563	0.387	4.646	4.258
	200	1.011	1.052	0.786	4.236	3.449	0.45	4.572	4.123
	400	0.986	0.987	0.867	4.105	3.238	0.551	4.421	3.869
	600	0.978	0.966	0.893	4.063	3.17	0.584	4.373	3.788
	800	0.978	0.965	0.896	4.06	3.164	0.588	4.369	3.781
	1000	0.977	0.961	0.902	4.052	3.151	0.594	4.36	3.765
$\hat{\nu}$	25	0.4	0.16	0.938	1.462	0.525	0.886	1.514	0.627
	50	0.4	0.16	0.938	1.462	0.525	0.886	1.514	0.627
	100	0.4	0.16	0.938	1.462	0.525	0.886	1.514	0.627
	200	0.4	0.16	0.938	1.462	0.525	0.886	1.514	0.627
	400	0.4	0.16	0.938	1.462	0.525	0.886	1.514	0.627
	600	0.4	0.16	0.938	1.462	0.525	0.886	1.514	0.627
	800	0.4	0.16	0.938	1.462	0.525	0.886	1.514	0.627
	1000	0.4	0.16	0.938	1.462	0.525	0.886	1.514	0.627

Table 2. Point and interval estimation of the parameters for $\beta = 0.4, \gamma = 1.0, \omega = 1.7, \delta = 1.5$, and $\nu = 1.8$ with different values of n.

Par.	n	Bias	MSEs	90%L	90%U	90%W	95%L	95%U	95%W
$\hat{\beta}$	25	0.498	0.303	0.401	1.394	0.994	0.304	1.491	1.187
$\hat{\beta}$	25	0.498	0.303	0.401	1.394	0.994	0.304	1.491	1.187
	50	0.458	0.226	0.487	1.229	0.742	0.415	1.302	0.887
	100	0.437	0.197	0.514	1.161	0.647	0.451	1.224	0.774
	200	0.426	0.187	0.519	1.133	0.613	0.459	1.193	0.733
	400	0.41	0.177	0.519	1.101	0.582	0.462	1.158	0.695
	600	0.404	0.175	0.517	1.091	0.574	0.461	1.147	0.686
	800	0.396	0.171	0.516	1.077	0.561	0.461	1.131	0.67
	1000	0.385	0.165	0.514	1.055	0.541	0.461	1.108	0.647
$\hat{\gamma}$	25	-0.098	0.073	0.783	1.021	0.238	0.76	1.044	0.284
	50	-0.147	0.034	0.797	0.908	0.111	0.787	0.919	0.132
	100	-0.159	0.03	0.792	0.89	0.098	0.782	0.9	0.117
	200	-0.164	0.03	0.787	0.884	0.097	0.778	0.894	0.116
	400	-0.16	0.028	0.794	0.885	0.091	0.785	0.894	0.109
	600	-0.158	0.027	0.798	0.887	0.089	0.789	0.896	0.107
	800	-0.157	0.027	0.799	0.888	0.089	0.79	0.896	0.106
	1000	-0.154	0.027	0.802	0.89	0.088	0.794	0.898	0.105
$\hat{\omega}$	25	-0.796	0.691	0.001	2.037	2.037	0.001	2.258	2.258
	50	-0.836	0.716	0.001	2.038	2.038	0.001	2.267	2.267
	100	-0.855	0.741	0.001	2.06	2.06	0.001	2.297	2.297
	200	-0.849	0.737	0.001	2.059	2.059	0.001	2.295	2.295
	400	-0.83	0.723	0.001	2.057	2.057	0.001	2.288	2.288
	600	-0.815	0.709	0.001	2.049	2.049	0.001	2.276	2.276
	800	-0.802	0.699	0.001	2.044	2.044	0.001	2.268	2.268
	1000	-0.783	0.683	0.001	2.037	2.037	0.001	2.255	2.255
$\hat{\delta}$	25	-0.597	0.415	0.222	1.584	1.362	0.089	1.717	1.628
	50	-0.637	0.422	0.17	1.556	1.386	0.035	1.691	1.656
	100	-0.654	0.436	0.131	1.561	1.43	0.001	1.701	1.709
	200	-0.652	0.436	0.134	1.563	1.429	0.001	1.702	1.707
	400	-0.657	0.441	0.12	1.565	1.445	0.001	1.706	1.706
	600	-0.651	0.435	0.136	1.563	1.427	0.001	1.702	1.702
	800	-0.638	0.428	0.161	1.564	1.403	0.024	1.701	1.677
	1000	-0.627	0.421	0.183	1.563	1.38	0.048	1.698	1.65
$\hat{\nu}$	25	-0.899	0.863	0.001	2.316	2.316	0.001	2.592	2.592
	50	-0.941	0.904	0.001	2.341	2.341	0.001	2.63	2.63
	100	-0.953	0.921	0.001	2.356	2.356	0.001	2.651	2.651
	200	-0.948	0.918	0.001	2.358	2.358	0.001	2.652	2.652
	400	-0.924	0.897	0.001	2.347	2.347	0.001	2.635	2.635
	600	-0.905	0.876	0.001	2.332	2.332	0.001	2.612	2.612
	800	-0.894	2.33	0.001	2.848	2.848	0.001	2.6005	2.601
	1000	-0.874	2.319	0.001	2.785	2.785	0.001	2.59	2.59

Table 3. Reliability engineering data summary.

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.058	0.254	1.600	3.410	4.534	15.080

Table 4. Relative quality of the NG-MW vs competing models.

Model	AIC	CAIC	BIC	HQIC
GWMWD(0.058,9.986,0.005,1.239,0.056)	212.9574	214.3211	222.5175	216.5980
GWWD(1.916, 0.972, 0.326, 0.635)	213.6829	214.5718	221.3310	216.5953
GWRD(0.360, 0.246, 3.774)	221.4256	221.9473	227.1617	223.609
GLWEXPD(0.466, 0.455, 0.715)	213.2008	213.7226	218.9369	215.3852
GWEXPD(0.465, 0.640, 0.902, 0.007)	216.9577	217.8465	224.6058	219.8701

6. Reliability engineering application of the GWMWD model

This section is devoted for illustrating the GWMWD through the analysis of a reliability engineering application. The data set represents the failure times of 50 electronic components (per 1000 h); see Aryal and Elbatal [23]. Suppose that the data was observed from GWMWD under the censoring scheme C-UHCS$(m, r; T_1, T_2)$, and set the arbitrary values for termination $k = 45$ and $T = 10.943$. Table 3 shows a summary of the reliability engineering data. The boxplot and Q-Q plot for the reliability engineering data are shown in Figure 4. The estimated parameters are $\hat{\beta} = 0.058, \hat{\gamma} = 9.986, \hat{\omega} = 0.005, \hat{\delta} = 1.239$, and $\hat{\nu} = 0.056$. Plots of the fitted density and distribution functions of the GWMWD model are shown in Figure 5. The likelihood probability (PP) and Kaplan-Meier survival curve are shown in Figure 6.

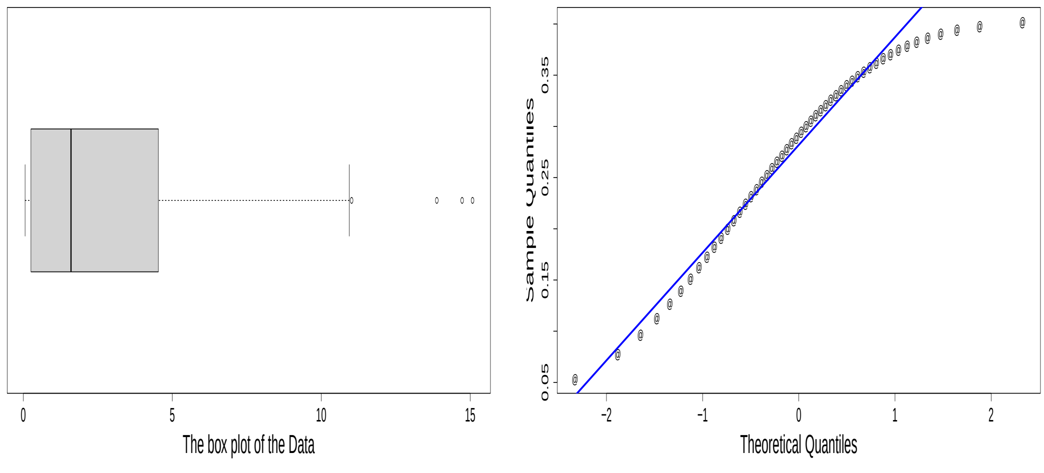

Figure 3. Boxplot and Q-Q plot for the reliability engineering data.

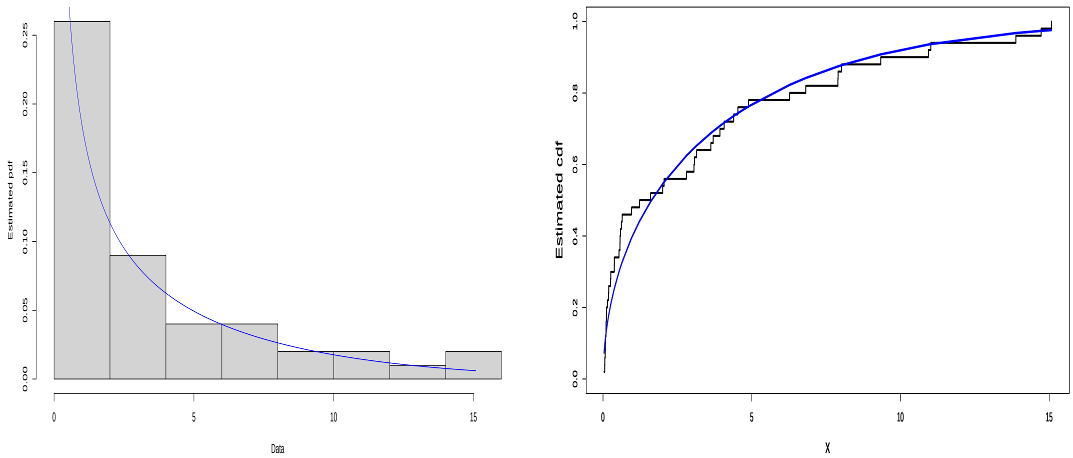

Figure 4. Plots of fitted PDF and CDF of the GWMWD.

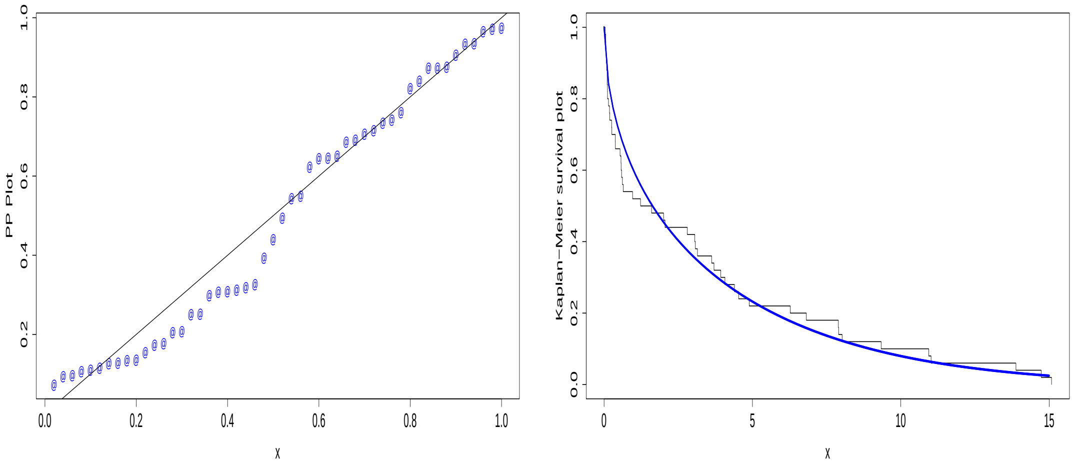

Figure 5. Plots of the PP and the Kaplan–Meier survival function of the GWMWD.

Table 4 compares the GWMWD based on some detection criteria, such as the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and consistent Akaike information criterion (CAIC). The goodness-of-fit results of the GWMWD model are compared with some other models, including the generalized Weibull distribution (GWWD), the generalized Weibull-Rayleigh distribution (GWRD), the generalized linear Weibull exponential distribution (GLWEXPD) and the generalized Weibull exponential distribution (GWEXPD). Table 5 compares the GWMWD with the Kolmogorov-Smirnov test for one sample. The results in Tables 4 and 5 suggest that the GWMWD provides a better fit than other competing models and could be chosen as a suitable model for analyzing heavy-tailed electronic data.

Table 5. One-sample Kolmogorov-Smirnov (KS) test.

Model	KS	p-value
GWMWD(0.058,9.986,0.005,1.239,0.056)	0.13263	0.3144
GWWD(1.916, 0.972, 0.326, 0.635)	0.16936	0.2247
GWRD(0.360, 0.246, 3.774)	0.14916	0.1952
GLWEXPD(0.466, 0.455, 0.715)	0.14875	0.1977
GWEXPD(0.465, 0.640, 0.902, 0.007)	0.17732	0.07589

7. Conclusions

A new extension of the Weibull distribution i.e., the generalized modified Weibull distribution with five parameters, is presented. The model has a high degree of flexibility to fit the data appropriately. The provided model exhibits strong tail-heavy behavior and has unimodal increasing failure rate functions. Based on a combined-unified hybrid sample, the maximum likelihood estimators of the intended model parameters and a Monte Carlo simulation study were obtained. To illustrate the applicability and potential of the intended distribution, a dataset of failure times of 50 electronic components was analyzed. The mean square errors and biases decrease with increasing sample size. It is clear that the proposed model agrees well with the estimated PDF and CDF plots. The boxplot shows that the electronic downtime data set has a highly right skewed tail. The new generalized modified Weibull distribution based on the one-sample Kolmogorov-Smirnov test provides a better fit than other competing models. The proposed model fits the Kaplan-Meier survival plot very well. The results indicate that the generalized Weibull distribution (modified Weibull distribution) is considered ideal for modeling the intended engineering data. For future studies, we hope to discuss the accelerated life testing based on the new distribution by using stress-strength models (Zhang et al. [24, 25]).

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Conflict of interest

The authors declare that they have no conflicts of interest.

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