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**Theory article**

## A new Tseng method for supply chain network equilibrium model

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**Abstract:** To solve the equilibrium problem of the supply chain network, a new subgradient extragradient method is introduced. The proposal achieves adaptive parameter selection, and supports a one-step subgradient projection operator, which can theoretically reduce the computational complexity of the algorithm. The introduction of subgradient projection operators makes the calculation of algorithms easier, and transforms the projection difficulty problem into how to find suitable sub-differential function problems. The given convergence proof further shows the advantages of the proposed algorithm. Finally, the presented algorithm is operated to a concrete supply chain network model. The comparisons show the proposed algorithm is better than other methods in term of CPU running time and iteration steps.

**Keywords:** supply chain network equilibrium model; tseng extragradient projection algorithm; accelerated technique; computational complexity; variational inequality problem

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### 1. Introduction

The main work of this paper is to construct a new iterative algorithm to solve the supply chain network equilibrium problem. In fact, we first transform the model into a variational inequality model [1], and then solve it with an algorithm. At present, there are many iterative algorithms for solving variational inequalities, but these algorithms rely on projection operators. The disadvantage of projection operators is that the calculation complexity is high, and the calculation difficulty is large. Therefore, it is the work of this paper to find an operation to replace the traditional projection operator to make the algorithm easy to calculate, and to reduce the computational complexity of the algorithm.

The concept of supply chain originated in the book *Competitive Advantage* in the last century [2], issued to Porter. Subsequently, the system of the supply chain was gradually established by scholars. Among them, Ma et al. [3] defined the supply chain as follows: The supply chain is around the core enterprise through the control of information flow, logistics and capital flow from the purchase of raw materials to make intermediate products and final products, and finally by the sales network to deliver

the products to consumers, suppliers, manufacturers, distributors, retailers and end users into a whole functional network structure model.

Supply chain management no longer only focuses on the optimization of the enterprise's own system, but extends the scope of management to suppliers and users, and emphasizes information sharing, integrated management and overall system optimization between multiple enterprises. Therefore, supply chain management can be considered as a modern management technology and management mode that integrates, plans, controls and coordinates the logistics, capital flow and information flow that occur in the entire network chain structure from the supplier's supplier to the customer [4].

According to relevant research [5], if enterprises want to gain an advantage in the competition, one of the most effective strategic means is to obtain a cost advantage, that is, a leading position in total costs. Therefore, in order to achieve a leading position in the total cost, it is necessary for the enterprise to reduce the price to the level below the same product through its own technical advantages, if the enterprise can maintain this advantage in the overall leading position. In the same industry, it will also achieve higher economic benefits than the same industry. Among them, supply chain management further integrates the industrial chain, eliminates a series of duplicate cost expenditures and effectively reduces cost consumption links, thereby reducing the total cost. The implementation of supply chain management can reduce the total cost of enterprises by 10%, increase the on-time delivery rate by 15%, shorten the production cycle by 25–35%, increase productivity by more than 10%, and improve asset operation performance by 15–20%. Many successful companies, such as Dell and some overseas multinational companies, have reduced their costs through supply chain management and increased their total profits. Therefore, the establishment and solution of supply chain management model is urgently needed in today's society.

Daniele and Maugeri [6] list multiple equilibrium models for supply chain management, establishes a optimization model and transforms it into a variational inequality problem for solving. The definition of the problem, or VIP for short, refers to finding a point  $u$  in a non-empty closed convex set  $\mathcal{S}$ , such that  $\langle \mathcal{T}u, v - u \rangle \geq 0, \forall v \in \mathcal{S}$ . Here,  $\langle \cdot, \cdot \rangle$  represents the inner product, and  $\mathcal{T}$  is an operator defined on the set  $\mathcal{S}$ . The VIP for supply chain management is commonly used by the one-step gradient projection method [7], that is

$$u_{i+1} = \Pi_{\mathcal{S}}(u_i - \varsigma \mathcal{T}u_i). \quad (1)$$

Here  $\varsigma \in_R (0, \mathcal{L}^{-1})$  is the parameter of the gradient projection,  $\mathcal{L}$  denotes the Lipschitz constant of  $\mathcal{T}$ , and  $\Pi_{\mathcal{S}} : \mathcal{H} \rightarrow \mathcal{S}$  is the projection operator, where  $\mathcal{H}$  is a real Hilbert space. Since the supply chain management model was transformed into a variational inequality model, people began to apply it to solve the supply chain management model. Solving the required operator  $\mathcal{T}$  is quasi-strong monotonic, which is more difficult. In actual management, supply chain models often fail to meet this requirement. In 2000, Tseng gave a class of extragradient methods [8]. Its specific form is as follows:

$$\begin{cases} y_i = \Pi_{\mathcal{S}}(u_i - \varsigma \mathcal{T}u_i), \\ u_{i+1} = y_i - \varsigma(\mathcal{T}y_i - \mathcal{T}u_i). \end{cases} \quad (2)$$

The algorithm requires a monotonic operator, which is relatively easy in actual supply chain management. The Tseng algorithm is a major leap forward in the construction of iterative methods for solving variational inequality model in human history, and it is also one of the most successful iterative algorithms; even today's iterative algorithms have the format of Tseng's algorithm.

At present, Censor et al. [9], Censor and Levy [10] and Cegielski and Censor [11] provide some exterior gradient iterative methods for solving variational inequality problems. An et al. [12] gave a new optimization iteration algorithm. Ceng and Yao [13], Ceng and Fong [14] and Ceng et al. [15, 16] constructed implicit iterative algorithms to solve variational inequality problems. Recently, Thong et al. [17] and Thong and Hieu [18] introduced an inertial term based on the Tseng method in the following form:

$$\begin{cases} z_i = u_i + \vartheta_i(u_i - u_{i-1}), \\ y_i = \Pi_{\mathcal{S}}(z_i - \varsigma \mathcal{T} z_i), \\ u_{i+1} = y_i - \varsigma(\mathcal{T} y_i - \mathcal{T} z_i). \end{cases} \quad (3)$$

Here,  $\{\vartheta_i\}$  is a non-increasing sequence,  $\vartheta_i \in (0, 1)$ ,  $i = 1, 2, \dots$ , and the form of  $z_i$  is the inertial term. The insertion of the inertia term makes the algorithm superior to the original Tseng algorithm in terms of convergence speed. However, the projection operator itself is difficult to calculate. In addition, the parameter of the algorithm is constant, if the constant parameter can be modified to the adaptive parameter, which is undoubtedly a further improvement of the algorithm iteration speed.

In order to overcome the difficulty of projection operator calculation, construct adaptive parameter to increase the iterative efficiency of the iterative algorithm and apply the variational inequality iterative algorithm to solve the equilibrium problem, we modify the constant parameters in the Thong method. In addition, the format of the Thong method is improved, that is, a subdifferentiable convex functional  $\phi(u)$  is found to replace the non-empty closed convex  $\mathcal{S}$  (Censor first proposed this idea in [19]), and the gradient projection format is modified to the subgradient projection format, thereby reducing the difficulty of projection calculation to a certain extent. Finally, this paper gives a specific example of the supply chain management, and uses our method, the Tseng method and Thong method to solve the example. Numerical analysis shows the algorithm in this article performs better than those available.

The rest of this paper is organized as follows: We introduce the supply chain network equilibrium model proposed by Zhou [20] and Li et al. [21], and transform the model into a variational inequality problem in order to facilitate the use of the iterative method in this paper to solve it in Section 2. Section 3 presents some important concepts and lemma to prove algorithmic convergence. In Section 4, the new Tseng subgradient extragradient projection iterative algorithm constructed in this paper, and the convergence proof of the algorithm are given. In Section 5, we give a specific example of the equilibrium model of the supply chain network based on the actual situation, and solve it by using the algorithm of this paper, the Tseng method and the Thong method. The calculation results and the performance data of the three algorithms are presented at the end of the section. Finally, a summary of the work in this article is given in Section 6.

## 2. Supply chain network equilibrium model

This section mainly introduces the equilibrium model of supply chain networks in [21], and the process of transforming this model into variational inequality problems. In fact, there is much more research on supply chain management. For example, Moosavi et al. gave recognizing potential disruption management strategies in the COVID-19 pandemic in [22]. This paper considers systematic literature survey studies that aim to identify promising supply chains disruption management strategies through bibliometric, network and thematic analyses. Soon after,

Soleimani et al. [23] discussed the design of a sustainable closed-loop supply chain, including suppliers, considering energy consumption. In addition, distribution centers also act as warehouses and collection centers. The issues involved remanufacturing, recycling and disposing of returned items. Goals include total profits, energy consumption and the number of jobs created. Fattahi and Govindan [24] address the dynamic design of supply chain networks, in which the moments of demand distribution function are uncertain and facilities' availability is stochastic because of possible disruptions, and proposes a data-driven rolling horizontal method that utilizes the observation of random parameters in stochastic optimization problems. Asghari et al. [25] has developed a new optimization model based on pricing and advertising decisions in the closed-loop supply chain of direct sales, and improved the standard particle swarm optimization algorithm using swarm learning theory. In the context of carbon emissions, the Nagurney et al. [26] study considers the decision-making research of the closed-loop supply chain with third-party recycling, and government rewards and punishments, which has certain reference value for enterprise competition decision-making and government policy formulation.

### 2.1. Manufacturer's profit model

Suppose  $m$  manufacturers can produce  $t$  products, as well as  $n$  distributors and  $o$  demand markets in the model. For convenience, we use  $i, j, k, l$  to indicate manufacturers, distributors, demand markets and products. The behavioral analysis of manufacturers and their profit model are as follows:

$$\max \sum_{j=1}^n \sum_{l=1}^t p_{1ijl} \mathfrak{Q}_{1ijl} - \sum_{l=1}^t f_{il}(\mathfrak{Q}_1) - \sum_{j=1}^n \sum_{l=1}^t c_{1ijl}(\mathfrak{Q}_{1ijl}) \text{ s.t. } \mathfrak{Q}_{1ijl} \geq 0, \forall j, l. \quad (4)$$

Here,  $p_{1ijl}$  represents price of the  $l$ th product offered by the  $i$ th manufacturer to the  $j$ th distributor.  $\mathfrak{Q}_{1ijl} \in R_+^{mnt}$  represents the quantity of the  $l$ th product supplied by the  $i$ th manufacturer to the  $j$ th distributor, which is the constituent vector  $\mathfrak{Q}_1$ .  $f_{il}(\mathfrak{Q}_1)$  represents the manufacturer's cost function.  $c_{1ijl}(\mathfrak{Q}_{1ijl})$  represents the shipping cost required to ship  $\mathfrak{Q}_{1ijl}$   $l$ th product from the  $i$ th manufacturer to the  $j$ th distributor.

### 2.2. Distributor's profit model

Suppose  $\mathfrak{Q}_{2jkl}$  represents the number of products offered by the  $j$ th distributor to the  $k$ th demand market,  $c_{2jl}(\mathfrak{Q}_1)$  represents the cost of the distributor,  $p_{2jkl}$  represents the selling price of the  $l$ th product offered in the  $k$ th market. The profit model is as follows:

$$\begin{aligned} \max & \sum_{k=1}^o \sum_{l=1}^t p_{2jkl} \mathfrak{Q}_{2jkl} - \sum_{l=1}^t c_{2jl}(\mathfrak{Q}_1) - \sum_{i=1}^m \sum_{l=1}^t p_{1ijl} \mathfrak{Q}_{1ijl} \\ \text{s.t. } & \sum_{i=1}^m \mathfrak{Q}_{1ijl} \geq \sum_{k=1}^o \mathfrak{Q}_{2jkl}, \mathfrak{Q}_{1ijl} \geq 0, \mathfrak{Q}_{2jkl} \geq 0, \forall i, k, l. \end{aligned} \quad (5)$$

### 2.3. Demand-market equilibrium model

Suppose  $c_{3jkl}$  is the cost incurred during transportation,  $p_{3kl}$  represents the unit selling price of the  $l$ th product in the  $k$ th demand market, and  $d_{3kl}$  represents the corresponding demand. The equilibrium

model is as follows:

$$p_{2jkl} + c_{3jkl}(\mathfrak{Q}_2) \begin{cases} \geq p_{3kl}, \text{ as } \mathfrak{Q}_{2jkl} = 0, \\ = p_{3kl}, \text{ as } \mathfrak{Q}_{2jkl} \geq 0. \end{cases} \quad (6)$$

$$d_{3jkl}(p_{3l}) \begin{cases} \leq \sum_{j=1}^n \mathfrak{Q}_{2jkl}, \text{ as } p_{3kl} = 0, \\ = \sum_{j=1}^n \mathfrak{Q}_{2jkl}, \text{ as } p_{3kl} \geq 0. \end{cases} \quad (7)$$

Now, we transform the equilibrium model of supply chain networks into a variational inequality model. First, we give the Lemma 1.

**Lemma 1** ([20], Lemma 1.3.1). *Let  $\mathcal{F}$  be continuous and differentiable, the set  $\mathcal{S}$  non-empty closed convex. Then,  $u$  is the solution to the optimization problem  $\mathcal{F}(u) = \max_{v \in \mathcal{S}} \mathcal{F}(v)$  if and only if  $u$  is also the solution of the variational inequality problem, that is,  $u \in VI(\mathcal{S}, \nabla \mathcal{F})$ . Here,  $VI(\mathcal{S}, \nabla \mathcal{F})$  denotes the solution set of the variational inequality problem.*

According to Lemma 1 and the equilibrium model of the supply chain network, we can also obtain the corresponding variational inequality model, which is Theorem 2.

**Theorem 2** ([20], Theorem 6.2.1). *Solving the Dr. Zhou's supply chain network equilibrium model in this section is equivalent to solving the variational inequality problem, that is, finding the  $u \in \mathcal{S} = R_+^{m+n+o}$ , such that  $\langle \mathcal{T}u, v - u \rangle \geq 0, \forall v \in \mathcal{S}$ . Here,*

$$\mathcal{T}v = \begin{pmatrix} \sum_{i=1}^m \sum_{j=1}^n \sum_{l=1}^t \left( \frac{\partial f_{il}(\mathfrak{Q}_1)}{\partial \mathfrak{Q}_{1ijl}} + \frac{\partial c_{1ijl}(\mathfrak{Q}_{1ijl})}{\partial \mathfrak{Q}_{1ijl}} + \frac{\partial c_{2jl}(\mathfrak{Q}_1)}{\partial \mathfrak{Q}_{1ijl}} - \rho_{jl} \right) \\ \sum_{j=1}^n \sum_{k=1}^o \sum_{l=1}^t \left( c_{3jkl}(\mathfrak{Q}_2) + \rho_{jl} - p_{3jkl} \right) \\ \sum_{j=1}^n \sum_{l=1}^t \left( \sum_{i=1}^m \mathfrak{Q}_{1ijl} - \sum_{k=1}^o \mathfrak{Q}_{2jkl} \right) \\ \sum_{k=1}^o \sum_{l=1}^t \left( \sum_{j=1}^n \mathfrak{Q}_{2jkl} - d_{kl}(p_{3kl}) \right) \end{pmatrix}, v = \begin{pmatrix} \mathfrak{Q}_{1ijl} \\ \mathfrak{Q}_{2jkl} \\ \rho_{jl} \\ p_{3kl} \end{pmatrix}. \quad (8)$$

The  $\rho_{jl}$  is a Lagrange multiplier of the profit model of distributors in the second part of the network equilibrium problem, and its meaning is equivalent to the price at which distributor  $j$  sells the product  $l$  in equilibrium.

### 3. Preliminaries

The section gives some definitions and lemmas, which will be used in Section 4. Let  $\|\cdot\|$  be norm in  $\mathcal{H}$ , then we have

$$\|ku + (1 - k)v\|^2 = k\|u\|^2 + (1 - k)\|v\|^2 - k(1 - k)\|u - v\|^2 \quad (9)$$

and

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v \rangle, \forall u, v \in \mathcal{H}, k \in R \quad (10)$$

in [17].

**Definition 3** ([27], Definition 2.1). *Let  $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$  be an operator.*

- $\mathcal{T}$  is said to be  $\mathcal{L}$ -Lipschitz continuous with  $\mathcal{L} > 0$  if

$$\|\mathcal{T}u - \mathcal{T}v\| \leq \mathcal{L}\|u - v\|, \forall u, v \in \mathcal{H}. \quad (11)$$

If  $\mathcal{L} < 1$ , then  $\mathcal{T}$  is contractive. If  $\mathcal{L} = 1$ , then  $\mathcal{T}$  is nonexpansive.

- $\mathcal{T}$  is said to be monotone if

$$\langle \mathcal{T}u - \mathcal{T}v, u - v \rangle \geq 0, \forall u, v \in \mathcal{H}. \quad (12)$$

In addition, the definition of the quasi-strong monotonic operator  $\mathcal{T}$  is as follows.

$$\langle \mathcal{T}u - \mathcal{T}v, u - v \rangle \geq \gamma \|\mathcal{T}u - \mathcal{T}v\|^2, \gamma > 0, \forall u, v \in \mathcal{H}. \quad (13)$$

**Lemma 4** ([28], Theorem 1.2.4). *Let  $\mathcal{S}$  be a nonempty closed convex subset of a real Hilbert space  $\mathcal{H}$ , and let  $\Pi_{\mathcal{S}}$  be a projection operator from  $\mathcal{H}$  to  $\mathcal{S}$ . Given  $u \in \mathcal{H}$  and  $z \in \mathcal{S}$ , then  $z = \Pi_{\mathcal{S}}u \Leftrightarrow \langle u - z, z - v \rangle \geq 0, \forall v \in \mathcal{S}$ , and  $\Pi_{\mathcal{S}}(u) = \arg \min_{c \in \mathcal{S}} \|u - c\|$ .*

**Lemma 5** ([29], Lemma 2.2, Opial's Theorem). *Let  $\emptyset \neq \mathcal{S} \subset \mathcal{H}$  and  $\{u_i\}$  be the sequence in  $\mathcal{H}$  such that*

- $\forall u \in \mathcal{S}, \lim_{i \rightarrow \infty} \|u_i - u\|$  exist.
- Every point of weakly gathering in the sequence  $\{u_i\}$  is in  $\mathcal{S}$ .

Then,  $u_i$  weakly converges to a point  $u$  in  $\mathcal{S}$ .

**Lemma 6** ([30], Lemma 1.2). *Let  $\{a_i\}$  and  $\{\varsigma_i\}$  be nonnegative real sequences with  $\varsigma_i \in (0, 1)$  and  $\sum_{i=0}^{\infty} \varsigma_i = \infty$ . If there exist a sequence  $\{b_i\}$  with  $\limsup_{i \rightarrow \infty} b_i < 0$ , and  $0 < M \in N$ , such that  $a_{i+1} \leq (1 - \varsigma_i)a_i + \varsigma_i b_i$  for all  $i \geq M$ , then  $\lim_{i \rightarrow \infty} a_i = 0$ .*

#### 4. New Tseng subgradient extragradient iterative method

Let  $\mathcal{T}$  be a non-expansion operator and  $\Pi_{\mathcal{S}_i}$  be a subgradient projection operator.  $\mathcal{S} := \{u \in \mathcal{H} | \phi(u) \leq 0\}$  and  $\mathcal{S}_i := \{a \in \mathcal{H} | \phi(u_i) + \langle g_{\phi_i}(u_i), a - u_i \rangle \leq 0\}$ ,  $g_{\phi_i} \in \partial\phi(u_i)$ . Here,

$$\partial\phi(u) := \{g_{\phi_i} \in \mathcal{H} | \phi(v) \geq \phi(u) + \langle g_{\phi_i}, v - u \rangle, \forall v \in \mathcal{H}\}.$$

First, we introduce the following algorithm.

**Algorithm 7.** *Let  $\{\vartheta_i\}_{i \geq 0}$  be a non-increasing positive sequence,  $\varsigma_0 \in_R (0, 1)$ .*

*Step 1. For given  $\{u_i\}$ , compute the sequence  $\{z_i\}$  by*

$$z_i = u_i + \vartheta_i(u_i - u_{i-1}).$$

*Step 2. Compute*

$$y_i = \Pi_{\mathcal{S}_i}(z_i - \varsigma_i \mathcal{T} z_i).$$

*If  $z_i = y_i$ , then  $z_i$  or  $y_i$  is the solution to the supply chain network equilibrium model. Otherwise, go to Step 3.*

Step 3. Calculate

$$u_{i+1} = y_i - \varsigma_i(\mathcal{T}y_i - \mathcal{T}z_i).$$

Step 4. Set  $i := i + 1$  and return to Step 1.

Here,

$$\varsigma_{i+1} = \begin{cases} \min \left\{ \frac{\mu \|y_i - z_i\|}{\|\mathcal{T}y_i - \mathcal{T}z_i\|}, \varsigma_i \right\}, & \text{as } \mathcal{T}y_i - \mathcal{T}z_i \neq 0, \\ \varsigma_i, & \text{as } \mathcal{T}y_i - \mathcal{T}z_i = 0, \end{cases} \quad \mu \in_R (0, 1).$$

**Lemma 8.** Let  $\{\varsigma_i\}$  be a sequence obtained by Algorithm 7, then the sequence  $\{\varsigma_i\}$  is non-increasing and its lower bound is  $\min \{\mu \mathcal{L}^{-1}, \varsigma_0\}$ .

*Proof.* According to the definition of  $\{\varsigma_i\}$  and operator  $\mathcal{T}$ , we have that

$$\frac{\mu \|y_i - z_i\|}{\|\mathcal{T}y_i - \mathcal{T}z_i\|} \geq \frac{\mu \|y_i - z_i\|}{\mathcal{L} \|y_i - z_i\|} = \frac{\mu}{\mathcal{L}}.$$

In addition, we know that  $\{\varsigma_i\}$  is non-increasing by the definition. So, lower bound of the sequence  $\{\varsigma_i\}$  is  $\min \{\mu \mathcal{L}^{-1}, \varsigma_0\}$ .  $\square$

**Theorem 9.** Let  $\{u_i\}$  be the sequence run by Algorithm 7, then  $\forall p \in VI(\mathcal{S}, \mathcal{T})$  has

$$\|u_{i+1} - p\|^2 \leq \|z_i - p\|^2 - \left(1 - \frac{\varsigma_i^2 \mu^2}{\varsigma_{i+1}^2}\right) \|y_i - z_i\|^2. \quad (14)$$

*Proof.* By the definition of the sequence  $\{u_i\}$ ,

$$\begin{aligned} \|u_{i+1} - p\|^2 &= \|y_i - \varsigma_i(\mathcal{T}y_i - \mathcal{T}z_i) - p\|^2 \\ &= \|y_i - p\|^2 + \varsigma_i^2 \|\mathcal{T}y_i - \mathcal{T}z_i\|^2 - 2\varsigma_i \langle y_i - p, \mathcal{T}y_i - \mathcal{T}z_i \rangle \\ &= \|z_i - p\|^2 + \|z_i - y_i\|^2 + 2\langle y_i - p, \mathcal{T}y_i - \mathcal{T}z_i \rangle \\ &\quad + \varsigma_i^2 \|\mathcal{T}y_i - \mathcal{T}z_i\|^2 - 2\varsigma_i \langle y_i - p, \mathcal{T}y_i - \mathcal{T}z_i \rangle \\ &= \|z_i - p\|^2 + \|z_i - y_i\|^2 - 2\langle y_i - z_i, y_i - z_i \rangle \\ &\quad + 2\langle y_i - z_i, y_i - p \rangle + \varsigma_i^2 \|\mathcal{T}y_i - \mathcal{T}z_i\|^2 \\ &\quad - 2\varsigma_i \langle y_i - p, \mathcal{T}y_i - \mathcal{T}z_i \rangle \\ &= \|z_i - p\|^2 - \|z_i - y_i\|^2 + 2\langle y_i - z_i, y_i - p \rangle \\ &\quad + \varsigma_i^2 \|\mathcal{T}y_i - \mathcal{T}z_i\|^2 - 2\varsigma_i \langle y_i - p, \mathcal{T}y_i - \mathcal{T}z_i \rangle, \end{aligned} \quad (15)$$

and

$$\langle y_i - z_i + \varsigma_i \mathcal{T}z_i, y_i - p \rangle \leq 0,$$

which is from the construction of  $y_i$ . Further,

$$\langle y_i - z_i, y_i - p \rangle \leq -\varsigma_i \langle \mathcal{T}z_i, y_i - p \rangle. \quad (16)$$

Considering (15) and (16), one can obtain

$$\begin{aligned}
\|u_{i+1} - p\|^2 &\leq \|z_i - p\|^2 - \|z_i - y_i\|^2 - 2\varsigma_i \langle \mathcal{T}z_i, y_i - p \rangle \\
&\quad + \varsigma_i^2 \|\mathcal{T}y_i - \mathcal{T}z_i\|^2 - 2\varsigma_i \langle y_i - p, \mathcal{T}y_i - \mathcal{T}z_i \rangle \\
&= \|z_i - p\|^2 - \|z_i - y_i\|^2 + \varsigma_i^2 \|\mathcal{T}y_i - \mathcal{T}z_i\|^2 - 2\varsigma_i \langle y_i - p, \mathcal{T}y_i \rangle \\
&\leq \|z_i - p\|^2 - \|z_i - y_i\|^2 + \frac{\mu^2 \varsigma_i^2}{\varsigma_{i+1}^2} \|y_i - z_i\|^2 \\
&\quad - 2\varsigma_i \langle y_i - p, \mathcal{T}y_i - \mathcal{T}p \rangle - 2\varsigma_i \langle y_i - p, \mathcal{T}p \rangle \\
&\leq \|z_i - p\|^2 - \left(1 - \frac{\mu^2 \varsigma_i^2}{\varsigma_{i+1}^2}\right) \|y_i - z_i\|^2.
\end{aligned}$$

□

**Theorem 10.** Let  $\{u_i\}$ ,  $\{y_i\}$  and  $\{z_i\}$  be the sequences run by Algorithm 7. If  $\{\vartheta_i\}$  is a non-incrementing sequence,  $0 \leq \vartheta_i \leq \vartheta$  and

$$\vartheta < \min \left\{ \frac{\sqrt{1 + 8\mathfrak{R}_i} - 1 - 2\mathfrak{R}_i}{2(1 - \mathfrak{R}_i)}, 1 \right\}, \mathfrak{R}_i := \frac{1 - \frac{\mu^2 \varsigma_i^2}{\varsigma_{i+1}^2}}{\left(1 + \frac{\mu \varsigma_i}{\varsigma_{i+1}}\right)^2}, \quad (17)$$

then  $\{u_i\}$ ,  $\{y_i\}$ ,  $\{z_i\}$ , and  $\{\|u_i - p\|\}$  are all bounded sequences, where  $p \in VI(C, \mathcal{T})$ .

*Proof.* The definition of the sequence  $\{u_i\}$  implies that

$$\|u_{i+1} - y_i\| = \|y_i - \varsigma_i(\mathcal{T}y_i - \mathcal{T}z_i) - y_i\| \leq \varsigma_i \|\mathcal{T}y_i - \mathcal{T}z_i\| \leq \frac{\mu \varsigma_i}{\varsigma_{i+1}} \|y_i - z_i\|,$$

so, we get

$$\|u_{i+1} - z_i\| \leq \|x_{i+1} - y_i\| + \|y_i - z_i\| \leq \left(1 + \frac{\mu \varsigma_i}{\varsigma_{i+1}}\right) \|y_i - z_i\|. \quad (18)$$

By Theorem 9, we derive

$$\|u_{i+1} - p\|^2 \leq \|z_i - p\|^2 - \left(1 - \frac{\mu^2 \varsigma_i^2}{\varsigma_{i+1}^2}\right) \|y_i - z_i\|^2. \quad (19)$$

Further, we have

$$\begin{aligned}
\|u_{i+1} - p\|^2 &\leq \|z_i - p\|^2 - \frac{1 - \frac{\mu^2 \varsigma_i^2}{\varsigma_{i+1}^2}}{\left(1 + \frac{\mu \varsigma_i}{\varsigma_{i+1}}\right)^2} \|u_{i+1} - z_i\|^2 \\
&= \|z_i - p\|^2 - \mathfrak{R}_i \|u_{i+1} - z_i\|^2,
\end{aligned} \quad (20)$$

and

$$\begin{aligned}
\|z_i - p\|^2 &= \|u_i + \vartheta_i(u_i - u_{i-1}) - p\|^2 \\
&= \|(1 + \vartheta_i)(u_i - p) - \vartheta_i(u_{i-1} - p)\|^2 \\
&\leq (1 + \vartheta_i) \|u_i - p\|^2 - \vartheta_i \|u_{i-1} - p\|^2 + \vartheta_i(1 + \vartheta_i) \|u_i - u_{i-1}\|^2,
\end{aligned} \quad (21)$$

which is known from the construction of  $\{z_i\}$ . In addition, considering (20) and (21), we deduce

$$\begin{aligned}\|u_{i+1} - p\|^2 &\leq (1 + \vartheta_i)\|u_i - p\|^2 - \vartheta_i\|u_{i-1} - p\|^2 + \vartheta_i(1 + \vartheta_i)\|u_i - u_{i-1}\|^2 \\ &\leq (1 + \vartheta_i)\|u_i - p\|^2 - \vartheta_i\|u_{i-1} - p\|^2 + 2\vartheta\|u_i - u_{i-1}\|^2,\end{aligned}\quad (22)$$

where,  $\vartheta_i(1 + \vartheta_i) \leq 2\vartheta$  is caused by the theorem condition  $0 \leq \vartheta_i \leq \vartheta \leq 1$  and

$$\vartheta_i \leq \vartheta, \vartheta_i^2 \leq \vartheta^2 \leq \vartheta.$$

On the other hand, we get

$$\begin{aligned}\|u_{i+1} - z_i\|^2 &= \|u_{i+1} - u_i - \vartheta_i(u_i - u_{i-1})\|^2 \\ &= \|u_{i+1} - u_i\|^2 + \vartheta_i^2\|u_i - u_{i-1}\|^2 - 2\vartheta_i\langle u_{i+1} - u_i, u_i - u_{i-1} \rangle \\ &\geq \|u_{i+1} - u_i\|^2 + \vartheta_i^2\|u_i - u_{i-1}\|^2 - 2\vartheta_i\|u_{i+1} - u_i\| \cdot \|u_i - u_{i-1}\| \\ &\geq (1 - \vartheta_i)\|u_{i+1} - u_i\|^2 + (\vartheta_i^2 - \vartheta_i)\|u_i - u_{i-1}\|^2.\end{aligned}\quad (23)$$

Substituting (21) and (23) into (20), we know that

$$\begin{aligned}\|u_{i+1} - p\|^2 &\leq (1 + \vartheta_i)\|u_i - p\|^2 - \vartheta_i\|u_{i-1} - p\|^2 + \vartheta_i(1 + \vartheta_i)\|u_i - u_{i-1}\|^2 \\ &\quad - \mathfrak{R}_i(1 - \vartheta_i)\|u_{i+1} - u_i\|^2 - \mathfrak{R}_i(\vartheta_i^2 - \vartheta_i)\|u_i - u_{i-1}\|^2 \\ &= (1 + \vartheta_i)\|u_i - p\|^2 - \vartheta_i\|u_{i-1} - p\|^2 - \mathfrak{R}_i(1 - \vartheta_i)\|u_{i+1} - u_i\|^2 \\ &\quad + [\vartheta_i(1 + \vartheta_i) - \mathfrak{R}_i(\vartheta_i^2 - \vartheta_i)]\|u_i - u_{i-1}\|^2 \\ &= (1 + \vartheta_i)\|u_i - p\|^2 - \vartheta_i\|u_{i-1} - p\|^2 - \gamma_i\|u_{i+1} - u_i\|^2 + \mu_i\|u_i - u_{i-1}\|^2.\end{aligned}\quad (24)$$

Here,  $\gamma_i := \mathfrak{R}_i(1 - \vartheta_i)$  and  $\mu_i := \vartheta_i(1 + \vartheta_i) - \mathfrak{R}_i(\vartheta_i^2 - \vartheta_i) \geq 0$ . Let  $\Phi_i := \|u_i - p\|^2 - \vartheta_i\|u_{i-1} - p\|^2 + \mu_i\|u_i - u_{i-1}\|^2$ . We obtain

$$\begin{aligned}\Phi_{i+1} - \Phi_i &= \|u_{i+1} - p\|^2 - (1 + \vartheta_{i+1})\|u_i - p\|^2 + \vartheta_i\|u_{i-1} - p\|^2 \\ &\quad + \mu_{i+1}\|u_{i+1} - u_i\|^2 - \mu_i\|u_i - u_{i-1}\|^2 \\ &\leq -(\gamma_i - \mu_{i+1})\|u_{i+1} - u_i\|^2.\end{aligned}\quad (25)$$

Because of  $0 \leq \vartheta_i \leq \vartheta_{i+1} \leq \vartheta$ , we derive

$$\begin{aligned}\gamma_i - \mu_{i+1} &= \mathfrak{R}_i(1 - \vartheta_i) - \vartheta_{i+1}(1 + \vartheta_{i+1}) + \mathfrak{R}_i(\vartheta_{i+1}^2 - \vartheta_{i+1}) \\ &\geq \mathfrak{R}_i(1 - \vartheta_{i+1}) - \vartheta_{i+1}(1 + \vartheta_{i+1}) + \mathfrak{R}_i(\vartheta_{i+1}^2 - \vartheta_{i+1}) \\ &\geq \mathfrak{R}_i(1 - \vartheta) - \vartheta(1 + \vartheta) + \mathfrak{R}_i(\vartheta^2 - \vartheta) \\ &\geq -(1 - \mathfrak{R}_i)\vartheta^2 - (1 + 2\mathfrak{R}_i)\vartheta + \mathfrak{R}_i.\end{aligned}\quad (26)$$

Additionally,

$$\vartheta < \min \left\{ \frac{\sqrt{1 + 8\mathfrak{R}_i} - 1 - 2\mathfrak{R}_i}{2(1 - \mathfrak{R}_i)}, 1 \right\} \leq \frac{\sqrt{1 + 8\mathfrak{R}_i} - 1 - 2\mathfrak{R}_i}{2(1 - \mathfrak{R}_i)},$$

so,  $-(1 - \mathfrak{R}_i)\vartheta^2 - (1 + 2\mathfrak{R}_i)\vartheta + \mathfrak{R}_i \geq 0$ . According to (25) and (26),

$$\Phi_{i+1} - \Phi_i \leq -\left[ -(1 - \mathfrak{R}_i)\vartheta^2 - (1 + 2\mathfrak{R}_i)\vartheta + \mathfrak{R}_i \right] \|u_{i+1} - u_i\|^2 \leq 0. \quad (27)$$

Therefore, the sequence  $\{\Phi_i\}$  is non-incremental. Also, due to  $\mu_i \geq 0$ ,

$$\begin{aligned}\Phi_i &= \|u_i - p\|^2 - \vartheta_i \|u_{i-1} - p\|^2 + \mu_i \|u_i - u_{i-1}\|^2 \\ &\geq \|u_i - p\|^2 - \vartheta_i \|u_{i-1} - p\|^2,\end{aligned}\quad (28)$$

or

$$\begin{aligned}\|u_i - p\|^2 &\leq \vartheta_i \|u_{i-1} - p\|^2 + \Phi_i \\ &\leq \vartheta \|u_{i-1} - p\|^2 + \Phi_1 \\ &\leq \dots \leq \vartheta^i \|u_0 - p\|^2 + \Phi_1(1 + \dots + \vartheta^{i-1}) \\ &\leq \vartheta^i \|u_0 - p\|^2 + \frac{\Phi_1}{1 - \vartheta}.\end{aligned}\quad (29)$$

Similarly,

$$\begin{aligned}\Phi_{i+1} &= \|u_{i+1} - p\|^2 - \vartheta_{i+1} \|u_i - p\|^2 + \mu_{i+1} \|u_{i+1} - u_i\|^2 \\ &\geq \|u_{i+1} - p\|^2 - \vartheta_{i+1} \|u_i - p\|^2 \\ &\geq -\vartheta_{i+1} \|u_i - p\|^2.\end{aligned}\quad (30)$$

By (28), (29) and (30), the following holds:

$$-\Phi_{i+1} \leq \vartheta_{i+1} \|u_i - p\|^2 \leq \vartheta \|u_i - p\|^2 \leq \vartheta^{i+1} \|u_0 - p\|^2 + \frac{\vartheta \Phi_1}{1 - \vartheta} \leq \|u_0 - p\|^2 + \frac{\vartheta \Phi_1}{1 - \vartheta}.$$

Combine the above equation and rewrite Eq (29), we get

$$\begin{aligned}\left[ -(1 - \vartheta_i) \vartheta^2 - (1 + 2\vartheta_i) \vartheta + \vartheta_i \right] \sum_{i=1}^k \|u_{i+1} - u_i\|^2 &\leq \Phi_1 - \Phi_{k+1} \\ &\leq \vartheta^{k+1} \|u_0 - p\|^2 + \frac{\vartheta \Phi_1}{1 - \vartheta} + \Phi_1 \\ &\leq \|u_0 - p\|^2 + \frac{\vartheta \Phi_1}{1 - \vartheta} + \Phi_1 \\ &= \|u_0 - p\|^2 + \frac{\Phi_1}{1 - \vartheta}.\end{aligned}$$

This means  $\sum_{i=1}^{\infty} \|u_{i+1} - u_i\|^2 < \infty$ . So, we have  $\|u_{i+1} - u_i\| \rightarrow 0$ , as  $i \rightarrow \infty$ , and

$$\lim_{i \rightarrow \infty} \|u_{i+1} - z_i\|^2 = \lim_{i \rightarrow \infty} \left( \|u_{i+1} - u_i\|^2 + \vartheta^2 \|u_i - u_{i-1}\|^2 - 2\vartheta_i \langle u_{i+1} - u_i, u_i - u_{i-1} \rangle \right) = 0. \quad (31)$$

In addition, from Eq (19),

$$\lim_{i \rightarrow \infty} \|y_i - z_i\|^2 = \lim_{i \rightarrow \infty} \left( 1 - \frac{\mu^2 \varsigma_i^2}{\varsigma_{i+1}^2} \right)^{-1} \left( \|z_i - p\|^2 - \|u_{i+1} - p\|^2 \right) = 0. \quad (32)$$

It is easy to obtain by (31) and (32) that

$$\lim_{i \rightarrow \infty} u_i = \lim_{i \rightarrow \infty} z_i = \lim_{i \rightarrow \infty} y_i. \quad (33)$$

Further,  $\{u_i\}$ ,  $\{y_i\}$ , and  $\{z_i\}$  are all bounded sequences, so the sequence  $\{\|u_i - p\|\}$  is also bounded.  $\square$

**Theorem 11.** Let  $\{u_i\}$  be the sequence run by Algorithm 7. If  $\{\vartheta_i\}$  is a non-incrementing sequence,  $0 \leq \vartheta_i \leq \vartheta$  and

$$\vartheta < \min \left\{ \frac{\sqrt{1 + 8\mathfrak{R}_i} - 1 - 2\mathfrak{R}_i}{2(1 - \mathfrak{R}_i)}, 1 \right\}, \mathfrak{R}_i := \frac{1 - \frac{\mu^2 \varsigma_i^2}{s_{i+1}^2}}{\left(1 + \frac{\mu \varsigma_i}{s_{i+1}}\right)^2},$$

then  $\{u_i\}$  weakly converges to  $z \in VI(\mathcal{S}, \mathcal{T})$ .

*Proof.* In fact, since the sequence  $\{u_i\}$  is bounded, suppose there is a sub-sequence of  $\{u_i\}$ , as  $i \rightarrow \infty$ , so that the sub-sequence weakly converges to a point  $z \in \mathcal{H}$ . Without losing its generality, this article still uses  $\{u_i\}$  to represent this sub-sequence, i.e.,  $u_i \rightharpoonup z$ , where " $\rightharpoonup$ " denotes weak convergence.

Since  $\lim_{i \rightarrow \infty} u_i = \lim_{i \rightarrow \infty} z_i = \lim_{i \rightarrow \infty} y_i$  by Theorem 10, there are sub-sequences of  $\{z_i\}$  and  $\{y_i\}$  that converge weakly to  $z$ . Similarly,  $\{z_i\}$  and  $\{y_i\}$  are still used to represent their sub-sequences, i.e.,  $z_i \rightharpoonup z$  and  $y_i \rightharpoonup z$ . Since  $\mathcal{T}$  is a monotonic operator and  $y_i = \Pi_{\mathcal{S}_i}(z_i - \varsigma_i \mathcal{T} z_i)$ ,  $\forall a \in \mathcal{S}$ , then there is

$$\begin{aligned} 0 &\leq \langle y_i - z_i + \varsigma_i \mathcal{T} z_i, a - y_i \rangle \\ &= \langle y_i - z_i, a - y_i \rangle + \varsigma_i \langle \mathcal{T} z_i, a - y_i \rangle \\ &= \langle y_i - z_i, a - y_i \rangle + \varsigma_i \langle \mathcal{T} z_i, a - z_i \rangle + \varsigma_i \langle \mathcal{T} z_i, z_i - y_i \rangle. \end{aligned}$$

Let  $i \rightarrow \infty$ , then  $\forall a \in \mathcal{S}$ , so there is

$$\varsigma_\infty \langle \mathcal{T} z, a - z \rangle \geq 0.$$

Because of  $\varsigma_\infty \geq \min \{\mu \mathcal{L}^{-1}, \varsigma_0\} > 0$  by the Lemma 8, we have

$$\langle \mathcal{T} z, a - z \rangle \geq 0, \forall a \in \mathcal{S}.$$

Therefore, there is  $z \in VI(\mathcal{S}, \mathcal{T})$ . Thus the Opial's Theorem (Lemma 5) derivable theorem holds.  $\square$

**Corollary 12.** Let  $\{u_i\}$  be the sequence run by Algorithm 7. If  $\{\vartheta_i\}$  is a non-incrementing sequence,  $0 \leq \vartheta_i \leq \vartheta$  and

$$\vartheta < \min \left\{ \frac{\sqrt{1 + 8\mathfrak{R}_i} - 1 - 2\mathfrak{R}_i}{2(1 - \mathfrak{R}_i)}, 1 \right\}, \mathfrak{R}_i := \frac{1 - \frac{\mu^2 \varsigma_i^2}{s_{i+1}^2}}{\left(1 + \frac{\mu \varsigma_i}{s_{i+1}}\right)^2},$$

then,  $\{u_i\}$  weakly converges to the solution of Dr. Zhou's supply chain network equilibrium model. That is,  $\{u_i\}$  weakly converges to  $z \in VI(\mathcal{S} = R_+^{mmt+not+nt+ot}, \mathcal{A})$ , and

$$\mathcal{A}x = \left( \begin{array}{l} \sum_{i=1}^m \sum_{j=1}^n \sum_{l=1}^t \left( \frac{\partial f_{il}(\mathfrak{Q}_1)}{\partial \mathfrak{Q}_{1ijl}} + \frac{\partial c_{1ijl}(\mathfrak{Q}_{1ijl})}{\partial \mathfrak{Q}_{1ijl}} + \frac{\partial c_{2jl}(\mathfrak{Q}_1)}{\partial \mathfrak{Q}_{1ijl}} - \rho_{jl} \right) \\ \sum_{j=1}^n \sum_{k=1}^o \sum_{l=1}^t \left( c_{3jkl}(\mathfrak{Q}_2) + \rho_{jl} - p_{3jkl} \right) \\ \sum_{j=1}^n \sum_{l=1}^t \left( \sum_{i=1}^m \mathfrak{Q}_{1ijl} - \sum_{k=1}^o \mathfrak{Q}_{2jkl} \right) \\ \sum_{k=1}^o \sum_{l=1}^t \left( \sum_{j=1}^n \mathfrak{Q}_{2jkl} - d_{kl}(p_{3kl}) \right) \end{array} \right), x = \begin{pmatrix} \mathfrak{Q}_{1ijl} \\ \mathfrak{Q}_{2jkl} \\ \rho_{jl} \\ p_{3kl} \end{pmatrix} \in \mathcal{H} = R_+^{mmt+not+nt+ot}.$$

## 5. Solve the supply chain network equilibrium model by the new Tseng method

In this section, we give a concrete example of the equilibrium model of the supply chain network and solve it using our method in Section 4, the Thong method in [18], and Tseng method in [8]. All the projection over  $\mathcal{S}$  are computed effectively by math, numpy and matplotlib.pyplot in Python 3.8. All the programs are performed on Vostro Dell PC Desktop 11th Gen Intel(R) Core(TM) i5-11400 @ 2.60 GHz 2.59 GHz, RAM 8.00 GB.

**Table 1.** The notations.

$f_{il}(\mathfrak{Q}_1)$	Manufacturer's cost function.
$\mathfrak{Q}_1$	The constituent vector of $\mathfrak{Q}_{1ijl}$ .
$\mathfrak{Q}_{1ijl}$	Quantity of the $l$ th product supplied by the $i$ th manufacturer to the $j$ th distributor,
$\mathfrak{Q}_2$	The constituent vector of $\mathfrak{Q}_{2jkl}$ .
$\mathfrak{Q}_{2jkl}$	The number of products offered by the $j$ th distributor to the $k$ th demand market.
$c_{1ijl}(\mathfrak{Q}_{1ijl})$	The shipping cost required to ship $\mathfrak{Q}_{1ijl}$ $l$ th product from the $i$ th manufacturer to the $j$ th distributor.
$c_{2jl}(\mathfrak{Q}_1)$	The cost of the distributor.
$p_{3l}$	$o$ -dimensional vector composed of the unit selling.
$p_{3kl}$	The unit selling price of the $l$ th product in the $k$ th demand market
	price of the $l$ th product in all demand markets.
$d_{3l}$	The corresponding demand.

This example is from [20] (Example 6 in Chapter 6, Section 4). It takes into account the competition in the model compared to other examples. There are 2 manufacturers, 2 distributors and 2 demand markets. In order to increase the competitiveness of the model, the manufacturer's production costs are not only related to the number of products they produce, but also to the production of other manufacturers:

$$\begin{aligned} f_{11}(\mathfrak{Q}_1) &= 0.5\mathfrak{Q}_{1111}^2 + \mathfrak{Q}_{1111}\mathfrak{Q}_{1121} + \mathfrak{Q}_{1111}, \\ f_{12}(\mathfrak{Q}_1) &= 0.5\mathfrak{Q}_{1112}^2 + \mathfrak{Q}_{1112}\mathfrak{Q}_{1122} + \mathfrak{Q}_{1112}, \\ f_{21}(\mathfrak{Q}_1) &= 0.5\mathfrak{Q}_{1121}^2 + \mathfrak{Q}_{1111}\mathfrak{Q}_{1121} + \mathfrak{Q}_{1121}, \\ f_{22}(\mathfrak{Q}_1) &= 0.5\mathfrak{Q}_{1122}^2 + \mathfrak{Q}_{1112}\mathfrak{Q}_{1122} + \mathfrak{Q}_{1122}. \end{aligned}$$

The manufacturer's shipping cost function are as follows:

$$\begin{aligned} c_{1111}(\mathfrak{Q}_{1111}) &= 0.05\mathfrak{Q}_{1111}^2 + 2\mathfrak{Q}_{1111}, c_{1112}(\mathfrak{Q}_{1112}) = 0.05\mathfrak{Q}_{1112}^2 + 2\mathfrak{Q}_{1112}, \\ c_{1121}(\mathfrak{Q}_{1121}) &= 0.05\mathfrak{Q}_{1121}^2 + 2\mathfrak{Q}_{1121}, c_{1122}(\mathfrak{Q}_{1122}) = 0.05\mathfrak{Q}_{1122}^2 + 2\mathfrak{Q}_{1122}, \\ c_{1211}(\mathfrak{Q}_{1211}) &= 0.05\mathfrak{Q}_{1211}^2 + 2\mathfrak{Q}_{1211}, c_{1212}(\mathfrak{Q}_{1212}) = 0.05\mathfrak{Q}_{1212}^2 + 2\mathfrak{Q}_{1212}, \\ c_{1221}(\mathfrak{Q}_{1221}) &= 0.05\mathfrak{Q}_{1221}^2 + 2\mathfrak{Q}_{1221}, c_{1222}(\mathfrak{Q}_{1222}) = 0.05\mathfrak{Q}_{1222}^2 + 2\mathfrak{Q}_{1222}. \end{aligned}$$

The holding costs of distributors are as follows:

$$c_{211}(\mathfrak{Q}_1) = 0.005 \left( \sum_i^2 \mathfrak{Q}_{1i11} \right)^2, c_{212}(\mathfrak{Q}_1) = 0.005 \left( \sum_i^2 \mathfrak{Q}_{1i12} \right)^2,$$

$$c_{221}(\mathfrak{Q}_1) = 0.005 \left( \sum_i^2 \mathfrak{Q}_{1i21} \right)^2, c_{222}(\mathfrak{Q}_1) = 0.005 \left( \sum_i^2 \mathfrak{Q}_{1i22} \right)^2.$$

Distributors need to market transportation costs are as follows:

$$c_{3111}(\mathfrak{Q}_2) = 0.5\mathfrak{Q}_{2111} + 50, c_{3112}(\mathfrak{Q}_2) = 0.5\mathfrak{Q}_{2112} + 50,$$

$$c_{3121}(\mathfrak{Q}_2) = 0.5\mathfrak{Q}_{2121} + 50, c_{3122}(\mathfrak{Q}_2) = 0.5\mathfrak{Q}_{2122} + 50,$$

$$c_{3211}(\mathfrak{Q}_2) = 0.5\mathfrak{Q}_{2211} + 50, c_{3212}(\mathfrak{Q}_2) = 0.5\mathfrak{Q}_{2212} + 50,$$

$$c_{3221}(\mathfrak{Q}_2) = 0.5\mathfrak{Q}_{2221} + 50, c_{3222}(\mathfrak{Q}_2) = 0.5\mathfrak{Q}_{2222} + 50.$$

In the demand market, the demand for a product is not only affected by its own price, but also related to the selling price in other demand markets:

$$d_{11}(p_{31}) = -2p_{311} + 1.5p_{321} + 1000,$$

$$d_{12}(p_{32}) = -2p_{312} + 1.5p_{322} + 1000,$$

$$d_{21}(p_{31}) = -2p_{321} + 1.5p_{311} + 1000,$$

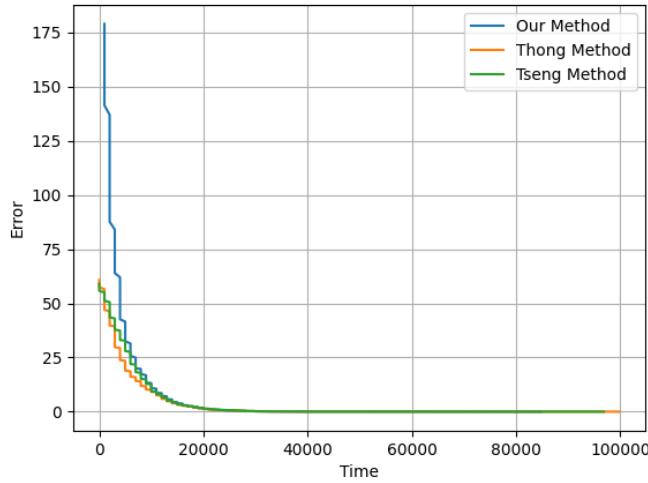
$$d_{12}(p_{32}) = -2p_{322} + 1.5p_{312} + 1000.$$

Set real Hilbert space  $\mathcal{H} := R^{2 \times 2 \times 2 + 2 \times 2 \times 2 + 2 \times 2 \times 2 = 24}$ . The subdifferential function of this experiment is  $\phi(u) := d_{R_+^{24}}^2(u)$ , and  $\mathcal{S} := \{u \in \mathcal{H} | \phi(u) = d_{R_+^{24}}^2(u) \leq 0\}$ , where  $d_{R_+^{24}}^2(u)$  represents the distance function from  $u$  to  $R_+^{24}$ . Given  $\varsigma = \mu = 0.03$ ,  $\varsigma_0 = 0.06$ ,  $\vartheta_i = n^{-1}$ , if  $\|z_i - y_i\| \leq 10^{-6}$ , then stop iteration.

Obviously, because the competition between the members of the model is considered, the balance problem solved is more complex and closer to the real situation in the real world. The optimal results obtained by our method, Thong method and Tseng method are as follows: The number of products supplied by the manufacturer to the distributor is  $\mathfrak{Q}_1^* : \mathfrak{Q}_{1111}^* = \mathfrak{Q}_{1211}^* = \mathfrak{Q}_{1121}^* = \mathfrak{Q}_{1221}^* = 225.87$ ,  $\mathfrak{Q}_{1112}^* = \mathfrak{Q}_{1212}^* = \mathfrak{Q}_{1122}^* = \mathfrak{Q}_{1222}^* = 183.33$ . The number of products offered by distributors to demand markets is  $\mathfrak{Q}_2^* : \mathfrak{Q}_{2111}^* = \mathfrak{Q}_{2211}^* = \mathfrak{Q}_{2121}^* = \mathfrak{Q}_{2221}^* = 225.87$ ,  $\mathfrak{Q}_{2112}^* = \mathfrak{Q}_{2212}^* = \mathfrak{Q}_{2122}^* = \mathfrak{Q}_{2222}^* = 183.33$ . The unit price of the product at the distributor is  $\rho_{11}^* = \rho_{21}^* = 933.58$  and  $\rho_{12}^* = \rho_{22}^* = 1125.00$ . The unit price of the product in the demand market is  $p_{311}^* = p_{321}^* = 1096.52$  and  $p_{312}^* = p_{322}^* = 1266.67$ . The data produced by the three algorithms is the same as in [20].

**Table 2.** Comparison of our method, Thong method, and Tseng method.

	CPU time(in milliseconds)	Iteration steps
Our method	80787	775
Thong method	99787	1447
Tseng method	96850	1480



**Figure 1.** Time error plot of the three methods.

The data obtained by the three methods and [20] are consistent, and the correctness of the algorithm is guaranteed. As can be seen from Figure 1, the iteration error of our algorithm lagged behind that of the Thong algorithm and the Tseng algorithm at the beginning, as our algorithm has a tendency to go beyond the Thong method and the Tseng method, as the iteration progressed. As we know from Table 2, our method does go beyond the Thong method and the Tseng method. Moreover, both CPU running time and iteration steps are better than the Thong method and the Tseng method.

This example contains two manufacturers, two distributors, and two demand markets. If it contains more elements, the set  $\mathcal{S}$  is more complex, the computational complexity of the projection operator will be higher, and the advantages of the subgradient projection operator in this paper will be more advantageous in theory.

## 6. Conclusions

We mainly study the Tseng extragradient projection iterative method for the solving supply chain network equilibrium problem, and we study and draw on the research results of variational inequality iterative algorithms in recent decades, especially the method constructed by Thong et al. In this paper, the constant parameter  $\varsigma$  in the Thong method is modified to the adaptive parameter  $\varsigma_i$ , a sub-differentiable convex operator  $\phi(u)$  is found instead of the non-empty closed convex  $\mathcal{S}$ , the gradient projection format is modified to the subgradient projection format, and the supply chain management model is derived into a variational inequality problem.

However, how to choose the appropriate sub-differential functionals to construct subgradient projection operators is also a research problem, and the sub-differential functional used in this paper is  $\phi(u) = d_{R^2}^2(u)$ . In fact, there is more than one suitable sub-differential functional. It can be seen from the examples in this paper that the speed of our algorithm is better than that of the two algorithms of the Thong method and the Tseng method, so it has certain practicality.

In this paper, the Algorithm 7 is constructed by considering the variational inequality problem itself.

In fact, we can also try to solve the supply chain network equilibrium problem through algorithms in other fields, such as online learning [31], scheduling [32, 33], multi-objective optimization [34] and others [35, 36].

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## Conflict of interest

The authors declare there is no conflict of interest.

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