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Research article

Blow-up and boundedness in quasilinear attraction-repulsion systems with nonlinear signal production

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Abstract: In this paper, we consider the quasilinear parabolic-elliptic-elliptic attraction-repulsion system

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t>0,\\ 0 = \Delta v - \mu_1(t) + f_1(u), & x\in\Omega,\ t>0,\\ 0 = \Delta w - \mu_2(t) + f_2(u), & x\in\Omega,\ t>0 \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset\mathbb{R}^n,\ n\geq 2$. The nonlinear diffusivity D and nonlinear signal productions f_1, f_2 are supposed to extend the prototypes

$$D(s) = (1+s)^{m-1},\ f_1(s) = (1+s)^{\gamma_1},\ f_2(s) = (1+s)^{\gamma_2},\ s\geq 0, \gamma_1, \gamma_2 > 0, m\in\mathbb{R}.$$

We proved that if $\gamma_1 > \gamma_2$ and $1+\gamma_1-m > \frac{2}{n}$, then the solution with initial mass concentrating enough in a small ball centered at origin will blow up in finite time. However, the system admits a global bounded classical solution for suitable smooth initial datum when $\gamma_2 < 1+\gamma_1 < \frac{2}{n}+m$.

Keywords: chemotaxis; quasilinear; attraction-repulsion; blow-up; boundedness

1. Introduction

Chemotaxis is the property of cells to move in an oriented manner in response to an increasing concentration of chemo-attractant or decreasing concentration of chemo-repellent, where the former is referred to as attractive chemotaxis and the later to repulsive chemotaxis. To begin with, it is important to study the quasilinear Keller-Segel system as follows

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(\phi(u)\nabla v), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - \alpha v + \beta u, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

subject to homogeneous Neumann boundary conditions, where the functions $D(u)$ and $\phi(u)$ denote the strength of diffusion and chemoattractant, respectively, and the function $u = u(x,t)$ idealizes the density of cell, $v = x(x,t)$ represents the concentration of the chemoattractant. Here the attractive (repulsive) chemotaxis corresponds to $\chi > 0$ ($\chi < 0$), and $|\chi| \in \mathbb{R}\backslash\{0\}$ measures the strength of chemotactic response. The parameters $\tau \in \{0,1\}$, and $\alpha, \beta > 0$ denote the production and degradation rates of the chemical. The above system describes the chemotactic interaction between cells and one chemical signal (either attractive or repulsive), and it has been investigated quite extensively on the existence of global bounded solutions or the occurrence of blow-up in finite time in the past four decades. In particular, the system (1.1) is the prototypical Keller-Segel model [1] when $D(u) = 1, \phi(u) = u$. In the case $\tau = 1$, there are many works to show that the solution is bounded [2–5], and blow-up in finite time [6–11]. If the cell's movement is much slower than the chemical signal diffusing, the second equation of (1.1) is reduced to $0 = \Delta v - M + u$, where $M := \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$ and the simplified system has many significant results [12–15].

For further information concerning nonlinear signal production, when the chemical signal function is denoted by $g(u)$, authors derived for more general nonlinear diffusive system as follows

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (\phi(u)\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - M + g(u), & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $M := \frac{1}{|\Omega|}\int_\Omega g(u(x,t))dx$. Recently, when $D(u) = u^{-p}, \phi(u) = u$ and $g(u) = u^l$, it has been shown that all solutions are global and uniformly bounded if $p + l < \frac{2}{n}$, whereas $p + l > \frac{2}{n}$ implies that the solution blows up in finite time [16]. What's more, there are many significant works [17–19] associated with this system.

Subsequently, the attraction-repulsion system has been introduced in ([20, 21]) as follows

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + \xi\nabla \cdot (u\nabla w), & x \in \Omega,\ t > 0, \\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \\ \tau_2 w_t = \Delta w + \delta u - \gamma w, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

subject to homogeneous Neumann boundary conditions, where $\chi, \xi, \alpha, \beta, \delta, \gamma > 0$ are constants, and the functions $u(x,t)$, $v(x,t)$ and $w(x,t)$ denote the cell density, the concentration of the chemoattractant and chemorepellent, respectively. The above attraction-repulsion chemotaxis system has been studied actively in recent years, and there are many significant works to be shown as follows.

For example, if $\tau_1 = \tau_2 = 0$, Perthame [22] investigated a hyperbolic Keller-Segel system with attraction and repulsion when $n = 1$. Subsequently, Tao and Wang [23] proved that the solution of (1.3) is globally bounded provided $\xi\gamma - \chi\alpha > 0$ when $n \geq 2$, and the solution would blow up in finite time provided $\xi\gamma - \chi\alpha < 0, \alpha = \beta$ when $n = 2$. Then, there is a blow-up solution when $\chi\alpha - \xi\gamma > 0, \delta \geq \beta$ or $\chi\alpha\delta - \xi\gamma\beta > 0, \delta < \beta$ for $n = 2$ [24]. Moreover, Viglialoro [25] studied the explicit lower bound of blow-up time when $n = 2$. In another hand, if $\tau_1 = 1, \tau_2 = 0$, Jin and Wang [26] showed that the solution is bounded when $n = 2$ with $\xi\gamma - \chi\alpha \geq 0$, and Zhong et al. [27] obtained the global existence of weak solution when $\xi\gamma - \chi\alpha \geq 0$ for $n = 3$. Furthermore, if $\tau_1 = \tau_2 = 1$, Liu and Wang [28] obtained the global existence of solutions, and Jin et al. [29–31] also showed a uniform-in-time bound for solutions. In addition, there are plenty of available results of the attraction-repulsion system with logistic terms [32–40], and for further information concerning (1.3) based on the nonlinear signal production, it was used to model the aggregation patterns formed by some bacterial chemotaxis in [41–43].

We turn our eyes into a multi-dimensional attraction-repulsion system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(\phi(u)\nabla v) + \xi\nabla\cdot(\psi(u)\nabla w), & x\in\Omega,\ t>0,\\ \tau_1 v_t = \Delta v - \mu_1(t) + f(u), & x\in\Omega,\ t>0,\\ \tau_2 w_t = \Delta w - \mu_2(t) + g(u), & x\in\Omega,\ t>0, \end{cases} \tag{1.4}$$

where $\Omega \in \mathbb{R}^n (n \geq 2)$ is a bounded domain with smooth boundary, $\mu_1(t) = \frac{1}{|\Omega|}\int_\Omega f(u)dx, \mu_2(t) = \frac{1}{|\Omega|}\int_\Omega g(u)dx$ and $\tau_1, \tau_2 \in \{0, 1\}$. Later on, the system (1.4) has attracted great attention of many mathematicians. In particular, when $\phi(u) = \psi(u) = u, f(u) = u^k$ and $g(u) = u^l$, Liu and Li [44] proved that all solutions are bounded if $k < \frac{2}{n}$, while blow-up occurs for $k > l$ and $k > \frac{2}{n}$ in the case $\tau_1 = \tau_2 = 0$.

Inspired by the above literature, we are devoted to deal with the quasilinear attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t>0,\\ 0 = \Delta v - \mu_1(t) + f_1(u), & x\in\Omega,\ t>0,\\ 0 = \Delta w - \mu_2(t) + f_2(u), & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.5}$$

in a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 2$ with smooth boundary, where $\frac{\partial}{\partial\nu}$ denotes outward normal derivatives on $\partial\Omega$. The function $u(x, t)$ denotes the cell density, $v(x, t)$ represents the concentration of an attractive signal (chemo-attractant), and $w(x, t)$ is the concentration of a repulsive signal (chemo-repellent). The parameters satisfy $\chi, \xi \geq 0$, which denote the strength of the attraction and repulsion, respectively. Here $\mu_1(t) = \frac{1}{|\Omega|}\int_\Omega f_1(u(x, t))dx$, $\mu_2(t) = \frac{1}{|\Omega|}\int_\Omega f_2(u(x, t))dx$, and f_1, f_2 are nonnegative Hölder continuous functions.

In the end, we propose the following assumptions on D, f_1, f_2 and u_0 for the system (1.5).

(I_1) The nonlinear diffusivity D is positive function satisfying

$$D \in C^2([0, \infty)). \tag{1.6}$$

(I_2) The function f_i is nonnegative and nondecreasing and satisfies

$$f_i \in \bigcup_{\theta\in(0,1)} C^\theta_{loc}([0, \infty)) \cap C^1((0, \infty)) \tag{1.7}$$

with $i \in \{1, 2\}$.

(I_3) The initial datum

$$u_0 \in \bigcup_{\theta\in(0,1)} C^\theta(\overline{\Omega}) \text{ is nonnegative and radially decreasing, } \frac{\partial u_0}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{1.8}$$

The goal of the article is twofold. On the one hand, we need to find out the mutual effect of the nonlinear diffusivity $D(u)$ and the nonlinear signal production $f_i(u)(i = 1, 2)$. On the other hand, we need to make a substantial step towards the dynamic of blowing up in finite time. Hence, we draw our main results concerning (1.5) read as follows.

Theorem 1.1. *Let $n \geq 2$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$ be a ball, and suppose that the function D satisfies (1.6) and f_1, f_2 are assumed to fulfill (1.7) as well as*

$$D(u) \leq d(1+u)^{m-1},\ f_1(u) \geq k_1(1+u)^{\gamma_1},\ f_2(u) \leq k_2(1+u)^{\gamma_2}\ for\ all\ u \geq 0,$$

with $m \in \mathbb{R}$, $k_1, k_2, \gamma_1, \gamma_2, d > 0$ and

$$\gamma_1 > \gamma_2 \text{ and } 1 + \gamma_1 - m > \frac{2}{n}. \tag{1.9}$$

For any $M > 0$ there exist $\varepsilon = \varepsilon(\gamma_1, M, R) \in (0, M)$ and $r^* = r^*(\gamma_1, M, R) \in (0, R)$ such that if u_0 satisfies (1.8) with

$$\int_\Omega u_0 = M \text{ and } \int_{B_{r^*}(0)} u_0 \geq M - \varepsilon,$$

then the corresponding solution of the system (1.5) blows up in finite time.

Theorem 1.2. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, and suppose that the function D satisfies (1.6) and f_1, f_2 are assumed to fulfill (1.7) as well as*

$$D(u) \geq d(1+u)^{m-1},\ f_1(u) \leq k_1(1+u)^{\gamma_1},\ f_2(u) = k_2(1+u)^{\gamma_2}\ for\ all\ u \geq 0,$$

with $m \in \mathbb{R}$, $k_1, k_2, \gamma_1, \gamma_2, d > 0$ and

$$\gamma_2 < 1 + \gamma_1 < \frac{2}{n} + m. \tag{1.10}$$

Then for each $u_0 \in \bigcup_{\theta\in(0,1)} C^\theta(\overline{\Omega})$, $u_0 \geq 0$ with $\frac{\partial u_0}{\partial \nu} = 0$ on $\partial\Omega$, and the system (1.5) admits a unique global classical solution (u, v, w) with

$$u, v, w \in C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap C^0(\overline{\Omega} \times [0, \infty)).$$

Furthermore, u, v and w are all non-negative and bounded.

The structure of this paper reads as follows: In section 2, we will show the local-in-time existence of a classical solution to the system (1.5) and some lemmas that we will use later. In section 3, we will prove Theorem 1.1 by establishing a superlinear differential inequality. In section 4, we will solve the boundedness of u in L^∞ and prove Theorem 1.2.

2. Preliminaries

Firstly, we state one result concerning local-in-time existence of a classical solution to the system (1.5). Then, we denote some new variables to transfer the original equations in (1.5) to a new system according to the ideas in [19–25]. In addition, in order to prove the main result, we will state some lemmas which will be needed later.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a bounded domain with smooth boundary. Assume that D fulfills (1.6), f_1, f_2 satisfy (1.7) and $u_0 \in \bigcup_{\theta\in(0,1)} C^\theta(\overline{\Omega})$ with $\frac{\partial u_0}{\partial \nu} = 0$ on $\partial\Omega$ as well as $u_0 \geq 0$, then there exist $T_{max} \in (0, \infty]$ and a classical solution (u, v, w) to (1.5) uniquely determined by*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ v \in \cap_{q>n} L^\infty((0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{max})), \\ w \in \cap_{q>n} L^\infty((0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{max})). \end{cases}$$

In addition, the function $u \geq 0$ in $\Omega \times (0, T_{max})$ and if $T_{max} < \infty$ then

$$\lim_{t \nearrow T_{max}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

Moreover,

$$\int_\Omega v(\cdot, t) = 0, \int_\Omega w(\cdot, t) = 0 \; for \; all \; t \in (0, T_{max}). \tag{2.2}$$

Finally, the solution (u, v, w) is radially symmetric with respect to $|x|$ if u_0 satisfies (1.8).

Proof. The proof of this lemma needs to be divided into four steps. Firstly, the method to solve the local time existence of the classical solution to the problem (1.5) is based on a standard fixed point theorem. Next, we will use the standard extension theorem to obtain (2.1). Then, we are going to use integration by parts to deduce (2.2). Finally, we would use the comparison principle to conclude that the solution is radially symmetric. For the details, we refer to [45–48]. □

For the convenience of analysis and in order to prove Theorem 1.1, we set $h = \chi v - \xi w$, then the system (1.5) is rewritten as

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u \nabla h), & x \in \Omega, \; t > 0, \\ 0 = \Delta h - \mu(t) + f(u), & x \in \Omega, \; t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial h}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.3}$$

where $\mu(t) = \chi \mu_1(t) - \xi \mu_2(t) = \frac{1}{|\Omega|} \int_\Omega f(u(x, t))dx$ and $f(u) = \chi f_1(u) - \xi f_2(u)$.

For the same reason, we will convert the system (2.3) into a scalar equation. Let us assume $\Omega = B_R(0)$ with some $R > 0$ is a ball and the initial data $u_0 = u_0(r)$ with $r = |x| \in [0, R]$ satisfies (1.8). In the radial framework, the system (2.3) can be transformed into the following form

$$\begin{cases} r^{n-1} u_t = (r^{n-1} D(u) u_r)_r - (r^{n-1} u h_r)_r, & r \in (0, R), \; t > 0, \\ 0 = (r^{n-1} h_r)_r - r^{n-1}\mu(t) + r^{n-1} f(u), & r \in (0, R), \; t > 0, \\ u_r = h_r = 0, & r = R, \; t > 0, \\ u(r, 0) = u_0(r), & r \in (0, R). \end{cases} \tag{2.4}$$

Lemma 2.2. *Let us introduce the function*

$$U(s, t) = n \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t) d\rho, \quad s = r^n \in [0, R^n], \; t \in (0, T_{max}),$$

then

$$U_s(t) = u(s^{\frac{1}{n}}, t), \; U_{ss}(t) = \frac{1}{n} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t), \tag{2.5}$$

and

$$U_t(s, t) = n^2 s^{2-\frac{2}{n}} D(U_s) U_{ss} - s\mu(t) U_s + U_s \cdot \int_0^s f(U_s(\sigma, t)) d\sigma. \tag{2.6}$$

Proof. Firstly, integrating the second equation of (2.4) over $(0, r)$, we have

$$r^{n-1}h_r(r,t) = \frac{r^n}{n}\mu(t) - \int_0^r \rho^{n-1} f(u(\rho,t))d\rho,$$

so

$$s^{1-\frac{1}{n}}h_r(s^{\frac{1}{n}},t) = \frac{s}{n}\mu(t) - \frac{1}{n}\int_0^s f(u(\sigma^{\frac{1}{n}},t))d\sigma, \quad \forall s \in (0,R^n),\ t \in (0,T_{max}).$$

Then, a direct calculation yields

$$U_s(s,t) = u(s^{\frac{1}{n}},t), \quad \forall s \in (0,R^n),\ t \in (0,T_{max}),$$

and

$$U_{ss}(s,t) = \frac{1}{n}s^{\frac{1}{n}-1}u_r(s^{\frac{1}{n}},t), \quad \forall s \in (0,R^n),\ t \in (0,T_{max}),$$

as well as

$$\begin{aligned} U_t(s,t) &= n\int_0^{s^{\frac{1}{n}}} \rho^{n-1}u_t(\rho,t)d\rho \\ &= n^2 s^{2-\frac{2}{n}}D(U_s)U_{ss} - ns^{1-\frac{1}{n}}U_s h_r \\ &= n^2 s^{2-\frac{2}{n}}D(U_s)U_{ss} - s\mu(t)U_s + U_s \cdot \int_0^s f(U_s(\sigma,t))d\sigma \end{aligned}$$

for all $s \in (0,R^n)$ and $t \in (0,T_{max})$. □

Furthermore, by a direct calculation and (1.7), we know that the functions U and f satisfy the following results

$$\begin{cases} U_s(s,t) = u(s^{\frac{1}{n}},t) > 0, & s \in (0,R^n), t \in (0,T_{max}), \\ U(0,t) = 0, \quad U(R^n,t) = \dfrac{n}{\omega_n}\displaystyle\int_\Omega u(\cdot,t) = \dfrac{nM}{\omega_n}, & t \in [0,T_{max}), \\ |f(s)|, f_1(s), f_2(s) \le C_0, & 0 \le s \le A, C_0 = C_0(A) > 0, \end{cases} \tag{2.7}$$

where $\omega_n = n|B_1(0)|$ and A is a positive constant.

Lemma 2.3. *Suppose that (1.7), (1.8) and (2.7) hold, then we have*

$$h_r(r,t) = \frac{1}{n}\mu(t)r - r^{1-n}\int_0^r \rho^{n-1}f(u(\rho,t))d\rho \qquad \text{for all } r \in (0,R), t \in (0,T_{max}).$$

In particular,

$$h_r(r,t) \le \frac{1}{n}(\mu(t) + C_0)r. \tag{2.8}$$

Proof. By integration the second equation in (2.4), we obtain that

$$r^{n-1}h_r = \mu(t)\cdot\int_0^r \rho^{n-1}d\rho - \int_0^r \rho^{n-1}f(u(\rho,t))d\rho \qquad \text{for all } r \in (0,R), t \in (0,T_{max}).$$

According to (1.9), we can easily get that $f(u) \geq 0$ if $u \geq C^* = \max\{0, (\frac{k_2\xi}{k_1\chi})^{\frac{1}{\gamma_1-\gamma_2}} - 1\}$, and split

$$\int_0^r \rho^{n-1} f(u(\rho,t))d\rho = \int_0^r \chi_{\{u(\cdot,t)\geq C^*\}}(\rho)\cdot\rho^{n-1}f(u(\rho,t))d\rho + \int_0^r \chi_{\{u(\cdot,t)<C^*\}}(\rho)\cdot\rho^{n-1}f(u(\rho,t))d\rho.$$

Combining these we have

$$\begin{aligned} h_r &= \frac{1}{n}\mu(t)r - r^{1-n}\int_0^r \chi_{\{u(\cdot,t)\geq C^*\}}(\rho)\cdot\rho^{n-1}f(u(\rho,t))d\rho - r^{1-n}\int_0^r \chi_{\{u(\cdot,t)<C^*\}}(\rho)\cdot\rho^{n-1}f(u(\rho,t))d\rho \\ &\leq \frac{1}{n}\mu(t)r - r^{1-n}\int_0^r \chi_{\{u(\cdot,t)<C^*\}}(\rho)\cdot\rho^{n-1}f(u(\rho,t))d\rho \\ &\leq \frac{1}{n}\mu(t)r + C_0r^{1-n}\int_0^r \chi_{\{u(\cdot,t)<C^*\}}(\rho)\cdot\rho^{n-1}d\rho \\ &\leq \frac{1}{n}\mu(t)r + C_0r^{1-n}\int_0^r \rho^{n-1}d\rho \\ &\leq \frac{1}{n}(\mu(t)+C_0)r, \end{aligned}$$

so we complete this proof. □

To show the existence of a finite-time blow-up solution of (2.4), we need to prove that U_{ss} is nonpositive by the following lemma. The proof follows the strategy in [48].

Lemma 2.4. *Suppose that D, f and u_0 satisfy $(I_1), (I_2)$ and (I_3) respectively. Then*

$$u_r(r,t) \leq 0 \; for \; all \; r \in (0,R), t \in (0,T_{max}). \tag{2.9}$$

Moreover,

$$U_{ss}(s,t) \leq 0 \; for \; all \; r \in (0,R), t \in (0,T_{max}). \tag{2.10}$$

Proof. Without loss of generality we may assume that $u_0 \in C^2([0,\infty))$ and $f \in C^2([0,\infty))$. Applying the regularity theory in ([49, 50]), we all know that u and u_r belong to $C^0([0,R]\times[0,T)) \cap C^{2,1}((0,R)\times(0,T))$ and we fixed $T \in (0,T_{max})$. From (2.4), we have for $r \in (0,R)$ and $t \in (0,T)$

$$h_{rr} + \frac{n-1}{r}h_r = \mu(t) - f(u), \tag{2.11}$$

and from (2.4) we obtain

$$\begin{aligned} u_{rt} &= \Big((D(u)u_r)_r + \frac{n-1}{r}D(u)u_r + uf(u) - u\mu(t) - u_rh_r\Big)_r \\ &= (D(u)u_r)_{rr} + a_1(D(u)u_r)_r + a_2u_{rr} + bu_r, \end{aligned}$$

for all $r \in (0,R)$ and $t \in (0,T)$, where

$$a_1(r,t) = \frac{n-1}{r}, \quad a_2(r,t) = -h_r, \quad b(r,t) = -\frac{n-1}{r^2}D(u) - \mu(t) - h_{rr} + f(u) + uf'(u),$$

for all $r \in (0, R)$ and $t \in (0, T)$. Moreover, we have $h_r \leq \frac{r}{n}(\mu(t) + C_0)$ by (2.8) and from (2.11) such that

$$-h_{rr} = \frac{n-1}{r}h_r - \mu(t) + f(u) \leq \frac{n-1}{n}\mu(t) + \frac{n-1}{n}C_0 - \mu(t) + f(u) \leq f(u) + C_0 \quad \text{for all } r \in (0, R) \text{ and } t \in (0, T),$$

then setting $c_1 := \sup_{(r,t)\in(0,R)\times(0,T)}(2f(u) + uf'(u) + C_0)$, we obtain

$$b(r, t) \leq c_1 \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T),$$

and we introduce

$$c_2 := \sup_{(r,t)\in(0,R)\times(0,T)} ((D(u))_{rr} + a_1(D(u))_r) < \infty,$$

and set $c_3 = 2(c_1 + c_2 + 1)$. Since $u_r(r, t) = 0$ for $r \in \{0, R\}, t \in (0, T)$ (because u is radially symmetric) and $u_{0_r} \leq 0$, the function $y : [0, R] \times [0, T] \to \mathbb{R}$, $(r, t) \mapsto u_r(r, t) - \varepsilon e^{c_3 t}$ belongs to $C^0([0, R] \times [0, T])$ and fulfills

$$\begin{cases} y_t = (D(u)(y + \varepsilon e^{c_3 t}))_{rr} + a_1(D(u)(y + \varepsilon e^{c_3 t}))_r + a_2 y_r + b(y + \varepsilon e^{c_3 t}) - c_3 \varepsilon e^{c_3 t} \\ \quad = (D(u)y)_{rr} + a_1(D(u)y)_r + a_2 y_r + by + \varepsilon e^{c_3 t}((D(u))_{rr} + a_1(D(u))_r + b - c_3) \\ \quad \leq (D(u)y)_{rr} + a_1(D(u)y)_r + a_2 y_r + by + \varepsilon e^{c_3 t}(c_1 + c_2 - c_3), & \text{in } (0, R) \times (0, T), \\ y < 0, & \text{on } \{0, R\} \times (0, T), \\ y(\cdot, 0) < 0, & \text{in } (0, R). \end{cases} \tag{2.12}$$

By the estimate for $y(\cdot, 0)$ in (2.12) and continuity of y, the time $t_0 := \sup\{t \in (0, T) : y \leq 0 \text{ in } [0, R] \times (0, T)\} \in (0, T]$ is defined. Suppose that $t_0 < T$, then there exists $r_0 \in [0, R]$ such that $y(r_0, t_0) = 0$ and $y(r, t) \leq 0$ for all $r \in [0, R]$ and $t \in [0, t_0]$; hence, $y_t(r_0, t_0) \geq 0$. As $D \geq 0$ in $[0, \infty)$, not only $y(\cdot, t_0)$ but also $z : (0, R) \to \mathbb{R}, r \longmapsto D(u(r, t_0))y(r, t_0)$ attains its maximum 0 at r_0. Since the second equality in (2.12) asserts $r_0 \in (0, R)$, we conclude $z_{rr}(r_0) \leq 0, z_r(r_0) = 0$ and $y_r(r_0, t_0) = 0$. Hence, we could obtain the contradiction

$$\begin{aligned} 0 &\leq y_t(r_0, t_0) \\ &\leq z_{rr}(r_0) + a_1(r_0, t_0)z_r(r_0) + a_2(r_0, t_0)y_r(r_0, t_0) + b(r_0, t_0)y(r_0, t_0) + \varepsilon e^{c_3 t_0}(c_1 + c_2 - c_3) \\ &\leq -\frac{c_3}{2}\varepsilon e^{c_3 t_0} < 0, \end{aligned}$$

since we have

$$c_1 + c_2 \leq \frac{c_3}{2}.$$

So that $t_0 = T$, implying $y \leq 0$ in $[0, R] \times [0, T]$ and hence $u_r \leq \varepsilon e^{c_3 t}$ in $[0, R] \times [0, T]$. Letting first $\varepsilon \searrow 0$ and then $T \nearrow T_{max}$, this proves that $u_r \leq 0$ in $[0, R] \times [0, T_{max})$, and we have $U_{ss} \leq 0$ because of (2.5). □

3. Finite-time blow-up

In this section our aim is to establish a function and to select appropriate parameters such that the function satisfies ODI, which means finiteness of T_{max} by counter evidence. Firstly, we introduce a moment-like functional as follows

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)U(s, t)ds, \quad t \in [0, T_{max}), \tag{3.1}$$

with $\gamma \in (-\infty, 1)$ and $s_0 \in (0, R^n)$. As a preparation of the subsequent analysis of ϕ, we denote

$$S_\phi := \left\{ t \in (0, T_{max}) | \phi(t) \geq \frac{nM - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \right\}. \tag{3.2}$$

The following lemma provides a lower bound for U.

Lemma 3.1. *Let $\gamma \in (-\infty, 1)$ and $s_0 \in (0, R^n)$, then*

$$U(\frac{s_0}{2}, t) \geq \frac{1}{\omega_n} \cdot (nM - \frac{4s_0}{2^\gamma(3-\gamma)}). \tag{3.3}$$

Proof. If (3.3) was false for some $t \in S_\phi$ such that $U(\frac{s_0}{2}, t) < \frac{1}{\omega_n} \cdot (nM - \frac{4s_0}{2^\gamma(3-\gamma)})$, then necessarily $\delta := \frac{4s_0}{2^\gamma(3-\gamma)} < nM$. By the monotonicity of $U(\cdot, t)$ we would obtain that $U(s,t) < \frac{nM-\delta}{\omega_n}$ for all $s \in (0, \frac{s_0}{2})$. Since $U(s,t) < \frac{nM}{\omega_n}$ for all $s \in (0, R^n)$, we have

$$\begin{aligned}
\phi(t) &< \frac{nM-\delta}{\omega_n} \cdot \int_0^{\frac{s_0}{2}} s^{-\gamma}(s_0 - s) ds + \frac{nM}{\omega_n} \cdot \int_{\frac{s_0}{2}}^{s_0} s^{-\gamma}(s_0 - s) ds \\
&= \frac{nM}{\omega_n} \cdot \int_0^{s_0} s^{-\gamma}(s_0 - s) ds - \frac{\delta}{\omega_n} \cdot \int_0^{\frac{s_0}{2}} s^{-\gamma}(s_0 - s) ds \\
&= \frac{nM}{\omega_n} \cdot \frac{s_0^{2-\gamma}}{(1-\gamma)(2-\gamma)} - \frac{\delta}{\omega_n} \cdot \frac{2^\gamma(3-\gamma)s_0^{2-\gamma}}{4(1-\gamma)(2-\gamma)}.
\end{aligned}$$

In view of the definition of S_ϕ, we find that $nM - s_0 < nM - \frac{2^\gamma(3-\gamma)\delta}{4}$, which contradicts our definition of δ. □

An upper bound for μ is established by the following lemma.

Lemma 3.2. *Let $\gamma \in (-\infty, 1)$ and $s_0 > 0$ such that $s_0 \leq \frac{R^n}{6}$. Then the function $\mu(t)$ has property that*

$$\mu(t) \leq C_1 + \frac{1}{2s} \int_0^s f(U_s(\sigma, t)) d\sigma \qquad \textit{for all } s \in (0, s_0) \textit{ and any } t \in S_\phi, \tag{3.4}$$

where $C_1 = \frac{\frac{\chi}{2}C_0 + C_0 + C_2}{3} + C_3 = \frac{1}{3}\left(\frac{\chi}{2}C_0 + C_0 + \frac{\chi k_1(\gamma_1 - \gamma_2)}{2\gamma_2}(\frac{2\xi k_2 \gamma_2}{\chi k_1 \gamma_1})^{\frac{\gamma_1}{\gamma_1 - \gamma_2}}\right) + \chi f_1\left(\frac{2\delta}{\omega_n s_0}\right)$.

Proof. First for any fixed $t \in S_\phi$, we may invoke Lemma 3.1 to see that

$$U(\frac{s_0}{2}, t) \geq \frac{nM - \delta}{\omega_n},$$

and thus, as $U \leq \frac{nM}{\omega_n}$,

$$\frac{U(s_0, t) - U(\frac{s_0}{2}, t)}{\frac{s_0}{2}} \leq \frac{\frac{nM}{\omega_n} - \frac{nM-\delta}{\omega_n}}{\frac{s_0}{2}} = \frac{2\delta}{\omega_n s_0}.$$

However, by concavity of $U(\cdot, t)$, as asserted by Lemma 2.4,

$$\frac{U(s_0, t) - U(\frac{s_0}{2}, t)}{\frac{s_0}{2}} \geq U_s(s_0, t) \geq U_s(s, t) \qquad \text{for all } s \in (s_0, R^n).$$

Then let $s_0 \in (0, R^n)$, we know that

$$\begin{aligned}\mu(t) &= \frac{1}{R^n}\int_0^{s_0} f(U_s(\sigma,t))d\sigma + \frac{1}{R^n}\int_{s_0}^{R^n} f(U_s(\sigma,t))d\sigma \\ &= \frac{1}{R^n}\int_0^{s} f(U_s(\sigma,t))d\sigma + \frac{1}{R^n}\int_{s}^{s_0} f(U_s(\sigma,t))d\sigma + \frac{1}{R^n}\int_{s_0}^{R^n} f(U_s(\sigma,t))d\sigma, \forall t \in (0, T_{max}). \quad (3.5)\end{aligned}$$

Since $\gamma_1 > \gamma_2$ and Young's inequality such that $\xi f_2(u) \le \xi k_2(1+u)^{\gamma_2} \le \frac{\chi k_1}{2}(1+u)^{\gamma_1} + C_2 \le \frac{\chi}{2} f_1(u) + C_2$ with $C_2 = \frac{\chi k_1(\gamma_1-\gamma_2)}{2\gamma_2}(\frac{2\xi k_2\gamma_2}{\chi k_1\gamma_1})^{\frac{\gamma_1}{\gamma_1-\gamma_2}}$ for $u \ge 0$, then for all $s \in (0, R^n)$ and $t \in (0, T_{max})$ we show that

$$\frac{\chi}{2} f_1(U_s(s,t)) - C_2 \le f(U_s(s,t)) \le \chi f_1(U_s(s,t)). \quad (3.6)$$

Accordingly, by the monotonicity of $U_s(\cdot, t)$ along with (1.7) and (3.6), we have

$$\begin{aligned}\int_s^{s_0} f(U_s(\sigma,t))d\sigma &\le \int_s^{s_0} \chi f_1(U_s(\sigma,t))d\sigma \\ &\le \int_s^{s_0} \chi f_1(U_s(s,t))d\sigma \\ &\le s_0\chi f_1(U_s(s,t)), \qquad \forall s \in (0, s_0),\ t \in (0, T_{max}).\end{aligned}$$

Since the condition of (2.7) implies that

$$\begin{aligned}\int_0^s f(U_s(\sigma,t))d\sigma &= \int_0^s \chi_{\{U_s(\cdot,t)\ge 1\}}(\sigma)\cdot f(U_s(\sigma,t)d\sigma + \int_0^s \chi_{\{U_s(\cdot,t)<1\}}(\sigma)\cdot f(U_s(\sigma,t)d\sigma \\ &\ge \int_0^s \chi_{\{U_s(\cdot,t)\ge 1\}}(\sigma)\cdot \Big(\frac{\chi}{2} f_1(U_s(\sigma,t)) - C_2\Big)d\sigma - C_0 s \\ &\ge \int_0^s \chi_{\{U_s(\cdot,t)\ge 1\}}(\sigma)\cdot \frac{\chi}{2} f_1(U_s(\sigma,t))d\sigma - (C_0 + C_2)s \\ &= \int_0^s \frac{\chi}{2} f_1(U_s(\sigma,t))d\sigma - \int_0^s \chi_{\{U_s(\cdot,t)<1\}}(\sigma)\cdot \frac{\chi}{2} f_1(U_s(\sigma,t))d\sigma - (C_0 + C_2)s \\ &\ge \int_0^s \frac{\chi}{2} f_1(U_s(s,t))d\sigma - \frac{\chi}{2}C_0 s - (C_0 + C_2)s \\ &\ge \frac{s\chi}{2} f_1(U_s(s,t)) - (\frac{\chi}{2}C_0 + C_0 + C_2)s.\end{aligned}$$

Therefore, we obtain

$$\int_s^{s_0} f(U_s(\sigma,t))d\sigma \le \frac{2s_0}{s}\int_0^s f(U_s(\sigma,t))d\sigma + 2(\frac{\chi}{2}C_0 + C_0 + C_2)s_0.$$

Since (3.5) we have for all $s \in (0, s_0)$

$$\begin{aligned}&\frac{1}{R^n}\int_0^s f(U_s(\sigma,t))d\sigma + \frac{1}{R^n}\int_s^{s_0} f(U_s(\sigma,t))d\sigma \\ &\le \frac{1}{R^n}\int_0^s f(U_s(\sigma,t))d\sigma + \frac{2s_0}{R^n s}\int_0^s f(U_s(\sigma,t))d\sigma + \frac{2(\frac{\chi}{2}C_0 + C_0 + C_2)s_0}{R^n}, \quad (3.7)\end{aligned}$$

where $s_0 \le \frac{R^n}{6}$ such that $\frac{1}{R^n} \le \frac{1}{6s_0} \le \frac{1}{6s}, \frac{2s_0}{R^n s} \le \frac{1}{3s}$ and $\frac{s_0}{R^n} \le \frac{1}{6}$ for all $s \in (0, s_0)$. Finally, we estimate the last summand of (3.5)

$$\frac{1}{R^n}\int_{s_0}^{R^n} f(U_s(\sigma,t))d\sigma \le \frac{1}{R^n}\int_{s_0}^{R^n} \chi f_1(U_s(\sigma,t))d\sigma \le \chi f_1\Big(\frac{2\delta}{\omega_n s_0}\Big) = C_3. \tag{3.8}$$

Together with (3.5), (3.7) and (3.8) imply (3.4). □

Lemma 3.3. *Assume that $\gamma \in (-\infty, 1)$ satisfying*

$$\gamma < 2 - \frac{2}{n},$$

and $s_0 \in (0, \frac{R^n}{6}]$. Then the function $\phi : [0, T_{max}) \to \mathbb{R}$ defined by (3.1) belongs to $C^0([0, T_{max})) \cap C^1((0, T_{max}))$ and satisfies

$$\begin{aligned}\phi'(t) \ge{}& n^2\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)U_{ss}D(U_s)ds\\ &+\frac{1}{2}\int_0^{s_0} s^{-\gamma}(s_0 - s)U_s\cdot\Big\{\int_0^s f(U_s(\sigma,t))d\sigma\Big\}ds - C_1\int_0^{s_0} s^{1-\gamma}(s_0 - s)U_s ds\\ =:{}& J_1(t) + J_2(t) + J_3(t),\end{aligned} \tag{3.9}$$

for all $t \in [0, T_{max})$, where C_1 is defined in Lemma 3.2.

Proof. Combining (2.6) and (3.4) we have

$$\begin{aligned}U_t(s,t) &= n^2 s^{2-\frac{2}{n}} D(U_s)U_{ss} - s\mu(t)U_s + U_s\cdot\int_0^s f(U_s(\sigma,t))d\sigma\\ &\ge n^2 s^{2-\frac{2}{n}} U_{ss}D(U_s) + \frac{1}{2}U_s\cdot\int_0^s f(U_s(\sigma,t))d\sigma - C_1 sU_s.\end{aligned}$$

Notice $\phi(t)$ conforms $\phi(t) = \int_0^{s_0} s^{-\gamma}(s_0 - s)U(s,t)ds$. So (3.9) is a direct consequence. □

Lemma 3.4. *Let $s_0 \in (0, \frac{R^n}{6}]$, and $\gamma \in (-\infty, 1)$ satisfying $\gamma < 2 - \frac{2}{n}$. Then $J_1(t)$ in* (3.9) *satisfies*

$$J_1(t) \ge -I, \tag{3.10}$$

where

$$I := \begin{cases} -\dfrac{n^2 d}{m}(2 - \dfrac{2}{n} - \gamma)\displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s), & m < 0,\\ n^2 d(2 - \dfrac{2}{n} - \gamma)\displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)\ln(U_s + 1), & m = 0,\\ \dfrac{n^2 d}{m}(2 - \dfrac{2}{n} - \gamma)\displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)(U_s + 1)^m, & m > 0,\end{cases} \tag{3.11}$$

for all $t \in S_\phi$.

Proof. Since $D \in C^2([0,\infty))$, suppose that

$$G(\tau) = \int_0^\tau D(\delta)d\delta,$$

then

$$0 < G(\tau) \le d\int_0^\tau (1+\delta)^{m-1}d\delta \le \begin{cases} \dfrac{-d}{m}, & m<0,\\ d\ln(\tau+1), & m=0,\\ \dfrac{d}{m}(\tau+1)^m, & m>0. \end{cases}$$

Here integrating by parts we obtain

$$\begin{aligned} J_1(t) &= n^2\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)dG(U_s)\\ &= n^2 s^{2-\frac{2}{n}-\gamma}(s_0-s)G(U_s)|_0^{s_0} + n^2\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}G(U_s)ds\\ &\quad - n^2(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)G(U_s)ds. \end{aligned}$$

Hence a direct calculation yields

$$J_1(t) \ge \begin{cases} \dfrac{n^2 d}{m}(2-\dfrac{2}{n}-\gamma)\displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s), & m<0,\\ -n^2 d(2-\dfrac{2}{n}-\gamma)\displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)\ln(U_s+1), & m=0,\\ -\dfrac{n^2 d}{m}(2-\dfrac{2}{n}-\gamma)\displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)(U_s+1)^m, & m>0, \end{cases}$$

for all $t \in S_\phi$. We conclude (3.10). □

Lemma 3.5. *Assume that $\gamma \in (-\infty, 1)$ satisfying $\gamma < 2-\frac{2}{n}$ and $s_0 \in (0, \frac{R^n}{6}]$. Then we have*

$$J_2(t) + J_3(t) \ge \frac{k_1\chi}{4}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{1+\gamma_1}ds - C_4\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s ds \tag{3.12}$$

for all $t \in S_\phi$, where $C_4 = C_1 + \frac{(\frac{\chi}{2}C_0+C_0+C_2)}{2}$.

Proof. Since Lemma 3.2 we have

$$\int_0^s f(U_s(\sigma,t))d\sigma \ge \frac{s}{2}\chi f_1(U_s(s,t)) - (\frac{\chi}{2}C_0 + C_0 + C_2)s \qquad \text{for all } s\in(0,s_0) \text{ and } t\in(0,T_{max}).$$

Therefore,

$$J_2(t) = \frac{1}{2}\int_0^{s_0} s^{-\gamma}(s_0-s)U_s\cdot\Big\{\int_0^s f(U_s(\sigma,t))d\sigma\Big\}ds$$

$$\geq \frac{\chi}{4}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s f_1(U_s(s,t))ds - \frac{(\frac{\chi}{2}C_0+C_0+C_2)}{2}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s ds$$
$$\geq \frac{k_1\chi}{4}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{1+\gamma_1}ds - \frac{(\frac{\chi}{2}C_0+C_0+C_2)}{2}\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s ds,$$

where $f_1(U_s(s,t)) \geq k_1(1+U_s)^{\gamma_1} \geq k_1(U_s)^{\gamma_1}$. Combining these inequalities we can deduce (3.12). □

Lemma 3.6. *Let* $\gamma_1 > \max\{0, m-1\}$. *For any* $\gamma \in (-\infty, 1)$ *satisfying*

$$\gamma \in \min\left\{2-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1},\ 2-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}\right\}, \tag{3.13}$$

and $s_0 \in (0, \frac{R^n}{6}]$, *the function* $\phi : [0, T_{max}) \to \mathbb{R}$ *defined in (3.1) satisfies*

$$\phi'(t) \geq \begin{cases} C\psi(t) - Cs_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}}, & m \leq 1, \\ C\psi(t) - Cs_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}}, & m > 1, \end{cases} \tag{3.14}$$

with $C > 0$ *for all* $t \in S_\phi$, *where* $\psi(t) := \int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{1+\gamma_1}ds$.

Proof. From (3.10) and (3.12) we have

$$\phi'(t) \geq \frac{k_1\chi}{4}\psi(t) - I - C_4\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s ds, \tag{3.15}$$

for all $t \in S_\phi$ and I is given by (3.11). In the case $m < 0$,

$$-\frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)ds \leq -\frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)s_0\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}ds$$
$$= -\frac{n^2 d}{m}s_0^{3-\frac{2}{n}-\gamma}.$$

If $m = 0$, we use the fact that $\frac{\ln(1+x)}{x} < 1$ for any $x > 0$ and Hölder's inequality to estimate

$$n^2 d(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)\ln(U_s+1)ds$$
$$= n^2 d(2-\frac{2}{n}-\gamma)\int_0^{s_0}[s^{1-\gamma}(s_0-s)U_s^{1+\gamma_1}]^{\frac{1}{1+\gamma_1}}\cdot s^{1-\frac{2}{n}-\gamma-\frac{1-\gamma}{1+\gamma_1}}(s_0-s)^{1-\frac{1}{1+\gamma_1}}\frac{\ln(1+U_s)}{U_s}ds$$
$$\leq n^2 d(2-\frac{2}{n}-\gamma)\left\{\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{1+\gamma_1}ds\right\}^{\frac{1}{1+\gamma_1}}\cdot\left\{\int_0^{s_0}\left(s^{1-\frac{2}{n}-\gamma-\frac{1-\gamma}{1+\gamma_1}}(s_0-s)^{\frac{\gamma_1}{1+\gamma_1}}\right)^{\frac{1+\gamma_1}{\gamma_1}}ds\right\}^{\frac{\gamma_1}{1+\gamma_1}}$$
$$\leq n^2 d(2-\frac{2}{n}-\gamma)s_0^{\frac{\gamma_1}{1+\gamma_1}}\left\{\int_0^{s_0} s^{\frac{(1-\frac{2}{n}-\gamma)\gamma_1-\frac{2}{n}}{\gamma_1}}ds\right\}^{\frac{\gamma_1}{1+\gamma_1}}\psi^{\frac{1}{1+\gamma_1}}(t)$$
$$= C_5\psi^{\frac{1}{1+\gamma_1}}(t)s_0^{\frac{(3-\frac{2}{n}-\gamma)\gamma_1-\frac{2}{n}}{1+\gamma_1}},$$

for all $t \in S_\phi$ with $C_5 := n^2 d(2-\frac{2}{n}-\gamma)\cdot\left(\frac{1}{2-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}}\right)^{\frac{\gamma_1}{1+\gamma_1}} > 0$ by (3.13). In the case $m > 0$, by using the elementary inequality $(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$ for all $a, b > 0$ and every $\alpha > 0$, we obtain

$$\frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)(U_s+1)^m ds$$

$$\leq 2^m \frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)U_s^m ds + 2^m \frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)ds, \qquad (3.16)$$

for all $t \in S_\phi$, and we first estimate the second term on the right of (3.16)

$$2^m \frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)ds \leq 2^m \frac{n^2 d}{m} s_0^{3-\frac{2}{n}-\gamma}.$$

Since $\gamma_1 > m-1$ and by Hölder's inequality we deduce that

$$\begin{aligned}
&2^m \frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)U_s^m ds \\
&= 2^m \frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\int_0^{s_0} s^{(1-\gamma)\cdot\frac{m}{1+\gamma_1}}(s_0-s)^{\frac{m}{1+\gamma_1}} U_s^m \cdot s^{1-\frac{2}{n}-\gamma-(1-\gamma)\cdot\frac{m}{1+\gamma_1}}(s_0-s)^{1-\frac{m}{1+\gamma_1}} ds \\
&\leq 2^m \frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\Big\{\int_0^{s_0} [s^{(1-\gamma)\cdot\frac{m}{1+\gamma_1}}(s_0-s)^{\frac{m}{1+\gamma_1}} U_s^m]^{\frac{1+\gamma_1}{m}} ds\Big\}^{\frac{m}{1+\gamma_1}} \\
&\times \Big\{\int_0^{s_0} [s^{1-\frac{2}{n}-\gamma-(1-\gamma)\cdot\frac{m}{1+\gamma_1}}(s_0-s)^{1-\frac{m}{1+\gamma_1}}]^{\frac{1+\gamma_1}{1+\gamma_1-m}} ds\Big\}^{\frac{1+\gamma_1-m}{1+\gamma_1}} \\
&\leq 2^m \frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\psi^{\frac{m}{1+\gamma_1}}(t) s_0^{\frac{1+\gamma_1-m}{1+\gamma_1}} \cdot \Big\{\int_0^{s_0} s^{\frac{(1+\gamma_1-m)(1-\gamma)-\frac{2}{n}(1+\gamma_1)}{1+\gamma_1-m}} ds\Big\}^{\frac{1+\gamma_1-m}{1+\gamma_1}} \\
&\leq C_6 \psi^{\frac{m}{1+\gamma_1}}(t) s_0^{\frac{(1+\gamma_1-m)(3-\gamma)-\frac{2}{n}(1+\gamma_1)}{1+\gamma_1}},
\end{aligned}$$

for all $t \in S_\phi$ with $C_6 = 2^m \frac{n^2 d}{m}(2-\frac{2}{n}-\gamma)\Big(\frac{1}{2-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}}\Big)^{\frac{1+\gamma_1-m}{1+\gamma_1}} > 0$ where $\gamma < 2-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}$ from (3.13).

Next, we can estimate the third expression on the right of (3.15) as follows

$$\begin{aligned}
&C_4 \int_0^{s_0} s^{1-\gamma}(s_0-s)U_s ds \\
&= C_4 \int_0^{s_0} [s^{1-\gamma}(s_0-s)U_s^{1+\gamma_1}]^{\frac{1}{1+\gamma_1}} \cdot s^{1-\gamma-\frac{1-\gamma}{1+\gamma_1}}(s_0-s)^{1-\frac{1}{1+\gamma_1}} ds \\
&\leq C_4 \Big\{\int_0^{s_0} s^{1-\gamma}(s_0-s)U_s^{1+\gamma_1} ds\Big\}^{\frac{1}{1+\gamma_1}} \cdot \Big\{\int_0^{s_0} [s^{1-\gamma-\frac{1-\gamma}{1+\gamma_1}}(s_0-s)^{1-\frac{1}{1+\gamma_1}}]^{\frac{1+\gamma_1}{\gamma_1}} ds\Big\}^{\frac{\gamma_1}{1+\gamma_1}} \\
&\leq C_4 \psi^{\frac{1}{1+\gamma_1}}(t) s_0^{\frac{\gamma_1}{1+\gamma_1}} \Big\{\int_0^{s_0} s^{1-\gamma} ds\Big\}^{\frac{\gamma_1}{1+\gamma_1}} \\
&= C_7 \psi^{\frac{1}{1+\gamma_1}}(t) s_0^{\frac{(3-\gamma)\gamma_1}{1+\gamma_1}},
\end{aligned}$$

where $C_7 = C_4(\frac{1}{2-\gamma})^{\frac{\gamma_1}{1+\gamma_1}}$ for all $t \in S_\phi$. By (3.15) and collecting the estimates above we have

$$\phi'(t) \geq \begin{cases}
\frac{k_1\chi}{4}\psi(t) + \frac{n^2 d}{m} s_0^{3-\frac{2}{n}-\gamma} - C_7 \psi^{\frac{1}{1+\gamma_1}}(t) s_0^{\frac{(3-\gamma)\gamma_1}{1+\gamma_1}}, & m < 0, t \in S_\phi, \\
\frac{k_1\chi}{4}\psi(t) - C_5 \psi^{\frac{1}{1+\gamma_1}}(t) s_0^{\frac{(3-\frac{2}{n}-\gamma)\gamma_1-\frac{2}{n}}{1+\gamma_1}} - C_7 \psi^{\frac{1}{1+\gamma_1}}(t) s_0^{\frac{(3-\gamma)\gamma_1}{1+\gamma_1}}, & m = 0, t \in S_\phi, \\
\frac{k_1\chi}{4}\psi(t) - 2^m \frac{n^2 d}{m} s_0^{3-\frac{2}{n}-\gamma} - C_6 \psi^{\frac{m}{1+\gamma_1}}(t) s_0^{\frac{(1+\gamma_1-m)(3-\gamma)-\frac{2}{n}(1+\gamma_1)}{1+\gamma_1}} - C_7 \psi^{\frac{1}{1+\gamma_1}}(t) s_0^{\frac{(3-\gamma)\gamma_1}{1+\gamma_1}}, & m > 0, t \in S_\phi.
\end{cases}$$

If $m = 0$, by Young's inequality we can find positive constants C_8, C_9 such that

$$C_5\psi^{\frac{1}{1+\gamma_1}}(t)s_0^{\frac{(3-\frac{2}{n}-\gamma)\gamma_1-\frac{2}{n}}{1+\gamma_1}} \le \frac{k_1\chi}{16}\psi(t) + C_8 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}}, \qquad \forall t \in S_\phi,$$

while as $m > 0$ we have

$$C_6\psi^{\frac{m}{1+\gamma_1}}(t)s_0^{\frac{(1+\gamma_1-m)(3-\gamma)-\frac{2}{n}(1+\gamma_1)}{1+\gamma_1}} \le \frac{k_1\chi}{16}\psi(t) + C_9 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}}, \qquad \forall t \in S_\phi.$$

On the other hand, we use Young's inequality again

$$C_7\psi^{\frac{1}{1+\gamma_1}}(t)s_0^{\frac{(3-\gamma)\gamma_1}{1+\gamma_1}} \le \frac{k_1\chi}{16}\psi(t) + C_{10}s_0^{3-\gamma}, \qquad \forall t \in S_\phi.$$

In the case $m < 0$, because of $s_0 \in (0, \frac{R^n}{6}]$, we have

$$s_0^{3-\frac{2}{n}-\gamma} = s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}} \cdot s_0^{\frac{2}{n\gamma_1}} \le (\frac{R^n}{6})^{\frac{2}{n\gamma_1}} s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}},$$

when $m > 0$ we have

$$s_0^{3-\frac{2}{n}-\gamma} = s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}} \cdot s_0^{\frac{2m}{n(1+\gamma_1-m)}} \le (\frac{R^n}{6})^{\frac{2m}{n(1+\gamma_1-m)}} s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}}.$$

All in all, we have

$$\phi'(t) \ge \begin{cases} \frac{k_1\chi}{8}\psi(t) - C_{11}s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}} - C_{10}s_0^{3-\gamma}, & m \le 0, \\ \frac{k_1\chi}{8}\psi(t) - C_{12}s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}} - C_{10}s_0^{3-\gamma}, & m > 0, \end{cases} \tag{3.17}$$

for all $t \in S_\phi$ with $C_{11} = C_8 - \frac{n^2 d}{m}(\frac{R^n}{6})^{\frac{2}{n\gamma_1}}$ and $C_{12} = C_9 + \frac{2^m n^2 d}{m}(\frac{R^n}{6})^{\frac{2m}{n(1+\gamma_1-m)}}$. When $0 < m \le 1$, we have $\frac{1+\gamma_1}{1+\gamma_1-m} \le \frac{1+\gamma_1}{\gamma_1}$ such that $s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}} = s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}} s_0^{\frac{2}{n}(\frac{1+\gamma_1}{\gamma_1}-\frac{1+\gamma_1}{1+\gamma_1-m})} \le (\frac{R^n}{6})^{\frac{2(1-m)(1+\gamma_1)}{n\gamma_1(1+\gamma_1-m)}} s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}}$. In the case $m \le 1$

$$s_0^{3-\gamma} = s_0^{\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}} \cdot s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}} \le (\frac{R^n}{6})^{\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}} \cdot s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}},$$

and if $m > 1$ we have

$$s_0^{3-\gamma} = s_0^{\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}} \cdot s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}} \le (\frac{R^n}{6})^{\frac{2}{n}\cdot\frac{1+\gamma_1}{(1+\gamma_1-m)}} \cdot s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}}.$$

Thus (3.17) turns into (3.14). □

Next, we need to build a connection between $\phi(t)$ and $\psi(t)$. Let us define

$$S_\psi := \left\{t \in (0, T_{max}) | \psi(t) \ge s_0^{3-\gamma}\right\}. \tag{3.18}$$

Lemma 3.7. *Let $\gamma \in (-\infty, 1)$ satisfying $\gamma > 1 - \gamma_1$ and (3.13). Then for any choice of $s_0 \in (0, \frac{R^n}{6}]$, the following inequality*

$$\phi'(t) \geq \begin{cases} Cs_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(t) - Cs_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}}, & m \leq 1, \\ Cs_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(t) - Cs_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}}, & m > 1, \end{cases} \tag{3.19}$$

holds for all $t \in S_\phi \cap S_\psi$ with $C > 0$.

Proof. We first split

$$\begin{aligned} U(s,t) = \int_0^s U_s(\sigma,t)d\sigma &= \int_0^s \chi_{\{U_s(\cdot,t)<1\}}(\sigma)\cdot U_s(\sigma,t)d\sigma + \int_0^s \chi_{\{U_s(\cdot,t)\geq 1\}}(\sigma)\cdot U_s(\sigma,t)d\sigma \\ &\leq s + \int_0^s \chi_{\{U_s(\cdot,t)\geq 1\}}(\sigma)\cdot\{\sigma^{1-\gamma}(s_0-\sigma)U_s^{1+\gamma_1}\}^{\frac{1}{1+\gamma_1}}\cdot\sigma^{-\frac{1-\gamma}{1+\gamma_1}}(s_0-\sigma)^{-\frac{1}{1+\gamma_1}}d\sigma \\ &\leq s + (s_0-s)^{-\frac{1}{1+\gamma_1}}\psi^{\frac{1}{1+\gamma_1}}(t)\cdot\Big\{\int_0^s \sigma^{-\frac{1-\gamma}{1+\gamma_1}\cdot\frac{1+\gamma_1}{\gamma_1}}d\sigma\Big\}^{\frac{\gamma_1}{1+\gamma_1}} \\ &= s + \Big(\frac{\gamma_1}{\gamma+\gamma_1-1}\Big)^{\frac{\gamma_1}{1+\gamma_1}}(s_0-s)^{-\frac{1}{1+\gamma_1}}s^{\frac{\gamma+\gamma_1-1}{1+\gamma_1}}\psi^{\frac{1}{1+\gamma_1}}(t), \end{aligned} \tag{3.20}$$

for all $s \in (0, s_0)$ and $t \in (0, T_{max})$ where $\frac{\gamma_1}{\gamma+\gamma_1-1} > 0$. According to the definition of S_ψ, we can find

$$\begin{aligned} \frac{s}{(s_0-s)^{-\frac{1}{1+\gamma_1}}s^{\frac{\gamma+\gamma_1-1}{1+\gamma_1}}\psi^{\frac{1}{1+\gamma_1}}(t)} &= s^{\frac{2-\gamma}{1+\gamma_1}}(s_0-s)^{\frac{1}{1+\gamma_1}}\psi^{-\frac{1}{1+\gamma_1}}(t) \\ &\leq s_0^{\frac{2-\gamma}{1+\gamma_1}}\cdot s_0^{\frac{1}{1+\gamma_1}}\cdot(s_0^{3-\gamma})^{-\frac{1}{1+\gamma_1}} = 1, \end{aligned} \tag{3.21}$$

for all $s \in (0, s_0)$ and $t \in S_\psi$. Combining (3.20) and (3.21) we have

$$U(s,t) \leq C_1 s^{\frac{\gamma+\gamma_1-1}{1+\gamma_1}}(s_0-s)^{-\frac{1}{1+\gamma_1}}\psi^{\frac{1}{1+\gamma_1}}(t),$$

where $C_1 = 1 + \Big(\frac{\gamma_1}{\gamma+\gamma_1-1}\Big)^{\frac{\gamma_1}{1+\gamma_1}}$ for all $s \in (0, s_0)$ and $t \in S_\psi$. Invoking Hölder's inequality, we get

$$\begin{aligned} \phi(t) &= \int_0^{s_0} s^{-\gamma}(s_0-s)U(s,t)ds \\ &\leq C_1\int_0^{s_0} s^{-\gamma+\frac{\gamma+\gamma_1-1}{1+\gamma_1}}(s_0-s)^{1-\frac{1}{1+\gamma_1}}ds\cdot\psi^{\frac{1}{1+\gamma_1}}(t) \\ &\leq C_1 s_0^{\frac{\gamma_1}{1+\gamma_1}}\int_0^{s_0} s^{-\gamma+\frac{\gamma+\gamma_1-1}{1+\gamma_1}}ds\cdot\psi^{\frac{1}{1+\gamma_1}}(t) \\ &= C_2 s_0^{\frac{\gamma_1(3-\gamma)}{1+\gamma_1}}\cdot\psi^{\frac{1}{1+\gamma_1}}(t), \end{aligned} \tag{3.22}$$

where $C_2 = C_1\frac{1+\gamma_1}{\gamma_1(2-\gamma)}$ for all $s \in (0, s_0)$ and $t \in S_\psi$. Employing these conclusion we deduce (3.19). □

These preparations above will enable us to establish a superlinear ODI for ϕ as mentioned earlier, and we prove our main result on blow-up based on a contradictory argument.

Proof of Theorem 1.1. Step 1. Assume on the contrary that $T_{max} = +\infty$, and we define the function

$$S := \left\{ T \in (0, +\infty) \middle| \phi(t) > \frac{nM - s_0}{\omega_n(1-\gamma)(2-\gamma)} \cdot s_0^{2-\gamma} \text{ for all } t \in [0, T] \right\}. \tag{3.23}$$

Let us choose $s_0 > 0$ such that

$$s_0 \le \min \left\{ \frac{R^n}{6}, \frac{nM}{2}, \frac{nM\gamma_1}{2(1-\gamma)\omega_n[(C_3+1)(1+\gamma_1)-1]} \right\}, \tag{3.24}$$

where M and ω_n were defined in (2.7) and $C_3 = (\frac{\gamma_1}{\gamma+\gamma_1-1})^{\frac{\gamma_1}{1+\gamma_1}}$ has been mentioned in (3.20). Then we pick $0 < \varepsilon(\gamma_1, M, R) = \varepsilon < \frac{s_0}{n}$ and $s^\star(\gamma_1, M, R) \in (0, s_0)$ wtih $r^\star(\gamma_1, M, R) = (s^\star)^{\frac{1}{n}} \in (0, R)$ such that

$$U(s, 0) \ge U(s^\star, 0) = \frac{n}{\omega_n} \int_{B_{r^\star(0)}} u_0 dx \ge \frac{n}{\omega_n}(M - \varepsilon), \qquad \forall s \in (s^\star, R^n).$$

Therefore it is possible to estimate

$$\begin{aligned}
\phi(0) &= \int_0^{s_0} s^{-\gamma}(s_0 - s)U(s, 0)ds \\
&\ge \int_0^{s_0} s^{-\gamma}(s_0 - s)U(s^\star, 0)ds \\
&> \frac{n}{\omega_n}(M - \frac{s_0}{n}) \int_0^{s_0} s^{-\gamma}(s_0 - s)ds \\
&= \frac{nM - s_0}{\omega_n(1-\gamma)(2-\gamma)} \cdot s_0^{2-\gamma}.
\end{aligned} \tag{3.25}$$

Then S is non-empty and denote $T = \sup S \in (0, \infty]$. Next, we need to prove $(0, T) \subset S_\phi \cap S_\psi \neq \emptyset$. Note that

$$\phi(t) > \frac{nM - s_0}{\omega_n(1-\gamma)(2-\gamma)} \cdot s_0^{2-\gamma}, \qquad \forall t \in (0, T), \tag{3.26}$$

we obtain $(0, T) \subset S_\phi$. From (3.20) we have

$$\begin{aligned}
\phi(t) &\le \int_0^{s_0} s^{-\gamma}(s_0 - s)[s + C_3 s^{\frac{\gamma+\gamma_1-1}{1+\gamma_1}}(s_0 - s)^{-\frac{1}{1+\gamma_1}} \psi^{\frac{1}{1+\gamma_1}}(t)]ds \\
&\le s_0 \int_0^{s_0} s^{1-\gamma} ds + C_3 \int_0^{s_0} s^{-\gamma + \frac{\gamma+\gamma_1-1}{1+\gamma_1}}(s_0 - s)^{\frac{\gamma_1}{1+\gamma_1}} \psi^{\frac{1}{1+\gamma_1}}(t)ds \\
&= \frac{s_0^{3-\gamma}}{2-\gamma} + \frac{C_3(1+\gamma_1)}{\gamma_1(2-\gamma)} s_0^{\frac{\gamma_1(3-\gamma)}{1+\gamma_1}} \cdot \psi^{\frac{1}{1+\gamma_1}}(t).
\end{aligned}$$

It follows from (3.24) and (3.26) that

$$\phi(t) \ge \frac{nM}{2(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \qquad \text{for all } t \in (0, T). \tag{3.27}$$

Then

$$\frac{C_3(1+\gamma_1)}{\gamma_1(2-\gamma)} s_0^{\frac{\gamma_1(3-\gamma)}{1+\gamma_1}} \cdot \psi^{\frac{1}{1+\gamma_1}}(t) \ge \frac{nM}{2(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} - \frac{s_0^{3-\gamma}}{2-\gamma}.$$

Note that (3.24) implies

$$\frac{nM\gamma_1}{2C_3(1-\gamma)\omega_n(1+\gamma_1)s_0} - \frac{\gamma_1}{C_3(1+\gamma_1)} \geq 1,$$

then we have

$$\begin{aligned}\psi(t) &\geq \Big[\Big(\frac{nMs_0^{2-\gamma}}{2(1-\gamma)(2-\gamma)\omega_n} - \frac{s_0^{3-\gamma}}{2-\gamma}\Big)\cdot\frac{\gamma_1(2-\gamma)}{C_3(1+\gamma_1)}s_0^{-\frac{\gamma_1(3-\gamma)}{1+\gamma_1}}\Big]^{1+\gamma_1} \\ &\geq \Big[\frac{nM\gamma_1}{2C_3(1-\gamma)\omega_n(1+\gamma_1)s_0} - \frac{\gamma_1}{C_3(1+\gamma_1)}\Big]^{1+\gamma_1}\cdot s_0^{3-\gamma} \\ &\geq s_0^{3-\gamma}.\end{aligned}$$

Therefore, $(0,T)\subset S_\phi\cap S_\psi\neq\emptyset$.

Step 2. Applying Lemma 3.7 we can find $\gamma\in(-\infty,1)$ and $C_1, C_2>0$ such that for all $s_0\in(0,\frac{R^n}{6}]$

$$\phi'(t) \geq \begin{cases} C_1 s_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(t) - C_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}}, & m\leq 1,\\ C_1 s_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(t) - C_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}}, & m>1,\end{cases}$$

for all $t\in S_\phi\cap S_\psi$ and with (3.22) we have

$$\psi(t)\geq C_3 s_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(t), \qquad \forall t\in S_\psi.$$

To specify our choice of s_0, for given $M>0$ we choose $s_0\in(0,\frac{R^n}{6}]$ small enough such that

$$s_0\leq\frac{nM}{2}, \tag{3.28}$$

and also

$$s_0^{\gamma_1} < \frac{TC_1\gamma_1}{4}\Big(\frac{nM}{2\omega_n(1-\gamma)(2-\gamma)}\Big)^{\gamma_1}, \tag{3.29}$$

as well as

$$s_0^{1+\gamma_1}\leq C_3\Big(\frac{nM}{2\omega_n(1-\gamma)(2-\gamma)}\Big)^{1+\gamma_1}. \tag{3.30}$$

From (3.23), (3.28) and (3.30) we have

$$\psi(t)\geq C_3 s_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(t) > C_3\Big(\frac{nM-s_0}{\omega_n(1-\gamma)(2-\gamma)}\cdot\frac{1}{s_0}\Big)^{1+\gamma_1}\cdot s_0^{3-\gamma}\geq s_0^{3-\gamma}, \quad \forall t\in S_\psi,$$

which shows that $S\subset S_\phi\cap S_\psi$. Since $1+\gamma_1-m>\frac{2}{n}$, we have $(1+\gamma_1)(1-\frac{2}{n(1+\gamma_1-m)})>0$ if $m>1$ so that we can choose s_0 sufficiently small satisfying (3.28)–(3.30) such that

$$s_0^{(1+\gamma_1)(1-\frac{2}{n(1+\gamma_1-m)})}\leq\frac{C_1}{2C_2}\Big(\frac{nM}{2\omega_n(1-\gamma)(2-\gamma)}\Big)^{1+\gamma_1},$$

while in the case $m\leq 1$, the condition $\gamma_1>m-1+\frac{2}{n}\geq\frac{2}{n}$ which infers that $(1+\gamma_1)(1-\frac{2}{n\gamma_1})>0$ and we select s_0 small enough fulfilling (3.28)–(3.30) such that

$$s_0^{(1+\gamma_1)(1-\frac{2}{n\gamma_1})}\leq\frac{C_1}{2C_2}\Big(\frac{nM}{2\omega_n(1-\gamma)(2-\gamma)}\Big)^{1+\gamma_1}.$$

It is possible to obtain

$$\frac{\frac{C_1}{2}s_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(0)}{C_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{1+\gamma_1-m}}} \geq \frac{C_1}{2C_2}\Big(\frac{nM}{2\omega_n(1-\gamma)(2-\gamma)}\Big)^{1+\gamma_1}\cdot s_0^{-(1+\gamma_1)+\frac{2}{n}\cdot\frac{1+\gamma_1}{(1+\gamma_1-m)}} \geq 1, \qquad \forall m > 1,$$

and we have

$$\frac{\frac{C_1}{2}s_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(0)}{C_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}}} \geq \frac{C_1}{2C_2}\Big(\frac{nM}{2\omega_n(1-\gamma)(2-\gamma)}\Big)^{1+\gamma_1}\cdot s_0^{-(1+\gamma_1)+\frac{2}{n}\cdot\frac{1+\gamma_1}{\gamma_1}} \geq 1, \qquad \forall m \leq 1.$$

All in all, for any $m \in \mathbb{R}$, we apply an ODI comparison argument to obtain that

$$\phi'(t) \geq \frac{C_1}{2}s_0^{-\gamma_1(3-\gamma)}\phi^{1+\gamma_1}(t), \qquad \forall t \in (0,T).$$

By a direct calculation we obtain

$$-\frac{1}{\gamma_1}\Big(\frac{1}{\phi^{\gamma_1}(t)}-\frac{1}{\phi^{\gamma_1}(0)}\Big) \geq \frac{C_1}{2}s_0^{-\gamma_1(3-\gamma)}t, \quad \forall t \in (0,T).$$

Hence, according to (3.25) and (3.29) we conclude

$$t < \frac{2}{C_1\gamma_1}s_0^{\gamma_1}\Big(\frac{2\omega_n(1-\gamma)(2-\gamma)}{nM}\Big)^{\gamma_1} \leq \frac{T}{2},$$

for all $t \in (0,T)$. As a consequence, we infer that T_{max} must be finite. □

4. Global boundedness

In this section, we are preparing to prove Theorem 1.2 by providing the L^p estimate of u and the Moser-type iteration.

Lemma 4.1. *Let (u,v,w) be a classical solution of the system (1.5) under the condition of Theorem* 1.2*. Suppose that*

$$\gamma_2 < 1+\gamma_1 < \frac{2}{n}+m. \tag{4.1}$$

Then for any $p > \max\{1, 2-m, \gamma_2\}$, there exists $C = C(p) > 0$ such that

$$\int_\Omega (1+u)^p(x,t)dx \leq C \quad on\ (0,T_{max}). \tag{4.2}$$

Proof. Notice $f_1(u) \leq k_1(1+u)^{\gamma_1}$, $f_2(u) = k_2(1+u)^{\gamma_2}$ for all $u \geq 0$. Multiplying the first equation of (1.5) by $p(1+u)^{p-1}$ and integrating by parts with the boundary conditions for u, v and w, we have

$$\frac{d}{dt}\int_\Omega (1+u)^p dx + p(p-1)\int_\Omega (1+u)^{p-2}D(u)|\nabla u|^2 dx$$
$$= \chi p(p-1)\int_\Omega u(1+u)^{p-2}\nabla u\cdot\nabla v dx - \xi p(p-1)\int_\Omega u(1+u)^{p-2}\nabla u\cdot\nabla w dx$$

$$
\begin{aligned}
&= -\chi(p-1)\int_\Omega (1+u)^p \Delta v dx + \chi p \int_\Omega (1+u)^{p-1}\Delta v dx \\
&\quad + \xi(p-1)\int_\Omega (1+u)^p \Delta w dx - \xi p \int_\Omega (1+u)^{p-1}\Delta w dx \\
&\le \chi(p-1)\int_\Omega (1+u)^p f_1(u) dx + \chi p \int_\Omega (1+u)^{p-1}\mu_1(t) dx + \xi(p-1)\int_\Omega (1+u)^p \mu_2(t) dx \\
&\quad - \xi(p-1)\int_\Omega (1+u)^p f_2(u) dx + \xi p \int_\Omega (1+u)^{p-1} f_2(u) dx \\
&\le k_1\chi(p-1)\int_\Omega (1+u)^{p+\gamma_1} dx + \chi p \int_\Omega (1+u)^{p-1}\mu_1(t) dx + \xi(p-1)\int_\Omega (1+u)^p \mu_2(t) dx \\
&\quad - k_2\xi(p-1)\int_\Omega (1+u)^{p+\gamma_2} dx + k_2\xi p \int_\Omega (1+u)^{p+\gamma_2-1} dx, \quad \forall t \in (0, T_{max}). \qquad (4.3)
\end{aligned}
$$

Firstly,

$$
\begin{aligned}
p(p-1)\int_\Omega (1+u)^{p-2} D(u)|\nabla u|^2 dx &\ge dp(p-1)\int_\Omega (1+u)^{p+m-3}|\nabla u|^2 dx \\
&= \frac{4dp(p-1)}{(p+m-1)^2}\int_\Omega |\nabla (1+u)^{\frac{p+m-1}{2}}|^2 dx.
\end{aligned}
$$

By Young's inequality and Hölder's inequality, we obtain

$$
\begin{aligned}
\chi p \int_\Omega (1+u)^{p-1}\mu_1(t) dx &\le C_1 \int_\Omega (1+u)^{p+\gamma_1} dx + C_2 \mu_1^{\frac{p+\gamma_1}{1+\gamma_1}}(t) \\
&= C_1 \int_\Omega (1+u)^{p+\gamma_1} dx + C_2\Big(\frac{1}{|\Omega|}\int_\Omega f_1(u) dx\Big)^{\frac{p+\gamma_1}{1+\gamma_1}} \\
&\le C_1 \int_\Omega (1+u)^{p+\gamma_1} dx + C_3\Big(\int_\Omega (1+u)^{1+\gamma_1} dx\Big)^{\frac{p+\gamma_1}{1+\gamma_1}} \\
&\le C_1 \int_\Omega (1+u)^{p+\gamma_1} dx + C_3\Big\{\Big(\int_\Omega (1+u)^{p+\gamma_1} dx\Big)^{\frac{1+\gamma_1}{p+\gamma_1}} \cdot |\Omega|^{\frac{p-1}{p+\gamma_1}}\Big\}^{\frac{p+\gamma_1}{1+\gamma_1}} \\
&= C_1 \int_\Omega (1+u)^{p+\gamma_1} dx + C_3 |\Omega|^{\frac{p-1}{1+\gamma_1}} \int_\Omega (1+u)^{p+\gamma_1} dx.
\end{aligned}
$$

for all $t \in (0, T_{max})$. Then by Hölder's inequality we obtain

$$
\begin{aligned}
\xi(p-1)\int_\Omega (1+u)^p \mu_2(t) dx &= \frac{k_2\xi(p-1)}{|\Omega|}\int_\Omega (1+u)^{\gamma_2} dx \int_\Omega (1+u)^p dx \\
&\le \frac{k_2\xi(p-1)}{|\Omega|}\Big\{\int_\Omega (1+u)^{p+\gamma_2} dx\Big\}^{\frac{\gamma_2}{p+\gamma_2}} |\Omega|^{\frac{p}{p+\gamma_2}} \times \Big\{\int_\Omega (1+u)^{p+\gamma_2} dx\Big\}^{\frac{p}{p+\gamma_2}} |\Omega|^{\frac{\gamma_2}{p+\gamma_2}} \\
&= k_2\xi(p-1)\int_\Omega (1+u)^{p+\gamma_2} dx, \qquad \forall t \in (0, T_{max}).
\end{aligned}
$$

Furthermore, by using Young's inequality and (4.1) we have

$$
k_2\xi p \int_\Omega (1+u)^{p+\gamma_2-1} dx \le C_4 \int_\Omega (1+u)^{p+\gamma_1} dx + C_5,
$$

for all $t \in (0, T_{max})$. Therefore, combining these we conclude

$$\frac{d}{dt}\int_\Omega (1+u)^p dx + \frac{4dp(p-1)}{(p+m-1)^2}\int_\Omega |\nabla(1+u)^{\frac{p+m-1}{2}}|^2 dx \le C_6\int_\Omega (1+u)^{p+\gamma_1}dx + C_5, \quad \forall t\in(0,T_{max}),$$

where $C_6 = C_1 + C_3|\Omega|^{\frac{p-1}{1+\gamma_1}} + C_4 + k_1\chi(p-1)$. By means of Gagliardo-Nirenberg inequality we can find C_7 such that

$$\begin{aligned} C_6\int_\Omega (1+u)^{p+\gamma_1}dx &= C_6\|(1+u)^{\frac{p+m-1}{2}}\|^{\frac{2(p+\gamma_1)}{p+m-1}}_{L^{\frac{2(p+\gamma_1)}{p+m-1}}(\Omega)} \\ &\le C_7\|\nabla(1+u)^{\frac{p+m-1}{2}}\|^{\frac{2(p+\gamma_1)}{p+m-1}\cdot a}_{L^2(\Omega)}\cdot\|(1+u)^{\frac{p+m-1}{2}}\|^{\frac{2(p+\gamma_1)}{p+m-1}\cdot(1-a)}_{L^{\frac{2}{p+m-1}}(\Omega)} \\ &\quad + C_7\|(1+u)^{\frac{p+m-1}{2}}\|^{\frac{2(p+\gamma_1)}{p+m-1}}_{L^{\frac{2}{p+m-1}}(\Omega)} \end{aligned}$$

for all $t \in (0, T_{max})$, where

$$a = \frac{\frac{p+m-1}{2} - \frac{p+m-1}{2(p+\gamma_1)}}{\frac{p+m-1}{2} - (\frac{1}{2}-\frac{1}{n})} \in (0,1).$$

Since $1 - m + \gamma_1 < \frac{2}{n}$, we have $\frac{2(p+\gamma_1)}{p+m-1}\cdot a < 2$, and we use Young's inequality to see that for all $t \in (0, T_{max})$

$$C_6\int_\Omega (1+u)^{p+\gamma_1}dx \le \frac{2dp(p-1)}{(p+m-1)^2}\|\nabla(1+u)^{\frac{p+m-1}{2}}\|^2_{L^2(\Omega)} + C_8.$$

In quite a similar manner, we obtain $C_9 = C_9(p) > 0$ fulfilling

$$\int_\Omega (1+u)^p dx \le \frac{2dp(p-1)}{(p+m-1)^2}\|\nabla(1+u)^{\frac{p+m-1}{2}}\|^2_{L^2(\Omega)} + C_9 \qquad \text{for all } t\in(0,T_{max}).$$

Finally, combining these to (4.3) we obtain

$$\frac{d}{dt}\int_\Omega (1+u)^p dx + \int_\Omega (1+u)^p dx \le C_5 + C_8 + C_9 \qquad \text{for all } t\in(0,T_{max}).$$

Thus,

$$\int_\Omega (1+u)^p dx \le \max\Big\{\int_\Omega (1+u_0)^p dx, C_5 + C_8 + C_9\Big\} \qquad \text{for all } t\in(0,T_{max}).$$

We have done the proof. □

Under the condition of Lemma 4.1 we can use the above information to prove Theorem 1.2.

Proof of Theorem 1.2. From Lemma 4.1, we let $p > \max\{\gamma_1 n, \gamma_2 n, 1\}$. By the elliptic L^p-estimate to the two elliptic equations in (1.5), we get that for all $t \in (0, T_{max})$ there exists some $C_{10}(p) > 0$ such that

$$\|v(\cdot,t)\|_{W^{2,\frac{p}{\gamma_1}}(\Omega)} \le C_{10}(p), \qquad \|w(\cdot,t)\|_{W^{2,\frac{p}{\gamma_2}}(\Omega)} \le C_{10}(p), \tag{4.4}$$

and hence, by the Sobolev embedding theorem, we get

$$\|v(\cdot,t)\|_{C^1(\overline{\Omega})} \le C_{10}(p), \qquad \|w(\cdot,t)\|_{C^1(\overline{\Omega})} \le C_{10}(p). \tag{4.5}$$

Now the Moser iteration technique ([3, 51]) ensures that $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C$ for any $t \in (0, T_{max})$. This concludes by Lemma 2.1 that $T_{max} = \infty$. □

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Conflict of interest

The authors declare that there is no conflict of interest.

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