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Research article

Dirichlet problems of fractional p-Laplacian equation with impulsive effects

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Abstract: The purpose of the article is to investigate Dirichlet boundary-value problems of the fractional p-Laplacian equation with impulsive effects. By using the Nehari manifold method, mountain pass theorem and three critical points theorem, some new results are achieved under more general growth conditions. In addition, this paper weakens the commonly used p-suplinear and p-sublinear growth conditions.

Keywords: fractional Dirichlet problem; p-Laplacian operator; impulsive effect; ground state solution; weak solution

1. Introduction

The article is concerned with the solvability of Dirichlet problems of the fractional p-Laplacian equation with impulsive effects, as follows:

$$\begin{cases} {}_tD_T^{\alpha}\phi_p({}_0D_t^{\alpha}u(t)) + a(t)\phi_p(u(t)) = \lambda f(t,u(t)), \ t \neq t_j, \ \text{a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), \ j = 1,2,\cdots,n, \ n \in \mathbb{N}, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where ${}_0^CD_t^{\alpha}$ is the left Caputo fractional derivative, ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives respectively, $\alpha \in (1/p, 1]$, $p > 1$, $\phi_p(x) = |x|^{p-2}x \ (x \neq 0)$, $\phi_p(0) = 0$, $\lambda > 0$, $\mu \in \mathbb{R}$, $a(t) \in C([0,T],\mathbb{R})$, $f \in C([0,T] \times \mathbb{R},\mathbb{R})$, $T > 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, $I_j \in C(\mathbb{R},\mathbb{R})$, and

$$\Delta({}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u))(t_j) = {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^+) - {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^-),$$

$${}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^+) = \lim_{t\to t_j^+} {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t), \ {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^-) = \lim_{t\to t_j^-} {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t).$$

Fractional calculus has experienced a growing focus in recent decades because of its application to real-world problems. This kind of problem has attracted the attention of many scholars and produced a series of excellent works [1–8]. In particular, left and right fractional differential operators have been widely used in the study of physical phenomena of anomalous diffusion, specifically, fractional convection-diffusion equations [9,10]. Recently, the equations containing left and right fractional differential operators have become a new field in the theory of fractional differential equations. For example, the authors of [11] first put forward the following fractional convection-diffusion equation:

$$\begin{cases} -aD\left(p_0 D_t^{-\beta} + q_t D_T^{-\beta}\right) Du(t) + b\,(t)\, Du(t) + c\,(t)\, u(t) = f, \quad \text{a.e. } t \in [0,T], \quad 0 \leq \beta < 1, \\ u\,(0) = u\,(T) = 0. \end{cases}$$

The authors gained the relevant conclusions about the solution of the above-mentioned problems by using the Lax-Milgram theorem. In [12], the authors discussed the following problem:

$$\begin{cases} \frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}\,(u'\,(t)) + \frac{1}{2}{}_tD_T^{-\beta}\,(u'\,(t))\right) + \nabla F(t, u\,(t)) = 0, \quad \text{a.e. } t \in [0,T], \quad 0 \leq \beta < 1, \\ u(0) = u(T) = 0. \end{cases}$$

By applying the minimization principle and mountain pass theorem, the existence results under the Ambrosetti-Rabinowitz condition were obtained. The following year, in [13], the authors made further research on the following issues:

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0,T], \quad \frac{1}{2} < \alpha \leq 1, \\ u(0) = u(T) = 0. \end{cases}$$

Use of impulsive differential equations is an effective method to describe the instantaneous change of the state of things, and it can reflect the changing law of things more deeply and accurately. It has practical significance and application value in many fields of science and technology, such as signal communication, economic regulation, aerospace technology, management science, engineering science, chaos theory, information science, life science and so on. Due to the application of impulsive differential equations to practical problems, more and more attention has been paid to them in recent years, and many scholars at home and abroad have studied such problems. For example, in [14,15], using the three critical points theorem, the authors discussed the impulse problems as follows:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^CD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t, u(t)), \ t \neq t_j, \ \text{a.e. t} \in [0,\text{T}], \ \alpha \in (\frac{1}{2}, 1], \\ \Delta({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), \ j = 1, 2, \cdots, n, \\ u(0) = u(T) = 0, \end{cases}$$

where $\lambda, \mu > 0$, $I_j \in C(\mathbb{R}, \mathbb{R})$, $a \in C([0,T])$ and there exist a_1 and a_2 such that $0 < a_1 \leq a(t) \leq a_2$. In addition,

$$\Delta({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u))(t_j) = {}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t_j^-),$$

$${}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t_j^+) = \lim_{t \to t_j^+}({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t)), \quad {}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t_j^-) = \lim_{t \to t_j^-}({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t)).$$

The p-Laplacian equation originated from the nonlinear diffusion equation proposed by Leibenson in 1983, when he studied the problem of one-dimensional variable turbulence of gas passing through porous media:

$$u_t = \frac{\partial}{\partial x}\left(\frac{\partial u^m}{\partial x}\left|\frac{\partial u^m}{\partial x}\right|^{\mu-1}\right), \quad m = n + 1.$$

When $m > 1$, the above equation is the porous medium equation; When $0 < m < 1$, the above equation is a fast diffusion equation; When $m = 1$, the above equation is a heat equation; However, when $m = 1, \mu \neq 1$, such equations often appear in the study of non-Newtonian fluids. In view of the importance of such equations, the above equation has been abstracted into the p-Laplacian equation:

$$(\phi_p(u'))' = f(t, u),$$

where $\phi_p(x) = |x|^{p-2}x\ (x \neq 0),\ \phi_p(0) = 0,\ p > 1$. When $p = 2$, the p-Laplacian equation is reduced to a classical second-order differential equation. Ledesma and Nyamoradi [16] researched the impulse problem with a p-Laplacian operator as below.

$$\begin{cases} {}_tD_T^{\alpha}\left(\left|{}_0D_t^{\alpha}u(t)\right|^{p-2}{}_0D_t^{\alpha}u(t)\right)+a(t)|u(t)|^{p-2}u(t)=f(t,u(t)),\, t\neq t_j, \text{a.e.} t\in[0,T],\\ \Delta\left({}_tI_T^{1-\alpha}\left(\left|{}_0D_t^{\alpha}u(t_j)\right|^{p-2}{}_0D_t^{\alpha}u(t_j)\right)\right)=I_j(u(t_j)),\ j=1,2,\cdots,n,\ n\in\mathbb{N},\\ u(0)=u(T)=0, \end{cases} \tag{1.2}$$

where $\alpha \in (\frac{1}{p}, 1]$, $p > 1$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$ and

$$\begin{aligned} &\Delta\left({}_tI_T^{1-\alpha}\left(\left|{}_0D_t^{\alpha}u(t_j)\right|^{p-2}{}_0D_t^{\alpha}u(t_j)\right)\right)\\ &= {}_tI_T^{1-\alpha}\left(\left|{}_0D_t^{\alpha}u(t_j^+)\right|^{p-2}{}_0D_t^{\alpha}u(t_j^+)\right) - {}_tI_T^{1-\alpha}\left(\left|{}_0D_t^{\alpha}u(t_j^-)\right|^{p-2}{}_0D_t^{\alpha}u(t_j^-)\right),\\ &{}_tI_T^{1-\alpha}\left(\left|{}_0D_t^{\alpha}u(t_j^+)\right|^{p-2}{}_0D_t^{\alpha}u(t_j^+)\right) = \lim_{t\to t_j^+}\ {}_tI_T^{1-\alpha}\left(\left|{}_0D_t^{\alpha}u(t)\right|^{p-2}{}_0D_t^{\alpha}u(t)\right),\\ &{}_tI_T^{1-\alpha}\left(\left|{}_0D_t^{\alpha}u(t_j^-)\right|^{p-2}{}_0D_t^{\alpha}u(t_j^-)\right) = \lim_{t\to t_j^-}\ {}_tI_T^{1-\alpha}\left(\left|{}_0D_t^{\alpha}u(t)\right|^{p-2}{}_0D_t^{\alpha}u(t)\right). \end{aligned}$$

By using the mountain pass theorem and the symmetric mountain pass theorem, the authors acquired the related results of Problem (1.2) under the Ambrosetti-Rabinowitz condition. If $\alpha = 1$ and $a(t) = 0$, then Problem (1.2) is reduced to the p-Laplacian equation with impulsive effects, as follows:

$$\begin{cases} -(|u'|^{p-2}u')' = f(t, u(t)),\quad t\neq t_j, \text{a.e. } t\in[0,1],\\ u(1)=u(0)=0,\\ u(t_j^+)=u(t_j^-),\ j=1,2,\cdots,n,\\ \Delta\left|u'(t_j)\right|^{p-2}u'(t_j)=I_j(u(t_j)),\ j=1,2,\cdots,n. \end{cases}$$

This problem has been studied in [17] and [18]. The main methods used in the above literature are the critical point theory and the topological degree theory. To show the major conclusions of literature [16], the following assumptions are first introduced below:

(F_1) There are $\theta > p$ and $r > 0$, so that $0 < \theta F(t,\xi) \leq \xi f(t,\xi),\ \forall t \in [0,T],\ |\xi| \geq r$;
(F_2) $f(t,\xi) = o(|\xi|^{p-1}),\ \xi \to 0$, for $\forall t \in [0,T]$;
(F_3) For $\forall j$, there are $c_j > 0$ and $\gamma_j \in (p-1,\ \theta-1)$ so that $\left|I_j(\xi)\right| \leq c_j|\xi|^{\gamma_j}$;
(F_4) For u large enough, one has $I_j(\xi)\xi \leq \theta\int_0^u I_j(\xi)d\xi,\ \forall j = 1,2,\cdots,n$.

Theorem 1. *([16]). If the conditions (F_1)–(F_4) hold, then the impulsive problem (1.2) possesses one weak solution.*

The research work of this paper is to further study the impulse problem (1.1) on the basis of the above work. To compare with Theorem 1, the supposed conditions and main results are given as below.

(H_0) $a(t) \in C([0,T],\mathbb{R})$ satisfies $\operatorname{essinf}_{t\in[0,T]} a(t) > -\lambda_1$, where $\lambda_1 = \inf\limits_{u \in E_0^{\alpha,p}\setminus\{0\}} \frac{\int_0^T |{}_0D_t^\alpha u(t)|^p dt}{\int_0^T |u(t)|^p dt} > 0$;

(H_1) For $\forall t \in \mathbb{R}$, $j = 1, 2, \cdots, m$, $m \in \mathbb{N}$, $I_j(t)$ satisfies $\int_0^t I_j(s)ds \geq 0$;

(H_2) There are $a_j,\ d_j > 0$ and $\gamma_j \in [0,\ p-1)$ so that $\left|I_j(t)\right| \leq a_j + d_j|t|^{\gamma_j},\ \forall t \in \mathbb{R}$;

(H_3) The map $s \to I_j(s)\big/|s|^{p-1}$ is strictly monotonically decreasing on $\mathbb{R}\setminus\{0\}$;

(H_4) The map $s \to f(t,s)/|s|^{p-1}$ is strictly monotonically increasing on $\mathbb{R}\setminus\{0\}$, for $\forall t \in [0,T]$;

(H_5) $f(t,u) = o(|u|^{p-1})$ $(|u| \to 0)$, uniformly for $\forall t \in [0,T]$;

(H_6) There are $M > 0$, $L > 0$ and $\theta > p$ so that

$$uf(t,u) - \theta F(t,u) \geq -M|u|^p,\ \forall t \in [0,T],\ |u| \geq L,$$

where $F(t,u) = \int_0^u f(t,s)ds$;

(H_7) $\lim\limits_{|u|\to\infty} \frac{F(t,u)}{|u|^\theta} = \infty$, uniformly for $\forall t \in [0,T]$.

Theorem 2. *Let $f \in C^1([0,T]\times\mathbb{R},\mathbb{R})$ and $I_j \in C^1(\mathbb{R},\mathbb{R})$. Assume that the conditions (H_0)–(H_7) hold. Then, Problem (1.1) with $\lambda = \mu = 1$ has at least one nontrivial ground-state solution.*

Remark 1. *Obviously, the conditions (H_6) and (H_7) are weaker than (F_1) of Theorem 1. In addition, for this kind of problem, the existence of solutions has been discussed in the past, while the ground-state solutions have been rarely studied. Therefore, our finding extends and enriches Theorem 1 in [16].*

Next, further research Problem (1.1) with the concave-convex nonlinearity. The function $f \in C([0,T]\times\mathbb{R},\mathbb{R})$ studied here satisfies the following conditions:

$$f(t,u) = f_1(t,u) + f_2(t,u), \tag{1.3}$$

where $f_1(t,u)$ is p-suplinear as $|u| \to \infty$ and $f_2(t,u)$ denotes p-sublinear growth at infinity. Below, some supposed conditions are given on f_1 and f_2, as below:

(H_8) $f_1(t,u) = o(|u|^{p-1})$ $(|u| \to 0)$, uniformly for $\forall t \in [0,T]$;

(H_9) There are $M > 0,\ L > 0$ and $\theta > p$ so that

$$uf_1(t,u) - \theta F_1(t,u) \geq -M|u|^p,\ \forall t \in [0,T],\ |u| \geq L,$$

where $F_1(t,u) = \int_0^u f_1(t,s)\,ds$;

(H_{10}) $\lim\limits_{|u|\to\infty} \frac{F_1(t,u)}{|u|^\theta} = \infty$, uniformly for $\forall t \in [0,T]$;

(H_{11}) There are $1 < r < p$ and $b \in C([0,T],\mathbb{R}^+)$, $\mathbb{R}^+ = (0,\infty)$, so that

$$F_2(t,u) \geq b(t)\,|u|^r,\ \forall (t,u) \in [0,T]\times\mathbb{R},$$

where $F_2(t,u) = \int_0^u f_2(t,s)\,ds$;

(H_{12}) There is $b_1 \in L^1([0,T],\mathbb{R}^+)$ so that $|f_2(t,u)| \leq b_1(t)|u|^{r-1},\ \forall (t,u) \in [0,T]\times\mathbb{R}$;

(H_{13}) There are $a_j,\ d_j > 0$ and $\gamma_j \in [0,\ \theta-1)$ so that $\left|I_j(t)\right| \leq a_j + d_j|t|^{\gamma_j},\ \forall t \in \mathbb{R}$;

(H_{14}) For t large enough, $I_j(t)$ satisfies $\theta\int_0^t I_j(s)ds \geq I_j(t)t$;

(H_{15}) For $\forall t \in \mathbb{R}$, $I_j(t)$ satisfies $\int_0^t I_j(s)ds \geq 0$.

Theorem 3. *Assume that the conditions* (H_0) *and* (H_8)–(H_{15}) *hold. Then, the impulse problem (1.1) with* $\lambda = \mu = 1$ *possesses at least two non-trivial weak solutions.*

Remark 2. *Obviously, the conditions* (H_9) *and* (H_{10}) *are weaker than* (F_1) *of Theorem 1. And, the condition* (H_{13}) *is weaker than the condition* (F_3) *of Theorem 1. Further, the function* f *studied in Theorem 3 contains both p-suplinear and p-sublinear terms, which is more general. Thus, our finding extends Theorem 1 in [16].*

Finally, the existence results of the three solutions of the impulse problem (1.1) in the case of the parameter $\mu \geq 0$ or $\mu < 0$ are considered respectively. We need the following supposed conditions.

(H_{16}) There are $L, L_1, \cdots, L_n > 0$, $0 < \beta \leq p$, $0 < d_j < p$ and $j = 1, \cdots, n$ so that

$$F(t, x) \leq L\left(1 + |x|^{\beta}\right), -J_j(x) \leq L_j\left(1 + |x|^{d_j}\right), \ \forall (t, x) \in [0, T] \times \mathbb{R}, \tag{1.4}$$

where $F(t, x) = \int_0^x f(t, s)ds$ and $J_j(x) = \int_0^x I_j(t)dt$;

(H_{17}) There are $r > 0$, and $\omega \in E_0^{\alpha,p}$ so that $\frac{1}{p}\|\omega\|^p > r$, $\int_0^T F(t, \omega(t))dt > 0$, $\sum\limits_{j=1}^{n} J_j(\omega(t_j)) > 0$ and

$$A_l := \frac{\frac{1}{p}\|\omega\|^p}{\int_0^T F(t, \omega(t))dt} < A_r := \frac{r}{\int_0^T \max\limits_{|x| \leq \Lambda_\infty (pr)^{1/p}} F(t, x)\, dt}. \tag{1.5}$$

Theorem 4. *Assume that the conditions* (H_0) *and* (H_{16})–(H_{17}) *hold. Then, for every* $\lambda \in \Lambda_r = (A_l, A_r)$*, there is*

$$\gamma := \min\left\{ \frac{r - \lambda \int_0^T \max\limits_{|x| \leq \Lambda_\infty (pr)^{1/p}} F(t, x)\, dt}{\max\limits_{|x| \leq \Lambda_\infty (pr)^{\frac{1}{p}}} \sum\limits_{j=1}^{n} \left(-J_j(x)\right)}, \frac{\lambda \int_0^T F(t, \omega)dt - \frac{1}{p}\|\omega\|^p}{\sum\limits_{j=1}^{n} J_j(\omega(t_j))} \right\} \tag{1.6}$$

so that, for each $\mu \in [0, \gamma)$*, the impulse problem (1.1) possesses at least three weak solutions.*

(H_{18}) There are $L, L_1, \cdots, L_n > 0$, $0 < \beta \leq p$, $0 < d_j < p$ and $j = 1, \cdots, n$ so that

$$F(t, x) \leq L\left(1 + |x|^{\beta}\right), J_j(x) \leq L_j\left(1 + |x|^{d_j}\right); \tag{1.7}$$

(H_{19}) There are $r > 0$ and $\omega \in E_0^{\alpha,p}$ so that $\frac{1}{p}\|\omega\|^p > r$, $\int_0^T F(t, \omega(t))dt > 0$, $\sum\limits_{j=1}^{n} J_j(\omega(t_j)) < 0$ and (1.5) hold.

Theorem 5. *Assume that the conditions* (H_0) *and* (H_{18})-(H_{19}) *hold. Then, for every* $\lambda \in \Lambda_r = (A_l, A_r)$*, there is*

$$\gamma^* := \max\left\{ \frac{\lambda \int_0^T \max\limits_{|x| \leq \Lambda_\infty (pr)^{1/p}} F(t, x)\, dt - r}{\max\limits_{|x| \leq \Lambda_\infty (pr)^{\frac{1}{p}}} \sum\limits_{j=1}^{n} J_j(x)}, \frac{\lambda \int_0^T F(t, \omega)dt - \frac{1}{p}\|\omega\|^p}{\sum\limits_{j=1}^{n} J_j(\omega(t_j))} \right\}$$

so that, for each $\mu \in (\gamma^*, 0]$*, the impulse problem (1.1) possesses at least three weak solutions.*

Remark 3. *The assumptions* (H_{16}) *and* (H_{18}) *study both* $0 < \beta < p$ *and* $\beta = p$. *When* $p = 2$, *the assumptions* (H_{16}) *and* (H_{18}) *contain the condition* $0 < \beta < 2$ *in [14,15]. In addition, this paper allows* $a(t)$ *to have a negative lower bound, satisfying* $\operatorname{essinf}_{t\in[0,T]} a(t) > -\lambda_1$, *where* $\lambda_1 = \inf_{u\in E_0^{\alpha,p}\setminus\{0\}} \frac{\int_0^T |{}_0D_t^\alpha u(t)|^p dt}{\int_0^T |u(t)|^p dt} > 0$, *and* $a(t)$ *in [14,15] has a positive lower bound satisfying* $0 < a_1 \le a(t) \le a_2$. *Thus, our conclusions extend the existing results.*

This paper studies Dirichlet boundary-value problems of the fractional p-Laplacian equation with impulsive effects. By using the Nehari manifold method, the existence theorem of the ground-state solution of the above impulsive problem is given. At the same time, the p-suplinear condition required for the proof is weakened. This is the research motivation for this paper. There is no relevant research work on this result. In addition, the existence and multiplicity theorems of nontrivial weak solutions to the impulsive problem are given by means of a variational method. In the process of building the proof, the conditions of nonlinear functions with the concave-convex terms are weakened and the conditions of impulsive terms and variable coefficient terms are weakened. Our work extends and enriches the existing results in [14–16], which is the innovation of this paper.

2. Preliminaries

Here are some definitions and lemmas of fractional calculus. For details, see [19].

Definition 1. *([19]). Let* u *be a function defined on* $[a, b]$. *The left and right Riemann-Liouville fractional derivatives of order* $0 \le \gamma < 1$ *for a function* u *denoted by* ${}_aD_t^\gamma u(t)$ *and* ${}_tD_b^\gamma u(t)$, *respectively, are defined by*

$$ {}_aD_t^\gamma u(t) = \frac{d}{dt}{}_aD_t^{\gamma-1} u(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_a^t (t-s)^{-\gamma} u(s) ds\right), $$

$$ {}_tD_b^\gamma u(t) = -\frac{d}{dt}{}_tD_b^{\gamma-1} u(t) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_t^b (s-t)^{-\gamma} u(s) ds\right), $$

where $t \in [a, b]$.

Definition 2. *([19]). Let* $0 < \gamma < 1$ *and* $u \in AC([a, b])$; *then, the left and right Caputo fractional derivatives of order* γ *for a function* u *denoted by* ${}_a^CD_t^\gamma u(t)$ *and* ${}_t^CD_b^\gamma u(t)$, *respectively, exist almost everywhere on* $[a, b]$. ${}_a^CD_t^\gamma u(t)$ *and* ${}_t^CD_b^\gamma u(t)$ *are respectively represented by*

$$ {}_a^CD_t^\gamma u(t) = {}_aD_t^{\gamma-1} u'(t) = \frac{1}{\Gamma(1-\gamma)}\int_a^t (t-s)^{-\gamma} u'(s) ds, $$

$$ {}_t^CD_b^\gamma u(t) = -{}_tD_b^{\gamma-1} u'(t) = -\frac{1}{\Gamma(1-\gamma)}\int_t^b (s-t)^{-\gamma} u'(s) ds, $$

where $t \in [a, b]$.

Definition 3. *([20]). Let* $0 < \alpha \le 1$ *and* $1 < p < \infty$. *Define the fractional derivative space* $E^{\alpha,p}$ *as follows:*

$$ E^{\alpha,p} = \{u \in L^p([0,T], \mathbb{R}) \,|\, {}_0D_t^\alpha u \in L^p([0,T], \mathbb{R})\}, $$

with the norm

$$\|u\|_{E^{\alpha,p}} = \left(\|u\|_{L^p}^p + \left\|{}_0D_t^\alpha u\right\|_{L^p}^p\right)^{\frac{1}{p}}, \tag{2.1}$$

where $\|u\|_{L^p} = (\int_0^T |u(t)|^p dt)^{1/p}$ *is the norm of* $L^p([0,T],\mathbb{R})$. $E_0^{\alpha,p}$ *is defined by closure of* $C_0^\infty([0,T],\mathbb{R})$ *with respect to the norm* $\|u\|_{E^{\alpha,p}}$.

Proposition 1 ([19]). Let u be a function defined on $[a,b]$. If ${}_a^cD_t^\gamma u(t)$, ${}_t^cD_b^\gamma u(t)$, ${}_aD_t^\gamma u(t)$ and ${}_tD_b^\gamma u(t)$ all exist, then

$$ {}_a^cD_t^\gamma u(t) = {}_aD_t^\gamma u(t) - \sum_{j=0}^{n-1} \frac{u^j(a)}{\Gamma(j-\gamma+1)}(t-a)^{j-\gamma}, \quad t \in [a,\ b], $$

$$ {}_t^cD_b^\gamma u(t) = {}_tD_b^\gamma u(t) - \sum_{j=0}^{n-1} \frac{u^j(b)}{\Gamma(j-\gamma+1)}(b-t)^{j-\gamma}, \quad t \in [a,\ b], $$

where $n \in \mathbb{N}$ and $n-1 < \gamma < n$.

Remark 4. *For any* $u \in E_0^{\alpha,p}$, *according to Proposition 1, when* $0 < \alpha < 1$ *and the boundary conditions* $u(0) = u(T) = 0$ *are satisfied, we can get* ${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t)$ *and* ${}_t^cD_T^\alpha u(t) = {}_tD_T^\alpha u(t)$, $t \in [0,\ T]$.

Lemma 1. *([20]). Let* $0 < \alpha \le 1$ *and* $1 < p < \infty$. *The fractional derivative space* $E_0^{\alpha,p}$ *is a reflexive and separable Banach space.*

Lemma 2. *([13]). Let* $0 < \alpha \le 1$ *and* $1 < p < \infty$. *If* $u \in E_0^{\alpha,p}$, *then*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\left\|{}_0D_t^\alpha u\right\|_{L^p}. \tag{2.2}$$

If $\alpha > 1/p$, *then*

$$\|u\|_\infty \le C_\infty \left\|{}_0D_t^\alpha u\right\|_{L^p}, \tag{2.3}$$

where $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$ *is the norm of* $C([0,T],\mathbb{R})$, $C_\infty = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q-q+1)^{\frac{1}{q}}} > 0$ *and* $q = \frac{p}{p-1} > 1$.

Combined with (2.2), we think over $E_0^{\alpha,p}$ with the norm as below.

$$\|u\|_{E^{\alpha,p}} = (\int_0^T \left|{}_0D_t^\alpha u(t)\right|^p dt)^{\frac{1}{p}} = \left\|{}_0D_t^\alpha u\right\|_{L^p},\ \forall u \in E_0^{\alpha,p}. \tag{2.4}$$

Lemma 3. *([13]). If* $1/p < \alpha \le 1$ *and* $1 < p < \infty$, *then* $E_0^{\alpha,p}$ *is compactly embedded in* $C([0,T],\mathbb{R})$.

Lemma 4. *([13]). Let* $1/p < \alpha \le 1$ *and* $1 < p < \infty$. *If the sequence* $\{u_k\}$ *converges weakly to u in* $E_0^{\alpha,p}$, *i.e.,* $u_k \rightharpoonup u$, *then* $u_k \to u$ *in* $C([0,T],\mathbb{R})$, *i.e.,* $\|u_k - u\|_\infty \to 0,\ k \to \infty$.

To investigate Problem (1.1), this article defines a new norm on the space $E_0^{\alpha,p}$, as follows:

$$\|u\| = (\int_0^T \left|{}_0D_t^\alpha u(t)\right|^p dt + \int_0^T a(t)|u(t)|^p dt)^{\frac{1}{p}}.$$

Lemma 5. *([16]). If* $\operatorname{essinf}_{t\in[0,T]} a(t) > -\lambda_1$, *where* $\lambda_1 = \inf\limits_{u\in E_0^{\alpha,p}\setminus\{0\}} \frac{\int_0^T \left|{}_0D_t^\alpha u(t)\right|^p dt}{\int_0^T |u(t)|^p dt} > 0$, *then* $\|u\|$ *is equivalent to* $\|u\|_{E^{\alpha,p}}$, *i.e., there are* $\Lambda_1,\ \Lambda_2 > 0$, *so that* $\Lambda_1\|u\|_{E^{\alpha,p}} \le \|u\| \le \Lambda_2\|u\|_{E^{\alpha,p}}$ *and* $\forall u \in E_0^{\alpha,p}$, *where* $\|u\|_{E^{\alpha,p}}$ *is defined in (2.4).*

Lemma 6. *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For $u \in E_0^{\alpha,p}$, by Lemmas 2 and 5 and (2.4), we have*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|u\|_{E^{\alpha,p}} \le \Lambda_p \|u\|, \tag{2.5}$$

where $\Lambda_p = \frac{T^\alpha}{\Lambda_1\Gamma(\alpha+1)}$. If $\alpha > 1/p$, then

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}\|u\|_{E^{\alpha,p}} \le \Lambda_\infty \|u\|, \tag{2.6}$$

where $\Lambda_\infty = \frac{T^{\alpha-\frac{1}{p}}}{\Lambda_1\Gamma(\alpha)(\alpha q-q+1)^{\frac{1}{q}}}$, $\quad q = \frac{p}{p-1} > 1$.

Lemma 7. *([19]). Let $\alpha > 0$, $p \ge 1$, $q \ge 1$ and $1/p + 1/q < 1+\alpha$, or $p \ne 1$, $q \ne 1$ and $1/p + 1/q = 1+\alpha$. Assume that the function $u \in L^p([a,b],\mathbb{R})$ and $v \in L^q([a,b],\mathbb{R})$; then,*

$$\int_a^b [{}_aD_t^{-\alpha}u(t)]v(t)\,dt = \int_a^b u(t)[{}_tD_b^{-\alpha}v(t)]dt. \tag{2.7}$$

By multiplying the equation in Problem (1.1) by $\forall v \in E_0^{\alpha,p}$ and integrating on $[0,T]$, one has

$$\int_0^T {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t))v(t)dt + \int_0^T a(t)\phi_p(u(t))v(t)dt - \lambda\int_0^T f(t,u(t))v(t)dt = 0.$$

According to Lemma 7, we can get

$$\begin{aligned}
&\int_0^T {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t))v(t)dt = -\sum_{j=0}^n \int_{t_j}^{t_{j+1}} v(t)d[{}_tD_T^{\alpha-1}\phi_p({}_0D_t^\alpha u(t))]\\
&= -\sum_{j=0}^n {}_tD_T^{\alpha-1}\phi_p({}_0D_t^\alpha u(t))v(t)|_{t_j}^{t_{j+1}} + \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \phi_p({}_0D_t^\alpha u(t))_0D_t^\alpha v(t)dt\\
&= \sum_{j=1}^n [{}_tD_T^{\alpha-1}\phi_p({}_0D_t^\alpha u(t_j^+))v(t_j) - {}_tD_T^{\alpha-1}\phi_p({}_0D_t^\alpha u(t_j^-))v(t_j)] + \int_0^T \phi_p({}_0D_t^\alpha u(t))_0D_t^\alpha v(t)dt\\
&=\mu\sum_{j=1}^n I_j(u(t_j))v(t_j) + \int_0^T \phi_p({}_0D_t^\alpha u(t))_0D_t^\alpha v(t)dt.
\end{aligned}$$

Definition 4. *Let $u \in E_0^{\alpha,p}$ be one weak solution of the impulse problem (1.1), if*

$$\int_0^T \phi_p({}_0D_t^\alpha u(t))_0D_t^\alpha v(t)dt + \int_0^T a(t)\phi_p(u(t))v(t)dt + \mu\sum_{j=1}^n I_j(u(t_j))v(t_j) - \lambda\int_0^T f(t,u(t))v(t)dt = 0$$

holds for $\forall v \in E_0^{\alpha,p}$.

Define a functional $\varphi : E_0^{\alpha,p} \to \mathbb{R}$ as below:

$$\varphi(u) = \frac{1}{p}\|u\|^p + \mu\sum_{j=1}^m \int_0^{u(t_j)} I_j(t)dt - \lambda\int_0^T F(t,u(t))dt, \tag{2.8}$$

where $F(t,u) = \int_0^u f(t,s)ds$. According to the continuity of the functions f and I_j, it is easy to prove that $\varphi \in C^1(E_0^{\alpha,p}, \mathbb{R})$. In addition,

$$\begin{aligned}\langle \varphi'(u), v\rangle = &\int_0^T \phi_p({}_0D_t^\alpha u(t))\,{}_0D_t^\alpha v(t)dt + \int_0^T a(t)\phi_p(u(t))v(t)dt \\ &+ \mu \sum_{j=1}^n I_j(u(t_j))v(t_j) - \lambda \int_0^T f(t,u(t))v(t)dt, \quad \forall u, v \in E_0^{\alpha,p}.\end{aligned} \tag{2.9}$$

Thus, the critical point of $\varphi(u)$ corresponds to a weak solution of the impulse problem (1.1). The ground-state solution here refers to the minimum energy solution of the functional φ.

Definition 5. *([21]). Let X be a real Banach space, $\varphi \in C^1(X, \mathbb{R})$. For $\forall \{u_n\}_{n\in\mathbb{N}} \subset X$, $\{u_n\}_{n\in\mathbb{N}}$ possesses one convergent subsequence if $\varphi(u_n) \to c\ \ (n \to \infty)$ and $\varphi'(u_n) \to 0\ (n \to \infty)$. Then, $\varphi(u)$ satisfies the* $(\mathrm{PS})_c$ *condition.*

Lemma 8. *([21]). Let X be a real Banach space and $\varphi \in C^1(X, \mathbb{R})$ satisfy the* $(\mathrm{PS})_c$ *condition. Assume that $\varphi(0) = 0$ and*

(i) there exist $\rho, \eta > 0$ such that $\varphi|_{\partial B_\rho} \geq \eta > 0$;

(ii) there exists an $e \in X\big/\overline{B_\rho}$ such that $\varphi(e) \leq 0$.

Then, φ has one critical value $c \geq \eta$. Moreover, c can be described as $c = \inf\limits_{g\in\Gamma} \max\limits_{s\in[0,1]} \varphi(g(s))$, where $\Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}$.

Lemma 9. *([22]). Let X be one reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be one sequentially weakly lower semi-continuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits one continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ be one continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\inf\limits_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$. Suppose there are $r > 0$ and $\overline{x} \in X$ with $r < \Phi(\overline{x})$ so that*

(i) $\sup\{\Psi(x) : \Phi(x) \leq r\} < r\frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(ii) for each $\lambda \in \Lambda_r = \left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\{\Psi(x):\Phi(x)\leq r\}}\right)$, the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ possesses at least three distinct critical points in X.

3. Main results

3.1. Proof of Theorem 2

Define $\mathcal{N} = \left\{u \in E_0^{\alpha,p} \setminus \{0\} \,|\, G(u) = 0\right\}$, where $G(u) = \langle \varphi'(u), u\rangle = \|u\|^p + \sum\limits_{j=1}^m I_j(u(t_j))u(t_j) - \int_0^T f(t,u(t))u(t)dt$. Then, any non-zero critical point of φ must be on $\mathcal{N}$. For $j = 1, 2, \cdots, m$ and $t \in [0,T]$, by (H_3) and (H_4), one has

$$I'_j(u(t_j))u^2(t_j) < (p-1)I_j(u(t_j))u(t_j),\ (p-1)f(t,u(t))u(t) < \frac{\partial f(t,u(t))}{\partial u}u^2(t). \tag{3.1}$$

So, for $u \in \mathcal{N}$, by (3.1), we get

$$\langle G'(u), u\rangle = p\|u\|^p + \sum_{j=1}^{m}(I'_j(u(t_j))u^2(t_j) + I_j(u(t_j))u(t_j)) - \int_0^T \left(\frac{\partial f(t,u(t))}{\partial u}\cdot u^2(t) + f(t,u(t))u(t)\right)dt$$

$$= \sum_{j=1}^{m}(I'_j(u(t_j))u^2(t_j) - (p-1)I_j(u(t_j))u(t_j)) + \int_0^T \left((p-1)f(t,u(t))u(t) - \frac{\partial f(t,u(t))}{\partial u}\cdot u^2(t)\right)dt < 0. \tag{3.2}$$

The formula indicates that $\mathcal{N}$ has one C^1 structure, which is a Nehari manifold. Here are some necessary lemmas to verify Theorem 2.

Lemma 10. *Let the assumptions given in* (H_3) *and* (H_4) *be satisfied. Additionally, we assume that* $u \in \mathcal{N}$ *is one critical point of* $\varphi|_{\mathcal{N}}$*; then,* $\varphi'(u) = 0$*. In other words,* $\mathcal{N}$ *is one natural constraint on* $\varphi(u)$*.*

Proof. If $u \in \mathcal{N}$ is one critical point of $\varphi|_{\mathcal{N}}$, there is one Lagrange multiplier $\lambda \in \mathbb{R}$ such that $\varphi'(u) = \lambda G'(u)$. Therefore, $\langle \varphi'(u), u\rangle = \lambda \langle G'(u), u\rangle = 0$. Combining with (3.2), we know that $\lambda = 0$, so $\varphi'(u) = 0$.

To discuss the critical point of $\varphi|_{\mathcal{N}}$, let us examine the structure of $\mathcal{N}$.

Lemma 11. *Let the assumptions given in* (H_0) *and* (H_7) *be satisfied. For* $\forall u \in E_0^{\alpha,p}\setminus\{0\}$*, there is one unique* $y = y(u) > 0$ *so that* $yu \in \mathcal{N}$*.*

Proof. The first step is to show that there are $\rho, \sigma > 0$ such that

$$\varphi(u) > 0,\ \forall u \in B_\rho(0)\backslash\{0\},\ \varphi(u) \geq \sigma,\ \forall u \in \partial B_\rho(0). \tag{3.3}$$

It is easy to know that 0 is one strict local minimizer of φ. By (H_5), for $\forall\varepsilon > 0$, there is $\delta > 0$ so that $F(t,u) \leq \varepsilon|u|^p,\ |u| \leq \delta$. So, for $u \in E_0^{\alpha,p},\ \|u\| = \rho,\ \|u\|_\infty \leq \Lambda_\infty\|u\| = \delta$, by (H_1), one has

$$\varphi(u) = \frac{1}{p}\|u\|^p + \sum_{j=1}^{m}\int_0^{u(t_j)} I_j(\mathrm{t})dt - \int_0^T F(t,u(t))dt \geq \frac{1}{p}\|u\|^p - \int_0^T F(t,u(t))dt \geq \frac{1}{p}\|u\|^p - \varepsilon T\Lambda_\infty^p\|u\|^p.$$

Select $\varepsilon = \frac{1}{2pT\Lambda_\infty^p}$; one has $\varphi(u) \geq \frac{1}{2p}\|u\|^p$. Let $\rho = \frac{\delta}{\Lambda_\infty}$ and $\sigma = \frac{\delta^p}{2p\Lambda_\infty^p}$. Therefore, we can conclude that there are $\rho, \sigma > 0$ so that, for $\forall u \in B_\rho\setminus\{0\}$, one has $\varphi(u) > 0$, and for $\forall u \in \partial B_\rho$, one has $\varphi(u) \geq \sigma$.

Second, we prove that $\varphi(yu) \to -\infty$ as $y \to \infty$. In fact, by (H_7), there exist $c_1,\ c_2 > 0$ so that

$$F(t,u) \geq c_1|u|^\theta - c_2,\ (t,u) \in [0,T]\times\mathbb{R}.$$

By (H_2), we have that $\varphi(yu) \leq \frac{y^p}{p}\|u\|^p + \sum_{j=1}^{m} a_jC_\infty y\|u\| + \sum_{j=1}^{m} d_jy^{\gamma_j+1}\frac{C_\infty^{\gamma_j+1}}{\gamma_j+1}\|u\|^{\gamma_j+1} - c_1y^\theta\|u\|_{L^\theta}^\theta + Tc_2$. Because $\gamma_j \in [0, p-1),\ p > 1,\ \theta > p,\ \varphi(yu) \to -\infty$, as $y \to \infty$. Let $g(y) := \varphi(yu)$, where $y > 0$. From the above proof, it can be seen that there exists at least one $yu = y(u) > 0$ so that $g(y_u) = \max_{y\geq 0} g(y) = \max_{y\geq 0}\varphi(yu) = \varphi(y_uu)$. Next, we show that, when $y > 0$, $g(y)$ possesses one unique critical point, which must be the global maximum point. In fact, if y is the critical point of g, then

$$g'(y) = \langle\varphi'(yu), u\rangle = y^{p-1}\|u\|^p + \sum_{j=1}^{m} I_j(yu(t_j))u(t_j) - \int_0^T f(t,yu(t))u(t)dt = 0.$$

By (3.1), we obtain

$$\begin{aligned} g''(y) &= (p-1)y^{p-2}\|u\|^p + \sum_{j=1}^{m} I'_j(yu(t_j))u^2(t_j) - \int_0^T \frac{\partial f(t,yu(t))}{\partial(yu)} \cdot u^2(t)dt \\ &= \frac{1}{y^2}\sum_{j=1}^{m} (I'_j(yu(t_j))(yu(t_j))^2 - (p-1)I_j(yu(t_j))yu(t_j)) \\ &+ \frac{1}{y^2}\int_0^T \left((p-1)f(t,yu(t))yu(t) - \frac{\partial f(t,yu(t))}{\partial yu}\cdot(yu(t))^2\right)dt \\ &< 0. \end{aligned} \tag{3.4}$$

Therefore, if y is one critical point of g, then it must be one strictly local maximum point, and the critical point is unique. In addition, according to

$$g'(y) = \langle\varphi'(yu), u\rangle = \frac{1}{y}\langle\varphi'(yu), yu\rangle, \tag{3.5}$$

if $yu \in \mathcal{N}$, then y is one critical point of g. Define $m = \inf_{\mathcal{N}}\varphi$. By (3.3), we have that $m \geq \inf_{\partial B_\rho}\varphi \geq \sigma > 0$.

Lemma 12. *Assume that the conditions (H_0) and (H_7) hold; then, there is $u \in \mathcal{N}$ so that $\varphi(u) = m$.*

Proof. According to the continuity of I_j and f and Lemma 4, it is easy to verify that φ is weakly lower semi-continuous. Let $\{u_k\} \subset \mathcal{N}$ be the minimization sequence of φ that satisfies $\varphi(u_k) \to \inf_{\mathcal{N}}\varphi = m$, so

$$\varphi(u_k) = m + o(1),\ G(u_k) = 0. \tag{3.6}$$

Now, we show that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. Otherwise, $\|u_k\| \to \infty$ as $k \to \infty$. For $u \in E_0^{\alpha,p}\setminus\{0\}$, choose $v_k = \frac{u_k}{\|u_k\|}$; then, $\|v_k\| = 1$. Since $E_0^{\alpha,p}$ is one reflexive Banach space, there is one subsequence of $\{v_k\}$ (still denoted as $\{v_k\}$) such that $v_k \rightharpoonup v$ in $E_0^{\alpha,p}$; then, $v_k \to v$ in $C([0,T],\mathbb{R})$. On the one hand, combining (2.8) and (H_2), one has

$$\int_0^T F(t,u_k)dt = \frac{1}{p}\|u_k\|^p + \sum_{j=1}^{m}\int_0^{u_k(t_j)} I_j(t)dt - \varphi(u_k) \leq \frac{1}{p}\|u_k\|^p + \sum_{j=1}^{m} a_j\Lambda_\infty\|u_k\| + \sum_{j=1}^{m} d_j\frac{\Lambda_\infty^{\gamma_j+1}}{\gamma_j+1}\|u\|^{\gamma_j+1} + M_1,$$

where $M_1 > 0$. Because $\gamma_j \in [0, p-1),\ p > 1,\ \theta > p$, we have that

$$\int_0^T \frac{F(t,u_k)}{\|u_k\|^\theta}dt \leq o(1),\ k \to \infty. \tag{3.7}$$

On the other side, according to the continuity of f, there is $M_2 > 0$ so that

$$|uf(t,u) - \theta F(t,u)| \leq M_2,\ \forall\, |u| \leq L,\ t \in [0,T].$$

Combining the condition (H_6), we have

$$uf(t,u) - \theta F(t,u) \geq -M|u|^p - M_2,\ \forall\, |u| \in \mathbb{R},\ t \in [0,T]. \tag{3.8}$$

Combining the conditions (H_1) and (H_2), we get

$$\begin{aligned} m + o(1) = \varphi(u_k) &= \frac{1}{p}\|u_k\|^p + \sum_{j=1}^{m}\int_0^{u_k(t_j)} I_j(t)dt - \int_0^T F(t, u_k(t))dt \\ &\geq \frac{1}{p}\|u_k\|^p - \frac{1}{\theta}\int_0^T u_k(t)f(t, u_k(t))dt - \frac{M}{\theta}\int_0^T |u(t)|^p dt - \frac{M_2 T}{\theta} \\ &\geq (\frac{1}{p} - \frac{1}{\theta})\|u_k\|^p - \frac{1}{\theta}\sum_{j=1}^{m} I_j(u_k(t_j))u_k(t_j) - \frac{MT}{\theta}\|u\|_\infty^p - \frac{M_2 T}{\theta} \\ &\geq (\frac{1}{p} - \frac{1}{\theta})\|u_k\|^p - \frac{1}{\theta}\sum_{j=1}^{m} a_j\|u_k\|_\infty - \frac{1}{\theta}\sum_{j=1}^{m} d_j\|u_k\|_\infty^{\gamma_j+1} - \frac{MT}{\theta}\|u\|_\infty^p - \frac{M_2 T}{\theta}. \end{aligned}$$

This means that there is $M_3 > 0$ so that $\lim_{k\to\infty}\|v_k\|_\infty = \lim_{k\to\infty}\frac{\|u_k\|_\infty}{\|u_k\|} \geq M_3 \geq 0$. Therefore, $v \neq 0$. Let $\Omega_1 = \{t \in [0, T] : v \neq 0\}$ and $\Omega_2 = [0, T]\backslash\Omega_1$. According to the condition (H_7), there exists $M_4 > 0$ so that $F(t, u) \geq 0$, $\forall t \in [0, T]$ and $|u| \geq M_4$. Combining with the condition (H_5), there exist $M_5, M_6 > 0$ so that $F(t, u) \geq -M_5 u^p - M_6$, $\forall t \in [0, T]$, $u \in \mathbb{R}$. According to the Fatou lemma, one has $\liminf_{k\to\infty}\int_{\Omega_2}\frac{F(t,u_k)}{\|u_k\|^\theta}dt > -\infty$. Combining with the condition (H_7), for $t \in [0, T]$, one has

$$\liminf_{k\to\infty}\int_0^T \frac{F(t, u_k)}{\|u_k\|^\theta}dt = \liminf_{k\to\infty}\int_{\Omega_1}\frac{F(t, u_k)}{|u_k|^\theta}|v_k|^\theta dt + \liminf_{k\to\infty}\int_{\Omega_2}\frac{F(t, u_k)}{|u_k|^\theta}|v_k|^\theta dt \to \infty.$$

This contradicts (3.7). So, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded. Assume that $\{u_k\}_{k\in\mathbb{N}}$ possesses one subsequence, still recorded as $\{u_k\}_{k\in\mathbb{N}}$; there exists $u \in E_0^{\alpha,p}$ so that $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$, so $u_k \to u$ in $C([0, T], \mathbb{R})$. For the last step, we show that $u \neq 0$. According to the condition (H_5), for $\forall\varepsilon > 0$, there exists $\delta > 0$ so that

$$f(t, u)u \leq \varepsilon|u|^p, \ \forall(t, u) \in [0, T] \times [-\delta, \delta]. \tag{3.9}$$

Suppose that $\|u_k\|_\infty \leq \delta$; for $u_k \in \mathcal{N}$, by (H_2) and (3.9), we obtain

$$\begin{aligned} \Lambda_\infty^{-p}\|u_k\|_\infty^p \leq \|u_k\|^p &= \int_0^T f(t, u_k(t))u_k(t)dt - \sum_{j=1}^{m} I_j(u_k(t_j))u_k(t_j) \\ &\leq \varepsilon T\|u_k\|_\infty^p - \sum_{j=1}^{m} a_j\|u_k\|_\infty - \sum_{j=1}^{m} d_j\|u_k\|_\infty^{\gamma_j+1}. \end{aligned}$$

There is one contradiction in the above formula, so the hypothesis is not valid, namely, $\|u\|_\infty = \lim_{k\to\infty}\|u_k\|_\infty \geq \delta > 0$, so $u \neq 0$. According to Lemma 11, there is one unique $y > 0$ so that $yu \in \mathcal{N}$. Because φ is weakly lower semi-continuous,

$$m \leq \varphi(yu) \leq \varliminf_{k\to\infty}\varphi(yu_k) \leq \lim_{k\to\infty}\varphi(yu_k). \tag{3.10}$$

For $\forall u_k \in \mathcal{N}$, by (3.4) and (3.5), we get that $y_k = 1$ is one global maximum point of g, so $\varphi(yu_k) \leq \varphi(u_k)$. Combined with (3.10), one has $m \leq \varphi(yu) \leq \lim_{k\to\infty}\varphi(u_k) = m$. Therefore, m is obtained at $yu \in \mathcal{N}$.

The proof process of Theorem 2 is given below.

Proof of Theorem 2. By Lemmas 11 and 12, we know that there exists $u \in \mathcal{N}$ so that $\varphi(u) = m = \inf_{\mathcal{N}}\varphi > 0$, i.e., u is the non-zero critical point of $\varphi|_{\mathcal{N}}$. By Lemma 10, one has $\varphi'(u) = 0$; thus, u is the non-trivial ground-state solution of Problem (1.1).

3.2. Proof of Theorem 3

Lemma 13. *Let $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ $I_j \in C(\mathbb{R}, \mathbb{R})$. Assume that the conditions (H_0) and (H_8)–(H_{15}) hold. Then, φ satisfies the $(\mathrm{PS})_c$ condition.*

Proof. Assume that there is the sequence $\{u_n\}_{n\in\mathbb{N}} \subset E_0^{\alpha,p}$ so that $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ $(n \to \infty)$; then, there is $c_1 > 0$ so that, for $n \in \mathbb{N}$, we have

$$|\varphi(u_n)| \le c_1, \ \|\varphi'(u_n)\|_{\left(E_0^{\alpha,p}\right)^*} \le c_1, \tag{3.11}$$

where $\left(E_0^{\alpha,p}\right)^*$ is the conjugate space of $E_0^{\alpha,p}$. Next, let us verify that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $E_0^{\alpha,p}$. If not, we assume that $\|u_n\| \to +\infty$ as $(n \to \infty)$. Let $v_n = \frac{u_n}{\|u_n\|}$; then, $\|v_n\| = 1$. Since $E_0^{\alpha,p}$ is one reflexive Banach space, there is one subsequence of $\{v_n\}$ (still denoted as $\{v_n\}$), so that $v_n \rightharpoonup v$ $(n \to \infty)$ in $E_0^{\alpha,p}$; then, $v_n \to v$ in $C([0,T], \mathbb{R})$. By (H_{11}) and (H_{12}), we get

$$|f_2(t,u) \cdot u| \le b_1(t)|u|^r, \ \ |F_2(t,u)| \le \frac{1}{r} b_1(t)|u|^r. \tag{3.12}$$

Two cases are discussed below.

Case 1: $v \ne 0$. Let $\Omega = \{t \in [0,T] \,|\, |v(t)| > 0\}$; then, meas$(\Omega) > 0$. Because $\|u_n\| \to +\infty$ $(n \to \infty)$ and $|u_n(t)| = |v_n(t)| \cdot \|u_n\|$, so for $t \in \Omega$, one has $|u_n(t)| \to +\infty$ $(n \to \infty)$. On the one side, by (2.6), (2.8), (3.11), (3.12) and (H_{13}), one has

$$\begin{aligned}\int_0^T F_1(t,u_n)dt &= \frac{1}{p}\|u_n\|^p + \sum_{j=1}^m \int_0^{u_n(t_j)} I_j(t)dt - \int_0^T F_2(t,u_n)dt - \varphi(u_n)\\ &\le \frac{1}{p}\|u_n\|^p + \sum_{j=1}^m a_j\Lambda_\infty\|u_n\| + \sum_{j=1}^m d_j \Lambda_\infty^{\gamma_j+1}\|u_n\|^{\gamma_j+1} + \frac{T}{r}\Lambda_\infty^r \|b_1\|_\infty \|u_n\|^r + c_1.\end{aligned}$$

Since $\gamma_j \in [0, \theta - 1)$, $\theta > p > r > 1$,

$$\lim_{n\to\infty} \int_0^T \frac{F_1(t,u_n)}{\|u_n\|^\theta} dt \le o(1), \ n \to \infty. \tag{3.13}$$

On the other side, Fatou's lemma combines with the properties of Ω and (H_{10}), so we get

$$\lim_{n\to\infty} \int_0^T \frac{F_1(t,u_n)}{\|u_n\|^\theta} dt \ge \lim_{n\to\infty} \int_\Omega \frac{F_1(t,u_n)}{\|u_n\|^\theta} dt = \lim_{n\to\infty} \int_\Omega \frac{F_1(t,u_n)}{|u_n(t)|^\theta} |v_n(t)|^\theta dt = +\infty.$$

This contradicts (3.13).

Case 2: $v \equiv 0$. From (H_8), for $\forall \varepsilon > 0$, there is $L_0 > 0$, so that $|f_1(t,u)| \le \varepsilon|u|^{p-1}$, $|u| \le L_0$. So, for $|u| \le L_0$, there is $\varepsilon_0 > 0$ so that $|uf_1(t,u) - \theta F_1(t,u)| \le \varepsilon_0(1+\theta)u^p$. For $(t,u) \in [0,T] \times [L_0, L]$, there is $c_2 > 0$ so that $|uf_1(t,u) - \theta F_1(t,u)| \le c_2$. Combined with the condition (H_9), one has

$$uf_1(t,u) - \theta F_1(t,u) \ge -\varepsilon_0(1+\theta)u^p - c_2, \ \forall (t,u) \in [0,T] \times \mathbb{R}. \tag{3.14}$$

By (H_{14}), we obtain that there exists $c_3 > 0$, such that

$$\theta \sum_{j=1}^m \int_0^{u_n(t_j)} I_j(t)dt - \sum_{j=1}^m I_j(u_n(t_j))u_n(t_j) \ge -c_3. \tag{3.15}$$

By (2.6), (2.8), (2.9), (3.11), (3.12), (3.14) and (3.15), we get that there exists $c_4 > 0$ such that

$$
\begin{aligned}
o(1) &= \frac{\theta c_1 + c_1 \|u_n\|}{\|u_n\|^p} \geq \frac{\theta\varphi(u_n) - \langle \varphi'(u_n), u_n\rangle}{\|u_n\|^p} \\
&= \left(\frac{\theta}{p} - 1\right) + \frac{1}{\|u_n\|^p}\left[\theta \sum_{j=1}^{m} \int_0^{u_n(t_j)} I_j(t)dt - \sum_{j=1}^{m} I_j(u_n(t_j))u_n(t_j)\right] \\
&+ \frac{1}{\|u_n\|^p}\int_0^T \left[u_n f_1(t,u_n) - \theta F_1(t,u_n)\right]dt + \frac{1}{\|u_n\|^p}\int_0^T \left[u_n f_2(t,u_n) - \theta F_2(t,u_n)\right]dt \\
&\geq \left(\frac{\theta}{p} - 1\right) + \frac{1}{\|u_n\|^p}\int_0^T \left[-\varepsilon_0(1+\theta)u_n^p - c_2\right]dt - \frac{1}{\|u_n\|^p}\left(\frac{\theta}{r} + 1\right)\int_0^T b_1(t)|u_n|^r dt - \frac{1}{\|u_n\|^p}c_3 \\
&\geq \left(\frac{\theta}{p} - 1\right) - \varepsilon_0(1+\theta)\int_0^T \frac{|u_n|^p}{\|u_n\|^p}dt - \frac{Tc_2}{\|u_n\|^p} - \frac{1}{\|u_n\|^p}\left(\frac{\theta}{r} + 1\right)\|b_1\|_{L^1}\|u_n\|_\infty^r - \frac{1}{\|u_n\|^p}c_3 \\
&\geq \left(\frac{\theta}{p} - 1\right) - \varepsilon_0(1+\theta)\int_0^T |v_n|^p dt - \frac{Tc_2}{\|u_n\|^p} - \left(\frac{\theta}{r} + 1\right)\|b_1\|_{L^1}\Lambda_\infty^r\|u_n\|^{r-p} \geq \left(\frac{\theta}{p} - 1\right),\ n \to \infty.
\end{aligned}
$$

It is a contradiction. Thus, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $E_0^{\alpha,p}$. Assume that the sequence $\{u_n\}_{n\in\mathbb{N}}$ possesses one subsequence, still recorded as $\{u_n\}_{n\in\mathbb{N}}$; there exists $u \in E_0^{\alpha,p}$ so that $u_n \rightharpoonup u$ in $E_0^{\alpha,p}$; then, $u_n \to u$ in $C([0,T],\mathbb{R})$. Therefore,

$$
\begin{cases}
\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle \to 0,\ n \to \infty, \\
\displaystyle\int_0^T \left[f(t,u_n(t)) - f(t,u(t))\right]\left[u_n(t) - u(t)\right]dt \to 0,\ n \to \infty, \\
\displaystyle\sum_{j=1}^{m} \Big(I_j(u_n(t_j)) - I_j(u(t_j))\Big)\Big(u_n(t_j) - u(t_j)\Big) \to 0,\ n \to \infty, \\
\displaystyle\int_0^T a(t)\Big(\phi_p(u_n(t)) - \phi_p(u(t))\Big)(u_n(t) - u(t))dt \to 0,\ n \to \infty.
\end{cases}
\tag{3.16}
$$

Through (2.9), we can get

$$
\begin{aligned}
\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle &= \int_0^T \Big(\phi_p({}_0D_t^\alpha u_n(t)) - \phi_p({}_0D_t^\alpha u(t))\Big)({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t))dt \\
+ \int_0^T a(t)\Big(\phi_p(u_n(t)) - \phi_p(u(t))\Big)&(u_n(t) - u(t))dt + \sum_{j=1}^{m}\Big(I_j(u_n(t_j)) - I_j(u(t_j))\Big)\Big(u_n(t_j) - u(t_j)\Big) \\
- \int_0^T \left[f(t,u_n(t)) - f(t,u(t))\right]&\left[u_n(t) - u(t)\right]dt.
\end{aligned}
\tag{3.17}
$$

From [23], we obtain

$$
\begin{aligned}
&\int_0^T \Big(\phi_p({}_0D_t^\alpha u_n(t)) - \phi_p({}_0D_t^\alpha u(t))\Big)\left({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\right)dt \\
&\geq \begin{cases} c\int_0^T \left|{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\right|^p dt,\ p \geq 2, \\ c\int_0^T \frac{\left|{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\right|^2}{\left(\left|{}_0D_t^\alpha u_n(t)\right| + \left|{}_0D_t^\alpha u(t)\right|\right)^{2-p}}dt,\ 1 < p < 2. \end{cases}
\end{aligned}
\tag{3.18}
$$

If $p \geq 2$, by (3.16)–(3.18), one has $\|u_n - u\| \to 0$ $(n \to \infty)$. If $1 < p < 2$, by the Hölder inequality, one has $\int_0^T \left|{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\right|^p dt \leq c\left(\int_0^T \frac{\left|{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\right|^2}{\left(\left|{}_0D_t^\alpha u_n(t)\right| + \left|{}_0D_t^\alpha u(t)\right|\right)^{2-p}} dt\right)^{\frac{p}{2}} (\|u_n\| + \|u\|)^{\frac{p(2-p)}{2}}$. Thus,

$$\begin{aligned}&\int_0^T \left(\phi_p({}_0D_t^\alpha u_n(t)) - \phi_p({}_0D_t^\alpha u(t))\right)({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t))\,dt \\ &\geq \frac{c}{(\|u_n\| + \|u\|)^{2-p}}\left(\int_0^T \left|{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\right|^p dt\right)^{\frac{2}{p}}.\end{aligned} \tag{3.19}$$

By (3.16), (3.17) and (3.19), one has $\|u_n - u\| \to 0$ $(n \to \infty)$. Hence, φ satisfies the $(\mathrm{PS})_c$ condition.

The proof of Theorem 3.

Step 1. Clearly, $\varphi(0) = 0$. Lemma 13 implies that $\varphi \in C^1\left(E_0^{\alpha,p}, \mathbb{R}\right)$ satisfies the $(\mathrm{PS})_c$ condition.

Step 2. For $\forall\varepsilon_1 > 0$, we know from (H_8) that there is $\delta > 0$ so that

$$F_1(t,u) \leq \varepsilon_1|u|^p, \ \forall t \in [0,T], \ |u| \leq \delta. \tag{3.20}$$

For $\forall u \in E_0^{\alpha,p}$, by (2.5), (2.6), (2.8), (3.12) and (H_{15}), we get

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t)dt - \int_0^T F(t,u(t))dt \geq \frac{1}{p}\|u\|^p - \int_0^T F(t,u(t))\,dt \\ &\geq \frac{1}{p}\|u\|^p - \varepsilon_1\int_0^T |u|^p dt - \frac{1}{r}\int_0^T b_1(t)|u|^r dt \geq \frac{1}{p}\|u\|^p - \varepsilon_1\Lambda_p^p\|u\|^p - \frac{1}{r}\|b_1\|_{L^1}\|u\|_\infty^r \\ &\geq \left(\frac{1}{p} - \varepsilon_1\Lambda_p^p - \frac{\Lambda_\infty^r}{r}\|b_1\|_{L^1}\|u\|^{r-p}\right)\|u\|^p.\end{aligned} \tag{3.21}$$

Choose $\varepsilon_1 = \frac{1}{2p\Lambda_p^p}$; one has $\varphi(u) \geq \left(\frac{1}{2p} - \frac{\Lambda_\infty^r}{r}\|b_1\|_{L^1}\|u\|^{r-p}\right)\|u\|^p$. Let $\rho = \left(\frac{r}{4p\Lambda_\infty^r\|b_1\|_{L^1}}\right)^{\frac{1}{r-p}}$ and $\eta = \frac{1}{4p}\rho^p$; then, for $u \in \partial B_\rho$, we obtain $\varphi(u) \geq \eta > 0$.

Step 3. From (H_{10}), for $|u| \geq L_1$, there exist $\varepsilon_2, \ \varepsilon_3 > 0$ such that

$$F_1(t,u) \geq \varepsilon_2|u|^\theta - \varepsilon_3. \tag{3.22}$$

By (H_8), for $|u| \leq L_1$, there exist $\varepsilon_4, \ \varepsilon_5 > 0$ such that

$$F_1(t,u) \geq -\varepsilon_4 u^p - \varepsilon_5. \tag{3.23}$$

From (3.22) and (3.23), we obtain that there exist $\varepsilon_6, \ \varepsilon_7 > 0$ so that

$$F_1(t,u) \geq \varepsilon_2|u|^\theta - \varepsilon_6 u^p - \varepsilon_7, \ \forall t \in [0,T], \ u \in \mathbb{R}, \tag{3.24}$$

where $\varepsilon_6 = \varepsilon_2 L_1^{\theta-p} + \varepsilon_4$. For $\forall u \in E_0^{\alpha,p}\backslash\{0\}$ and $\xi \in \mathbb{R}^+$, by (H_{13}), (2.5), (2.6), (2.8), (3.24) and the Hölder inequality, we have

$$\begin{aligned}\varphi(\xi u) &\leq \frac{\xi^p}{p}\|u\|^p + \sum_{j=1}^m a_j\xi\Lambda_\infty\|u\| + \sum_{j=1}^m d_j\xi^{\gamma_j+1}\Lambda_\infty^{\gamma_j+1}\|u\|^{\gamma_j+1} - \varepsilon_2\xi^\theta\int_0^T |u|^\theta dt + \varepsilon_6\Lambda_p^p\xi^p\|u\|^p + \varepsilon_7 T \\ &\leq \left(\frac{1}{p} + \varepsilon_6\Lambda_p^p\right)\xi^p\|u\|^p + \sum_{j=1}^m a_j\xi\Lambda_\infty\|u\| + \sum_{j=1}^m d_j\xi^{\gamma_j+1}\Lambda_\infty^{\gamma_j+1}\|u\|^{\gamma_j+1} - \varepsilon_2\xi^\theta\left(T^{\frac{p-\theta}{\theta}}\int_0^T |u(t)|^p dt\right)^{\frac{\theta}{p}} + \varepsilon_7 T \\ &\leq \left(\frac{1}{p} + \varepsilon_6\Lambda_p^p\right)\xi^p\|u\|^p + \sum_{j=1}^m a_j\xi\Lambda_\infty\|u\| + \sum_{j=1}^m d_j\xi^{\gamma_j+1}\Lambda_\infty^{\gamma_j+1}\|u\|^{\gamma_j+1} - \varepsilon_2\xi^\theta T^{\frac{p-\theta}{p}}\|u\|_{L^p}^\theta + \varepsilon_7 T.\end{aligned}$$

Since $\theta > p > 1$ and $\gamma_j + 1 \in [1, \theta)$, the above inequality indicates that $\varphi(\xi_0 u) \to -\infty$ when ξ_0 is large enough. Let $e = \xi_0 u$; one has $\varphi(e) < 0$. Thus, the condition (ii) in Lemma 8 holds. Lemma 8 implies that φ possesses one critical value $c^{(1)} \geq \eta > 0$. The specific form is $c^{(1)} = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi(g(s))$, where $\Gamma = \left\{g \in C\left([0,1], E_0^{\alpha,p}\right) : g(0) = 0, g(1) = e\right\}$. Hence, there is $0 \neq u^{(1)} \in E_0^{\alpha,p}$ so that

$$\varphi\left(u^{(1)}\right) = c^{(1)} \geq \eta > 0,\ \varphi'\left(u^{(1)}\right) = 0. \tag{3.25}$$

Step 4. Equation (3.21) implies that φ is bounded below in $\overline{B_\rho}$. Choose $\sigma \in E_0^{\alpha,p}$ so that $\sigma(t) \neq 0$ in $[0, T]$. For $\forall\ l \in (0, +\infty)$, by (2.6), (2.8), (H_{10}), (H_{11}) and (H_{13}), we have

$$\begin{aligned} \varphi(l\sigma) &\leq \frac{l^p}{p}\|\sigma\|^p + \sum_{j=1}^{m} a_j l \Lambda_\infty \|\sigma\| + \sum_{j=1}^{m} d_j l^{\gamma_j+1} \Lambda_\infty^{\gamma_j+1} \|\sigma\|^{\gamma_j+1} - \int_0^T F_2(t, l\sigma(t))\, dt \\ &\leq \frac{l^p}{p}\|\sigma\|^p + \sum_{j=1}^{m} a_j l \Lambda_\infty \|\sigma\| + \sum_{j=1}^{m} d_j l^{\gamma_j+1} \Lambda_\infty^{\gamma_j+1} \|\sigma\|^{\gamma_j+1} - l^r \int_0^T b(t) |\sigma(t)|^r dt. \end{aligned} \tag{3.26}$$

Thus, from $1 < r < p$ and $\gamma_j \in [0, \theta - 1)$, we know that for a small enough l_0 satisfying $\|l_0\sigma\| \leq \rho$, one has $\varphi(l_0\sigma) < 0$. Let $u = l_0\sigma$; we have that $c^{(2)} = \inf \varphi(u) < 0, \quad \|u\| \leq \rho$. Ekeland's variational principle shows that there is one minimization sequence $\{v_k\}_{k\in\mathbb{N}} \subset \overline{B_\rho}$ so that $\varphi(v_k) \to c^{(2)}$ and $\varphi'(v_k) \to 0,\ k \to \infty$, i.e., $\{v_k\}_{k\in\mathbb{N}}$ is one $(\mathrm{PS})_c$ sequence. Lemma 13 shows that φ satisfies the $(\mathrm{PS})_c$ condition. Thus, $c^{(2)} < 0$ is another critical value of φ. So, there exists $0 \neq u^{(2)} \in E_0^{\alpha,p}$ so that $\varphi\left(u^{(2)}\right) = c^{(2)} < 0,\ \left\|u^{(2)}\right\| < \rho$.

3.3. Proof of Theorem 4

Proof. The functionals $\Phi : E_0^{\alpha,p} \to \mathbb{R}$ and $\Psi : E_0^{\alpha,p} \to \mathbb{R}$ are defined as follows:

$$\Phi(u) = \frac{1}{p}\|u\|^p, \Psi(u) = \int_0^T F(t, u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{n} J_j(u(t_j));$$

then, $\varphi(u) = \Phi(u) - \lambda\Psi(u)$. We can calculate that

$$\inf_{u\in E_0^{\alpha,p}} \Phi(u) = \Phi(0) = 0, \Psi(0) = \int_0^T F(t, 0)dt - \frac{\mu}{\lambda}\sum_{j=1}^{n} J_j(0) = 0.$$

Furthermore, Φ and Ψ are continuous Gâteaux differentiable and

$$\langle \Phi'(u), v\rangle = \int_0^T \phi_p({}_0D_t^\alpha u(t))\,{}_0D_t^\alpha v(t)dt + \int_0^T a(t)\phi_p(u(t))v(t)dt, \tag{3.27}$$

$$\langle \Psi'(u), v\rangle = \int_0^T f(t, u(t))v(t)dt - \frac{\mu}{\lambda}\sum_{j=1}^{n} I_j(u(t_j))v(t_j), \forall u, v \in E_0^{\alpha,p}. \tag{3.28}$$

In addition, $\Phi' : E_0^{\alpha,p} \to \left(E_0^{\alpha,p}\right)^*$ is continuous. It is proved that $\Psi' : E_0^{\alpha,p} \to \left(E_0^{\alpha,p}\right)^*$ is a continuous compact operator. Suppose that $\{u_n\} \subset E_0^{\alpha,p}$, $u_n \rightharpoonup u$, $n \to \infty$; then, $\{u_n\}$ uniformly converges to u

on $C([0,T])$. Owing to $f \in C([0,T]\times\mathbb{R},\mathbb{R})$ and $I_j \in C(\mathbb{R},\mathbb{R})$, we have that $f(t,u_n) \to f(t,u)$ and $I_j(u_n(t_j)) \to I_j(u(t_j))$, $n \to \infty$. Thus, $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Then, Ψ' is strongly continuous. According to Proposition 26.2 in [24], Ψ' is one compact operator. It is proved that Φ is weakly semi-continuous. Suppose that $\{u_n\} \subset E_0^{\alpha,p}$, $\{u_n\} \rightharpoonup u$; then, $\{u_n\} \to u$ on $C([0,T])$, and $\liminf\limits_{n\to\infty}\|u_n\| \geq \|u\|$. So, $\liminf\limits_{n\to\infty}\Phi(u_n) = \liminf\limits_{n\to\infty}\left(\frac{1}{p}\|u_n\|^p\right) \geq \frac{1}{p}\|u\|^p = \Phi(u)$. Thus, Φ is weakly semi-continuous. Because $\Phi(u) = \frac{1}{p}\|u\|^p \to +\infty$ and $\|u\| \to +\infty$, Φ is coercive. By (3.27), we obtain

$$\begin{aligned}\langle \Phi'(u) - \Phi'(v), u - v\rangle &= \int_0^T \left(\phi_p({}_0D_t^\alpha u(t)) - \phi_p({}_0D_t^\alpha v(t))\right)({}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t))dt \\ &+ \int_0^T a(t)\left(\phi_p(u(t)) - \phi_p(v(t))\right)(u(t) - v(t))dt,\ \forall u, v \in E_0^{\alpha,p}.\end{aligned}$$

From [23], we know that there is $c > 0$ so that

$$\begin{aligned}&\int_0^T \left(\phi_p({}_0D_t^\alpha u(t)) - \phi_p({}_0D_t^\alpha v(t))\right)({}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t))\,dt \\ &\geq \begin{cases} c\int_0^T \left|{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)\right|^p dt, & p \geq 2, \\ c\int_0^T \frac{\left|{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)\right|^2}{\left(\left|{}_0D_t^\alpha u(t)\right| + \left|{}_0D_t^\alpha v(t)\right|\right)^{2-p}} dt, & 1 < p < 2.\end{cases}\end{aligned} \tag{3.29}$$

If $p \geq 2$, then $\langle \Phi'(u) - \Phi'(v), u - v\rangle \geq c\|u - v\|^p$. Thus, Φ' is uniformly monotonous. When $1 < p < 2$, the Hölder inequality implies

$$\begin{aligned}&\int_0^T \left|{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)\right|^p dt \\ &\leq \left(\int_0^T \frac{\left|{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)\right|^2}{\left(\left|{}_0D_t^\alpha u(t)\right| + \left|{}_0D_t^\alpha v(t)\right|\right)^{2-p}} dt\right)^{\frac{p}{2}} \left(\int_0^T \left(\left|{}_0D_t^\alpha u(t)\right| + \left|{}_0D_t^\alpha v(t)\right|\right)^p dt\right)^{\frac{2-p}{2}} \\ &\leq c_1\left(\int_0^T \frac{\left|{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)\right|^2}{\left(\left|{}_0D_t^\alpha u(t)\right| + \left|{}_0D_t^\alpha v(t)\right|\right)^{2-p}} dt\right)^{\frac{p}{2}} (\|u\|^p + \|v\|^p)^{\frac{2-p}{2}},\end{aligned}$$

where $c_1 = 2^{\frac{(p-1)(2-p)}{2}} > 0$. Then,

$$\int_0^T \frac{\left|{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)\right|^2}{\left(\left|{}_0D_t^\alpha u(t)\right| + \left|{}_0D_t^\alpha v(t)\right|\right)^{2-p}} dt \geq \frac{c_2}{(\|u\| + \|v\|)^{2-p}}\left(\int_0^T \left|{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)\right|^p dt\right)^{\frac{2}{p}},$$

where $c_2 = \frac{1}{c_1^{\frac{2}{p}}}$. Combined with (3.29), we can get

$$\int_0^T \left(\phi_p({}_0D_t^\alpha u(t)) - \phi_p({}_0D_t^\alpha v(t))\right)({}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t))\,dt \geq \frac{c}{(\|u\| + \|v\|)^{2-p}}\left(\int_0^T \left|{}_0D_t^\alpha u(t) - {}_0D_t^\alpha v(t)\right|^p dt\right)^{\frac{2}{p}}. \tag{3.30}$$

Thus, $\langle \Phi'(u) - \Phi'(v), u - v\rangle \geq \frac{c\|u-v\|^2}{(\|u\|+\|v\|)^{2-p}}$. So, Φ' is strictly monotonous. Theorem 26.A(d) in [24] implies that $(\Phi')^{-1}$ exists and is continuous. If $x \in E_0^{\alpha,p}$ satisfies $\Phi(x) = \frac{1}{p}\|x\|^p \leq r$, then, by (2.6), we obtain $\Phi(x) \geq \frac{1}{p\Lambda_\infty^p}\|x\|_\infty^p$, and

$$\left\{x \in E_0^{\alpha,p} : \Phi(x) \leq r\right\} \subseteq \left\{x : \frac{1}{p\Lambda_\infty^p}\|x\|_\infty^p \leq r\right\} = \{x : \|x\|_\infty^p \leq pr\Lambda_\infty^p\} = \left\{x : \|x\|_\infty \leq \Lambda_\infty(pr)^{\frac{1}{p}}\right\}.$$

Therefore, from $\lambda > 0$ and $\mu \geq 0$, we have

$$\begin{aligned}
&\sup\{\Psi(x) : \Phi(x) \leq r\} = \sup\left\{\int_0^T F(t, x(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n J_j(x(t_j)) : \Phi(x) \leq r\right\}\\
&\leq \int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}} F(t,x)dt + \frac{\mu}{\lambda}\max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}}\sum_{j=1}^n\left(-J_j(x)\right).
\end{aligned}$$

If $\max\limits_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}}\sum\limits_{j=1}^n\left(-J_j(x)\right) = 0$, by $\lambda < A_r$, we get

$$\sup\{\Psi(x) : \Phi(x) \leq r\} < \frac{r}{\lambda}. \tag{3.31}$$

If $\max\limits_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}}\sum\limits_{j=1}^n\left(-J_j(x)\right) > 0$, for $\mu \in [0,\gamma)$, $\gamma = \min\left\{\frac{r-\lambda\int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{1/p}} F(t,x)dt}{\max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}}\sum_{j=1}^n\left(-J_j(x)\right)}, \frac{\lambda\int_0^T F(t,\omega)dt - \frac{1}{p}\|\omega\|^p}{\sum_{j=1}^n J_j(\omega(t_j))}\right\}$, we have

$$\begin{aligned}
\sup\{\Psi(x) : \Phi(x) \leq r\} &\leq \int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}} F(t,x)dt + \frac{\mu}{\lambda}\max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}}\sum_{j=1}^n\left(-J_j(x)\right)\\
&< \int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}} F(t,x)dt + \frac{\frac{r-\lambda\int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}} F(t,x)dt}{\max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}}\sum_{j=1}^n\left(-J_j(x)\right)}}{\lambda} \times \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}}\sum_{j=1}^n\left(-J_j(x)\right)\\
&< \int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}} F(t,x)dt + \frac{r-\lambda\int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}} F(t,x)\,dt}{\lambda}\\
&= \int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}} F(t,x)dt + \frac{r}{\lambda} - \int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{\frac{1}{p}}} F(t,x)\,dt\\
&= \frac{r}{\lambda}.
\end{aligned}$$

Thus, (3.31) is also true. On the other side, for $\mu < \gamma$, one has

$$\Psi(\omega) = \int_0^T F(t,\omega(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n J_j(\omega(t_j)) > \frac{\Phi(\omega)}{\lambda}. \tag{3.32}$$

By combining (3.31) and (3.32), we obtain that $\frac{\Psi(\omega)}{\Phi(\omega)} > \frac{1}{\lambda} > \frac{\sup\{\Psi(x):\Phi(x)\leq r\}}{r}$. This means that the condition (i) of Lemma 9 holds.

Finally, for the third step, we show that, for any $\lambda \in \Lambda_r = (A_l, A_r)$, the functional $\Phi - \lambda\Psi$ is coercive. By (1.4), we obtain

$$\int_0^T F(t, x(t))dt \leq L \int_0^T \left(1 + |x(t)|^\beta\right)dt \leq LT + LT\,\|x\|_\infty^\beta \leq LT + LT\Lambda_\infty^\beta \|x\|^\beta,\ x \in E_0^{\alpha,p}. \tag{3.33}$$

and

$$-J_j(x(t_j)) \leq L_j\left(1 + \left|x(t_j)\right|^{d_j}\right) \leq L_j\left(1 + \|x\|_\infty^{d_j}\right) \leq L_j\left(1 + \Lambda_\infty^{d_j}\|x\|^{d_j}\right). \tag{3.34}$$

By (3.34), we get

$$\sum_{j=1}^n \left(-J_j(x(t_j))\right) \leq \sum_{j=1}^n L_j\left(1 + \Lambda_\infty^{d_j}\|x\|^{d_j}\right). \tag{3.35}$$

If $\frac{\mu}{\lambda} \geq 0$, for $x \in E_0^{\alpha,p}$, by (3.33) and (3.35), we have

$$\Psi(x) \leq LT + LT\Lambda_\infty^\beta\|x\|^\beta + \frac{\mu}{\lambda}\sum_{j=1}^n L_j\left(1 + \Lambda_\infty^{d_j}\|x\|^{d_j}\right) = LT + \frac{\mu}{\lambda}\sum_{j=1}^n L_j + LT\Lambda_\infty^\beta\|x\|^\beta + \frac{\mu}{\lambda}\sum_{j=1}^n L_j\Lambda_\infty^{d_j}\|x\|^{d_j}.$$

Thus, $\Phi(x) - \lambda\Psi(x) \geq \frac{1}{p}\|x\|^p - \lambda\left(LT + \frac{\mu}{\lambda}\sum_{j=1}^n L_j + LT\Lambda_\infty^\beta\|x\|^\beta + \frac{\mu}{\lambda}\sum_{j=1}^n L_j\Lambda_\infty^{d_j}\|x\|^{d_j}\right),\ \forall x \in E_0^{\alpha,p}$. If $0 < \beta$ and $d_j < p$, then $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda\Psi(x)) = +\infty,\ \lambda > 0$. Thus, $\Phi - \lambda\Psi$ is coercive. When $\beta = p$, $\Phi(x) - \lambda\Psi(x) \geq \left(\frac{1}{p} - \lambda LT\Lambda_\infty^p\right)\|x\|^p - \lambda\left(LT + \frac{\mu}{\lambda}\sum_{j=1}^n L_j + \frac{\mu}{\lambda}\sum_{j=1}^n L_j\Lambda_\infty^{d_j}\|x\|^{d_j}\right)$. Choose $L < \frac{\int_0^T \max_{|x|\leq\Lambda_\infty(pr)^{1/p}} F(t,x)dt}{prT\Lambda_\infty^p}$. We have that $\frac{1}{p} - \lambda LT\Lambda_\infty^p > 0$, for all $\lambda < A_r$. If $0 < d_j < p$, we have that $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda\Psi(x)) = +\infty$, for all $\lambda < A_r$. Obviously, the functional $\Phi - \lambda\Psi$ is coercive. Lemma 9 shows that $\varphi = \Phi - \lambda\Psi$ possesses at least three different critical points in $E_0^{\alpha,p}$.

4. Conclusions

This paper studies the solvability of Dirichlet boundary-value problems of the fractional p-Laplacian equation with impulsive effects. For this kind of problems, the existence of solutions has been discussed in the past, while the ground-state solutions have been rarely studied. By applying the Nehari manifold method, we have obtained the existence result of the ground-state solution (see Theorem 2). At the same time, by the mountain pass theorem and three critical points theorem, some new existence results on this problem were achieved (see Theorems 3–5). In particular, this paper weakens the commonly used p-suplinear and p-sublinear growth conditions, to a certain extent, and expands and enriches the results of [14–16]. This theory can provide a solid foundation for studying similar fractional impulsive differential equation problems. For example, one can consider the solvability of Sturm-Liouville boundary-value problems of fractional impulsive equations with the p-Laplacian operator. In addition, the proposed theory can also be used to study the existence of solutions to the periodic boundary-value problems of the fractional p-Laplacian equation with impulsive effects and their corresponding coupling systems.

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Conflict of interest

The authors declare that there is no conflict of interest.

References

1. J. L. Zhou, S. Q. Zhang, Y. B. He, Existence and stability of solution for nonlinear differential equations with ψ-Hilfer fractional derivative, *Appl. Math. Lett.*, **121** (2021), 107457. https://doi.org/10.1016/j.aml.2021.107457
2. J. Li, X. N. Su, K. Y. Zhao, Barycentric interpolation collocation algorithm to solve fractional differential equations, *Math. Comput. Simul.*, **205** (2023), 340–367. https://doi.org/10.1016/j.matcom.2022.10.005
3. T. T. Xue, W. B. Liu, T. F. Shen, Extremal solutions for p-Laplacian boundary value problems with the right-handed Riemann-Liouville fractional derivative, *Math. Methods Appl. Sci.*, **42** (2019), 4394–4407. https://doi.org/10.1002/mma.5660
4. J. L. Zhou, S. Q. Zhang, Y. B. He, Existence and stability of solution for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **498** (2021), 124921. https://doi.org/10.1016/j.jmaa.2020.124921
5. C. L. Tian, T. Jin, X. F. Yang, Q. Y. Liu, Reliability analysis of the uncertain heat conduction model, *Comput. Math. Appl.*, **119** (2022), 131–140. https://doi.org/10.1016/J.CAMWA.2022.05.033
6. T. T. Xue, X. L. Fan, J. Zhu, A class of deterministic and stochastic fractional epidemic models with vaccination, *Comput. Math. Methods Med.*, **1797258** (2022), 1–22. https://doi.org/10.1155/2022/1797258
7. T. Jin, X. F. Yang, Monotonicity theorem for the uncertain fractional differential equation and application to uncertain financial market, *Math. Comput. Simul.*, **190** (2021), 203–221. https://doi.org/10.1016/J.MATCOM.2021.05.018
8. T. Jin, X. F. Yang, H. X. Xia, D. Hui, Reliability index and option pricing formulas of the first-hitting time model based on the uncertain fractional-order differential equation with Caputo type, *Fractals*, **29** (2021). https://doi.org/10.1142/S0218348X21500122
9. D. A. Benson, S. W. Wheatcraft, M. M. Meerschaert, Application of a fractional advection-dispersion equation, *Water Resour. Res.*, **36** (2000), 1403–1412. https://doi.org/10.1029/2000WR900031
10. D. A. Benson, S. W. Wheatcraft, M. M. Meerschaert, The fractional-order governing equation of levy motion, *Water Resour. Res.*, **36** (2000), 1413–1423. https://doi.org/10.1029/2000WR900032

11. V. J. Ervin, J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Methods Partial Differ. Equations*, **22** (2006), 558–576. https://doi.org/10.1002/num.20112

12. F. Jiao, Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, *Comput. Math. Appl.*, **62** (2011), 1181–1199. https://doi.org/10.1016/j.camwa.2011.03.086

13. F. Jiao, Y. Zhou, Existence results for fractional boundary value problem via critical point theory, *Int. J. Bifurcation Chaos*, **22** (2012), 1–17. https://doi.org/10.1142/S0218127412500861

14. G. Bonanno, R. Rodríguez-López, S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.*, **17** (2014), 717–744. https://doi.org/10.2478/s13540-014-0196-y

15. R. Rodríguez-López, S. Tersian, Multiple solutions to boundary value problem for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.*, **17** (2014), 1016–1038. https://doi.org/10.2478/s13540-014-0212-2

16. C. E. T. Ledesma, N. Nyamoradi, Impulsive fractional boundary value problem with p-Laplace operator, *J. Appl. Math. Comput.*, **55** (2017), 257–278. https://doi.org/10.1007/s12190-016-1035-6

17. I. Bogun, Existence of weak solutions for impulsive p-Laplacian problem with superlinear impulses, *Nonlinear Anal. Real World Appl.*, **13** (2012), 2701–2707. https://doi.org/10.1016/j.nonrwa.2012.03.014

18. J. F. Xu, Z. L. Wei, Y. Z. Ding, Existence of weak solutions for p-Laplacian problem with impulsive effects, *Taiwanese J. Math.*, **17** (2013), 501–515. https://doi.org/10.11650/tjm.17.2013.2081

19. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Chichester: Elsevier, Amsterdam, 2006.

20. D. Idczak, S. Walczak, Fractional Sobolev spaces via Riemann-Liouville derivatives, *J. Funct. Spaces*, **2013** (2013), 1–15. https://doi.org/10.1155/2013/128043

21. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, *CBMS Regional Conference Series in Mathematics*, **65** (1986).

22. G. Bonanno G, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, **89** (2010), 1–10. https://doi.org/10.1080/00036810903397438

23. J. Simon, Régularité de la solution d'un problème aux limites non linéaires, *Ann. Fac. Sci. Toulouse*, **3** (1981), 247–274. https://doi.org/10.5802/AFST.569

24. E. Zeidler, Nonlinear functional analysis and its applications, Springer New York, NY, 1990. https://doi.org/10.1007/978-1-4612-0981-2

Appendix

Proof of Theorem 5

Proof. This is similar to the proof process of Theorem 4. Since $\lambda > 0$, $\mu \in (\gamma^*, 0]$, one has

$$\sup\{\Psi(x) : \Phi(x) \le r\} = \sup\left\{\int_0^T F(t, x(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{n} J_j(x(t_j)) : \Phi(x) \le r\right\}$$
$$\le \int_0^T \max_{|x| \le \Lambda_\infty (pr)^{\frac{1}{p}}} F(t, x)dt - \frac{\mu}{\lambda} \max_{|x| \le \Lambda_\infty (pr)^{\frac{1}{p}}} \sum_{j=1}^{n} J_j(x).$$

If $\max\limits_{|x| \le \Lambda_\infty (pr)^{\frac{1}{p}}} \sum\limits_{j=1}^{n} J_j(x) = 0$, by $\lambda < A_r$, we obtain

$$\sup\{\Psi(x) : \Phi(x) \le r\} < \frac{r}{\lambda}. \tag{A.1}$$

For $\mu \in (\gamma^*, 0]$, if $\max\limits_{|x| \le \Lambda_\infty (pr)^{\frac{1}{p}}} \sum\limits_{j=1}^{n} J_j(x) > 0$, then (A.1) is also true. On the other hand, for $\mu \in (\gamma^*, 0]$, we have

$$\Psi(\omega) = \int_0^T F(t, \omega(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{n} J_j(\omega(t_j)) > \frac{\Phi(\omega)}{\lambda}. \tag{A.2}$$

Combining (A.1) and (A.2), we get $\frac{\Psi(\omega)}{\Phi(\omega)} > \frac{1}{\lambda} > \frac{\sup\{\Psi(x):\Phi(x) \le r\}}{r}$, which shows that the condition (i) of Lemma 9 holds. Finally, we show that $\Phi - \lambda\Psi$ is coercive for $\forall \lambda \in \Lambda_r = (A_l, A_r)$. For $x \in E_0^{\alpha,p}$, by (1.7), we get

$$\int_0^T F(t, x(t))dt \le LT + LT\Lambda_\infty^{\beta}\|x\|^{\beta}, \ J_j(x(t_j)) \le L_j\left(1 + \Lambda_\infty^{d_j}\|x\|^{d_j}\right). \tag{A.3}$$

So,

$$\sum_{j=1}^{n} J_j(x(t_j)) \le \sum_{j=1}^{n} L_j\left(1 + \Lambda_\infty^{d_j}\|x\|^{d_j}\right). \tag{A.4}$$

For $x \in E_0^{\alpha,p}$, if $-\frac{\mu}{\lambda} \ge 0$, then, by (A.3) and (A.4), we have

$$\Psi(x) \le LT + LT\Lambda_\infty^{\beta}\|x\|^{\beta} - \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j\left(1 + \Lambda_\infty^{d_j}\|x\|^{d_j}\right) = LT - \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j + LT\Lambda_\infty^{\beta}\|x\|^{\beta} - \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j\Lambda_\infty^{d_j}\|x\|^{d_j}.$$

Thus, for $\forall x \in E_0^{\alpha,p}$, we get

$$\Phi(x) - \lambda\Psi(x) \ge \frac{1}{p}\|x\|^p - \lambda\left(LT - \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j + LT\Lambda_\infty^{\beta}\|x\|^{\beta} - \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j\Lambda_\infty^{d_j}\|x\|^{d_j}\right).$$

If $0 < \beta$ and $d_j < p$, then $\lim\limits_{\|x\| \to +\infty} (\Phi(x) - \lambda\Psi(x)) = +\infty$, $\lambda > 0$. Thus, $\Phi - \lambda\Psi$ is coercive. When $\beta = p$, $\Phi(x) - \lambda\Psi(x) \ge \left(\frac{1}{p} - \lambda LT\Lambda_\infty^{p}\right)\|x\|^p - \lambda\left(LT - \frac{\mu}{\lambda}\sum\limits_{j=1}^{n} L_j - \frac{\mu}{\lambda}\sum\limits_{j=1}^{n} L_j\Lambda_\infty^{d_j}\|x\|^{d_j}\right)$. Choose $L <$

$\frac{\int_0^T \max_{|x| \le \Lambda_\infty (pr)^{1/p}} F(t,x)dt}{prT\Lambda_\infty^p}$. We have that $\frac{1}{p} - \lambda LT\Lambda_\infty^p > 0$ for $\lambda < A_r$. If $0 < d_j < p$ for all $\lambda < A_r$, one has $\lim_{\|x\| \to +\infty} (\Phi(x) - \lambda\Psi(x)) = +\infty$. Obviously, the functional $\Phi - \lambda\Psi$ is coercive. Lemma 9 shows that $\varphi = \Phi - \lambda\Psi$ possesses at least three different critical points in $E_0^{\alpha,p}$.