Research article

On the stability of the diffusive and non-diffusive predator-prey system with consuming resources and disease in prey species

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Abstract: This research deals with formulating a multi-species eco-epidemiological mathematical model when the interacting species compete for the same food sources and the prey species have some infection. It is assumed that infection does not spread vertically. Infectious diseases severely affect the population dynamics of prey and predator. One of the most important factors in population dynamics is the movement of species in the habitat in search of resources or protection. The ecological influences of diffusion on the population density of both species are studied. The study also deals with the analysis of the effects of diffusion on the fixed points of the proposed model. The fixed points of the model are sorted out. The Lyapunov function is constructed for the proposed model. The fixed points of the proposed model are analyzed through the use of the Lyapunov stability criterion. It is proved that coexisting fixed points remain stable under the effects of self-diffusion, whereas, in the case of cross-diffusion, Turing instability exists conditionally. Moreover, a two-stage explicit numerical scheme is constructed, and the stability of the said scheme is found by using von Neumann stability analysis. Simulations are performed by using the constructed scheme to discuss the model's phase portraits and time-series solution. Many scenarios are discussed to display the present study's significance. The impacts of the transmission parameter γ and food resource f on the population density of species are presented in plots. It is verified that the availability of common food resources greatly influences the dynamics of such models. It is shown that all three classes, i.e., the predator, susceptible prey and infected prey, can coexist in the habitat, and this coexistence has a stable nature. Hence, in the realistic scenarios of predator-prey ecology, the results of the study show the importance of food availability for the interacting species.

1. Introduction

Competition is the fundamental relation of ecological systems in which species of one kind have to fight for limited resources in some particular habitat. These sorts of competitions become severe when the potential biological needs exceed the resources. The predator-prey populations are reasonably the building blocks for dynamic ecosystems. In ecology, among different kinds of relationships between species, the most important relationship is the predator-prey relationship. Mathematical models governed by differential equations are more appropriate for modeling the interaction in which populations are overlapped. Lotka and Volterra first put forward the fundamental predator-prey model in the early twentieth century. This basic model comprises two first-order nonlinear ordinary differential equations. Since then, researchers have presented many models covering several issues regarding the complex natural relationship. Consider

$$\frac{dU}{dt} = \alpha_1 \ U - \alpha_2 \ UV \tag{1}$$

$$\frac{dV}{dt} = \alpha_3 \ UV - \alpha_4 \ V \tag{2}$$

where U and V respectively indicate the populations of prey and predator; $\alpha_1, \alpha_2, \alpha_3$ and α_4 are the positive real parameters. Holling introduced the density-dependent response [1], after which several researchers came forward to contribute their valuable research using density-dependent responses. The main concern in the population dynamical models is the stability of fixed points. Many mathematical models have been presented and fixed-point stability has been studied [2]. The stability of the predator-prey model has been brought into the discussion by many researchers [3–5]. The other targeted area is the positivity of the solution of the prey-predator relation [6]. During the last few decades, many species have faced extinction due to limited resources, over-exploitation, pollution and predation exercise. Mostly, the extinction of some species occurs due to transgression of the environment or ecological structure. To impede species from extinction, external supporting trends are the refuge and restriction of the population to a specific area [7–9]. Considering the issue of extinction, many researchers have formulated models to deal with the problem and discussed the factors in detail. A mathematical model of two prey and one predator was considered by Takeuchi and Adachi [10]. Equilibrium points were investigated, and a mathematical analysis of equilibrium points was performed. Another important issue is bifurcation, which is the critical behavior of fixed points. Bifurcation analysis for interacting predator-prey models has been provided by many investigators [10–13]. In [14], the authors investigated the influence of nonlocal competition among the species. The study was further enhanced to investigate the effects of such competition on the stability of equilibrium points. Due to the overuse of chemicals for the higher production of crops and other factors, infectious diseases are spreading among the human population and animals. Researchers have published several articles on the problem of population density dynamics. Whenever a disease spreads in an environment, it seriously affects the population dynamics and disturbs the whole ecosystem. A model was presented by Haque [15] to discuss the impact of the

diseased predator on the population dynamics of predator and prey. He showed that the absence of prey has a moderately strong impact on the predator population when there exists an infection in the predator. The mathematical model for infected predator and prey, as presented by the author, is as follows:

$$\frac{dU}{dt} = aU(1 - \frac{U}{M}) - cV_1U - cqV_2 \tag{3}$$

$$\frac{dF}{dt} = rF(1 - \frac{F}{K}) + ecV_1U + ecqV_2U \tag{4}$$

$$\frac{dV_1}{dt} = (b - \frac{\theta rF}{K})F - (d + \frac{(1-\theta)rF}{k})V_1 - \frac{\beta V_1 V_2}{F} + \gamma V_2 + ecV_1U \tag{5}$$

$$\frac{dV_2}{dt} = FV_2(1 - \frac{V_2}{K}) + \frac{\beta V_1 V_2}{F} - \gamma V_2 - (d + \frac{(1-\theta)rF}{k})V_2 + ecqV_2U \tag{6}$$

Here, P and $F = V_1 + V_2$ respectively represent the prey and predator populations (susceptible and infected). Fang and Wang discussed the interaction between two species when the predator and prey use common food resources. Moreover, the predator is also consuming the prey species. The model suggested by them is as follows [16]:

$$\frac{dU}{dt} = U(\frac{af}{aU+bV} + \alpha_1 V - m) \tag{7}$$

$$\frac{dV}{dt} = V(\frac{bf}{aU+bV} - \alpha_1 U - n) \tag{8}$$

where U and V respectively represent the population of predator and prey, f is the common resource, a and b are the consumption rates of resources by predator and prey, respectively, and m and n are the natural death rates of predator and prey, respectively. Fang and his co-author discussed the global stability of the above dynamical system with the additional effects of diffusion. A detailed discussion was provided on the effects of diffusion by the authors. They discussed the instability of equilibrium points due to the involvement of diffusion. Moreover, to carry out the stability of the system the Lyapunov method was employed. Inseba et al. studied the nonlinear interaction of multi-species, taking into account diffusion and prey taxis. The authors also investigated the linearized stability of the system [17]. Chen and Yu discussed the multi-species predator-prey diffusive system. They showed the pattern formation of the bifurcating system in which the conversion rate was taken as a bifurcation parameter [18]. Djilali discussed the effect of the prey's social behavior on the diffusive predator-prey system. He also discussed the bifurcation and stability analysis [19]. In [20], Ferreira et al. studied the stability of the cross-diffusive system for the three-species interacting model. The equilibrium points were examined to perform a local stability analysis. Kant and Kumar studied the interacting model of species in which predator and prey are diseased. They showed that, for the basic reproductive number $R_0 > 1$, the disease spread [21]. Owolabi formulated a numerical scheme to deal with the fractional diffusion system of interacting species. He applied the Caputo fractional operator rather than the first-order time derivative [22]. Song et al. presented a qualitative approach to study the diffusive predator-prey system. They also discussed the system's instability when cross-diffusion was introduced [23]. Zhang et al. discussed

the influence of diffusion on predator-prey predators' hyperbolic mortality. The turning instability region was also found by linear stability [24]. Ghosh et al. investigated the stability and performed a bifurcation analysis of Bozykin's prey-predator model. The authors also discussed some properties of the solution, like uniqueness and boundedness [25]. In [26], Owolabi formulated a numerical scheme for the computation of fractional (time) derivatives by using the finite difference and Fourier algorithm. Many researchers have discussed the effects of diseased species on the evolution of population dynamics when prey, predator or both have some infection. The authors have investigated the models by using core characteristics like the existence and positivity of equilibria and local and global stability. Some authors have also discussed the bifurcation parameters in the model. Such studies have been supported by simulating the models via numerical schemes [27–31].

Recent years have seen a rise in the application of fractional differential calculus to solve critical and practical problems. In [32], a prey-predator model with three species is examined within the framework of a fractional operator. Two species, both of which grow logistically, are represented in the model. A competitor is considered to fall behind the third group due to their Holling type II functional response. To represent the interplay between tumor growth and the immune system, the authors of [33] provided some new approximate solutions to a computational formulation by using numerous fractional and fractal operators.

In [34], the researcher looked at a computer model to investigate the spread of a viral infectious disease that is more common in children, i.e., hand, foot, and mouth syndrome. A contribution of [35] is the development of a new, time-saving method for obtaining exact fractional solutions to local fractional equations. Equation of Gardner on Cantor sets defined by efficient numerical methods are studied.

In [36], the authors study the stability of a disease model with susceptible prey and infected predators around an internal steady state. The Mittag-Leffler kernels from the Liouville-Caputo idea, which are used to calculate fractional derivatives, have been considered for this purpose. In [37], the researchers looked at one realistic application from the current state of edge detection research. To achieve this goal, they first suggested two overarching structures that can be used to create brand-new fractional masks. Then, the Atangana-Baleanu operator, a fractional integral, was used to assess the roles of the various parts in these two architectures.

In [38], the authors suggested a predator-prey model with a Michaelis–Menten functional response and split prey into susceptible and infected subpopulations. The researchers presented an eco-epidemiological model of an infected predator-prey system [39]. Incorporating prey refuge ensures that a portion of the diseased prey is available for ingestion by the predator. The authors of [40] suggested a predator-prey model in which the prey population is affected by a disease. Here, healthy prey species exhibited defense mechanisms in response to a predator attack. A Leslie-Gower predator-prey model incorporating disease in the predator has been developed [41]. The genetic repercussions of the Lesli-Gower model allow it to be considered an evolutionary version of the Lotka-Volterra model. In [42], the authors created an eco-epidemic model with two prey populations and one predator population, where only the first prey population is affected by an infectious disease. More literature on predator-prey models can be seen in [43–47].

In the present research work, we formulate a multi-species eco-epidemiological model. Two major aspects of ecology are discussed, i.e., food resources and infection in the species. These factors bring drastic changes in the population density of the species under consideration. It is shown that this kind of interaction leads to system stability even under self-diffusion. It is also proved that the

system becomes unstable when cross-diffusion is considered. A two-stage numerical scheme is also applied in the present work, and numerical simulations support the theoretical findings. The stability of the proposed scheme is presented by using the von Neumann stability criterion. It is proved that the scheme is conditionally stable.

2. Model formulation

The ecosystem refers to a natural complex structure in which interacting species affect the population density of one another on a large scale. At the same time, the interacting species are in danger of infectious diseases in the real environment. Numerous mathematical models present the interaction among species with infection in some species. The infections are so critical that they disturb the whole ecosystem. The population density of all species is affected badly, as they depend on one another directly or indirectly. This problem has motivated several investigators to develop their model to study population dynamics effectively. Here, we propose a model to study the predator-prey system in which both species use the same food resources and there is a fatal infectious disease in the prey population. Suppose that U, V respectively represent the predator and prey populations at any time t. Here, we assume that $V = V_S + V_I$, where $V_S \wedge V_I$ respectively denote the susceptible and infected prey.

$$\frac{dU}{dt} = U\left(\frac{af}{aU+bV_S+cV_I} + \alpha_1 V_S + \alpha_2 V_I - m\right) \tag{9}$$

$$\frac{dV_S}{dt} = V_S\left(\frac{bf}{aU+bV_S+cV_I} - \gamma V_I - \alpha_1 U - n_1\right) \tag{10}$$

$$\frac{dV_I}{dt} = V_I\left(\frac{cf}{aU+bV_S+cV_I} + \gamma V_S - \alpha_2 U - n_2\right) \tag{11}$$

where

$$\phi_0 = \{U(0) = U_0 \geq 0, V_S(0) = V_{S,0} \geq 0, V_I(0) = V_{I,0} \geq 0\} \tag{12}$$

is the set of initial conditions. Here, m, n_1 and n_2 are the death rates of the predator and preys, and $n_2 = n_1 + \delta$ displays the sum for the related natural and infectious disease.

3. Stability for the predator-prey system of equations

The present section deals with the computation and stability of the equilibria of the system described by (9)–(11). By solving the system when the rate of change of population becomes zero, it is easy to obtain the equilibria of the system, which are seven in number for the present case. The equilibria of the system are described as follows:

$$E_1 = (0, \frac{f}{n_1}, 0), E_2 = (0, 0, \frac{f}{n_2}), E_3 = (\frac{f}{m}, 0, 0)$$

$$E_4 = (U_4, V_{S,4}, V_{I,4}), E_5 = (U_5, V_{S,5}, V_{I,5}), E_6 = (U_6, V_{S,6}, V_{I,6}), E_7 = (U^*, V_S^*, V_I^*) \tag{13}$$

where

$$U_4 = 0, \quad V_{S,4} = -\frac{-bn_2{}^2 + cf\gamma + cn_1\ n_2}{\gamma\ (bn_2 - cn_1)}, \quad V_{I,4} = \frac{bf\gamma - bn_1\ n_2 + cn_1{}^2}{\gamma\ (bn_2 - cn_1)}$$

$$U_5 = -\frac{an_1{}^2 + bf\alpha_1 - bmn_1}{\alpha_1\ (an_1 - bm)}, V_{S,5} = \frac{af\alpha_1 + am_{\ 1} - bm^2}{\alpha_1\ (an_1 - bm)}, V_{I,5} = 0$$

$$U_6 = -\frac{an_2{}^2 + cf\alpha_2 - cmn_2}{\alpha_2\ (an_2 - cm)}, V_{S,6} = 0, V_{I,6} = \frac{af\alpha_2 + amn_2 - cm^2}{\alpha_2\ (an_2 - cm)}$$

$$U^* = \frac{cn_1 - bn_2}{a\gamma - c\alpha_1 + b\alpha_2} + \frac{f\gamma}{m\gamma - n_2\alpha_1 + n_1\alpha_2} \tag{14}$$

$$V_S^* = \frac{-cm + an_2}{a\gamma - c\alpha_1 + b\alpha_2} + \frac{f\alpha_2}{m\gamma - n_2\alpha_1 + n_1\alpha_2} \tag{15}$$

$$V_I^* = \frac{bm - an_1}{a\gamma - c\alpha_1 + b\alpha_2} - \frac{f\alpha_1}{m\gamma - n_2\alpha_1 + n_1\alpha_2} \tag{16}$$

In the present section, we study the stability of the fixed point, which deals with the coexistence of all species. The existence condition for the coexistence equilibrium point E_7 is

$$f\gamma(\chi_1 + a\gamma) > \delta_1(\gamma\ m + \xi_1), \quad f\alpha_2(\chi_2 + b\alpha_2) > -\delta_2(n_1\ \alpha_2 + \xi_2),$$
$$f\alpha_1(\chi_3 + c\alpha_1) > -\delta_3\ (n_2\quad \alpha_1 - \xi_3)\ and\ \alpha\beta > 0 \tag{17}$$

where

$$\delta_1 = bn_2 - cn_1, \ \delta_2 = an_2 - cm, \ \delta_3 = an_1 - bm, \xi_1 = n_1\ \alpha_2 - n_2\ \alpha_1,$$
$$\xi_2 = \gamma\ m - n_2\ \alpha_1, \ \xi_3 = \gamma\ m + n_1\ \alpha_2, \ \alpha = \gamma\ m + \xi_1, \ \beta = a\gamma + \chi_1,$$
$$\chi_1 = b\alpha_2 - c\alpha_1, \quad \chi_2 = a\gamma - c\alpha_1, \ \chi_3 = -b\alpha_2 - a\gamma$$

Theorem 1: Let $a, b, c, f, \alpha_1, \alpha_2, m, n_1$ and n_2 be positive; then, (U^*, V_S^*, V_I^*) is globally stable in $\omega = \{(U, V_S, V_I): U > 0, V_S > 0, V_I > 0\}$.

Proof: We construct the Lyapunov function to determine the stability of the coexistence equilibrium point. Consider the following

$$W(U, V_S, V_I) = \int_{U^*}^{U} \frac{\eta - U^*}{\eta} d\eta + \int_{V_S^*}^{V_S} \frac{\kappa - V_S^*}{\kappa} d\kappa + \int_{V_I^*}^{V_I} \frac{\xi - z^*}{\xi} d\xi \tag{18}$$

By taking derivative with respect to *t*, we have

$$\frac{dW}{dt} = \frac{U - U^*}{U} \frac{dU}{dt} + \frac{V_S - V_S^*}{V_S} \frac{dV_S}{dt} + \frac{V_I - V_I^*}{V_I} \frac{dV_I}{dt} \tag{19}$$

$$\frac{dW}{dt} = (U - U^*) \left(\frac{af}{aU + bV_S + cV_I} + \alpha_1 V_S + \alpha_2 V_I - m \right)$$

$$+ (V_S - V_{S^*}) \left(\frac{bf}{aU + bV_S + cV_I} - \alpha_1 U - \gamma V_I - n_1 \right)$$

$$+ (V_I - V_{I^*}) \left(\frac{cf}{aU + bV_S + cV_I} - \alpha_2 U + \gamma V_S - n_2 \right) \tag{20}$$

After some simplification, we get

$$\frac{dW}{dt} = (U - U^*)af \frac{aU^* + bV_S^* + cV_I^* - aU - bV_S - cV_I}{(aU + bV_S + cV_I)(aU^* + bV_S^* + cV_I^*)}$$

$$+ (V_S - V_S^*)bf \frac{aU^* + bV_S^* + cV_I^* - aU - b_{\ \ S} - cV_I}{(aU + bV_S + cV_I)(aU^* + bV_S^* + cV_I^*)}$$

$$+ (V_I - V_I^*)cf \frac{aU^* + bV_S^* + cV_I^* - aU - bV_S - cV_I}{(aU + bV_S + cV_I)(aU^* + bV_S^* + cV_I^*)}$$

$$\frac{dW}{dt} = -f \frac{(aU + bV_S + cV_I - (aU^* + bV_S^* + cV_I^*))^2}{(aU + bV_S + cV_I)(aU^* + bV_S^* + cV_I^*)} \leq 0 \tag{21}$$

$$\frac{dW}{dt} = 0 \ \text{iff} \ aU + bV_S + cV_I = aU^* + bV_S^* + cV_I^*$$

Using the values from (16)–(18) and applying further simplification of (9) leads to the following result

$$\frac{dU}{dt} = U(\frac{\alpha_1 a}{b}(U^* - U) + (\frac{\alpha_1 c}{b} - \alpha_2)(V_I^* - V_I)) \tag{22}$$

Then $U \to U^*$ and $V_I \to V_I^*$ when t$\to\infty$ provided that $U(0) \geq 0$. Similarly $V_S \to V_S^*$; hence, the largest invariant set $\{U, V_I, V_S \in \omega: \frac{dW}{dt} = 0\}$ has the unique positive equilibrium (U^*, V_S^*, V_I^*). According to the LaSalle theorem (U^*, V_S^*, V_I^*) is globally stable in ω.
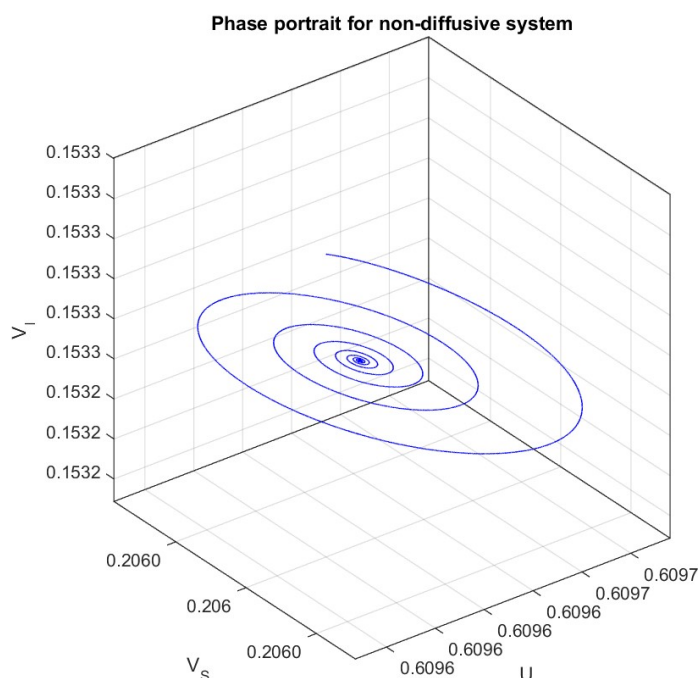

Figure 1. Phase portrait for the systems (9–11) at $U(0) = 0.60, V_S(0) = 0.06, V_I(0) = 0.73$.

Corollary 1. For the equilibrium points E_i ($i = 1,2,3,\ldots,6$), we have the following results.

(a). E_1, E_2 and E_3 always exist and are globally stable.

(b). E_4 exists if $n_2\delta_1 > cf\gamma$ and $n_2\delta_1 > cf\gamma$, and it is globally stable.

(c). E_5 and E_6 do not exist.

Proof:

(a). The results are straightforward from Eqs (13) and (22).

(b). Consider $E_4 = (U_4, V_{S,4}, V_{I,4})$, where

$$U_4 = 0, \quad V_{S,4} = -\frac{-bn_2{}^2 + cf\gamma + c\ _1\ n_2}{\gamma\ (bn_2 - cn_1)}, \quad V_{I,4} = \frac{bf\gamma - bn_1\ n_2 + cn_1{}^2}{\gamma\ (bn_2 - cn_1)}$$

$$V_{S,4} = -\frac{-bn_2{}^2 + cf\gamma + cn_1\ n_2}{\gamma\ (bn_2 - cn_1)}$$

$$= \frac{bn_2{}^2 - cf\gamma -\ _1\ n_2}{\gamma\ (bn_2 - cn_1)}$$

We can rewrite as

$$V_{S,4} = \frac{n_2\delta_1 - cf\gamma}{\gamma\ \delta_1} \quad \Rightarrow \quad n_2\delta_1 > cf\gamma$$

Similarly,

$$V_{I,4} = \frac{bf\gamma - b\ _1\ n_2 + cn_1{}^2}{\gamma\ (bn_2 - cn_1)} \quad \Rightarrow \quad bf\gamma > n_1\delta_1$$

The global stability is obvious from Eq (22).

(c). Consider $E_5 = (U_5, V_{S,5}, V_{I,5})$, where

$$U_5 = -\frac{an_1{}^2 + bf\alpha_1 - bmn_1}{\alpha_1\ (an_1 - bm)}, V_{S,5} = \frac{af\ _1 + amn_1 - bm^2}{\alpha_1\ (an_1 - bm)}, V_{I,5} = 0$$

$$U_5 = -\frac{an_1{}^2 + bf\alpha_1 - bmn_1}{\alpha_1\ (an_1 - bm)} \quad \Rightarrow \quad U_5 = -bf\alpha_1 - n_1\delta_3 < 0$$

Hence, the result, similarly, E_6 does not exist.

4. Global stability for a diffusive system

In the dynamical population model, random walking plays a central role in the structure of the habitat. The movement of species can vary from region to region depending on the distribution of food resources in the habitat. In addition, the species can vary with respect to density in the area where they live. The following section deals with the influence of diffusion on the coexistence of fixed points.

4.1. Global stability for the self-diffusive system

The self-diffusive system for (9)–(11) can be written as

$$\frac{dU}{dt} = D_{11} \triangle U + U(\frac{af}{aU+bV_S+cV_I} + \alpha_1 V_S + \alpha_2 V_I - m), \qquad x \in \Omega, t > 0 \tag{23}$$

$$\frac{dV_S}{dt} = D_{22} \triangle V_S + V_S(\frac{bf}{aU+bV_S+cV_I} - \gamma V_I - \alpha_1 U - n_1), \qquad x \in \Omega, t > 0 \tag{24}$$

$$\frac{dV_I}{dt} = D_{33} \triangle V_I + V_I(\frac{cf}{aU+bV_S+cV_I} + \gamma V_S - \alpha_2 U - n_2), \qquad x \in \Omega, t > 0 \tag{25}$$

$$\frac{\partial U}{\partial \vartheta} = \frac{\partial V_S}{\partial \vartheta} = \frac{\partial V_I}{\partial \vartheta} = 0, x \in \partial\Omega, t > 0 \tag{26}$$

with the following initial condition

$$\phi(0) = \{U(0) = U_0 \geq 0, V_S(0) = V_{S,0} \geq 0, V_I(0) = V_{I,0} \geq 0\} \tag{27}$$

where the self-diffusion coefficients D_{11}, D_{22}, D_{33} are non-negative. According to Hollis' results, the solution to the above system has global existence and boundedness [47]. The self-diffusive system has an equilibrium point $E_7 = (U^*, V_S^*, V_I^*)$ under the assumption of (17)

Theorem 2: Let $a, b, c, D_{11}, D_{22}, D_{33}, f, g, m, n_1, n_2, \alpha_1, \alpha_2$ and γ be positive; then, (U^*, V_S^*, V_I^*) is globally stable in $\omega = \{(U, V_S, V_I) : U > 0, V_S, V_I > 0\}$.

Proof. It is obvious that $\{(U, V_S, V_I) : U = 0\}$, $\{(U, V_S, V_I) : V_S = 0\}$ and $\{(U, V_S, V_I) : V_I = 0\}$ are the invariant manifolds. We establish the Lyapunov function to determine the global stability of the self-diffusive system as follows

$$W(U, V_S, V_I) = \int_\Omega \int_{U^*}^{U} \frac{\eta - U^*}{\eta} d\eta dX + \int_\Omega \int_{N_S^*}^{N_S} \frac{\kappa - V_S^*}{\kappa} d\kappa dX + \int_\Omega \int_{N_I^*}^{N_I} \frac{\xi - V_I^*}{\xi} d\xi dX \tag{28}$$

Now, taking derivative with respect to *t* on both sides, we get

$$\frac{dW}{dt} = \int_\Omega \frac{U - U^*}{U} \frac{\partial U}{\partial t} dX + \int_\Omega \frac{V_S - V_S^*}{V_S} \frac{\partial V_S}{\partial t} dX + \int_\Omega \frac{V_I - V_I^*}{V_I} \frac{\partial V_I}{\partial t} dX$$

$$= \int_\Omega \frac{U - U^*}{U} \left(D_{11} \triangle U + U \left(\frac{af}{aU+bV_S+cV_I} + \alpha_1 V_S + \alpha_2 V_I - m \right) \right)$$

$$+ \int_\Omega \frac{V_S - V_S^*}{V_S} \left(D_{22} \triangle V_S + V_S \left(\frac{bf}{aU+bV_S+cV_I} - \gamma V_I - \alpha_1 U - n_1 \right) \right)$$

$$+ \int_\Omega \frac{V_I - V_I^*}{V_I} \left(D_{33} \triangle V_I + V_I \left(\frac{cf}{aU+bV_S+cV_I} + \gamma V_S - \alpha_2 U - n_2 \right) \right)$$

$$\frac{dW}{dt} =: M_1 + M_2 + M_3 + M_4 \tag{29}$$

where

$$M_1 = -D_{11} U^* \int_\Omega \frac{1}{U^2} |\nabla U|^2 dX \leq 0 \tag{30}$$

$$M_2 = -D_{22}V_S^* \int_\Omega \frac{1}{V_S^2} |\nabla V_S^2| dX \leq 0 \tag{31}$$

$$M_3 = -D_{33}V_I^* \int_\Omega \frac{1}{V_I^2} |\nabla V_I^2| dX \leq 0 \tag{32}$$

$$M_4 = \int_\Omega (U - U^*)\left[\left(\frac{af}{aU+bV_S+cV_I} + \alpha_1 V_S + \alpha_2 V_I - m\right)\right] dX$$

$$+ \int_\Omega (V_S - V_S^*)\left[\left(\frac{bf}{aU+bV_S+cV_I} - \gamma V_I - \alpha_1 U - n_1\right)\right] dX$$

$$+ \int_\Omega (V_I - V_I^*)[(\frac{cf}{aU+bV_S+cV_I} + \gamma V_S - \alpha_2 U - n_2)]dX \leq 0 \tag{33}$$

$$M_4 = -\int_\Omega f \frac{(aU+bV_S+cV_I-(aU^*+bV_S^*+cV_I^*))^2}{(aU+b\ s+cV_I)(aU^*+bV_S^*+cV_I^*)} \leq 0 \tag{34}$$

It is obvious that

$$\frac{dW}{dt} \leq 0 \tag{35}$$

$$\frac{dW}{dt} = 0 \quad \text{iff} \quad aU + bV_S + cV_I = aU^* + bV_S^* + cV_I^*$$

Further simplification of (23) by using the values of the coexistence equilibrium point (U^*, V_S^*, V_I^*) leads to the following

$$\frac{dU}{dt} = D_{11} + U(\frac{\alpha_1 a}{b}(U^* - U) + \left(\frac{\alpha_1 c}{b} - \alpha_2\right)(V_I^* - V_I))$$

It is well known that the above equation represents a gradient system, where every orbit converges to a unique steady state $U = U^*$ and $V_I = V_I^*$ [47]. Thus, $V_S \rightarrow V_S^*$ since (U, V_S, V_I) is on $\Gamma :=$ $\{(U, V_S, V_I): aU + bV_S + cV_I = aU^* + bV_S^* + cV_I^*, U \geq 0, V_S \geq 0, V_I \geq 0\}$. Therefore, positive solution (U^*, V_S^*, V_I^*) is globally asymptotically stable.
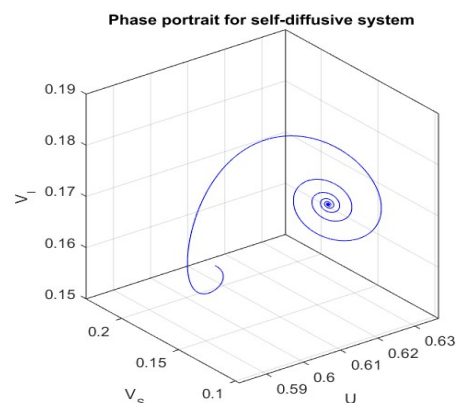

Figure 2. Phase portrait for the self-diffusive system at $U(0) = 0.53, V_S(0) = 0.0101, V_I(0) = 0.70$.

4.2. Induced stability for the cross-diffusive system

The predator is attracted to its prey in search of food resources. This attraction causes prey taxis, which is the predator's movement as controlled by the density of prey, and on the contrary, the prey is repelled by the predator. This results in the cross-diffusion of species in the habitat. Consider the following system of equations:

$$\frac{dU}{dt} = D_{11} \triangle U + D_{12} \triangle V_S + D_{13} \triangle V_I + U\left(\frac{af}{aU+bV_S+cV_I} + \alpha_1 V_S + \alpha_2 V_I - m\right),$$

$$x \in \Omega, t > 0 \tag{36}$$

$$\frac{dV_S}{dt} = D_{21} \triangle U + D_{22} \triangle V_S + V_S\left(\frac{bf}{aU+bV_S+cV_I} - \gamma V_I - \alpha_1 U - n_1\right),$$

$$x \in \Omega, t > 0 \tag{37}$$

$$\frac{dV_I}{dt} = D_{31} \triangle U + D_{33} \triangle V_I + V_I\left(\frac{cf}{aU+bV_S+cV_I} + \gamma V_S - \alpha_2 U - n_2\right),$$

$$x \in \Omega, t > 0 \tag{38}$$

$$\frac{\partial U}{\partial \vartheta} = \frac{\partial V_S}{\partial \vartheta} = \frac{\partial V_I}{\partial \vartheta} = 0 \ , \ x \in \partial\Omega, t > 0$$

$$\phi(0) = \{U(0) = U_0 \geq 0, V_S(0) = V_{S,0} \geq 0, V_I(0) = V_{I,0} \geq 0\}, x \in \Omega, t > 0 \tag{39}$$

One can see that the coexisting equilibrium point also holds for the cross-diffusive system. The self-diffusive system does not induce instability, but the Turing instability can be seen in the following section. The above system, as described by Eqs (36)–(38), can be written in the following way:

$$R\Phi = D \triangle \Phi + J\Phi \tag{40}$$

$$\Phi(t,x) = \begin{bmatrix} \phi(t,x) \\ \varphi(t,x) \\ \psi(t,x) \end{bmatrix}, D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & 0 \\ D_{31} & 0 & D_{33} \end{bmatrix} \text{ and } J = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where

$$A_{11} = -\frac{a^2 f U^*}{\left(aU^*+cV_I^*+bV_S^*\right)^2},$$

$$A_{12} = -\frac{abf U^*}{\left(aU^*+cV_I^*+bV_S^*\right)^2} + \alpha_1 \ U^*,$$

$$A_{13} = -\frac{acf U^*}{\left(aU^*+cV_I^*+bV_S^*\right)^2} + \alpha_2 \ U^*,$$

$$A_{21} = -\frac{abf V_S^*}{\left(aU^*+cV_I^*+bV_S^*\right)^2} - \alpha_1 \ V_S^*,$$

$$A_{22} = -\frac{b^2 f V_S^*}{(aU^* + cV_I^* + bV_S^*)^2},$$

$$A_{23} = -\frac{bcf V_S^*}{(aU^* + cV_I^* + bV_S^*)^2} - \gamma \ V_S^*,$$

$$A_{31} = -\frac{acf V_I^*}{(aU + cV_I + bV_S)^2} - \alpha_2 \ V_I^*,$$

$$A_{32} = -\frac{bcf V_I^*}{(aU^* + cV_I^* + bV_S^*)^2} + \gamma \ V_I^*,$$

$$A_{33} = -\frac{c^2 f V_I^*}{(aU^* + cV_I^* + bV_S^*)^2} \tag{41}$$

If all of the eigenvalues of the operator R are negative, then $(U^*, V_S^*, \ V_I^*)$ is asymptotically stable [47]. Next, it is clear that $-\triangle$ under the Neumann boundary condition has eigenvalues

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$$

and $\mu_i \to \infty$ when $t \to \infty$. It is also known that ν is the eigenvalue of the operator R if and only if it is the eigenvalue of $J_i = -\mu_i D + J$ for $i \geq 0$. Moreover, we obtain

$$\text{trace}(J_i) = -\mu_i (D_{11} + D_{22} + D_{33}) + \text{trace}(J)$$

It is obvious from (41) that

$$\text{trace}(J) \ = \ A_{11} + A_{22} + \ A_{33} \ < 0.$$

Next, we prove Turing instability for the cross-diffusive system described by (36)–(38).

Theorem 3: Let $a, b, c, D_{11}, D_{22}, D_{33}, f, g, m, n_1, n_2, \alpha_1, \alpha_2$ and γ be positive and

$$det(D) = (D_{11} \ D_{22} \ D_{33} - D_{12} \ D_{21} \ D_{33} + D_{13} \ D_{22} \ D_{31}) > 0$$

then, there exists an unbounded region where $s_1 < s_2 + s_3 + s_4$
such that (U^*, V_S^*, V_I^*) is unstable in the region where the above condition holds.
Additionally,

$$s_1 = -[\frac{1}{3}\frac{F(J,D) - \sqrt{3 \ det(D) f(J,D) + (F(J,D))^2}}{det(D)}]^3 det(D)$$

$$s_2 = [\frac{1}{3}\frac{F(J,D) - \sqrt{3 \ det(D) f(J,D) + (F(J,D))^2}}{det(D)}]^2 F(J,D)$$

$$s_3 = [1/3 \ \frac{F(J,D) - \sqrt{3 \ det(D) f(J,D) + (F(J,D))^2}}{det(D)}]f(J,D)$$

$$s_4 = A_{11} \ A_{22} \ A_{33} - A_{11} \ A_{23} \ A_{32} - A_{12} \ A_{21} \ A_{33}$$

$$+A_{12} \ A_{23} \ A_{31} + A_{13} \ A_{21} \ A_{32} - A_{13} \ A_{22} \ A_{31}$$

Proof. It is clear from the value of J_i that $trace(J_i)$ is negative. To show the instability of (U^*, V_S^*, V_I^*), we shall prove that $det(J_i) > 0$ for some i, where $i \geq 1$. As

$$\det(J_0) = -A_{11}\ A_{21}\ A_{33} + A_{11}\ A_{22}\ A_{33}$$

$$+A_{11}\ A_{23}\ A_{31} - A_{11}\ A_{23}\ A_{32}$$

$$+A_{13}\ A_{21}\ A_{32} - A_{13}\ A_{22}\ A_{31} \tag{42}$$

$$det(J_i) = \begin{bmatrix} -\mu_i\ D_{11} + A_{11} & -\mu_i\ D_{12} + A_{12} & -\mu_i\ D_{13} + A_{13} \\ -\mu_i\ D_{21} + A_{21} & -\mu_i\ D_{22} + A_{22} & A_{23} \\ -\mu_i\ D_{31} + A_{31} & A_{32} & -\mu_i\ D_{33} + A_{33} \end{bmatrix} \tag{43}$$

$$\begin{aligned}
det(J_i) = &-D_{11}\ D_{22}\ D_{33}\ \mu_i{}^3 + A_{11}\ D_{22}\ D_{33}\ \mu_i{}^2 + A_{22}\ D_{11}\ D_{33}\ \mu_i{}^2 + A_{33}\ D_{11}\ D_{22}\ \mu_i{}^2 \\
&- A_{11}\ A_{22}\ D_{33}\ \mu_i - A_{11}\ A_{33}\ D_{22}\ \mu_i - A_{22}\ A_{33}\ D_{11}\ \mu_i + A_{23}\ A_{32}\ D_{11}\ \mu_i \\
&+ D_{12}\ D_{21}\ D_{33}\ \mu_i{}^3 - A_{12}\ D_{21}\ D_{33}\ \mu_i{}^2 - A_{21}\ D_{12}\ D_{33}\ \mu_i{}^2 + A_{32}\ D_{13}\ D_{21}\ \mu_i{}^2 \\
&- A_{33}\ D_{12}\ D_{21}\ \mu_i{}^2 + A_{12}\ A_{21}\ D_{33}\ \mu_i + A_{12}\ A_{33}\ D_{21}\ \mu_i - A_{13}\ A_{32}\ D_{21}\ \mu_i \\
&- A_{21}\ A_{32}\ D_{13}\ \mu_i + A_{21}\ A_{33}\ D_{12}\ \mu_i + D_{13}\ D_{22}\ D_{31}\ \mu_i{}^3 - A_{13}\ D_{22}\ D_{31}\ \mu_i{}^2 \\
&- A_{22}\ D_{13}\ D_{31}\ \mu_i{}^2 + A_{23}\ D_{12}\ D_{31}\ \mu_i{}^2 - A_{31}\ D_{13}\ D_{22}\ \mu_i{}^2 - A_{12}\ A_{23}\ D_{31}\ \mu_i \\
&+ A_{13}\ A_{22}\ D_{31}\ \mu_i + A_{13}\ A_{31}\ D_{22}\ \mu_i + A_{22}\ A_{31}\ D_{13}\ \mu_i - A_{23}\ A_{31}\ D_{12}\ \mu_i \\
&- A_{12}\ A_{21}\ A_{33} + A_{13}\ A_{21}\ A_{32} + A_{12}\ A_{23}\ A_{31} - A_{13}\ A_{22}\ A_{31} + A_{11}\ A_{22}\ A_{33} \\
&- A_{11}\ A_{23}\ A_{32}
\end{aligned}$$

By collecting the terms containing μ_i, we obtain

$$\begin{aligned}
det(J_i) = &\ (-D_{11}\ D_{22}\ D_{33} + D_{12}\ D_{21}\ D_{33} + D_{13}\ D_{22}\ D_{31})\mu_i{}^3 \\
&+ (A_{11}\ D_{22}\ D_{33} + A_{22}\ D_{11}\ D_{33} + A_{33}\ D_{11}\ D_{22} - A_{12}\ D_{21}\ D_{33} - A_{21}\ D_{12}\ D_{33} \\
&+ A_{32}\ D_{13}\ D_{21} - A_{33}\ D_{12}\ D_{21} - A_{13}\ D_{22}\ D_{31} - A_{22}\ D_{13}\ D_{31} + A_{23}\ D_{12}\ D_{31} \\
&- A_{31}\ D_{13}\ D_{22})\mu_i{}^2 \\
&+ (-A_{11}\ A_{22}\ D_{33} - A_{11}\ A_{33}\ D_{22} - A_{22}\ A_{33}\ D_{11} + A_{23}\ A_{32}\ D_{11} + A_{12}\ A_{21}\ D_{33} \\
&+ A_{12}\ A_{33}\ D_{21} - A_{13}\ A_{32}\ D_{21} - A_{21}\ A_{32}\ D_{13} + A_{21}\ A_{33}\ D_{12} - A_{12}\ A_{23}\ D_{31} \\
&+ A_{13}\ A_{22}\ D_{31} + A_{13}\ A_{31}\ D_{22} + A_{22}\ A_{31}\ D_{13} - A_{23}\ A_{31}\ D_{12})\mu_i + A_{11}\ A_{22}\ A_{33} \\
&- A_{11}\ A_{23}\ A_{32} - A_{12}\ A_{21}\ A_{33} + A_{12}\ A_{23}\ A_{31} + A_{13}\ A_{21}\ A_{32} - A_{13}\ A_{22}\ A_{31}
\end{aligned}$$

$$det(J_i) = -\mu_i{}^3 \det(D) + \mu_i{}^2 F(J, D)$$

$$+\mu_i\ f(J, D) + A_{11}\ A_{22}\ A_{33} - A_{11}\ A_{23}\ A_{32} - A_{12}\ A_{21}\ A_{33}$$

$$+A_{12}\ A_{23}\ A_{31} + A_{13}\ A_{21}\ A_{32} - A_{13}\ A_{22}\ A_{31} \tag{44}$$

where

$$det(D) = (D_{11}\ D_{22}\ D_{33} - D_{12}\ D_{21}\ D_{33} + D_{13}\ D_{22}\ D_{31}) > 0$$

$$F(J, D) = \begin{pmatrix} A_{11}\ D_{22}\ D_{33} + A_{22}\ D_{11}\ D_{33} \\ +A_{33}\ D_{11}\ D_{22} - A_{12}\ D_{21}\ D_{33} - A_{21}\ D_{12}\ D_{33} \\ +A_{32}\ D_{13}\ D_{21} - A_{33}\ D_{12}\ D_{21} - A_{13}\ D_{22}\ D_{31} - A_{22}\ D_{13}\ D_{31} \\ +A_{23}\ D_{12}\ D_{31} - A_{31}\ D_{13}\ D_{22} \end{pmatrix} \tag{45}$$

$$f(J, D) = -A_{11}\ A_{22}\ D_{33} - A_{11}\ A_{33}\ D_{22} - A_{22}\ A_{33}\ D_{11}$$

$$+A_{23}\ A_{32}\ D_{11} + A_{12}\ A_{21}\ D_{33}$$

$$+A_{12}\ A_{33}\ D_{21} - A_{13}\ A_{32}\ D_{21} - A_{21}\ A_{32}\ D_{13}$$

$$+A_{21}\ A_{33}\ D_{12} - A_{12}\ A_{23}\ D_{31} + A_{13}\ A_{22}\ D_{31}$$

$$+A_{13}\ A_{31}\ D_{22} + A_{22}\ A_{31}\ D_{13} - A_{23}\ A_{31}\ D_{12} \tag{46}$$

The above equality of (44) leads to its minimum value $min_{\mu_i \in R^+} \det(J_i)$ at

$$\mu = \ 1/3 \ \frac{F(J,D) - \sqrt{3\ \det(D)f(J,D) + \left(F(J,D)\right)^2}}{\det(D)}$$

After substituting the above value in (44), we get

$$\text{Det}(J_i) = \ -[1/3\ \frac{F(J,D) - \sqrt{3\ \det(D)f(J,D) + \left(F(J,D)\right)^2}}{\det(D)}]^3 \det(D)$$

$$+[1/3\ \frac{F(J,D) - \sqrt{3\ \det(D)f(J,D) + \left(F(J,D)\right)^2}}{\det(D)}]^2 F(J,D)$$

$$+\left[\frac{1}{3}\frac{F(J,D) - \sqrt{3\ \det(D)f(J,D) + \left(F(J,D)\right)^2}}{\det(D)}\right] f(J,D)$$

$$+A_{11}\ A_{22}\ A_{33} - A_{11}\ A_{23}\ A_{32} - A_{12}\ A_{21}\ A_{33}$$

$$+A_{12}\ A_{23}\ A_{31} + A_{13}\ A_{21}\ A_{32} - A_{13}\ A_{22}\ A_{31}$$

$$\det(J_i) = -s_1 + s_2 + s_3 + s_4 \tag{47}$$

For the instability of the system, we must have $det(J_i) > 0$. Therefore above the equilibrium point (U^*, V_S^*, V_I^*) is unstable in the region where $s_1 \ < \ s_2 + \ s_3 + \ s_4$.

5. Construction of numerical scheme

A numerical scheme is constructed for solving Eqs (23)–(27). The scheme's construction discretizes the time variable, and suitable difference formula can be adopted to discretize the space variable. The scheme is the two-stage explicit scheme. The scheme is constructed on three-time levels. Consider the following equation:

$$\frac{\partial V}{\partial t} = G\left(V_1 \frac{\partial^2 V}{\partial x^2}\right) \tag{48}$$

The first stage of the scheme is written as

$$\bar{V}_i^{n+1} = V_i^n + \Delta t \left(\frac{\partial V}{\partial t}\right)_i^n \tag{49}$$

The second stage of the scheme contains three unknowns. Their values will be found later. The

second stage is written as

$$V_i^{n+1} = \frac{1}{2}(\bar{V}_i^{n+1} + V_i^{n-1}) + \Delta t \left\{ a_1 \left(\frac{\partial V}{\partial t}\right)_i^{n+1} + b_1 \left(\frac{\partial V}{\partial t}\right)_i^n + c_1 \left(\frac{\partial V}{\partial t}\right)_i^{n-1} \right\} \tag{50}$$

To find the values of unknowns a_1, b_1 and c_1, the Taylor series expansions for $V_i^{n+1}, V_i^{n-1}, \left(\frac{\partial V}{\partial t}\right)_i^{n+1}$ and $\left(\frac{\partial V}{\partial t}\right)_i^{n-1}$ are given as

$$V_i^{n+1} = V_i^n + \Delta t \left(\frac{\partial V}{\partial t}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 V}{\partial t^2}\right)_i^n + \frac{(\Delta t)^3}{6}\left(\frac{\partial^3 v}{\partial t^3}\right)_i^n + O((\Delta t)^4) \tag{51}$$

$$V_i^{n-1} = V_i^n - \Delta t \left(\frac{\partial V}{\partial t}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 V}{\partial t^2}\right)_i^n - \frac{(\Delta t)^3}{6}\left(\frac{\partial^3 V}{\partial t^3}\right)_i^n + O((\Delta t)^4) \tag{52}$$

$$\left(\frac{\partial V}{\partial t}\right)_i^{n+1} = \left(\frac{\partial V}{\partial t}\right)_i^n + \Delta t \left(\frac{\partial^2 V}{\partial t^2}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^3 V}{\partial t^3}\right)_i^n + O((\Delta t)^3) \tag{53}$$

$$\left(\frac{\partial V}{\partial t}\right)_i^{n-1} = \left(\frac{\partial V}{\partial t}\right)_i^n - \Delta t \left(\frac{\partial^2 V}{\partial t^2}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^3 V}{\partial t^3}\right)_i^n + O((\Delta t)^3) \tag{54}$$

Substituting Taylor series expansions (51)–(54) into Eq (50), we obtain the following Comparison of coefficients of $u_i^n, \Delta t \left(\frac{\partial u}{\partial t}\right)_i^n, (\Delta t)^2 \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n$ and $(\Delta t)^3 \left(\frac{\partial^3 u}{\partial t^3}\right)_i^n$ leads to the following

$$\begin{cases} 1 = a_1 + b_1 + c_1 \\ \frac{1}{2} = \frac{1}{4} + a_1 - c_1 \\ \frac{1}{6} = -\frac{1}{12} + \frac{c_1}{2} \end{cases} \tag{55}$$

Solving the above system of equations gives the values of the unknown parameters a_1, b_1 and c_1 as

$$a_1 = \frac{3}{4}, b_1 = -\frac{1}{4}, c_1 = \frac{1}{2} \tag{56}$$

5.1. Stability analysis

The stability condition of the proposed scheme for the parabolic Eq (48) von Neumann stability criterion is applied. For applying the stability analysis, some transformations are substituted into difference equations. For the considered problem, the transformations are given as follows.

Using a second-order central difference formula for the diffusion term, the first stage of the proposed scheme for the linearized equation is given as

$$\bar{V}_i^{n+1} = V_i^n + \frac{1}{4}u_i^{n-1} + d(V_{i+1}^n - 2V_i^n - V_{i-1}^n) \tag{57}$$

where

$$d = \frac{\Delta t}{(\Delta x)^2}$$

According to von Neumann's stability criterion, consider the following transformations

$$V_i^{n+1} = \bar{E}^{n+1}e^{iI\theta}, V_{i\pm1}^n = E^n e^{(i\pm1)I\theta}, u_{i\pm1}^{n-1} = E^{n+1}e^{(i\pm1)I\theta}$$

$$\text{where } I = \sqrt{-1} \tag{58}$$

Substituting the transformations of (50) into Eq (49), the following is obtained:

$$\bar{E}^{n+1} = E^n + d(e^{I\theta} - 2 + e^{-I\theta})E^n$$

$$\bar{E}^{n+1} = (1 + 2d(cos\theta - 1))E^n \tag{59}$$

By using the second-order central difference formula for the diffusion term, the second stage of the proposed scheme for the linearized equation is given as

$$V_i^{n+1} = \frac{1}{2}(\bar{V}_i^{n+1} + V_i^{n-1}) + d \left\{ \begin{array}{l} a_1(\bar{V}_{i+1}^{n+1} - 2\bar{V}_i^{n+1} + \bar{V}_{i-1}^{n+1}) \\ +b_1(V_{i+1}^n - 2V_i^n + V_{i-1}^n) \\ +c_1(V_{i+1}^{n-1} - 2 \end{array} \right.$$

$$V_i^{n-1} + V_{i-1}^{n-1})\} \tag{60}$$

Substituting the transformations of (58) into (60) and dividing the resulting equation by $e^{iI\theta}$, we get

$$E^{n+1} = \frac{1}{2}(\bar{E}_i^{n+1} + E_i^{n-1}) + d \left\{ \begin{array}{l} a_1(e^{I\theta} - 2 + e^{-I\theta})\bar{E}^{n+1} + \\ b_1(e^{I\theta} - 2 + e^{-I\theta})E^n + \\ c_1(e^{I\theta} - 2 + e^{-I\theta})E^{n-1} \end{array} \right.$$

Collecting the coefficients of E^{n+1} on the left-hand side of Eq (61) provides

$$E^{n+1} = \left(\frac{1}{2} + 2a_1 d(cos\theta - 1)\right)\bar{E}^{n+1} + 2bd(cos\theta - 1)E^n + \left(\frac{1}{2} + 2c_1 d(cos\theta - 1)\right)E^{n-1}$$

$$E^{n+1} = A_1 E^n + B_1 E^{n-1} \tag{61}$$

$$A_1 = \left(\frac{1}{2} + 2a_1 d(cos\theta - 1)\right)(1 + 2d(cos\theta - 1)) + 2b_1 d(cos\theta - 1)$$

$$B_1 = \frac{1}{2} + 2c_1 d(cos\theta - 1)$$

$$E^n = E^n + 0E^{n-1} \tag{62}$$

Equations (61) and (62) can be written in matrix form as

$$\begin{bmatrix} E^{n+1} \\ E^n \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} E^n \\ E^{n-1} \end{bmatrix} \tag{63}$$

The stability conditions can be imposed on the eigenvalue of the coefficient matrix, and these conditions are expressed as

$$\left| \frac{A_1 + \sqrt{A_1{}^2 + 4B_1}}{2} \right| \leq 1 \quad \text{and} \quad \left| \frac{A_1 - \sqrt{A_1{}^2 + 4B_1}}{2} \right| \leq 1 \tag{64}$$

The proposed scheme will be stable if it satisfies inequalities of (64).

6. Numerical simulations

The spatial terms in (23)–(25) are discretized by using the classical central difference formula to apply the proposed scheme for the considered diffusive model. The discretized equations for the model are given as

$$\overline{U}_i^{n+1} = U_i^n + \Delta t \left\{ \begin{array}{l} D_{11}\,(U_{i+1}^n - 2U_i^n + U_{i-1}^n)/(\Delta x)^2 \\ +U_i^n \left(\frac{af}{aU_i^n + bV_{S,i}^n + cV_{I,i}^n} + \alpha_1 V_{S,i}^n + \alpha_2 V_{I,i}^n - m \right) \end{array} \right\} \tag{65}$$

$$\overline{V}_{S,i}^{n+1} = V_{S,i}^n + \Delta t \left\{ \begin{array}{l} D_{22}\,\left(V_{S,i+1}^n - 2V_{S,i}^n + V_{S,i-1}^n\right)/(\Delta x)^2 \\ +U_{S,i}^n \left(\frac{bf}{aU_i^n + bV_{S,i}^n + cV_{I,i}^n} - \alpha_1 U_i^n - \gamma V_{I,i}^n - n_1 \right) \end{array} \right\} \tag{66}$$

$$\overline{V}_{S,i}^{n+1} = V_{I,i}^n + \Delta t \left\{ \begin{array}{l} D_{33}\,\left(V_{I,i+1}^n - 2V_{I,i}^n + V_{I,i-1}^n\right)/(\Delta x)^2 \\ +U_{I,i}^n \left(\frac{cf}{aU_i^n + bV_{S,i}^n + cV_{I,i}^n} - \alpha_2 U_i^n - \gamma V_{S,i}^n - n_2 \right) \end{array} \right\} \tag{67}$$

$$U_i^{n+1} = U_i^n + \Delta t \left[\begin{array}{l} a_1 \left\{ \begin{array}{l} D_{11}\,(\overline{U}_{i+1}^{n+1} - 2\overline{U}_i^{n+1} + \overline{U}_{i-1}^{n+1})/(\Delta x)^2 + \\ \overline{U}_i^{n+1} \left(\frac{af}{a\overline{U}_i^{n+1} + b\overline{V}_{S,i}^{n+1} + c\overline{V}_{I,i}^{n+1}} + \alpha_1 \overline{V}_{S,i}^{n+1} + \alpha_2 \overline{V}_{I,i}^{n+1} - m \right) \end{array} \right\} + \\[2em] b_1 \left\{ \begin{array}{l} D_{11}\,(U_{i+1}^n - 2U_i^n + U_{i-1}^n)/(\Delta x)^2 + \\ U_i^n \left(\frac{af}{aU_i^n + bV_{S,i}^n + cV_{I,i}^n} + \alpha_1 V_{S,i}^n + \alpha_2 V_{I,i}^n - m \right) \end{array} \right\} + \\[2em] c_1 \left\{ \begin{array}{l} D_{11}\,(U_{i+1}^{n-1} - 2U_i^{n-1} + U_{i-1}^{n-1})/(\Delta x)^2 + \\ U_i^{n-1} \left(\frac{af}{aU_i^{n-1} + bU_i^{n-1} + cV_{I,i}^{n-1}} + \alpha_1 V_{S,i}^{n-1} + \alpha_2 V_{I,i}^{n-1} - m \right) \end{array} \right\} \end{array} \right]$$

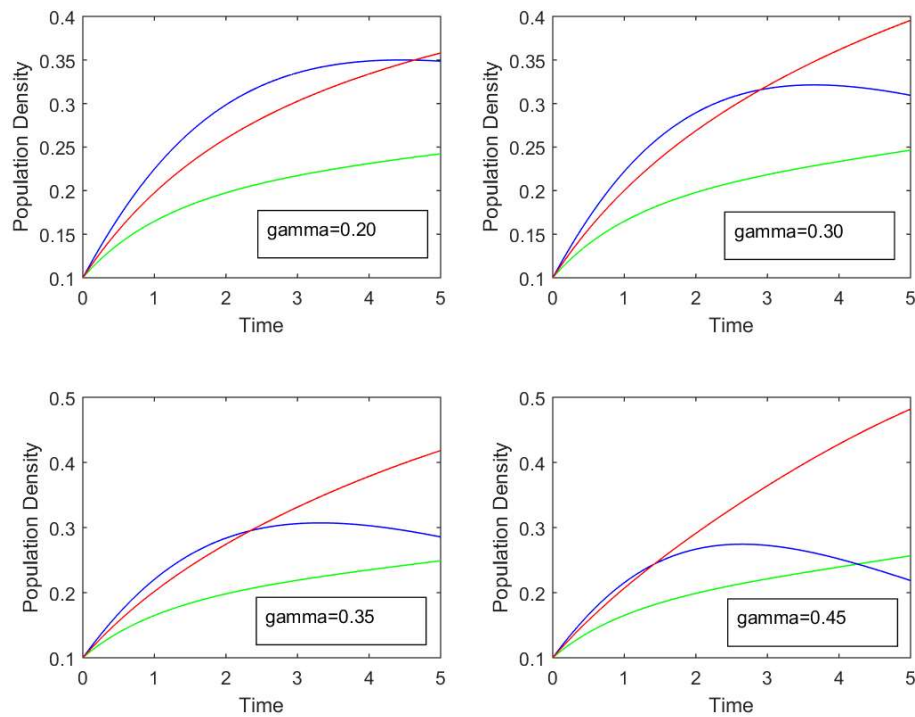$$V_{S,i}^{n+1} = V_{S,i}^{n+1} + \Delta t \left[\begin{array}{l} a_1 \left\{ \begin{array}{l} D_{22}\,(\overline{V}_{S,i+1}^{n+1} - 2\overline{V}_{S,i}^{n+1} + \overline{V}_{S,i-1}^{n+1})/(\Delta x)^2 + \\ \overline{U}_i^{n+1} \left(\frac{bf}{a\overline{U}_i^{n+1} + b\overline{V}_{S,i}^{n+1} + c\overline{V}_{I,i}^{n+1}} + \alpha_1 \overline{V}_{S,i}^{n+1} + \alpha_2 \overline{V}_{I,i}^{n+1} - n_1 \right) \end{array} \right\} + \\[2em] b_1 \left\{ \begin{array}{l} D_{22}\,(V_{S,i+1}^n - 2V_{S,i}^n + V_{S,i-1}^n)/(\Delta x)^2 + \\ U_i^n \left(\frac{bf}{aU_i^n + bV_{S,i}^n + cV_{I,i}^n} + \alpha_1 V_{S,i}^n + \alpha_2 V_{I,i}^n - n_2 \right) \end{array} \right\} + \\[2em] c_1 \left\{ \begin{array}{l} D_{22}\,(V_{S,i+1}^{n-1} - 2V_{S,i}^{n-1} + V_{S,i-1}^{n-1})/(\Delta x)^2 + \\ U_i^{n-1} \left(\frac{bf}{aU_i^{n-1} + bU_i^{n-1} + cV_{I,i}^{n-1}} + \alpha_1 V_{S,i}^{n-1} + \alpha_2 V_{I,i}^{n-1} - n_2 \right) \end{array} \right\} \end{array} \right]$$

$$V_{I,i}^{n+1} = V_{I,i}^{n} + \Delta t \begin{bmatrix} a_1 \begin{Bmatrix} D_{33}\left(\bar{V}_{I,i+1}^{n+1} - 2\bar{V}_{I,i}^{n+1} + \bar{V}_{I,i-1}^{n+1}\right)/(\Delta x)^2 + \\ \bar{V}_{I,i}^{n+1}\left(\dfrac{cf}{a\bar{U}_i^{n+1}+b\bar{V}_{S,i}^{n+1}+c\bar{V}_{I,i}^{n+1}} + \gamma\bar{V}_{S,i}^{n+1} - \alpha_2 U_i^{n+1} - n_2\right) \end{Bmatrix} + \\ b_1 \begin{Bmatrix} D_{33}\left(V_{I,i+1}^{n} - 2V_{I,i}^{n} + V_{I,i-1}^{n}\right)/(\Delta x)^2 + \\ V_{I,i}^{n}\left(\dfrac{cf}{aU_i^{n}+bV_{S,i}^{n}+cV_{I,i}^{n}} + \gamma V_{S,i}^{n} - \alpha_2 U_i^{n} - n_2\right) \end{Bmatrix} + \\ c_1 \begin{Bmatrix} D_{33}\left(V_{I,i+1}^{n-1} - 2V_{I,i}^{n-1} + V_{I,i-1}^{n-1}\right)/(\Delta x)^2 + \\ V_{I,i}^{n-1}\left(\dfrac{cf}{aU_i^{n-1}+bV_{S,i}^{n-1}+cV_{I,i}^{n-1}} + \alpha_1 V_{S,i}^{n-1} + \alpha_2 V_{I,i}^{n-1} - n_2\right) \end{Bmatrix} \end{bmatrix}$$

Figure 3. Solution for $t = 5$ and $\gamma = 0.20, 0.30, 0.35, 0.45$.
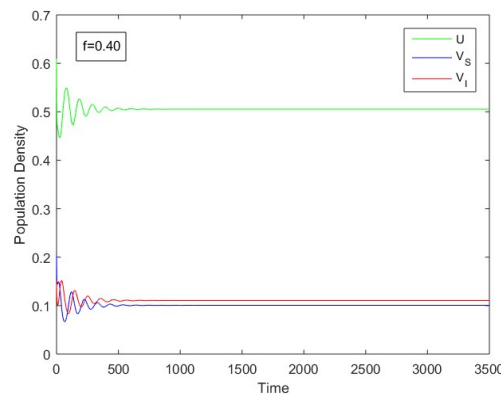
Figure 4. Time series solution for $t = 3500$ and $f = 0.40$.
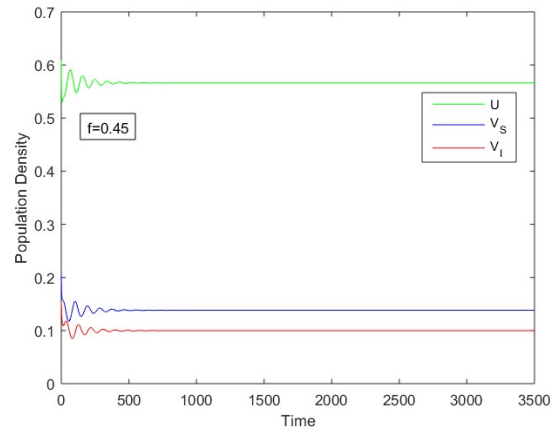
Figure 5. Time series solution for $t = 3500$ and $f = 0.40$.
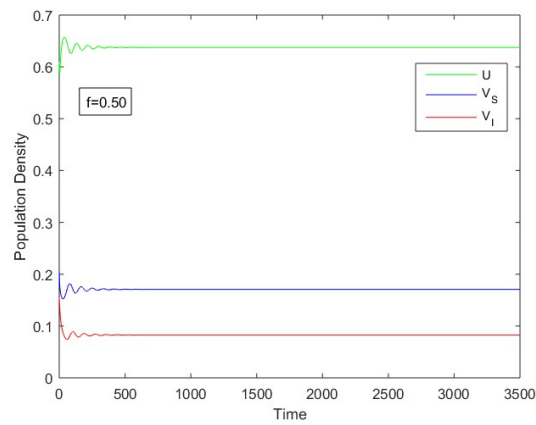
Figure 6. Time series solution for $t = 3500$ and $f = 0.40$.
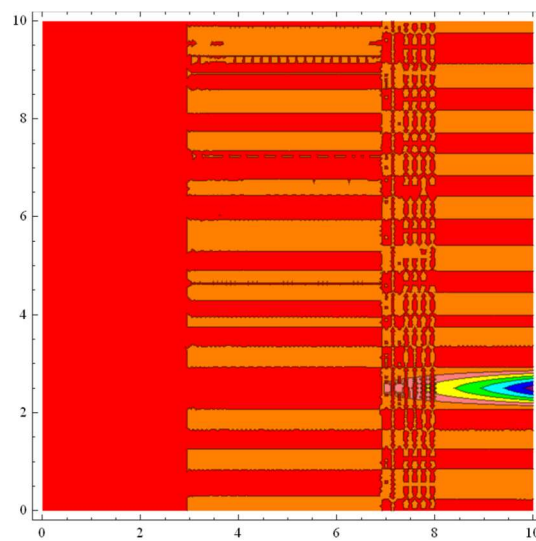
Figure 7. Contour plot for the predator.
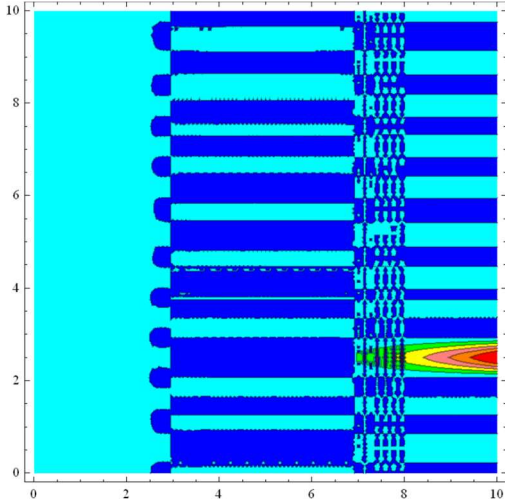
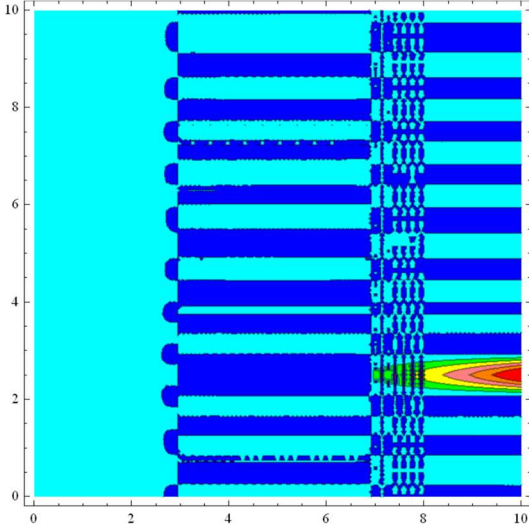**Figure 8**. Contour plot for the susceptible prey.

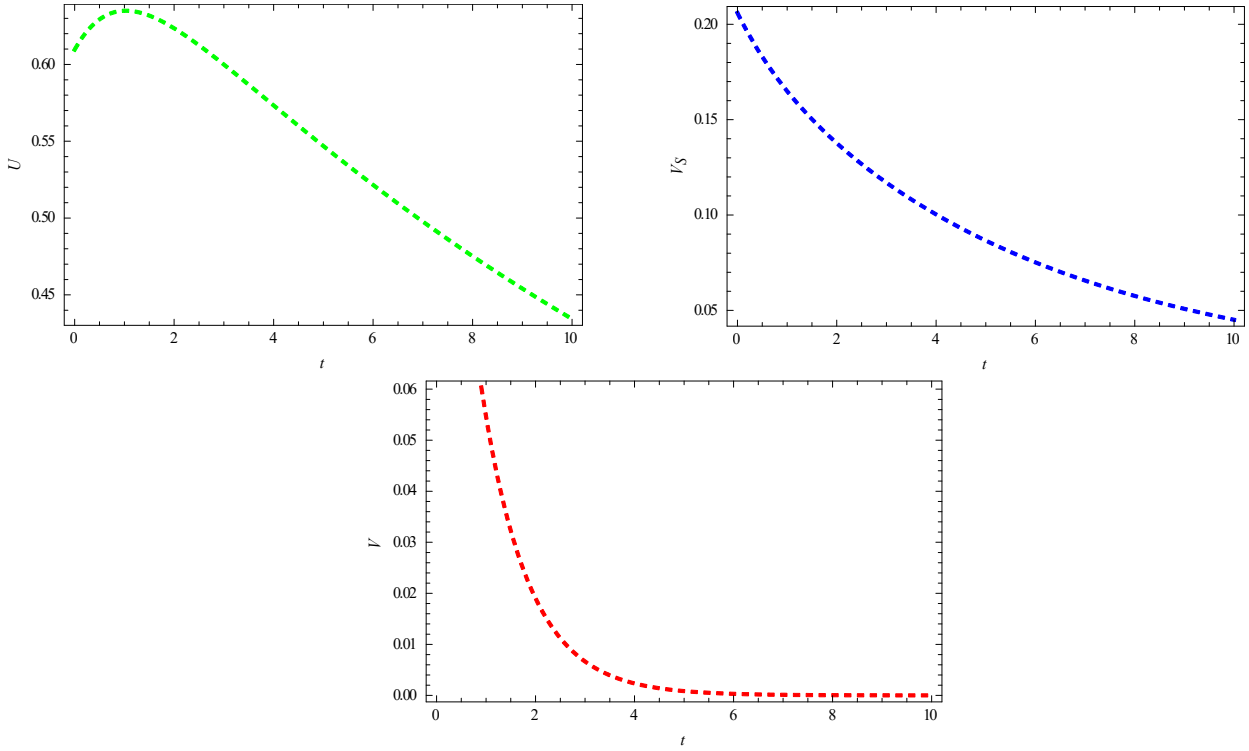**Figure 9.** Contour plot for the infected prey.

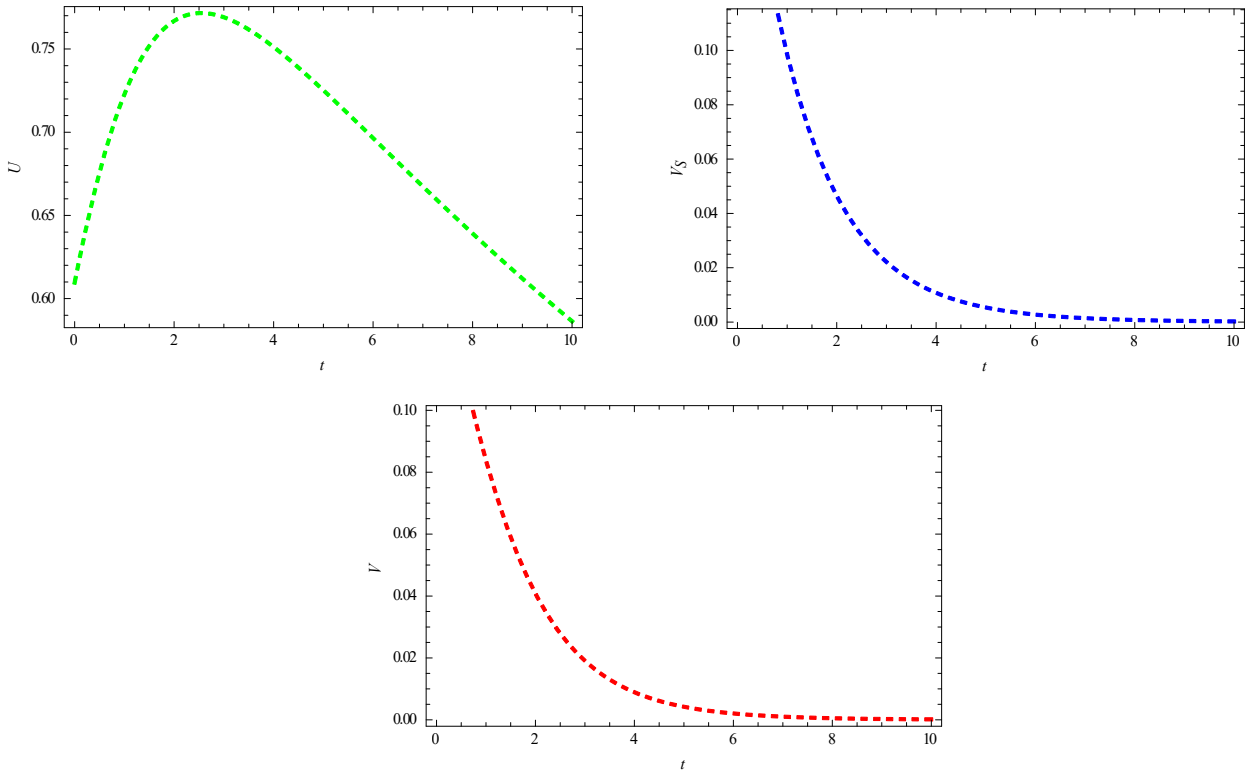**Figure 10**. Population density for non-diffusive system.

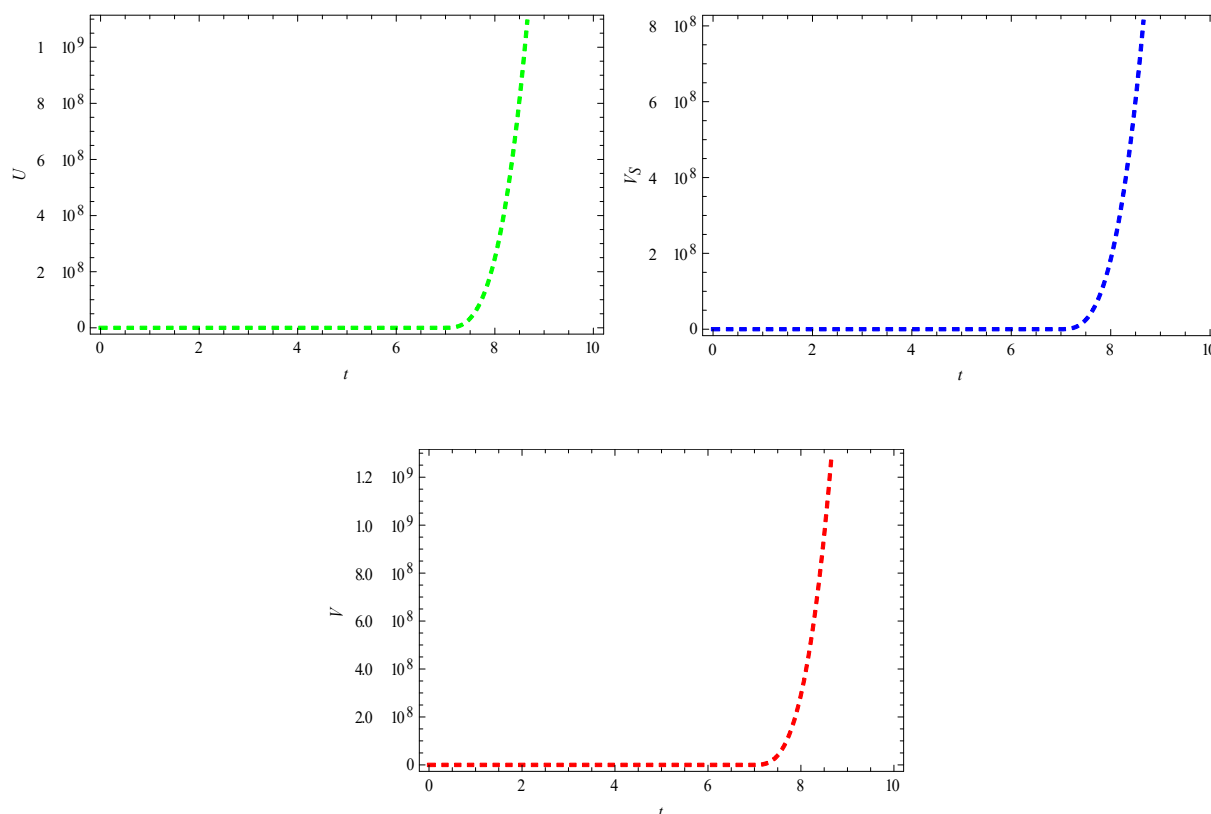
Figure 11. Population density for self-diffusive system.

Figure 12. Population density for self-diffusive system.

7. Results and discussion

T he present research yielded an eco-epidemiological model for the predator-prey system. We considered two classes of prey species, i.e., susceptible and infected prey. It is supposed in this study that the transmission of infection is non-vertical. Predation activity influences the biological environment on a large scale. Here, we considered that there is no disease transmission due to predation. The populations of two classes of prey species face disturbance due to infection. Figures 1 and 2 show the stability of the coexisting equilibrium point for the original and self-diffusive systems. The parameters used for the simulation in Figure 1 are $a = 0.45, b = 0.64, c = 0.53, f = 0.53, \alpha_1 = 0.14, \alpha_2 = 0.27, m = 0.56, n_1 = 0.55, n_2 = 0.49, \gamma = 0.49$, whereas, in Figure 2, the values of all of the parameters are the same as in Figure 1, with the diffusion coefficients $D_{11} = 0.10, D_{22} = 0.20 \wedge D_{33} = 0.10$. Figure 3 shows the solution of the system for $\gamma = 0.20, 0.30, 0.35, \wedge 0.45$. Other parameters have the same values as in Figure 1. Figures 4–6 show time-series solutions for the self-diffusive system with $D_{11} = 0.01, D_{22} = 0.06 \wedge D_{33} = 0.07$ for different resource values ($f = 0.40, 0.45, \wedge 0.50$); the values of the other parameters were the same as in Figure 1. For Figure 3, the initial condition was (0.1, 0.1, 0.1), whereas, for all Figures 4–6, the initial condition was (0.6096, 0.2060, 0.1533). The Figures 4–6 results indicate that the oscillation produced in the system dies out, resulting in the stability of the species' coexistence. Figures 7–9 present contour plots for the predator, susceptible prey and infected prey, respectively. The values of the parameters were $a = 0.91, b = 0.50, c = 0.55, f = 0.91, \alpha_1 = 0.77, \alpha_2 = 0.95, m = 0.06, n_1 = 0.85, n_2 = 0.055, \gamma = 0.60$. The values of the diffusion parameters were $D_{11} = 0.02, D_{12} = 0.84, D_{13} = 0.5$, $D_{21} = 0.4, D_{22} = 0.07, D_{23} = 0$, $D_{31} = 0.2, D_{32} = 0 \wedge$

$D_{33} = 0.01$. In Figures 1 and 2, it is evident that the systems described by (9)–(11) and (23)–(25) turn to the coexistence of the fixed point, which is biologically important. Figure 3 shows the direct impact of γ on the infected prey population. When the value of γ is raised from 0.20 to 0.45, there is a significant increase in the said population. In the time-series plots, it is interesting to note that increased values of the food parameter f lead to a remarkable increase in the predator and susceptible prey populations. Still, a reverse effect is obvious on the infected population of prey species. Figures 10–12 show a comparison among the systems presented in (9)–(11), (23)–(25) and (36)–(38). The population density of all species is remarkably affected by the inclusion of self- and cross-diffusion. The values of the parameters taken in Figure 9 are $a = 0.1, b = 0.50, c = 0.05, f = 0.01, \alpha_1 = 0.0177, \alpha_2 = 0.95, m = 0.06, n_1 = 0.185, n_2 = 0.055, \wedge \gamma = 0.60$. In Figure 10, all of the parameters have the same value, except $\alpha_1 = 0.0177$, and the diffusion coefficients are $D_{11} = 0.46, D_{22} = 0.7 \wedge D_{33} = 0.71$. All of the parameters have the same values as those applied in Figure 9, except $D_{12} = 0.31, D_{13} = 0.75 \wedge D_{21} = 0.14, D_{23} = 0, D_{31} = 0.65 \wedge D_{32} = 0$.

8. Conclusions

In the present article, we have formulated a dynamical nonlinear eco-epidemiological population model considering two species, i.e., the predator and prey, taking into account the species' dependence on the same food resources. Moreover, it is assumed that a fatal infectious, transmissible disease is present in the prey species. The equilibria of the proposed model have been found. We analyzed the global stability of the coexistence equilibrium point for the non-diffusive and diffusive cases by establishing the Lyapunov function. We have proved in Theorems 1 and 2 that the coexisting equilibrium point (U^*, V_S^*, V_I^*) is globally stable for the original and self-diffusive models, respectively. Also, it is proved in Theorem 3 that Turing instability of the system for the cross-diffusive case exists conditionally. Phase portraits were drawn to support the theoretical results and show the equilibrium point's stability for the non-diffusive and self-diffusive cases. Moreover, a two-stage explicit numerical scheme was constructed, and the stability of the proposed scheme was evaluated by using the von Neumann stability criterion. The impacts of parameters like the disease transmission parameter γ and food resource f have been analyzed through the use of plots. It is obvious from the plots that increasing the disease transmission parameter γ enhances the infected population density, and vice versa. The solution of the self-diffusive system for different values of the resource parameter f has been obtained, and it is shown that the increment in the food resource affects the population size.

Conflict of Interest

The authors declare that they have no conflict of interest to report regarding the present study

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Appendix A

For the proof of (22):

$$\frac{dU}{dt} = U(\frac{\alpha_1 a}{b}(U^* - U) + (\frac{\alpha_1 c}{b} - \alpha_2)(V_I^* - V_I))$$

We consider

$$aU + bV_S + cV_I = aU^* + bV_S^* + cV_I^*$$

$$aU + bV_S + cV_I = a\left(\frac{af\gamma^2 + bf\gamma\ \alpha_2 - b\gamma\ mn_2 - bn_1\ n_2\ \alpha_2 + bn_2^2\alpha_1 - cf\gamma\ \alpha_1 + c\gamma\ mn_1 + cn_1^2\alpha_2 - cn_1\ n_2\alpha_1}{(a\gamma + b\alpha_2 - c\alpha_1)(\gamma\ m + n_1\ \alpha_2 - n_2\ \alpha_1)}\right)$$

$$+b\left(\frac{af\gamma\ \alpha_2 + a\gamma\ mn_2 + an_1\ n_2\ \alpha_2 - an_2^2\alpha_1 + bf\alpha_2^2 - cf\alpha_1\ \alpha_2 - c\gamma\ m^2 - cmn_1\ \alpha_2 + cmn_2\ \alpha_1}{(a\gamma + b\alpha_2 - c\alpha_1)(\gamma\ m + n_1\ \alpha_2 - n_2\ \alpha_1)}\right)$$

$$+c(-\frac{af\gamma\ \alpha_1 + a\gamma\ mn_1 + an_1^2\alpha_2 - an_1\ n_2\ \alpha_1 + bf\alpha_1\ \alpha_2 - b\gamma\ m^2 - bmn_1\ \alpha_2 + bmn_2\ \alpha_1 - cf\alpha_1^2}{(a\gamma + b\alpha_2 - c\alpha_1)(\gamma\ m + n_1\ \alpha_2 - n_2\ \alpha_1)})$$

$$= \frac{f(a\gamma + b\alpha_2 - c\alpha_1)}{(\gamma\ m + n_1\ \alpha_2 - n_2\ \alpha_1)}$$

$$bV_S = \frac{f(a\gamma + b\alpha_2 - c\alpha_1)}{(\gamma\ m + n_1\ \alpha_2 - n_2\ \alpha_1)} - aU - cV_I$$

$$\alpha_1 V_S = \frac{\alpha_1 f(a\gamma + b\alpha_2 - c\alpha_1)}{(\gamma\ m + n_1\ \alpha_2 - n_2\ \alpha_1)} - \frac{\alpha_1 aU}{b} - \frac{\alpha_1 cV_I}{b}$$

Using above value in (9) and applying simplification leads us to the following result:

$$\frac{dU}{dt} = U(\frac{\alpha_1 a}{b}(U^* - U) + (\frac{\alpha_1 c}{b} - \alpha_2)(V_I^* - V_I))$$

Appendix B

To prove the value of $M_1 = -D_{11}U^* \int_\Omega \frac{1}{U^2}|\nabla U|^2 dX \leq 0$ in (28), we proceed as follows:

$$\text{As} \qquad M_1 = \int_\Omega \frac{U - U^*}{U} D_{11} \triangle U dX$$

By using Green's first identity, we have

$$\int_\Omega \frac{U - U^*}{U} D_{11} \triangle U dX$$

$$= D_{11}\ (\int_\Omega \frac{U - U^*}{U} \frac{\partial U}{\partial \vartheta} dX - \int_\Omega \nabla_x \frac{U - U^*}{U} . \nabla_x U dX)$$

$$= -D_{11} \int_\Omega \nabla_x \frac{U - U^*}{U} . \nabla_x U dX$$

$$= \text{-}D_{11} \int_{\Omega} \nabla_{\text{x}} \left(1 - \frac{U^*}{U} \right) \nabla_{\text{x}} U dX$$

Further simplification leads to the following:

$$= -D_{11} \int_{\Omega} \frac{U^*}{U^2} |\nabla U|^2 dX$$

So, we have

$$M_1 = -D_{11} U^* \int_{\Omega} \frac{1}{U^2} |\nabla U|^2 dX \le 0$$

Similarly, (31) and (32) can be proved.