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Research article

Improved decay of solution for strongly damped nonlinear wave equations

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Abstract: In this work, we deal with the initial boundary value problem of solutions for a class of linear strongly damped nonlinear wave equations $u_{tt} - \Delta u - \alpha\Delta u_t = f(u)$ in the frame of a family of potential wells. For this strongly damped wave equation, we not only prove the global-in-time existence of the solution, but we also improve the decay rate of the solution from the polynomial decay rate to the exponential decay rate.

Keywords: wave equations; linear strong damping; global existence; decay

1. Introduction

In this paper, we study the initial boundary value problem of solutions for the following strongly damped nonlinear wave equations:

$$u_{tt} - \Delta u - \alpha\Delta u_t = f(u),\ x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0, \tag{1.3}$$

where $\alpha > 0$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and f is a given nonlinear function satisfying the following hypothesis.

$$(\mathbf{H}) \begin{cases} (\mathrm{i}) f(s) \in C^1, s\,(sf'(s) - f(s)) \geq 0,\ \text{and the equality holds only for } u = 0; \\ (\mathrm{ii}) |f(s)| \leq a|s|^q,\ a > 0,\ 1 < q < \infty \text{ for } n = 1,2; 1 < q \leq \frac{n+2}{n-2} \text{ for } n \geq 3; \\ (\mathrm{iii}) (p+1)F(s) \leq sf(s),\ F(s) = \int_0^s f(\tau)\mathrm{d}\tau \text{ for some } p > 1. \end{cases}$$

In the one-dimensional case, Eq (1.1) models the longitudinal vibration of a uniform, homogeneous bar with the nonlinear stress law given by the function in [1]. In the two-dimensional and three-dimensional cases, Eq (1.1) describes antiplane shear motions of viscoelastic solids [2]. For $f(u) =$

$\sin u$, Eq (1.1) can be used to describe the propagation of fluxons in the Josephson junction between two superconductors [3,4]. The concept of a strongly damped nonlinear wave equation was introduced to describe many nonlinear phenomena described by Eqs [5–9], and it has attracted a lot of attention from the mathematical perspective [10–13]. In [14] the problem described by Eqs (1.1)–(1.3) for $n \leq 3$ was considered, and the global existence and asymptotic behavior of the strong solution were obtained for the positive-definite initial energy by properly adjusting the nonlinearity $f(u)$. Then, the authors of [15] further considered this problem and proved that the local existence of the solution is Lipschitz-continuous on a bounded domain, and that the solution is global and decays exponentially to zero as $t \to \infty$ under the assumption that $f(u)$ ensures the positive energy. In [16], the authors further showed the asymptotic behavior of the solution for the problem described by Eqs (1.1)–(1.3) by assuming that the nonlinearity takes the form to ensure the positive-definite initial energy.

Here, we would like to mention that the above results are based on the assumptions ensuring that the corresponding problem including Eq (1.1) has a positive-definite initial energy, or that the nonlinearity takes the linear form. In [17], the authors considered the nonlinear version of the model with non-positive initial energy. They showed the global existence of both weak and strong solutions to problem described by Eqs (1.1)–(1.3) by using the potential well method. It is natural to consider how the global solution behaves after we get the existence of the global solution, which involves describing the long time behavior of the solution as the time $t \to +\infty$.

Pata and Squassina [18] investigated the initial boundary value problem of the three-dimensional wave equation $u_{tt} - \Delta u - \alpha\Delta u_t + \phi(u) = f(u)$, and carefully described the long time behavior of the solution. The techniques and conclusions there depend on the the damping coefficient α, which means that the weakly damping term plays a crucial role in the arguments. For the initial boundary value problem of the strongly damped nonlinear wave equation given by

$$u_{tt} - \Delta u - \omega\Delta u_t + \mu u_t = |u|^{p-2}u, \tag{1.4}$$

Gazzola and Squassina in [19] conducted a kind of comprehensive study that involved proving the global well-posedness, finite time blowup and long time behavior of the solution; here, we focus on the conclusion concerning the decay as follows.

Theorem 1.1. *(Polynomial decay obtained in [19]) Let u be the unique local solution to (1.4). Assume that*

$$\omega \geq 0, \ \ \mu > -\omega\lambda_1,$$

λ_1 being the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions, and assume that

$$2 < p \leq \begin{cases} \frac{2n}{n-2} & for \ \ \omega > 0 \\ \frac{2n-2}{n-2} & for \ \ \omega = 0 \end{cases} \quad if\ n \geq 3, \ \ 2 < p < \infty \ \ if\ n = 1, 2. \tag{1.5}$$

In addition, assume that there exists $t_0 \in [0, T_{max})$ such that $u(t_0) \in W'$ and $E\left(u(t_0), u_t(t_0)\right) \leq d$. Then, for $T_{max} = \infty$ and every $t > t_0$, we have

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq \frac{\Theta(\omega, \mu)}{t},$$

where

$$\Theta(\omega,\mu)=\begin{cases}C_\nu\left(1+\frac{1}{\omega}+\omega\right) \text{ for } \omega>0,\\ C\left(1+\frac{1}{\mu}+\mu\right) \text{ for } \omega=0\end{cases}$$

and C is independent of μ, whereas C_μ only depends on μ.

The above theorem tells that the global solution decays polynomially in the form of $\frac{1}{t}$ for the initial data trapped within the potential well and $E(0) < d$, where d is the depth of the potential well, or as it is also called, the mountain pass level.

Later, Gerbi and Houari [20] again investigated the asymptotic behavior of solutions of Eq (1.4); they obtained the following theorem to improve the decay rate obtained in [19] by showing the exponential decay compared with the polynomial decay.

Theorem 1.2. *(Exponential decay obtained in [20]) Assume that $u_0 \in W'$, $E(0) < d$ and (1.5) hold. Then, there exist two positive constants $\bar{C}$ and ξ independent of t such that*

$$0 < E(t) \leq \bar{C}e^{-\xi t}, \quad \forall\, t \geq 0.$$

The key tool in the proof of Theorem 1.2 is the construction of a suitable Lyapunov function to make a small perturbation of the energy. The method introduced by Gerbi and Houari [20] strongly depends on the weakly damping term u_t. In other words, an interesting question is if such a method may work for the case without the weakly damping term, i.e., $\mu = 0$ in Theorem 1.2, and a similar case can be found in [21].

The interesting point of the present paper is to consider the asymptotic behavior of the solution to the strongly damped nonlinear wave equation without the weakly damping term in the framework of the potential well theory. It is well known that, in order to prove the global existence of solution for the nonlinear hyperbolic equation, Sattinger [22] introduced the potential well method to treat the problem without positive-definite energy. Later, in [23], the potential well method was applied to prove the global nonexistence of the solution for semilinear hyperbolic and parabolic equations. In the recent decades, the theory relying on the single potential well was improved by proposing the theory of a family of potential wells [24]. Both of these two theories have been improved and applied in studies of many important mathematical models (see the theory of the single potential well and the family of potential wells [25–29]).

In the present paper, first, we define a family of potential wells, and then, by using them, we give the existence of the global weak solution of the problem described by Eqs (1.1)–(1.3). Although, for the problem described by Eqs (1.1)–(1.3), there have been some conclusions about the global well-posedness of the solution, most of them were not established in the framework of the family of potential wells; hence, it is necessary to rebuild it again to give a strict argument. Further we prove the asymptotic behavior of the solution, i.e., the solution of the problem described by Eqs (1.1)– (1.3) decays exponentially to zero as $t \to \infty$.

2. Preliminary conclusions of setting up the potential wells theory

We shall introduce some necessary functionals and manifolds in order to setup the variational structure. Throughout the paper, we use the following denotations: $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p \mathrm{d}x\right)^{\frac{1}{p}}$,

$\|\cdot\| := \|\cdot\|_2 := \left(\int_\Omega u^2 \mathrm{d}x\right)^{\frac{1}{2}}$ and $(u, v) := \int_\Omega uv\mathrm{d}x$. First, for the problem described by Eqs (1.1)–(1.3), we define the potential energy:

$$J(u) := \frac{1}{2}\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x,$$

where

$$F(u) := \int_0^u f(s)\mathrm{d}s.$$

Here, we also need the Nehari functional:

$$I(u) := \|\nabla u\|^2 - \int_\Omega uf(u)\mathrm{d}x,$$

as well as its family version:

$$I_\delta(u) := \delta\|\nabla u\|^2 - \int_\Omega uf(u)\mathrm{d}x, \ \delta > 0.$$

Next, in aid of the Nehari functional and the family of Nehari functionals, we can define the depth of a single potential well and the depths of a family of potential wells, respectively, as follows:

(i) for a single potential well,

$$d := \inf_{u\in\mathcal{N}} J(u),$$

$$\mathcal{N} := \{u \in H_0^1(\Omega) | I(u) = 0, u \neq 0\}.$$

(ii) for family of potential wells,

$$d(\delta) := \inf_{u\in\mathcal{N}_\delta} J(u),$$

$$\mathcal{N}_\delta := \{u \in H_0^1(\Omega) | I_\delta(u) = 0, u \neq 0\}.$$

Lemma 2.1. *Let $f(u)$ satisfy* (**H**)*, $u \in H_0^1(\Omega)$ and $w(u) := \frac{f(u)}{u}$, $u \neq 0$. Then, it holds that*

(i) $\lim_{u\to 0} w(u) = 0$*;*
(ii) $w(u)$ is an increasing function on $(0, \infty)$ and a decreasing function on $(-\infty, 0)$;
(iii) $f(u)u > 0$ for $u \neq 0$;
(iv) $f(u)$ is strictly increasing on $(-\infty, \infty)$;
(v) $0 \le F(u) \le A_1|u|^{q+1}$ for $u \in \mathbb{R}$ and some positive constant A_1.

Proof. (i) The assumption (ii) in (**H**) can directly give this conclusion.

(ii) In view of the assumption (i) in (**H**), we can prove this point by considering the following two cases. If $u > 0$, we see

$$w'(u) = \frac{uf'(u) - f(u)}{u^2} > 0,$$

and if $u < 0$, we see

$$w'(u) = \frac{uf'(u) - f(u)}{u^2} < 0,$$

which is the conclusion we desired here.

(iii) By the already established conclusions (i) and (ii) proved above, we can easily find that $w(u) > 0$ for any $u \neq 0$, which proves that $f(u)u > 0$ for any $u \neq 0$.

(iv) By the assumption (i) in (**H**) and the conclusion (iii) proved just now, we obtain

$$f'(u) \geq \frac{f(u)}{u} > 0, \text{ for } u \neq 0,$$

which proves the conclusion claimed in this item.

(v) By the fact that $f(0) = 0$, with the item (ii) in this lemma, we get $F(u) \geq 0$. Then, $F(u) \leq A_1|u|^{q+1}$ follows from (ii) in (**H**).

Lemma 2.2 (Calculus in an abstract space [30]). *Let $u \in W^{1,p}(0,T;X)$ for some $1 \leq p < \infty$. Then, $u \in C([0,T];X)$.*

Now, let us introduce the following four sets:

$$\begin{aligned} W &:= \{u \in H_0^1(\Omega) | I(u) > 0, J(u) < d\} \cup \{0\}, \\ W_\delta &:= \{u \in H_0^1(\Omega) | I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\}, \ 0 < \delta < \delta_0, \\ W' &:= \{u \in H_0^1(\Omega) | I(u) > 0\} \cup \{0\}, \\ W'_\delta &:= \{u \in H_0^1(\Omega) | I_\delta(u) > 0\} \cup \{0\}, \ 0 < \delta < \delta_0. \end{aligned}$$

3. Global-in-time existence of solution

In this section, we prove the existence of a global weak solution for the problem described by Eqs (1.1)–(1.3) under the assumption (**H**).

Definition 3.1 The function $u = u(x,t)$ is called a weak solution of the problem described by Eqs (1.1)–(1.3) on $\Omega\times[0,T)$, where T is the maximum existence time of the solution if $u \in C((0,T); H_0^1(\Omega))$ and $u_t \in L^\infty((0,T); L^2(\Omega))$ satisfy

(i)

$$\begin{aligned} &(u_t, v) + \alpha (\nabla u, \nabla v) + \int_0^t (\nabla u, \nabla v) \, d\tau \\ = &\int_0^t (f(u), v) \, d\tau + (u_1, v) + \alpha (\nabla u_0, \nabla v), \ \forall v \in H_0^1(\Omega), \ t \in [0,T), \end{aligned}$$

(ii)

$$u(x,0) = u_0(x) \text{ in } H_0^1(\Omega), \ u_t(x,0) = u_1(x) \text{ in } L^2(\Omega).$$

Theorem 3.1. *Let $f(u)$ satisfy (**H**), $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$ and $u_0(x) \in W'$. Then, the problem described by Eqs (1.1)–(1.3) admits a global weak solution $u \in C((0,\infty); H_0^1(\Omega))$ with $u_t \in L^\infty\left((0,\infty); L^2(\Omega)\right) \cap L^2\left((0,\infty); H_0^1(\Omega)\right)$ and $u \in W$ for $0 \leq t < \infty$.*

Proof. Let $\{w_j(x)\}$ be a system of base functions in $H_0^1(\Omega)$. Construct the approximate solutions of the problem described by Eqs (1.1)–(1.3)

$$u_m(x,t) = \sum_{j=1}^m g_{jm}(t) w_j(x), \quad m = 1, 2, \cdots,$$

satisfying

$$\langle u_{mtt}, w_s\rangle + (\nabla u_m, \nabla w_s) + \alpha(\nabla u_{mt}, \nabla w_s) = (f(u_m), w_s), \tag{3.1}$$
$$s = 1, 2, \cdots, m,$$
$$u_m(x, 0) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0(x) \text{ in } H_0^1(\Omega), \tag{3.2}$$
$$u_{mt}(x, 0) = \sum_{j=1}^{m} b_{jm} w_j(x) \to u_1(x) \text{ in } L^2(\Omega). \tag{3.3}$$

Testing the both sides of (3.1) by $g'_{sm}(t)$ and taking the sum for s, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_m(t) + \alpha\|\nabla u_{mt}\|^2 = 0 \tag{3.4}$$

and

$$E_m(t) + \alpha \int_0^t \|\nabla u_{m\tau}\|^2 \mathrm{d}\tau = E_m(0), \ \ 0 \le t < \infty, \tag{3.5}$$

where

$$E_m(t) := \frac{1}{2}\|u_{mt}\|^2 + J(u_m).$$

Now, we need a conclusion about the approximate solution that $E_m(0) < d$ and $u_m(0) \in W'$ by a limit, which can be derived from the assumptions $E(0) < d$ and $u_0(x) \in W'$ with the settings of (3.2) and (3.3). Hence, from (3.5), we have

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) + \alpha \int_0^t \|\nabla u_{m\tau}\|^2 \mathrm{d}\tau < d, \ \ 0 \le t < \infty. \tag{3.6}$$

From (3.6) and an argument similar to that in [25–27], for $0 \le t < \infty$ and a sufficiently large m, we can obtain

$$u_m(t) \in W'. \tag{3.7}$$

From this and

$$\begin{aligned} &\frac{1}{2}\|u_{mt}\|^2 + \frac{p-1}{2(p+1)}\|\nabla u_m\|^2 + \frac{1}{p+1} I(u_m) + \alpha \int_0^t \|\nabla u_{m\tau}\|^2 \mathrm{d}\tau \\ =&\frac{1}{2}\|u_{mt}\|^2 + J(u_m) + \alpha \int_0^t \|\nabla u_{m\tau}\|^2 \mathrm{d}\tau < d, \ \ 0 \le t < \infty, \end{aligned}$$

we can get

$$\|\nabla u_m\|^2 \le \frac{2(p+1)}{p-1} d, \ \ 0 \le t < \infty,$$
$$\|u_{mt}\|^2 < 2d, \ 0 \le t < \infty,$$
$$\int_0^t \|\nabla u_{m\tau}\|^2 \mathrm{d}\tau \le \frac{d}{\alpha}, \ 0 \le t < \infty,$$

$$\|f(u_m)\|_r^r \le \int_\Omega (a|u_m|)^{qr}\,\mathrm{d}x \le C,\ 0 \le t < \infty,\ r = \frac{q+1}{q}.$$

Hence, there exists a u and a subsequence $\{u_\nu\}$ of $\{u_m\}$ such that, as $\nu \to \infty$,
$u_\nu \to u$ in $L^\infty((0,\infty); H_0^1(\Omega))$ weakly star and a.e. in $Q = \Omega \times [0,\infty)$,
$u_{\nu t} \to u_t$ in $L^\infty((0,\infty); L^2(\Omega))$ weakly star,
$f(u_\nu) \to f(u)$ in $L^\infty((0,\infty); L^r(\Omega))$ weakly star.
Further, for an s that is fixed, letting $m \to \infty$ in (3.1), we obtain

$$\langle u_{tt}, \omega_s\rangle + (\nabla u, \nabla\omega_s) + \alpha\,(\nabla u_t, \omega_s) = (f(u), \omega_s)\,.$$

Since $\omega_j(x)$ is a system of base functions in $H_0^1(\Omega)$, for any $v \in H_0^1(\Omega)$, we deduce

$$\langle u_{tt}, v\rangle + (\nabla u, \nabla v) + \alpha\,(\nabla u_t, v) = (f(u), v)\,.$$

Moreover, (3.2) indicates that $u_m(0) \to u_0 \in H_0^1(\Omega)$ and (3.3) gives $u_{mt}(0) \to u_1 \in L^2(\Omega)$. Hence, from the above disscussion, we know that the problem described by Eqs (1.1)–(1.3) admits a weak solution

$$u \in L^\infty\left((0,\infty); H_0^1(\Omega)\right) \tag{3.8}$$

with

$$u_t \in L^\infty\left((0,\infty); L^2(\Omega)\right) \cap L^2\left((0,\infty); H_0^1(\Omega)\right). \tag{3.9}$$

Regarding the continuity in time t, from (3.8) and (3.9), we get

$$u \in H^1\left((0,\infty); H_0^1(\Omega)\right),$$

and then we obtain

$$u \in C\left((0,\infty); H_0^1(\Omega)\right)$$

by Lemma 2.2. By (3.7) and the compactness method, for $t \in [0,\infty)$, we derive $u(t) \in W$.

4. Improved decay of solution

In this section, we discuss the asymptotic behavior of a solution for the problem described by Eqs (1.1)–(1.3). We prove that the global weak solution given in Theorem 3.1 decays exponentially to zero as $t \to \infty$.

First, we give the following lemma, which will be used to prove the decay of the solution. And, this lemma will be proved in aid of the family of potential wells.

Lemma 4.1. *Let $f(u)$ satisfy* (**H**)*, $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$ and u_m be the approximate solutions. Then, the following holds:*

(i)

$$I(u_m) = \|u_{mt}\|^2 - \frac{\mathrm{d}}{\mathrm{d}t}(u_{mt}, u_m) - \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_m\|^2. \tag{4.1}$$

(ii) For $0 < E(0) < d$*,* $u_0(x) \in W'$ *and* $0 \le t < \infty$*, it holds that*

$$I(u_m) \ge (1 - \delta_1)\|\nabla u_m\|^2,$$

where (δ_1, δ_2) *is the maximal interval such that* $d(\delta) > E(0)$*.*

Proof. (i) Multiplying (3.1) by $g_{sm}(t)$ and summing for s, we get (4.1).

(ii) First, from $E(0) < d(\delta)$ for $\delta \in (\delta_1, \delta_2)$, (3.2) and (3.3), it follows that $E_m(0) < d(\delta)$ for $\delta \in (\delta_1, \delta_2)$. Next, from (3.5), for $\delta \in (\delta_1, \delta_2)$ and $0 \le t < \infty$, we get

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) + \alpha \int_0^t \|\nabla u_{m\tau}\|^2 \mathrm{d}\tau < d(\delta). \tag{4.2}$$

From (4.2) and an argument similar to that in [25–27], we can prove that $u_m(t) \in W'_\delta$ for $\delta \in (\delta_1, \delta_2)$, $0 \le t < \infty$ and a sufficiently large m. Hence, we have that $I_\delta(u_m) \ge 0$ for $\delta \in (\delta_1, \delta_2)$ and $I_{\delta_1}(u_m) \ge 0$ for $0 \le t < \infty$ and a sufficiently large m. Thereby, for $0 \le t < \infty$, we get

$$\begin{aligned} I(u_m) &= \|\nabla u_m\|^2 - \int_\Omega u_m f(u_m)\mathrm{d}x \\ &= (1 - \delta_1)\|\nabla u_m\|^2 + I_{\delta_1}(u_m) \\ &\ge (1 - \delta_1)\|\nabla u_m\|^2. \end{aligned}$$

Theorem 4.1. *Let* $f(u)$ *satisfy* (**H**)*,* $u_0(x) \in H_0^1(\Omega)$ *and* $u_1(x) \in L^2(\Omega)$*. Assume that* $0 < E(0) < d$ *and* $u_0(x) \in W'$*. Then, for the global weak solution, it holds that*

$$\|u_t\|^2 + \|\nabla u\|^2 \le \frac{2(p+1)}{p-1}\left(C_2 e^{-\gamma t} + C_3 t e^{-\lambda t}\right),\ 0 \le t < \infty \tag{4.3}$$

for some positive constants C_2*,* C_3*,* γ *and* λ*.*

Proof. We aim to prove the approximate solution

$$E_m(t) \le C_2 e^{-\gamma t} + C_3 t e^{-\lambda t},\ 0 \le t < \infty \tag{4.4}$$

for $C_2 > 0$, $C_3 > 0$ and $\lambda > 0$. To do this, multiplying (3.4) by $e^{\gamma t}(\gamma > 0)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\gamma t} E_m(t)\right) + \alpha e^{\gamma t}\|\nabla u_{mt}\|^2 = \gamma e^{\gamma t} E_m(t)$$

and

$$\begin{aligned} &e^{\gamma t} E_m(t) + \alpha \int_0^t e^{\gamma \tau}\|\nabla u_{m\tau}\|^2 \mathrm{d}\tau \\ =&E_m(0) + \gamma \int_0^t e^{\gamma \tau} E_m(\tau)\mathrm{d}\tau,\ 0 \le t < \infty. \end{aligned} \tag{4.5}$$

By Lemmas 2.1 and 4.1, we get

$$\int_0^t e^{\gamma \tau} E_m(\tau)\mathrm{d}\tau = \int_0^t e^{\gamma \tau}\left(\frac{1}{2}\|u_{m\tau}\|^2 + \frac{1}{2}\|\nabla u_m\|^2 - \int_\Omega F(u_m)\,\mathrm{d}x\right)\mathrm{d}\tau$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_0^t e^{\gamma\tau}\left(\|u_{m\tau}\|^2 + \|\nabla u_m\|^2\right)\mathrm{d}\tau \\
&\leq \frac{1}{2}\int_0^t e^{\gamma\tau}\left(\|u_{m\tau}\|^2 + \frac{1}{1-\delta_1}I(u_m)\right)\mathrm{d}\tau \\
&= \frac{1}{2}\left(1 + \frac{1}{1-\delta_1}\right)\int_0^t e^{\gamma\tau}\|u_{m\tau}\|^2\mathrm{d}\tau \\
&\quad - \frac{1}{2(1-\delta_1)}\int_0^t e^{\gamma\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left((u_{m\tau}, u_m) + \frac{\alpha}{2}\|\nabla u_m\|^2\right)\mathrm{d}\tau
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
& -\int_0^t e^{\gamma\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\left((u_{m\tau}, u_m) + \frac{\alpha}{2}\|\nabla u_m\|^2\right)\mathrm{d}\tau \\
=& (u_{mt}(0), u_m(0)) + \frac{\alpha}{2}\|\nabla u_m(0)\|^2 - e^{\gamma t}\left((u_{mt}, u_m) + \frac{\alpha}{2}\|\nabla u_m\|^2\right) \\
& + \gamma\int_0^t e^{\gamma\tau}\left((u_{m\tau}, u_m) + \frac{\alpha}{2}\|\nabla u_m\|^2\right)\mathrm{d}\tau \\
\leq & \frac{1}{2}\left(\|u_{mt}(0)\|^2 + \|u_m(0)\|^2 + \alpha\|\nabla u_m(0)\|^2\right) \\
& + \frac{1}{2}e^{\gamma t}\left(\|u_{mt}\|^2 + \|u_m\|^2 + \alpha\|\nabla u_m\|^2\right) \\
& + \frac{\gamma}{2}\int_0^t e^{\gamma\tau}\left(\|u_{m\tau}\|^2 + \|u_m\|^2 + \alpha\|\nabla u_m\|^2\right)\mathrm{d}\tau, \ 0 \leq t < \infty.
\end{aligned}
\tag{4.7}
$$

From

$$
\begin{aligned}
E_m(t) &= \frac{1}{2}\|u_{mt}\|^2 + J(u_m) \\
&= \frac{1}{2}\|u_{mt}\|^2 + \frac{p-1}{2(p+1)}\|\nabla u_m\|^2 + \frac{1}{p+1}I(u_m) \\
&\geq \frac{1}{2}\|u_{mt}\|^2 + \frac{p-1}{2(p+1)}\|\nabla u_m\|^2,
\end{aligned}
$$

we have

$$
\|u_{mt}\|^2 + \|\nabla u_m\|^2 \leq \frac{2(p+1)}{p-1}E_m(0), \ 0 \leq t < \infty. \tag{4.8}
$$

Hence, there exits a $C > 0$ such that

$$
\frac{1}{2}\left(\|u_{mt}\|^2 + \|u_m\|^2 + \alpha\|\nabla u_m\|^2\right) \leq CE_m(t), \ 0 \leq t < \infty. \tag{4.9}
$$

From (4.5)–(4.7) and (4.9), it follows that there exist some positive constants C_0 and C_1 such that

$$
\begin{aligned}
& e^{\gamma t}E_m(t) + \alpha\int_0^t e^{\gamma\tau}\|\nabla u_{m\tau}\|^2\mathrm{d}\tau \\
\leq & C_0E_m(0) + \frac{\lambda_0\gamma}{2}\left(1 + \frac{1}{1-\delta_1}\right)\int_0^t e^{\gamma\tau}\|\nabla u_{m\tau}\|^2\mathrm{d}\tau + C_1\gamma e^{\gamma\tau}E_m(t) \\
& + C_1\gamma^2\int_0^t e^{\gamma\tau}E_m(\tau)\mathrm{d}\tau, \ \ 0 \leq t < \infty,
\end{aligned}
\tag{4.10}
$$

where

$$\lambda_0 = \sup_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|\nabla u\|^2}.$$

Take γ such that

$$0 < \gamma < \min\left\{\frac{1}{2C_1}, \frac{2\alpha}{\lambda_0\left(1 + \frac{1}{1-\delta_1}\right)}\right\}.$$

Then, (4.10) gives

$$\begin{aligned} e^{\gamma t}E_m(t) &\le 2C_0E_m(0) + 2C_1\gamma^2 \int_0^t e^{\gamma\tau}E_m(\tau)\mathrm{d}\tau \\ &< 2C_0d + 2C_1\gamma^2 \int_0^t e^{\gamma\tau}E_m(\tau)\mathrm{d}\tau. \end{aligned}$$

And, by the Gronwall inequality, we can obtain

$$e^{\gamma t}E_m(t) \le 2C_0d\left(1 + 2C_1\gamma^2 te^{2C_1\gamma^2 t}\right),\ 0 \le t < \infty,$$

i.e.,

$$E_m(t) \le C_2e^{-\gamma t} + C_3te^{-\lambda t},\ 0 \le t < \infty$$

and (4.4), where $C_2 := 2C_0d$, $C_3 := 4C_0C_1d\gamma^2$ and $\lambda := \gamma(1 - 2C_1\gamma) > 0$. Finally, let $\{u_\nu\}$ be the subsequence of $\{u_m\}$ given in the proof of Theorem 3.1. Then, from (4.8) and (4.4), we get

$$\begin{aligned} \|u_t\|^2 + \|\nabla u\|^2 &\le \liminf_{\nu\to\infty}\left(\|u_{\nu t}\|^2 + \|\nabla u_\nu\|^2\right) \\ &\le \liminf_{\nu\to\infty} \frac{2(p+1)}{p-1}E_\nu(t) \\ &\le \frac{2(p+1)}{p-1}\left(C_2e^{-\gamma t} + C_3te^{-\lambda t}\right),\ 0 \le t < \infty, \end{aligned}$$

which gives (4.3).

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Conflict of interest

The authors declare that there is no conflict of interest.

References

1. V. P. Maslov, P. P. Mosolov, *Nonlinear Wave Equations Perturbed by Viscous Terms*, Walter de Gruyter, Berlin, New York, 2000.

2. W. N. Findleky, J. S. Lai, K. O. Onaran, *Creep and Relaxation of Nonlinear Viscoelastic Materials*, North-Holland, Amsterdam, New York, 1976.

3. J. B. V. D. Berg, S. A. V. Gils, T. P. P. Visser, Parameter dependence of homoclinic solutions in a single long Josephson junction, *Nonlinearity*, **16** (2003), 707–717. https://doi.org/10.1088/0951-7715/16/2/320

4. W. X. Qin, P. L. Zhang, Discrete rotating waves in a ring of coupled mechanical oscillators with strong damping, *J. Math. Phys.*, **50** (2009), 052701. https://doi.org/10.1063/1.3122772

5. C. Dafermos, R. Diperna, The Riemann problem for certain classes of hyperbolic systems of conservation laws, *J. Differ. Equations*, **20** (1976), 90–114.https://doi.org/10.1016/0022-0396(76)90098-X

6. J. Greenberg, Smooth and time-periodic solutions to the quasilinear wave equation, *Arch. Rational Mech. Anal.*, **60** (1975), 29–50. https://doi.org/10.1137/S0036141000369344

7. J. Keller, On solutions of nonlinear wave equations, *Comm. Pure Appl. Math.*, **10** (1957), 523–530. https://doi.org/10.1002/cpa.3160100404

8. T. Cazenave, Uniform estimates for solutions of nonlinear Klein-Gordon equations, *J. Funct. Anal.*, **60** (1985), 36–55. https://doi.org/10.1016/0022-1236(85)90057-6

9. J. Ball, Global attractors for damped semilinear wave equations, *Discrete Contin. Dyn. Syst.*, **10** (2004), 31–52. https://doi.org/10.3934/dcds.2004.10.31

10. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source term, *J. Differ. Equations*, **109** (1994), 295–308. https://doi.org/10.1006/jdeq.1994.1051

11. A. Haraux, M. A. Jendoubi, Convergence of bounded weak solutions of the wave equation with dissipation and analytic nonlinearity, *Calc. Var. Partial Differ. Equations*, **9** (1999), 95–124. https://doi.org/10.1007/s005260050133

12. P. Pucci, J. Serrin, Global nonexistence for abstract evolution equations with positive initial energy, *J. Differ. Equations*, **150** (1998), 203–214. https://doi.org/10.1006/jdeq.1998.3477

13. E. Vitillaro, Global existence theorems for a class of evolution equations with dissipation, *Arch. Rational Mech. Anal.*, **149** (1999), 155–182. https://doi.org/10.1007/s002050050171

14. G. F. Webb, Existence and asypmtotic behavior for a strongly damped nonlinear wave equation, *Canad. J. Math.*, **32** (1980), 631–643. https://doi.org/10.4153/CJM-1980-049-5

15. D. D. Ang, A. P. N. Dinh, On the strongly damped wave equation $u_{tt} - \Delta u - \Delta u_t + f(u) = 0$, *SIAM J. Math. Anal.*, **19** (1988), 1409–1417. https://doi.org/10.1137/0519103

16. R. Z. Xu, Y. C. Liu, Asymptotic behavior of solutions for initial-boundary value problems for strongly damped nonlinear wave equations, *Nonlinear Anal.*, **69** (2008), 2492–2495. https://doi.org/10.1016/j.na.2007.08.027

17. Y. Liu, P. Liu, On potential well and application to strongly damped nonlinear wave equations, *Acta Math. Appl. Sin.*, **27** (2004). 710–722. https://doi.org/10.2116/analsci.20.717

18. V. Pata, M. Squassina, On the strongly damped wave equation, *Comm. Math. Phys.*, **253** (2005), 511–533. https://doi.org/10.1007/S00220-004-1233-1

19. F. Gazzola, M. Squassina, Global solutions and finite time blow up for damped semi-linear wave equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **23** (2006), 185–207. https://doi.org/10.1016/j.anihpc.2005.02.007
20. S. Gerbi, B. S. Houari, Exponential decay for solutions to semilinear damped wave equation, *Discrete Contin. Dyn. Syst. Ser. S*, **5** (2008), 559–566. https://doi.org/10.1002/mma.306
21. R. Z. Xu, Y. B. Yang, Global existence and asymptotic behaviour of solutions for a class of fourth order strongly damped nonlinear wave equations, *Q. Appl. Math.*, **71** (2013), 401–415. https://doi.org/10.1090/s0033-569x-2012-01295-6
22. D. H. Sattinger, On global solution of nonlinear hyperbolic equations, *Arch. Rat. Mech. Anal.*, **30** (1968), 148–172. https://doi.org/10.1007/BF00250942
23. L. E. Payne, D. H. Sattinger, Sadle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, **22** (1975), 273–303. https://doi.org/10.1007/bf02761595
24. Y. C. Liu, On potential and vacuum isolating of solutions for semilinear wave equations, *J. Differ. Equations*, **192** (2003), 155–169. https://doi.org/10.1016/S0022-0396(02)00020-7
25. R. Z. Xu, J. Su, Global existence and finite time blow-up for a class of semi-linear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763. https://doi.org/10.1016/j.jfa.2013.03.010
26. R. Z. Xu, W. Lian, Y. Niu, Global well-posedness of coupled parabolic systems, *Sci. China Math.*, **63** (2020), 321–356. https://doi.org/10.1007/s11425-017-9280-x
27. W. Lian, R. Z. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear Anal.*, **9** (2020), 613–632. https://doi.org/10.1515/anona-2020-0016
28. W. Lian, J. Wang, R. Z. Xu, Global existence and blow up of solutions for pseudo-parabolic equation with singular potential, *J. Differ. Equations*, **269** (2020), 4914–4959. https://doi.org/10.1016/j.jde.2020.03.047
29. C. Yao, V. D. Radulescu, R. Z. Xu, M. Y. Zhang, Global well-posedness analysis for the nonlinear extensible beam equations in a class of modified Woinowsky-Krieger models, *Adv. Nonlinear Stud.*, **22** (2022), 436–468. https://doi.org/10.1515/ans-2022-0024
30. L. C. Evans, *Partial Differential Equations*, 2nd edition, American Mathematical Society, 2010.

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