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Research article

Regularization effect of the mixed-type damping in a higher-dimensional logarithmic Keller-Segel system related to crime modeling

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Abstract: We study a logarithmic Keller-Segel system proposed by Rodríguez for crime modeling as follows:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v) - \kappa uv + h_1, \\ v_t = \Delta v - v + u + h_2, \end{cases}$$

in a bounded and smooth spatial domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, with the parameters $\chi > 0$ and $\kappa > 0$, and with the nonnegative functions h_1 and h_2. For the case that $\kappa = 0$, $h_1 \equiv 0$ and $h_2 \equiv 0$, recent results showed that the corresponding initial-boundary value problem admits a global generalized solution provided that $\chi < \chi_0$ with some $\chi_0 > 0$.

In the present work, our first result shows that for the case of $\kappa > 0$ such problem possesses global generalized solutions provided that $\chi < \chi_1$ with some $\chi_1 > \chi_0$, which seems to confirm that the mixed-type damping $-\kappa uv$ has a regularization effect on solutions. Besides the existence result for generalized solutions, a statement on the large-time behavior of such solutions is derived as well.

Keywords: Keller-Segel system; crime modelling; generalized solution; large-time behavior

1. Introduction and main results

Let $u(x,t)$ denote the density of criminals, and let $v(x,t)$ represent the abstract so-called attractiveness value. A class of logarithmic Keller-Segel models of the following form

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v) - \kappa uv + h_1, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u + h_2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

with the parameters $\chi > 0$ and $\kappa > 0$, was introduced in [1] to model the propagation of criminal activities, where $\Omega \subset \mathbb{R}^n$ are bounded and smooth spatial domains. In the model (1.1), the given functions $h_1(x,t)$ and $h_2(x,t)$ describe the density of additional criminals and the source of attractiveness, respectively.

When $+u$ in the second equation in (1.1) is replaced by $+uv$, it arrives at the original Short et al. crime model [2, 3], which is rewritten as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v) - \kappa uv + h_1, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + uv + h_2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

with the particular value $\chi = 2$. Note that results on related stationary problems, as in [4–12], strongly support that the model (1.2) is adequate to describe the formation of crime hotspots encountered in reality. As for the corresponding initial-boundary value problems, the understanding of them are incomplete. The local-in-time classical solution established in [13] is global provided that either $n = 1$ [14, 15] or $n \geq 2$ and $\chi < \frac{2}{n}$ [16, 17] or both the initial data and the given functions h_1 and h_2 are appropriately small [18, 19]. For larger ranges of χ, global existence results, without imposing smallness on the initial data and on the given functions, are only available for either certain types of weak solutions or certain modified versions which contain additional regularizing ingredients: the globally radial renormalized solution was obtained for $n = 2$ and any $\chi > 0$ [20], which was extended to $n = 3$ with restriction that $\chi \in (0, \sqrt{3})$ [21]; the global weak solution was established in [22] for $n = 2$ and $\chi > 0$ by nonlinear diffusion enhancement (i.e., Δu is replaced by Δu^m with $m > \frac{3}{2}$); the global generalized solution was structured in [23] for $n = 2$ and $\chi > 0$ by incorporating the logistic source (i.e., $au - bu^2$), which was extended to the case without incorporating the logistic source in [24]. Moreover, to suppress the formation of crime hotspots, the effects of law enforcement agents can be incorporated into (1.2) [3, 11, 25–27], and we also refer to [28–31] for the existence and stability of the related steady states.

Note that, whenever $\kappa = 0$, $h_1 \equiv 0$ and $h_2 \equiv 0$, the model (1.1) becomes the celebrated logarithmic Keller-Segel model [32]:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

in which u and v respectively represent the density of chemotactic cells and the chemoattractant concentration. To motivate our study, we also recall some results on (1.3). As to the global solvability of (1.3), various thresholds of χ have been introduced. Namely, the initial-boundary value problem possesses a global bounded classical solution for suitably regular initial data (u_0, v_0), provided that either $\chi < \sqrt{\frac{2}{n}}$ [33, 34], or $n = 2$ and $\chi < \hat{\chi}$ with some $\hat{\chi} \in (1, 2)$ [35], or $\chi \leq \frac{4}{n}$ [18]. Beyond this, the restrictions on χ have been relaxed within suitably generalized solution frameworks, for instance, $\chi < \sqrt{\frac{n+2}{3n-4}}$ in the weak sense [34], $\chi < \frac{n}{n-2}$ in radially symmetric setting [36], $\chi < \chi_0$ with $\chi_0 = \infty$ for $n = 2$ and

$$\chi_0 = \begin{cases} \sqrt{8}, & n = 3, \\ \dfrac{n}{n-2}, & n \geq 4, \end{cases} \tag{1.4}$$

in the integrable sense [37], and $\chi > 0$ in the measure-valued sense [38]. In the case that u_t in the first equation in (1.3) is replaced by εu_t with appropriately small ε, there exists an unbounded solution for large initial data, provided that $\chi > \frac{n}{n-2}$ with $n \geq 3$ [39]. As to the asymptotic stability of constant steady states, for a variant of (1.3) in more general non-normalized parameter settings it was established in [40] under the smallness of the domain size $|\Omega|$, and later on, this restriction was removed out in [41] by assuming $\chi \le \frac{1}{2}$ and the convexity of Ω. In addition, when the second equation in (1.3) is replaced by $v_t = \Delta v - uv$, the corresponding model is known as the logarithmic Keller-Segel model with signal absorption, which has also been studied in a series of papers, see for instance [42–49].

Concerning the mathematical analysis, the model (1.1) is expected to have better solution properties than that of the models (1.2) and (1.3). However, to the best of our knowledge, the analysis results on model (1.1) are very sparse: Rodríguez in [1] presented that the corresponding initial-boundary value problem admits a global classical solution for the case that $\kappa = 1, \chi = 1$ and $n = 2$, which was extended to the case that $\chi \le \frac{4}{n}$, $\kappa \ge 0$ and $n \ge 2$ in [18]; very recently, we showed in [50] that such problem possesses globally generalized solutions in the two-dimensional setting for any $\chi > 0$, and investigated the eventual smoothness of these generalized solutions. Compared these to aforementioned results related to (1.3), an appealing problem naturally appears: Does the mixed-type damping term $-\kappa uv$ possess some regularization effect that contributes to enlarging the range of the parameter χ within which the higher-dimensional initial-boundary value problem of (1.1) admits global solvability at least within some generalized framework?

To reveal it, the purpose of the present work is to explore the regularization effect of the quadratic absorption term $-\kappa uv$ with $\kappa > 0$ in the following initial-boundary value problem related to (1.1):

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v) - \kappa uv + h_1, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u + h_2, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

with the parameters $\chi > 0$ and $\kappa > 0$, where $\Omega \subset \mathbb{R}^n (n \ge 3)$ are bounded and smooth domains, and ν denotes the exterior normal vector to the boundary $\partial\Omega$.

To specify the setup for our analysis, we assume throughout the sequel that the initial data (u_0, v_0) fulfill that

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \ \text{ with } \ u_0 \ge 0 \ \text{ and } \ u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\overline{\Omega}) \ \text{ with } \ \inf\limits_{x \in \overline{\Omega}} v_0 > 0, \end{cases} \tag{1.6}$$

and the given functions h_1 and h_2 satisfy that

$$0 \le h_i \in C^1(\overline{\Omega} \times [0,\infty)) \cap L^\infty(\Omega \times (0,\infty)), \quad i = 1, 2. \tag{1.7}$$

The first attempt is to show that the initial-boundary value problem (1.5) admits some global generalized solutions for general initial data and arbitrary $\chi < \chi_1$ with some $\chi_1 > \chi_0$, where χ_0 is given in (1.4). For any given (u_0, v_0, h_1, h_2) obeying (1.6) and (1.7), the global generalized solution of the problem (1.5) can be defined as follows:

Definition 1.1. A pair (u, v) is called a global generalized solution to the initial-boundary value problem (1.5) if for any $T > 0$,

(1) it holds that for some $r > 1$, $p \in (0,1)$ and $q \in (0,1)$

$$\begin{cases} u \in L^r(\Omega \times (0,T)), \ \ \nabla \ln(1+u) \in L^2(\Omega \times (0,T)), \\ uv \in L^1(\Omega \times (0,T)), \ \ u^{p+1}v^{q-1} \in L^1(\Omega \times (0,T)), \\ v \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \ \ \nabla \ln v \in L^2(\Omega \times (0,T)) \\ u(x,t) \geq 0, \ \ v(x,t) > 0, \ \ a.e. \text{ in } \Omega \times [0,T], \\ \int_\Omega u(\cdot,t)dx + \kappa \int_0^t \int_\Omega uv dx ds \leq \int_\Omega u_0 dx + \int_0^t \int_\Omega h_1 dx ds, \ \ a.e. \text{ in } [0,T]; \end{cases} \tag{1.8}$$

(2) it holds that for each nonnegative $\varphi(x,t) \in C_0^\infty(\overline{\Omega} \times [0,T))$

$$\int_0^T \int_\Omega \Big(-\ln(u+1)\varphi_t + \nabla \ln(1+u) \cdot \nabla\varphi - \varphi |\nabla \ln(1+u)|^2 - \chi \frac{u}{1+u} \nabla\varphi \cdot \nabla \ln v \\ + \chi \frac{u\varphi}{1+u} \nabla \ln(1+u) \cdot \nabla \ln v + \frac{\kappa uv}{1+u}\varphi - \frac{h_1}{1+u}\varphi \Big) dxdt \geq \int_\Omega \ln(u_0+1)\varphi|_{t=0} dx; \tag{1.9}$$

(3) it holds that for any $\varphi(x,t) \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T; H^1(\Omega))$ having compact support in $\overline{\Omega} \times [0,T)$ with $\varphi_t \in L^2(\Omega \times (0,T))$

$$\int_0^T \int_\Omega \Big(-v\varphi_t + \nabla v \cdot \nabla\varphi + v\varphi - u\varphi - h_2\varphi \Big) dxdt = \int_\Omega v_0 \varphi|_{t=0} dx. \tag{1.10}$$

We would like to remark that such concept of generalized solutions resembles those for the (logarithmic) Keller-Segel system with signal absorption used in [49, 51], but is different from that proposed in [37] by Lankeit and Winkler for the model (1.3). The first result on the global existence of such generalized solutions can be stated as follows.

Theorem 1.2. *Let* (1.6) *and* (1.7) *hold, and let* $\chi > 0$ *fulfill that*

$$\chi < \chi_1 := \begin{cases} 2\sqrt{3}, & n = 3, \\ 2\sqrt{1 + \frac{4}{n}}, & n \geq 4. \end{cases} \tag{1.11}$$

Then the initial-boundary value problem (1.5) *possesses at least one global generalized solution* (u, v) *in the sense of Definition 1.1.*

Remark 1.1. *Simple computation shows that* χ_1 *defined in* (1.11) *is larger than* χ_0 *provided by* (1.4). *This reveals in some sense that* $-\kappa uv$ *with* $\kappa > 0$ *indeed has some regularization effect on solutions.*

With the global existence statement at hand, it is natural to focus on the large-time behavior of generalized solutions. To achieve it, we need the following additional assumptions on h_1 and h_2:

$$\inf_{t>0} \int_\Omega h_2(x,t)dx > 0, \tag{1.12}$$

$$\int_t^{t+1} \int_\Omega h_1(\cdot,t) dx ds \to 0, \ \ \text{as } t \to \infty, \tag{1.13}$$

$$\int_t^{t+1} \int_\Omega |h_2(\cdot,t) - h_{2,\infty}(\cdot)|^2 dx ds \to 0, \ \ \text{as } t \to \infty, \tag{1.14}$$

with some $0 \not\equiv h_{2,\infty} \in C^1(\overline{\Omega})$. The corresponding result can be stated as follows.

Theorem 1.3. *Let Ω be convex and the extra assumptions* (1.12)–(1.14) *hold. Then, for the global generalized solution of the initial-boundary value problem* (1.5) *from Theorem 1.2, there exists a null set $\mathcal{N} \subset (0,\infty)$ such that*

$$\|u(\cdot,t)\|_{L^1} + \|v(\cdot,t) - v_\infty(\cdot)\|_{L^2} \to 0, \quad \text{as } (0,\infty) \setminus \mathcal{N} \ni t \to \infty, \tag{1.15}$$

where v_∞ denotes the solution of the boundary value problem

$$\begin{cases} 0 = \Delta v_\infty - v_\infty + h_{2,\infty}, & x \in \Omega, \\ \nabla v_\infty \cdot \nu = 0, & x \in \partial\Omega. \end{cases} \tag{1.16}$$

Let's state of the art and strategy of our proofs:

The main objective of this paper is to present that $-\kappa uv$ with $\kappa > 0$ has a regularization effect on the solution of the problem (1.5) in the n-dimensional settings with $n \geq 3$. Precisely, we prove that the initial-boundary value problem (1.5) possesses a global generalized solution for any $\chi < \chi_1$ (given in (1.11)), where χ_1 is greater than χ_0 (given in (1.4)). Note that the condition that $\chi < \chi_0$ is required in [37] to guarantee the global existence of generalized solutions to the initial-boundary value problem of (1.3). Usually, to get the generalized solvability, one should seek an appropriate generalized framework, and thereby obtain the global existence of generalized solutions by an appropriate approximation procedure. Here, our novelty of analysis consists of further developing the generalized framework given in [49] by Winkler for the logarithmic Keller-Segel system with signal absorption, and using the coupled quantity $u_\varepsilon^p v_\varepsilon^q$ with some $p, q \in (0,1)$ introduced in [37] by Lankeit and Winkler for the model (1.3) to derive the uniform in ε bound of

$$\int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} dxdt, \quad T > 0,$$

see Lemma 3.1, in which, in contrast with [37], we must deal with the additional term

$$2p\kappa \int_0^T \int_\Omega u_\varepsilon^p v_\varepsilon^{q+1} dxdt.$$

To this end, we need some additional assumption on p to obtain some now uniform in ε estimates by using the benefit of $-\kappa u_\varepsilon v_\varepsilon$, see Lemma 2.2. After this, by taking advantage of Lemma 2.2 again, we address ourselves to the uniform in ε bound of $\|u_\varepsilon\|_{L^r(\Omega\times(0,T))}$ with some $r > 1$, see Lemma 3.3. We would like to remark that, in contrast with [37], the additional condition that $q < \frac{4}{n}$ will be required in our situation when $n \geq 5$, which results in that [37, Lemma 5.1] is no longer in force. Here, our novelty of analysis contains establishing some fragile estimates, by which we can get the core of our requirement (1.11) on χ. Based on the above processes, we can employ the result that $u_\varepsilon \to u$ a.e. in $\Omega \times (0,T)$ and the Vitali convergence theorem to get the strong convergence of $\{u_\varepsilon\}$ in $L^1(\Omega \times (0,T))$, see Lemma 3.5. Moreover, by establishing a series of uniform *a-priori* estimates as desired, we can get the global existence of generalized solutions to the initial-boundary value problem (1.5) via passing to limit, and subsequently complete the proof of Theorem 1.2 in Section 3.

The second objective of this paper is to show the large-time behavior of such generalized solutions, under the additional assumptions on h_1 and h_2. Here, our novelty of analysis consists of tracking the time evolution of the combinational functional of the form

$$\mathcal{E}_\varepsilon(t) := \int_\Omega |v_\varepsilon - v_\infty|^2 + \mu u_\varepsilon dx, \quad t > 0,$$

with some $\mu > 0$, where v_∞ is a classical solution of the boundary value problem (1.16). A fragile calculation yields that

$$\int_\Omega |v_\varepsilon - v_\infty|^2(\cdot,t) + u_\varepsilon(\cdot,t)dx \to 0 \quad \text{as} \quad t\to\infty \quad \text{uniformly in} \quad \varepsilon,$$

see Lemma 4.5 for details. This, together with the Fubini-Tonelli theorem and Fatou's lemma, ensures the desired results in Theorem 1.3, see Section 4 for details.

2. Global approximate solutions and basic estimates

To structure the generalized solution of the initial-boundary value problem (1.5) by an approximation procedure, for any $\varepsilon \in (0,1)$ we shall consider the following approximate problem

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi\nabla\cdot\left(\dfrac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla \ln v_\varepsilon\right) - \kappa u_\varepsilon v_\varepsilon + h_1, & x\in\Omega,\ t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon + h_2, & x\in\Omega,\ t>0,\\ \dfrac{\partial u_\varepsilon}{\partial\nu} = \dfrac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

and then first obtain the following.

Lemma 2.1. *Assume that the assumptions* (1.6) *and* (1.7) *hold. For each* $\varepsilon \in (0,1)$ *and any* $\chi > 0$, *there exists a unique pair* $(u_\varepsilon, v_\varepsilon)$ *of positive functions with the properties that for any* $T > 0$

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega}\times[0,T]) \cap C^{2,1}(\overline{\Omega}\times(0,T)),\\ v_\varepsilon \in \bigcap_{r>n} C^0(0,T;W^{1,r}(\overline{\Omega})) \cap C^{2,1}(\overline{\Omega}\times(0,T))' \end{cases}$$

such that $(u_\varepsilon, v_\varepsilon)$ *solves the approximate problem* (2.1) *classically in* $\Omega\times[0,T)$. *Moreover, the following two statements are true:*

$$v_\varepsilon(\cdot,t) \geq e^{-t}\inf_{x\in\Omega} v_0(x), \quad t>0, \tag{2.2}$$

and

$$\|u_\varepsilon(\cdot,t)\|_{L^1} + \int_0^t\int_\Omega u_\varepsilon v_\varepsilon(\cdot,s)dxds \leq C(1+t), \quad t>0, \tag{2.3}$$

for some $C > 0$, *independent of* ε.

Proof. An application of the well-known strategy, as in [34, 52], implies that there exist time $T_{\max,\varepsilon} \in (0,\infty]$ and a unique pair $(u_\varepsilon, v_\varepsilon)$ of positive functions with the properties that

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega}\times[0,T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max,\varepsilon})),\\ v_\varepsilon \in \bigcap_{r>n} C^0([0,T_{\max,\varepsilon});W^{1,r}(\overline{\Omega})) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max,\varepsilon})), \end{cases}$$

such that $(u_\varepsilon, v_\varepsilon)$ solves the approximate problem (1.5) classically in $\Omega \times [0, T_{\max,\varepsilon})$. Moreover, if $T_{\max,\varepsilon} < \infty$, then for any $q > n$

$$\limsup_{t\to T_{\max,\varepsilon}} (\|u_\varepsilon(\cdot,t)\|_{L^\infty} + \|\nabla v_\varepsilon(\cdot,t)\|_{L^q} + \|v_\varepsilon^{-1}(\cdot,t)\|_{L^\infty}) = \infty. \tag{2.4}$$

To show that $T_{\max,\varepsilon} = \infty$, let us start with the pointwise lower bound for the solution component v_ε and the bound of $\|u_\varepsilon\|_{L^1}$. Indeed, according to the variation-of-constants formula for v_ε

$$v_\varepsilon(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,s)ds + \int_0^t e^{(t-s)(\Delta-1)}h_2(\cdot,s)ds, \tag{2.5}$$

the comparison principle for the Neumann problem associated with the heat equation and the facts that $h_2 \geq 0$ and $u_\varepsilon > 0$, we have

$$v_\varepsilon(\cdot,t) \geq e^{t(\Delta-1)}v_0 \geq e^{-t}\inf_{x\in\Omega} v_0(x), \quad t \in (0, T_{\max,\varepsilon}). \tag{2.6}$$

To get the bound of $\|u_\varepsilon\|_{L^1}$, we integrate the first equation in (2.1) over Ω to obtain

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx + \kappa\int_\Omega u_\varepsilon v_\varepsilon dx = \int_\Omega h_1 dx, \tag{2.7}$$

which, integrating over $[0, t]$, implies that

$$\|u_\varepsilon(\cdot,t)\|_{L^1} + \kappa\int_0^t\int_\Omega u_\varepsilon v_\varepsilon dxds \leq \|u_0\|_{L^1} + \|h_1\|_{L^\infty(\Omega\times(0,\infty))}t, \quad t \in (0, T_{\max,\varepsilon}). \tag{2.8}$$

We now estimate $\|u_\varepsilon\|_{L^\infty}$. Indeed, according to the variation-of-constants formula for u_ε, we can infer from the maximum principle and the nonnegativity of $\kappa u_\varepsilon v_\varepsilon$ that

$$\begin{aligned} u_\varepsilon(\cdot,t) &\leq e^{t(\Delta-1)}u_0 + \int_0^t e^{(t-s)(\Delta-1)}\left\{-\chi\nabla\cdot\left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla\ln v_\varepsilon\right) + u_\varepsilon + h_1\right\}ds \\ &\leq \|e^{t(\Delta-1)}u_0\|_{L^\infty} + \int_0^t \left\|e^{(t-s)(\Delta-1)}\left\{-\chi\nabla\cdot\left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla\ln v_\varepsilon\right) + u_\varepsilon + h_1\right\}\right\|_{L^\infty} ds, \end{aligned}$$

which, combining with the properties of the Neumann heat semigroup (see [53, 54]), (2.6) and the nonnegativity of u_ε, leads to that for any $r > n$

$$\begin{aligned} &\|u_\varepsilon(\cdot,t)\|_{L^\infty} \\ \leq &\|u_0\|_{L^\infty} + C_\varepsilon e^t\int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2r}} + (t-s)^{-\frac{n}{2r}}\right)e^{-(t-s)}\left(\|\nabla v_\varepsilon\|_{L^r} + \|u_\varepsilon + h_1\|_{L^r}\right)ds. \end{aligned} \tag{2.9}$$

By means of the interpolation inequality and (2.8), we obtain

$$\|u_\varepsilon\|_{L^r} \leq \|u_\varepsilon\|_{L^1}^{\frac{1}{r}}\|u_\varepsilon\|_{L^\infty}^{1-\frac{1}{r}} \leq C(1+t^{\frac{1}{r}})\|u_\varepsilon\|_{L^\infty}^{1-\frac{1}{r}}, \quad t \in (0, T_{\max,\varepsilon}). \tag{2.10}$$

In addition, the application of the properties of the Neumann heat semigroup to (2.5) entails that for any $r > n$

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^r} \leq \|\nabla e^{t(\Delta-1)}v_0\|_{L^r} + \int_0^t \left\|\nabla e^{(t-s)(\Delta-1)}(u_\varepsilon + h_2)\right\|_{L^r} ds$$

$$\leq C\|\nabla v_0\|_{L^r} + C\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(t-s)}\|u_\varepsilon + h_2\|_{L^r}\,ds.$$

Using (2.10) and letting $K(T) := \sup_{t\in(0,T)}\|u_\varepsilon(\cdot,t)\|_{L^\infty}$ for any $T \in (0, T_{\max,\varepsilon})$, we arrive at

$$\sup_{t\in(0,T)}\|\nabla v_\varepsilon(\cdot,t)\|_{L^r} \leq C + C(1+T^{\frac{1}{r}})K^{1-\frac{1}{r}}(T). \tag{2.11}$$

Substituting this and (2.10) into (2.9), we conclude that

$$K(T) \leq C + C_\varepsilon e^T\left(1+(1+T^{\frac{1}{r}})K^{1-\frac{1}{r}}(T)\right).$$

Since $0 < 1-\frac{1}{r} < 1$, an application of Young's inequality implies that $K(T) \leq C_\varepsilon(T)$. Hence for any $T \in (0, T_{\max,\varepsilon})$ we infer that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} \leq C_\varepsilon(T), \quad t \in (0,T)\,.$$

This, together with (2.6) and (2.11), establishes a contradiction to (2.4) and thereby ensures that actually we must have $T_{\max,\varepsilon} = \infty$.

Moreover, using (2.6) and (2.8) with $T_{\max,\varepsilon} = \infty$, the assertions (2.2) and (2.3) hold as desired.

The estimate (2.3) turns out to be sufficient for the derivation of the bound of $\|v_\varepsilon\|_{L^2}$.

Lemma 2.2. *Let $r \in [1, \frac{n}{n-2})$. Assume that $(u_\varepsilon, v_\varepsilon)$ is taken from Lemma 2.1. Then there exists $C = C(r) > 0$, with the property that*

$$\|v_\varepsilon(\cdot,t)\|_{L^r} \leq C(1+t), \quad t > 0 \text{ and } \varepsilon \in (0,1). \tag{2.12}$$

Moreover, there is $C > 0$ such that

$$\int_\Omega v_\varepsilon^2(\cdot,t)dx \leq C(1+t), \quad t > 0 \text{ and } \varepsilon \in (0,1), \tag{2.13}$$

and

$$\int_0^t \|\nabla v_\varepsilon(\cdot,s)\|_{L^2}^2 ds \leq C(1+t), \quad t > 0 \text{ and } \varepsilon \in (0,1). \tag{2.14}$$

Proof. Recalling (2.5) and invoking the properties of the Neumann heat semigroup (see [53, 54]), for any $r \in [1, \frac{n}{n-2})$ and $t > 0$ we have

$$\|v_\varepsilon(\cdot,t)\|_{L^r} \leq C_1\|v_0\|_{L^r} + C_1\int_0^t \left(1+(t-s)^{-\frac{n}{2}(1-\frac{1}{r})}\right)e^{-(t-s)}\left(\|u_\varepsilon(\cdot,s)\|_{L^1} + \|h_2(\cdot,s)\|_{L^1}\right)ds,$$

which, using (1.7) and (2.3), leads to (2.12) as desired.

Next, we test the second equation in (2.1) by v_ε and use the integration by part to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2 dx + \int_\Omega |\nabla v_\varepsilon|^2 dx + \int_\Omega v_\varepsilon^2 dx = \int_\Omega u_\varepsilon v_\varepsilon dx + \int_\Omega h_2 v_\varepsilon dx, \ t > 0.$$

By means of (1.7), the applications of Young's inequality and Hölder's inequality yield

$$\int_\Omega h_2 v_\varepsilon dx \le \frac{1}{2}\|v_\varepsilon\|_{L^2}^2 + \frac{1}{2}\|h_2\|_{L^2}^2 \le \frac{1}{2}\|v_\varepsilon\|_{L^2}^2 + C_3.$$

Invoking this we arrive at

$$\frac{d}{dt}\int_\Omega v_\varepsilon^2 dx + 2\int_\Omega |\nabla v_\varepsilon|^2 dx + \int_\Omega v_\varepsilon^2 dx \le 2\int_\Omega u_\varepsilon v_\varepsilon dx + 2C_3,\ \ t > 0. \tag{2.15}$$

Integrating (2.15) over $[0, t]$ we obtain that

$$\int_\Omega v_\varepsilon^2(\cdot, t)dx + 2\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 dxds + \int_0^t\int_\Omega v_\varepsilon^2 dxds \le \int_\Omega v_0^2 dx + 2\int_0^t\int_\Omega u_\varepsilon v_\varepsilon dxds + 2C_3 t,$$

which, combined with (2.3), ensures (2.13) and (2.14).

In comparison to Lemma 2.2, deriving the bound for $\int_0^t\int_\Omega |\nabla u_\varepsilon|^2 dxds$ seems to be more delicate, due to the presence of the taxis-type term in the first equation in (2.1). Motivated by [49, 51], we resort to estimating $\nabla \ln(u_\varepsilon + 1)$ instead.

Lemma 2.3. *Let* $(u_\varepsilon, v_\varepsilon)$ *be given in Lemma 2.1. There exists* $C > 0$*, with the property that*

$$\int_0^t \|\nabla \ln(u_\varepsilon(\cdot, s) + 1)\|_{L^2}^2 ds \le C(1+t)e^t, \quad t > 0 \text{ and } \varepsilon \in (0, 1). \tag{2.16}$$

Proof. Multiplying the first equation in (2.1) by $\frac{1}{1+u_\varepsilon}$ and using the integration by parts, we see that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \ln(1+u_\varepsilon)dx &= \int_\Omega \frac{1}{1+u_\varepsilon}\left\{\Delta u_\varepsilon - \chi\nabla\cdot\left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla \ln v_\varepsilon\right) - \kappa u_\varepsilon v_\varepsilon + h_1\right\} \\
&= \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dx - \chi\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2(1+\varepsilon u_\varepsilon)}\nabla u_\varepsilon\cdot\nabla \ln v_\varepsilon dx \\
&\quad - \int_\Omega \frac{\kappa u_\varepsilon v_\varepsilon}{1+u_\varepsilon}dx + \int_\Omega \frac{h_1}{1+u_\varepsilon}dx,\ \ t > 0.
\end{aligned}$$

Invoking this, an application of Young's inequality yields that

$$\begin{aligned}
\frac{1}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dx \le & \frac{d}{dt}\int_\Omega \ln(1+u_\varepsilon)dx + \frac{\chi^2}{2}\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^2(\varepsilon u_\varepsilon+1)^2}|\nabla \ln v_\varepsilon|^2 dx \\
& + \int_\Omega \frac{\kappa u_\varepsilon v_\varepsilon}{1+u_\varepsilon}dx - \int_\Omega \frac{h_1}{1+u_\varepsilon}dx \\
\le & \frac{d}{dt}\int_\Omega \ln(1+u_\varepsilon)dx + \frac{\chi^2}{2}\int_\Omega |\nabla \ln v_\varepsilon|^2 dx + \kappa\int_\Omega v_\varepsilon dx,\ \ t > 0,
\end{aligned}$$

which, using (2.2) and integrating in time, leads to

$$\frac{1}{2}\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dxds \le \int_\Omega \ln(1+u_\varepsilon(\cdot,t))dx + Ce^t\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 dxds + \kappa\int_0^t\int_\Omega v_\varepsilon dxds,\ t > 0.$$

This, together with (2.12) and (2.14), ensures there exists $C > 0$ such that

$$\frac{1}{2}\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dxds \le \int_\Omega \ln(1+u_\varepsilon(\cdot,t))dx + C(1+t)e^t, \quad t>0.$$

Since $\zeta \ge \ln(1+\zeta) \ge 0$ for any $\zeta \ge 0$, we obtain that

$$\frac{1}{2}\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dxds \le \int_\Omega u_\varepsilon(\cdot,t)dx + C(1+t)e^t, \quad t>0,$$

which, in view of (2.3), entails (2.16).

3. Global generalized solutions

To obtain the desired integrability for u in Definition 1.1, a crucial step in our analysis will consist of deriving the uniform spatio-temporal integrability of u_ε. To this end, we further develop the framework presented in [37, Lemma 5.1] to get some essential *a-priori* estimates for (2.1).

Lemma 3.1. *Let $n \ge 3$, $p \in (0,1)$ satisfying $p < \frac{1}{\chi^2}$ and $p < \frac{4}{n}$, and $q \in (q_-(p), q_+(p))$ with*

$$q_\pm(p) := \frac{1-p}{2}\left(1 \pm \sqrt{1-p\chi^2}\right). \tag{3.1}$$

Assume that $(u_\varepsilon, v_\varepsilon)$ is given in Lemma 2.1. Then, there exists $C > 0$, with the property that for any $t > 0$ and $\varepsilon \in (0,1)$

$$\int_0^t\int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 dxds + \int_0^t\int_\Omega u_\varepsilon^{p-2} v_\varepsilon^{q}|\nabla u_\varepsilon|^2 dxds + \int_0^t\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} dxds \le C(1+t)^4. \tag{3.2}$$

Proof. Using the facts that $u_\varepsilon > 0$ and $v_\varepsilon > 0$, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p v_\varepsilon^q dx =& p\int_\Omega u_\varepsilon^{p-1}v_\varepsilon^q \partial_t u_\varepsilon dx + q\int_\Omega u_\varepsilon^p v_\varepsilon^{q-1}\partial_t v_\varepsilon dx\\ =& p\int_\Omega u_\varepsilon^{p-1}v_\varepsilon^q\left\{\Delta u_\varepsilon - \chi\nabla\cdot\left(\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\nabla \ln v_\varepsilon\right) - \kappa u_\varepsilon v_\varepsilon + h_1\right\}dx\\ &+ q\int_\Omega u_\varepsilon^p v_\varepsilon^{q-1}\left(\Delta v_\varepsilon - v_\varepsilon + u_\varepsilon + h_2\right)dx, \ t>0,\end{aligned}$$

which, using the integration by parts, leads to

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p v_\varepsilon^q dx =& p(1-p)\int_\Omega u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2dx - \int_\Omega\left(2pq + \frac{p(1-p)\chi}{1+\varepsilon u_\varepsilon}\right)u_\varepsilon^{p-1}v_\varepsilon^{q-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon dx\\ &+ \int_\Omega\left(\frac{pq\chi}{1+\varepsilon u_\varepsilon} + q(1-q)\right)u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2dx - p\kappa\int_\Omega u_\varepsilon^p v_\varepsilon^{q+1}dx\\ &+ p\int_\Omega u_\varepsilon^{p-1}v_\varepsilon^q h_1 dx - q\int_\Omega u_\varepsilon^p v_\varepsilon^q dx + q\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1}dx + q\int_\Omega u_\varepsilon^p v_\varepsilon^{q-1}h_2 dx, \ t>0.\end{aligned}$$

Thanks to the nonnegativity of h_1, h_2, u_ε and v_ε, we arrive at

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon^p v_\varepsilon^q dx \geq & p(1-p)\int_\Omega u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2 dx - \int_\Omega\left(2pq+\frac{p(1-p)\chi}{1+\varepsilon u_\varepsilon}\right)u_\varepsilon^{p-1}v^{q-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon dx \\ & + \int_\Omega\left(\frac{pq\chi}{1+\varepsilon u_\varepsilon}+q(1-q)\right)u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 dx + q\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1}dx \\ & - p\kappa\int_\Omega u_\varepsilon^p v_\varepsilon^{q+1}dx - q\int_\Omega u_\varepsilon^p v_\varepsilon^q dx \\ =: & P_1+P_2+P_3+P_4+P_5+P_6,\ \ t>0.\end{aligned}$$

A straightforward rearrangement in the first three integrands on the right entails

$$\begin{aligned}P_1+P_2+P_3 = & \, p(1-p)\int_\Omega\left|u_\varepsilon^{\frac{p}{2}-1}v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon - \frac{2q+\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon}}{2(1-p)}u_\varepsilon^{\frac{p}{2}}v_\varepsilon^{\frac{q}{2}-1}\nabla v_\varepsilon\right|^2 dx \\ & + \int_\Omega\left\{q\left(\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q\right)-\frac{p\left(2q+\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon}\right)^2}{4(1-p)}\right\}u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 dx \\ = & \int_\Omega q\left(\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q\right)\left|u_\varepsilon^{\frac{p}{2}}v_\varepsilon^{\frac{q}{2}-1}\nabla v_\varepsilon - \frac{p\left(2q+\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon}\right)}{2q\left(\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q\right)}v^{\frac{p}{2}-1}v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon\right| dx \\ & + \int_\Omega\left\{p(1-p)-\frac{p^2\left(2q+\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon}\right)^2}{4q\left(\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q\right)}\right\}u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2 dx.\end{aligned}$$

Invoking this, we obtain

$$\begin{aligned}& 2\frac{d}{dt}\int_\Omega u_\varepsilon^p v_\varepsilon^q dx + 2p\kappa\int_\Omega u_\varepsilon^p v_\varepsilon^{q+1}dx + 2q\int_\Omega u_\varepsilon^p v_\varepsilon^q dx \\ \geq & \int_\Omega c_1(x,t)u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 dx + \int_\Omega c_2(x,t)u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2 dx + 2q\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1}dx,\end{aligned}$$

where

$$\begin{aligned}c_1(x,t) &:= q\left(\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q\right)-\frac{p\left(2q+\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon}\right)^2}{4(1-p)}, \\ c_2(x,t) &:= p(1-p)-\frac{p^2\left(2q+\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon}\right)^2}{4q\left(\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q\right)}.\end{aligned}$$

We also note that the assumption (3.1) on q warrants that $q < 1-p$ and

$$4(1-p)q - 4q^2 - p(1-p)^2\chi^2 = -4(q-q_+(p))(q-q_-(p)) > 0,$$

which, due to $p \in (0,1)$, ensures

$$4(1-p)c_1(x,t) = 4q(1-p) - 4q^2 - \frac{p(1-p)^2\chi^2}{(1+\varepsilon u_\varepsilon)^2}$$

$$\geq 4(1-p)q - 4q^2 - p(1-p)^2\chi^2 > 0.$$

Similarly, we have

$$\begin{aligned}4q\,(p\chi + 1 - q)\,p^{-1}c_2(x,t) \geq & 4q\left(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q\right)p^{-1}c_2(x,t)\\ \geq & 4(1-p)q - 4q^2 - p(1-p)^2\chi^2 > 0.\end{aligned}$$

Collecting these, there exist two positive constants $\widehat{c_1}$ and $\widehat{c_2}$, denoted by

$$\widehat{c_1} := q(p\chi + 1 - q) - \frac{p(2q + (1-p)\chi)^2}{4(1-p)}, \quad \widehat{c_2} := p(1-p) - \frac{p^2(2q + (1-p)\chi)^2}{4q(p\chi + 1 - q)},$$

such that

$$\begin{aligned}&2\frac{d}{dt}\int_\Omega u_\varepsilon^p v_\varepsilon^q dx + 2p\kappa \int_\Omega u_\varepsilon^p v_\varepsilon^{q+1} dx + 2q\int_\Omega u_\varepsilon^p v_\varepsilon^q dx\\ &\geq \widehat{c_1}\int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 dx + \widehat{c_2}\int_\Omega u_\varepsilon^{p-2} v_\varepsilon^q |\nabla u_\varepsilon|^2 dx + 2q\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} dx.\end{aligned}$$

Hence, an integration in time shows

$$\begin{aligned}&\widehat{c_1}\int_0^t\int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 dxds + \widehat{c_2}\int_0^t\int_\Omega u_\varepsilon^{p-2} v_\varepsilon^q|\nabla u_\varepsilon|^2 dxds + 2q\int_0^t\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} dxds\\ &\leq 2\int_\Omega u_\varepsilon^p v_\varepsilon^q(x,t)dx + 2p\kappa\int_0^t\int_\Omega u_\varepsilon^p v_\varepsilon^{q+1} dxds + 2q\int_0^t\int_\Omega u_\varepsilon^p v_\varepsilon^q dxds. \end{aligned} \tag{3.3}$$

Using Hölder's inequality and the fact that $q < 1 - p$ again, we have

$$\int_\Omega u_\varepsilon^p v_\varepsilon^q dx \leq \|u_\varepsilon^p\|_{L^{\frac{1}{p}}}\|v_\varepsilon^q\|_{L^{\frac{1}{1-p}}} = \|u_\varepsilon\|_{L^1}^p \|v_\varepsilon\|^q_{L^{\frac{q}{1-p}}} \leq C\|u_\varepsilon\|_{L^1}^p\|v_\varepsilon\|_{L^2}^q,$$

which, together with (2.3) and (2.13), implies that there exists $C > 0$, independent of ε, such that

$$2\int_\Omega u_\varepsilon^p v_\varepsilon^q(x,t)dx + 2q\int_0^t\int_\Omega u_\varepsilon^p v_\varepsilon^q dxds \leq C(1+t)^{p+\frac{q}{2}+1} \leq C(1+t)^2, \quad t > 0. \tag{3.4}$$

Similarly, an application of Hölder's inequality and Young's inequality yields that

$$\begin{aligned}2p\kappa\int_0^t\int_\Omega u_\varepsilon^p v_\varepsilon^{q+1} dxds \leq & 2p\kappa\int_0^t\left(\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1}dx\right)^{\frac{p}{p+1}}\left(\int_\Omega v_\varepsilon^{2p+q+1}dx\right)^{\frac{1}{p+1}} ds\\ \leq & q\int_0^t\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1}dxds + C\int_0^t\int_\Omega v_\varepsilon^{2p+q+1}dxds.\end{aligned} \tag{3.5}$$

Since $q < 1 - p$ and $0 < p < 1$, we have $2p + q + 1 < 2 + p < 3$. In the case that $n = 3$, thanks to $\frac{n}{n-2} = 3$, it follows from (2.12) that there exists $C > 0$ such that

$$\int_0^t\int_\Omega v_\varepsilon^{2p+q+1}dxds \leq C\int_0^t(1+s)^{2p+q+1}ds \leq C(1+t)^4, \quad t > 0. \tag{3.6}$$

In the case that $n \geq 4$, if $2p + q + 1 \leq 2$, according to Young's inequality and (2.13) there exists $C > 0$ such that

$$\int_0^t \int_\Omega v_\varepsilon^{2p+q+1} dxds \leq \int_0^t \int_\Omega v_\varepsilon^2 + 1 dxds \leq C(1+t)^2, \quad t > 0. \tag{3.7}$$

In addition, if $2 < 2p + q + 1 \leq \frac{2n}{n-2}$, then we can infer from the Gagliardo-Nirenberg inequality that

$$\|v_\varepsilon\|_{L^{2p+q+1}} \leq C\left(\|v_\varepsilon\|_{L^2}^{1-\theta}\|\nabla v_\varepsilon\|_{L^2}^{\theta} + \|v_\varepsilon\|_{L^2}\right), \quad \theta := \frac{n(2p+q-1)}{2(2p+q+1)},$$

which, combined with (2.13), leads to

$$\|v_\varepsilon(\cdot,t)\|_{L^{2p+q+1}}^{2p+q+1} \leq C(1+t)^{2p+q+1}\left(\|\nabla v_\varepsilon(\cdot,t)\|_{L^2}^{\frac{n}{2}(2p+q-1)} + 1\right), \quad t > 0.$$

Recalling the fact that $2 < 2p + q + 1$, we have $0 < \frac{n}{2}(2p + q - 1) < \frac{n}{2}p < 2$ due to $p + q < 1$ and $p < \frac{4}{n}$, and thereby infer from Young's inequality and (2.14) that

$$\int_0^t \int_\Omega v_\varepsilon^{2p+q+1} dxds \leq C(1+t)^{p+\frac{q}{2}+\frac{1}{2}} \int_0^t \left(\|\nabla v_\varepsilon(\cdot,s)\|_{L^2}^2 + 1\right) ds \leq C(1+t)^3, \quad t > 0. \tag{3.8}$$

Collecting (3.6)–(3.8), it follows from (3.5) that, whenever $n \geq 3$,

$$2p\kappa \int_0^t \int_\Omega u_\varepsilon^p v_\varepsilon^{q+1} dxds \leq q \int_0^t \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} dxds + C(1+t)^4, \quad t > 0. \tag{3.9}$$

Substituting (3.4) and (3.9) into (3.3), we have (3.2) as desired, due to the facts that $\widehat{c_1}, \widehat{c_2} > 0$ and $q < 1$.

Indeed, using Lemma 3.1, we can get the bound of u_ε in some reflexive L^r spaces. To achieve it, we need to identify the minimal possible choice of an integrability exponent arising in (3.19) below, which will form the core of our requirement (1.11) on χ.

Lemma 3.2. *Let $\chi > 0$, and for $p \in \left(0, \min\{1, \frac{4}{n}, \frac{1}{\chi^2}\}\right)$ let $q_\pm(p)$ be defined in* (3.1). *If $n \in \{3, 4\}$, then*

$$\inf_{\substack{p\in(0,1),p<\frac{1}{\chi^2}\\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} = \begin{cases} 1 & \text{if } \chi \leq 1, \\ \chi & \text{if } \chi \in (1,2), \\ 1 + \frac{\chi^2}{4} & \text{if } \chi \geq 2, \end{cases} \tag{3.10}$$

and if $n \geq 5$, then

$$\inf_{\substack{p\in(0,\min\{\frac{4}{n},\frac{1}{\chi^2}\})\\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} = \begin{cases} \frac{n}{8} + \frac{1}{2} - \left(\frac{n}{8} - \frac{1}{2}\right)\sqrt{1 - \frac{4}{n}\chi^2} & \text{if } \chi \leq \frac{2n}{n+4}, \\ \chi & \text{if } \chi \in \left(\frac{2n}{n+4}, 2\right), \\ 1 + \frac{\chi^2}{4} & \text{if } \chi \geq 2. \end{cases} \tag{3.11}$$

Proof. If $n \in \{3,4\}$, then $p \in (0, \min\{1, \frac{1}{\chi^2}\})$. In this case, the assertion (3.11) directly follows from [37, Lemma 5.1].

If $n \geq 5$, then we have $p \in (0, \min\{\frac{4}{n}, \frac{1}{\chi^2}\})$. A straightforward calculation shows that

$$I(\chi) := \inf_{\substack{p\in(0,\min\{\frac{4}{n},\frac{1}{\chi^2}\})\\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} = \inf_{p\in(0,\min\{\frac{4}{n},\frac{1}{\chi^2}\})} \frac{1+p-(1-p)\sqrt{1-p\chi^2}}{2p},$$

and that $I(\chi) \geq 1$ for any $\chi > 0$. Setting $\xi := \sqrt{1-p\chi^2}$, we get that

$$\xi \in \begin{cases} \left(\sqrt{1-\frac{4}{n}\chi^2}, 1\right), & \text{if } \chi \leq \frac{\sqrt{n}}{2}, \\ (0,1) & \text{if } \chi > \frac{\sqrt{n}}{2}. \end{cases} \tag{3.12}$$

Note that $p = \frac{1-\xi^2}{\chi^2}$, simple computation shows that

$$\frac{1+p-(1-p)\sqrt{1-p\chi^2}}{2p} = \frac{1}{2}\left(\frac{\chi^2}{1+\xi} + 1 + \xi\right) =: g(\xi).$$

Accordingly, we have

$$I(\chi) = \begin{cases} \inf_{\xi\in\left[\sqrt{1-\frac{4}{n}\chi^2},1\right]} g(\xi), & \text{if } \chi \leq \frac{\sqrt{n}}{2}, \\ \inf_{\xi\in[0,1]} g(\xi), & \text{if } \chi > \frac{\sqrt{n}}{2}. \end{cases} \tag{3.13}$$

As

$$g'(\xi) := \frac{1}{2}\left(-\frac{\chi^2}{(1+\xi)^2} + 1\right),$$

which implies that $g(\xi)$ is strictly monotonely decreasing in the interval $[0, \chi-1]$ and strictly monotonely increasing in the intrerval $[\chi-1, +\infty)$, correspondingly, we have

$$I(\chi) = \begin{cases} g(\chi-1), & \text{if } \chi \in \left(1+\sqrt{1-\frac{4}{n}\chi^2}, 2\right] \cap \left(0, \frac{\sqrt{n}}{2}\right], \\ g(\sqrt{1-\frac{4}{n}\chi^2}), & \text{if } \chi \in \left(0, 1+\sqrt{1-\frac{4}{n}\chi^2}\right] \cap \left(0, \frac{\sqrt{n}}{2}\right], \\ g(1), & \text{if } \chi \in (2, \frac{\sqrt{n}}{2}],\ n > 16, \\ g(\chi-1), & \text{if } \chi \in [1,2] \cap (\frac{\sqrt{n}}{2}, +\infty) \\ g(1), & \text{if } \chi \in (2,+\infty) \cap (\frac{\sqrt{n}}{2}, +\infty). \end{cases} \tag{3.14}$$

Direct calculation shows that

$$\chi \in \left(1+\sqrt{1-\frac{4}{n}\chi^2}, 2\right] \cap \left(0, \frac{\sqrt{n}}{2}\right] \Leftrightarrow \chi \in \left(\frac{2n}{n+4}, \min\{2, \frac{\sqrt{n}}{2}\}\right],$$

and

$$\chi \in \left(0, 1+\sqrt{1-\frac{4}{n}\chi^2}\right] \cap (0, \frac{\sqrt{n}}{2}] \Leftrightarrow \chi \in (0, \frac{2n}{n+4}],$$

moreover,

$$(2,+\infty)\cap(\frac{\sqrt{n}}{2},+\infty)=(2,+\infty)\ \text{if}\ n\le 16$$

and

$$(2,\frac{\sqrt{n}}{2}]\cup\Big((2,+\infty)\cap(\frac{\sqrt{n}}{2},+\infty)\Big)=(2,+\infty)\ \text{if}\ n>16,$$

as well as

$$\Big(\frac{2n}{n+4},\min\{2,\frac{\sqrt{n}}{2}\}\Big]\cup\Big([1,2]\cap(\frac{\sqrt{n}}{2},+\infty)\Big)=(\frac{2n}{n+4},2].$$

Thus, it follows from (3.14) that

$$I(\chi)=\begin{cases} g(\sqrt{1-\frac{4}{n}\chi^2}), & \text{if}\ \chi\in(0,\frac{2n}{n+4}],\\ g(\chi-1), & \text{if}\ \chi\in(\frac{2n}{n+4},2],\\ g(1), & \text{if}\ \chi\in(2,+\infty)\end{cases} \tag{3.15}$$

Note that

$$g\left(\sqrt{1-\frac{4}{n}\chi^2}\right)=\frac{1}{2}\left(1+\frac{n}{4}+\left(1-\frac{n}{4}\right)\sqrt{1-\frac{4}{n}\chi^2}\right),$$

$$g(\chi-1)=\chi,$$

and

$$g(1)=1+\frac{1}{4}\chi^2,$$

these together with (3.15) gives us the desired (3.11).

Now under the assumptions on χ in Theorem 1.2, the interpolation argument, invoking Lemma 3.2, indeed bears fruit of the desired flavour.

Lemma 3.3. *Let p and q be taken from Lemma 3.1, and χ satisfy* (1.11). *Then, for $(u_\varepsilon,v_\varepsilon)$ given in Lemma 2.1, there exist $r>1$ and $C>0$, with the property that*

$$\int_0^t\int_\Omega u_\varepsilon^r(\cdot,s)dxds\le C(1+t)^4,\quad t>0\ \text{and}\ \varepsilon\in(0,1). \tag{3.16}$$

Proof. According to (3.10), we infer that if $n=3$, then we have, as long as $\chi<2\sqrt{3}$,

$$\inf_{\substack{p\in(0,1),p<\frac{1}{\chi^2}\\ q\in(q_-(p),q_+(p))}}\frac{1-q}{p}<4,$$

with $q_-(p)$ and $q_+(p)$ given by Lemma 3.1, which ensures that we can find $p\in(0,\min\{1,\frac{1}{\chi^2}\}$ and $q\in(q_-(p),q_+(p))$ such that

$$\frac{(1-q)}{p}<4. \tag{3.17}$$

Similarly, we can deduce from (3.10) and (3.11) that

$$\inf_{\substack{p\in(0,\min\{\frac{4}{n},\frac{1}{\chi^2}\})\\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} < 2+\frac{4}{n},$$

for any $n \geq 4$, as long as $\chi < 2\sqrt{1+\frac{4}{n}}$, which also guarantees that we can choose $p \in (0, \min\{\frac{4}{n}, \frac{1}{\chi^2}\}$ and $q \in (q_-(p), q_+(p))$ such that

$$\frac{(1-q)}{p} < 2+\frac{4}{n}. \tag{3.18}$$

Fix p and q in (3.17) and (3.18) respectively, utilizing a continuity argument we can further pick $r \in (1, 1+p)$ sufficiently close to 1 such that

$$\beta := \frac{(1-q)r}{p+1-r}\begin{cases} < 4, & n=3, \\ < 2+\frac{4}{n}, & n\geq 4. \end{cases}$$

For such r, an application of Young's inequality yields that

$$\int_0^t\int_\Omega u_\varepsilon^r dxds \leq \int_0^t\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1}dxds + \int_0^t\int_\Omega v_\varepsilon^{\frac{(1-q)r}{1+p-r}}dxds, \ \ t>0,$$

which, united (3.2), ensures that

$$\int_0^t\int_\Omega u_\varepsilon^r dxds \leq C(1+t)^4 + \int_0^t\int_\Omega v_\varepsilon^{\beta}dxds, \ \ t>0, \tag{3.19}$$

with $\beta = \frac{(1-q)r}{1+p-r}$ and $C > 0$ independent of ϵ.

In the case $n = 3$, if $\beta < \frac{n}{n-2}$, i.e, $\beta < 3$, then it follows from (2.12) that

$$\int_0^t\int_\Omega v_\varepsilon^{\beta}dxds \leq C(1+t)^{\beta+1} \leq C(1+t)^4, \quad t>0,$$

thus we have from (3.19) that

$$\int_0^t\int_\Omega u_\varepsilon^r dxds \leq C(1+t)^4, \ \ t>0,$$

namely, (3.16) is valid. Meanwhile, if $\beta \in [3, 4)$, then $\gamma := \frac{3\beta-6}{2} \in [\frac{3}{2}, 3)$ and $\frac{6(\beta-\gamma)}{6-\gamma} = 2$, the Gagliardo-Nirenberg inequality implies that

$$\|v_\varepsilon\|_{L^\beta}^\beta \leq C\left(\|v_\varepsilon\|_{L^\gamma}^{\beta-2}\|\nabla v_\varepsilon\|_{L^2}^2 + \|v_\varepsilon\|_{L^2}^\beta\right), \quad t>0,$$

which, together with (2.12)–(2.14), gives us

$$\int_0^t\int_\Omega v_\varepsilon^{\beta}dxds \leq C\int_0^t (1+s)^{\beta-2}\|\nabla v_\varepsilon(\cdot,s)\|_{L^2}^2 + (1+s)^{\frac{\beta}{2}}ds$$

$$\leq C\left\{(1+t)^{\beta-1}+(1+t)^{\frac{\beta}{2}+1}\right\},\ t>0.$$

This, combined with (3.19), also entails (3.16) due to $\beta < 4$.

In the case $n \geq 4$, if $\beta \leq 2$, then the Young inequality and (2.13) entail that

$$\int_0^t \int_\Omega v_\varepsilon^\beta dxds \leq \int_0^t \int_\Omega v_\varepsilon^2 + 1 dxds \leq C(1+t)^2,\ \ t>0.$$

This, together with (3.19), also guarantees the validity of (3.16). If $\beta \in (2, 2+\frac{4}{n})$, then $\frac{n(\beta-2)}{2} < 2$. Applications of the Gagliardo-Nirenberg inequality and the Young inequality imply that

$$\begin{aligned}\|v_\varepsilon\|_{L^\beta}^\beta \leq & C\left(\|v_\varepsilon\|_{L^2}^{\beta-\frac{n(\beta-2)}{2}}\|\nabla v_\varepsilon\|_{L^2}^{\frac{n(\beta-2)}{2}}+\|v_\varepsilon\|_{L^2}^\beta\right)\\ \leq & C\left(\|v_\varepsilon\|_{L^2}^{\beta-\frac{n(\beta-2)}{2}}\|\nabla v_\varepsilon\|_{L^2}^{2}+\|v_\varepsilon\|_{L^2}^\beta+1\right), \quad t>0.\end{aligned}$$

In this case, we can infer from (2.13) and (2.14) that

$$\int_0^t \int_\Omega v_\varepsilon^\beta dxds \leq C(1+t)^{\frac{\beta}{2}+1},\ \ t>0,$$

which, combined (3.19), implies that (3.16) is also valid due to $\beta < 2+\frac{4}{n}$.

As a final preparation for our limit procedure, we establish some regularity features of the time derivatives in (2.1).

Lemma 3.4. *Let $(u_\varepsilon, v_\varepsilon)$ be established in Lemma 2.1. For any $T>0$, there exists $C(T)>0$, with the property that for $r>n$*

$$\int_0^T \|v_{\varepsilon s}(\cdot,s)\|_{(W^{1,r})^\star}^2\, ds \leq C(T)\ \text{ for any } \varepsilon \in (0,1), \tag{3.20}$$

$$\int_0^T \|\partial_s \ln\left(u_\varepsilon(\cdot,s)+1\right)\|_{(W^{1,r})^\star}\, ds \leq C(T)\ \text{ for any } \varepsilon \in (0,1). \tag{3.21}$$

Proof. On the basis of the second equation in (2.1), we obtain from the integration by parts and Hölder's inequality that for any $\varphi \in C^\infty(\overline{\Omega})$ and $t>0$

$$|\langle v_{\varepsilon t}, \varphi\rangle| \leq \|\nabla v_\varepsilon\|_{L^2}\|\nabla\varphi\|_{L^2}+\|v_\varepsilon\|_{L^2}\|\varphi\|_{L^2}+\|u_\varepsilon\|_{L^1}\|\varphi\|_{L^\infty}+\|h_2\|_{L^\infty}\|\varphi\|_{L^1},$$

which, combined with the Sobolev embedding theorem, entails that for any $r>n$ there exists $C>0$ independent of ε such that for any $t>0$

$$|\langle v_{\varepsilon t}, \varphi\rangle| \leq C\left(\|v_\varepsilon\|_{H^1}+\|u_\varepsilon\|_{L^1}+\|h_2\|_{L^\infty}\right)\|\varphi\|_{W^{1,r}}.$$

This, in view of (2.3), (2.13), (2.14) and (1.7), in turn ensures (3.20).

Next, multiplying the first equation in (2.1) by $\frac{\varphi}{u_\varepsilon+1}$ for any $\varphi \in C^\infty(\overline{\Omega})$ we have for any $t>0$

$$\int_\Omega \partial_t \ln(1+u_\varepsilon)\varphi dx = \int_\Omega \frac{|\nabla u_\varepsilon|^2\varphi}{(u_\varepsilon+1)^2}dx - \chi\int_\Omega \frac{u_\varepsilon(\nabla u_\varepsilon\cdot\nabla\ln v_\varepsilon)\varphi}{(u_\varepsilon+1)^2(1+\varepsilon u_\varepsilon)}dx - \int_\Omega \frac{\nabla u_\varepsilon\cdot\nabla\varphi}{1+u_\varepsilon}dx$$

$$+\chi\int_\Omega \frac{u_\varepsilon\nabla\ln v_\varepsilon\cdot\nabla\varphi}{(1+u_\varepsilon)(1+\varepsilon u_\varepsilon)}dx-\kappa\int_\Omega\frac{u_\varepsilon v_\varepsilon\varphi}{1+u_\varepsilon}dx+\int_\Omega\frac{h_1\varphi}{1+u_\varepsilon}dx,$$

which, by using Hölder's inequality, Young's inequality and Sobolev's inequality, entails that for any $r>n$ there exists $C>0$ such that for any $t>0$

$$\begin{aligned}|\langle\partial_t\ln(1+u_\varepsilon),\varphi\rangle|\le&\|\varphi\|_{L^\infty}\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dx+\chi\left(\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dx\right)^{\frac12}\|\nabla\ln v_\varepsilon\|_{L^2}\|\varphi\|_{L^\infty}\\&+\left(\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dx\right)^{\frac12}\|\nabla\varphi\|_{L^2}+\chi\|\nabla\ln v_\varepsilon\|_{L^2}\|\nabla\varphi\|_{L^2}\\&+\kappa\|v_\varepsilon\|_{L^2}\|\varphi\|_{L^2}+\|h_1\|_{L^\infty}\|\varphi\|_{L^1}\\\le&C\|\varphi\|_{W^{1,r}}\left(\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dx+\|\nabla\ln v_\varepsilon\|_{L^2}^2+\|v_\varepsilon\|_{L^2}+\|h_1\|_{L^\infty}+1\right).\end{aligned}$$

After an integration in time, we infer from (1.7), (2.13), (2.14), (2.16) and (2.2) that (3.21) holds as desired.

On the basis of the standard compactness arguments, we can find a candidate (u,v) for a generalized solution.

Lemma 3.5. *Let $(u_\varepsilon,v_\varepsilon)$ be taken from Lemma 2.1. Then, for any $T>0$ there exist functions $u\ge0$ and $v>0$ defined on $\Omega\times(0,T)$ and a sequence $\{\varepsilon_j\}_{j=1}^\infty\subset(0,1)$ such that $\varepsilon_j\to0$ as $j\to\infty$, with the properties that as $\varepsilon=\varepsilon_j\to0$,*

$$\begin{aligned}
&\ln(1+u_\varepsilon)\to\ln(1+u)\quad\text{in}\quad L^2(\Omega\times(0,T)), &(3.22)\\
&\ln(1+u_\varepsilon)\rightharpoonup\ln(1+u)\quad\text{in}\quad L^2(0,T;H^1(\Omega)), &(3.23)\\
&u_\varepsilon\to u\quad a.e.\ \text{in}\quad\Omega\times(0,T), &(3.24)\\
&u_\varepsilon\to u\quad\text{in}\quad L^1(\Omega\times(0,T)), &(3.25)\\
&u_\varepsilon\rightharpoonup u\quad\text{in}\quad L^r(\Omega\times(0,T)), &(3.26)\\
&v_\varepsilon\to v\quad\text{in}\quad L^2(0,T;L^q(\Omega)), &(3.27)\\
&v_\varepsilon^{-1}\to v^{-1}\quad\text{in}\quad L^2(0,T;L^q(\Omega)), &(3.28)\\
&\ln v_\varepsilon\to\ln v\quad\text{in}\quad L^2(0,T;L^q(\Omega)), &(3.29)\\
&v_\varepsilon\overset{*}{\rightharpoonup}v\quad\text{in}\quad L^\infty(0,T;L^2(\Omega)), &(3.30)\\
&v_\varepsilon\rightharpoonup v\quad\text{in}\quad L^2(0,T;H^1(\Omega)), &(3.31)
\end{aligned}$$

where $q<\frac{2n}{n-2}$.

Proof. Since $\frac12\ln^2(1+\zeta)\le\zeta$ for any $\zeta\ge0$, we infer from the bounds (2.16) and (2.3) that

$$\begin{aligned}\int_0^t\|\ln(1+u_\varepsilon)\|_{H^1}^2ds\le&C\int_0^t\left(\|u_\varepsilon(\cdot,s)\|_{L^1}+\|\nabla\ln(1+u_\varepsilon)(\cdot,s)\|_{L^2}^2\right)ds\\\le&C(T),\quad t\in(0,T).\end{aligned}$$

Based on this, the bound (3.21), combined with the Aubin-Lions compactness theorem [55], implies that there exist a subsequence of $\{\varepsilon_j\}_{j=1}^{\infty}$ (still expressed as $\{\varepsilon_j\}_{j=1}^{\infty}$) and a function $w \in L^2(0,T;H^1(\Omega))$, fulfilling that as $\varepsilon = \varepsilon_j \to 0$,

$$\ln(u_\varepsilon + 1) \to w, \quad \nabla \ln(u_\varepsilon + 1) \rightharpoonup \nabla w \quad \text{in } L^2(\Omega \times (0,T),$$

and thereby

$$\ln(u_\varepsilon + 1) \to w \quad \text{and} \quad u_\varepsilon \to e^w - 1 \quad \text{a.e. in } \Omega \times (0,T).$$

Hence, denoting $u = e^w - 1$ and using the bound (2.3) again, we obtain the assertions (3.22)–(3.24). Due to (3.24), according to the uniform integrability property implied by Lemma 3.3 we may apply the Vitali convergence theorem to get that in fact (3.25) also holds. Meanwhile, using (3.24) and invoking Lemma 3.3, we arrive at (3.26).

According to the bounds (2.13), (2.14) and (3.20), and the Sobolev embedding theorem, a standard subsequence extraction procedure resorting to the Aubin-Lions compactness theorem (see [55]) entails model (3.27) immediately. Due to (2.2), we have

$$\left\| v_\varepsilon^{-1}(\cdot,t) \right\|_{L^2} \le C(T), \quad t \in (0,T),$$

and also infer from (2.14) that

$$\int_0^t \left\| \nabla v_\varepsilon^{-1}(\cdot,s) \right\|_{L^2}^2 ds \le C(T), \quad t \in (0,T).$$

Since $(v_\varepsilon^{-1})_t = -v_\varepsilon^{-2}(\Delta v_\varepsilon - v_\varepsilon + u_\varepsilon + h_2)$, similar to (3.20), using (2.2) again we get

$$\int_0^T \left\| v_{\varepsilon s}^{-1}(\cdot,s) \right\|_{(W^{1,r})^\star}^2 ds \le C(T), \quad r > n.$$

Hence, invoking the Aubin-Lions compactness theorem ([55]), there exists a subsequence of $\{\varepsilon_j\}_{j=1}^{\infty}$ (still expressed as $\{\varepsilon_j\}_{j=1}^{\infty}$) such that (3.28) holds as desired, as $\varepsilon = \varepsilon_j \to 0$. Similarly, (3.29) also holds. On the other hand, using the bounds (2.13) and (2.14) again yields the last two assertions in lemma.

Up to now, our knowledge on approximation of (u,v) by $(u_\varepsilon, v_\varepsilon)$ is enough to pass to the limit $\varepsilon = \varepsilon_j \to 0$ in the weak formulation of the second equation in the approximate problem (2.1), which also show that v is indeed a weak solution of the respective sub-problem of (1.5) in the sense of Definition 1.1.

Lemma 3.6. *Let u and v be given in Lemma 3.5. For any $T > 0$, the identity* (1.10) *in Definition 1.1 is valid for any $\varphi(x,t) \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T;H^1(\Omega))$ having compact support in $\overline{\Omega} \times [0,T)$ with $\varphi_t \in L^2(\Omega \times (0,T))$.*

Proof. For each φ from the class indicated in (1.10), it follows from (3.25) and the Lebesgue dominated convergence theorem that there exists a subsequence of $\{\varepsilon_j\}_{j=1}^{\infty}$ (still expressed as $\{\varepsilon_j\}_{j=1}^{\infty}$) such that for any $T > 0$, as $\varepsilon = \varepsilon_j \to 0$,

$$\int_0^T \int_\Omega u_\varepsilon \varphi dxds \to \int_0^T \int_\Omega u\varphi dxds.$$

Hence, we can take the limit $\varepsilon = \varepsilon_j \to 0$ on the second equation in (2.1) in the weak sense by employing Lemma 3.5. Moreover, the functions u and v obtained in Lemma 3.5 satisfy the identity (1.10) in Definition 1.1.

To take the limit also in the first equation in the approximate problem (2.1) in an appropriate manner, we shall obtain the strongly convergence of $\nabla \ln v_\varepsilon$ in $L^2(\Omega \times (0,T))$ for any $T > 0$.

Lemma 3.7. *Let* $(u_\varepsilon, v_\varepsilon)$ *be described in Lemma 2.1, and let u and v be established in Lemma 3.5. Then there exists a subsequence of* $\{\varepsilon_j\}_{j=1}^\infty$ *(still expressed as* $\{\varepsilon_j\}_{j=1}^\infty$*) such that for any* $T > 0$*, as* $\varepsilon = \varepsilon_j \to 0$,

$$\nabla \ln v_\varepsilon \to \nabla \ln v \quad \text{in} \quad L^2(\Omega \times (0,T)). \tag{3.32}$$

Proof. We can adopt a strategy similar to [49, Lemma 2.10] to get (3.32) as desired.

Invoking Lemma 3.7, we can present the validity of (1.9) in Definition 1.1.

Lemma 3.8. *Let u and v be given in Lemma 3.5. For any* $T > 0$*, the inequality* (1.9) *in Definition 1.1 is valid for each nonnegative* $\varphi(x,t) \in C_0^\infty(\overline{\Omega} \times [0,T))$.

Proof. Testing the first equation in (2.1) by $\frac{\varphi}{1+u_\varepsilon}$ with $0 \le \varphi \in C_0^\infty(\Omega \times [0,T))$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega |\nabla \ln(u_\varepsilon + 1)|^2 \varphi dxdt \\
&= -\int_0^T \int_\Omega \ln(u_\varepsilon + 1)\varphi_t dxdt - \int_\Omega \ln(u_0 + 1)\varphi(\cdot, 0)dx + \int_0^T \int_\Omega \nabla \ln(u_\varepsilon + 1) \cdot \nabla\varphi dxdt \\
&\quad + \chi \int_0^T \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + 1)(1 + \varepsilon u_\varepsilon)} (\nabla \ln(u_\varepsilon + 1) \cdot \nabla \ln v_\varepsilon)\varphi dxdt \\
&\quad - \chi \int_0^T \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + 1)(1 + \varepsilon u_\varepsilon)} \nabla \ln v_\varepsilon \cdot \nabla\varphi dxdt \\
&\quad + \kappa \int_0^T \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1 + u_\varepsilon}\varphi dxdt - \int_0^T \int_\Omega \frac{h_1}{1 + u_\varepsilon}\varphi dxdt.
\end{aligned}$$

We conclude from (3.23) that as $\varepsilon = \varepsilon_j \to 0$,

$$\int_0^T \int_\Omega \ln(u_\varepsilon + 1)\varphi_t dxdt \to \int_0^T \int_\Omega \ln(u + 1)\varphi_t dxdt,$$
$$\int_0^T \int_\Omega \nabla \ln(u_\varepsilon + 1) \cdot \nabla\varphi dxdt \to \int_0^T \int_\Omega \nabla \ln(u + 1) \cdot \nabla\varphi dxdt.$$

Since $\frac{u_\varepsilon}{(u_\varepsilon+1)(1+\varepsilon u_\varepsilon)} \to \frac{u}{u+1}$ a.e. in $\Omega \times (0,T)$ as $\varepsilon = \varepsilon_j \to 0$, we infer from (3.32) and [51, Lemma A.4] that, as $\varepsilon = \varepsilon_j \to 0$,

$$\frac{u_\varepsilon}{(u_\varepsilon + 1)(1 + \varepsilon u_\varepsilon)} \nabla \ln v_\varepsilon \to \frac{u}{u+1} \nabla \ln v \quad \text{in } L^2(\Omega \times (0,T)),$$

which, combined with (3.23), further implies that, as $\varepsilon = \varepsilon_j \to 0$,

$$\int_0^T \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + 1)(1 + \varepsilon u_\varepsilon)} (\nabla \ln(u_\varepsilon + 1) \cdot \nabla \ln v_\varepsilon)\varphi dxdt$$

$$\to \int_0^T \int_\Omega \frac{u}{u+1} \left(\nabla \ln (u+1) \cdot \nabla \ln v\right) \varphi dxdt$$

and

$$\int_0^T \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)(1+\varepsilon u_\varepsilon)} \nabla \ln v_\varepsilon \cdot \nabla\varphi dxdt \to \int_0^T \int_\Omega \frac{u}{u+1} \nabla \ln v \cdot \nabla\varphi dxdt.$$

Similarly, we obtain that $\varepsilon = \varepsilon_j \to 0$,

$$\kappa \int_0^T \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1+u_\varepsilon} \varphi dxdt \to \kappa \int_0^T \int_\Omega \frac{uv}{1+u} \varphi dxdt.$$

By using the Lebesgue dominated convergence theorem, we have

$$\int_0^T \int_\Omega \frac{h_1}{1+u_\varepsilon} \varphi dxdt \to \int_0^T \int_\Omega \frac{h_1}{1+u} \varphi dxdt.$$

Invoking (3.23), an application of the weak lower semicontinuity of the norm implies

$$\int_0^T \int_\Omega |\nabla \ln(u+1)|^2 \varphi dxdt \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_0^T \int_\Omega |\nabla \ln (u_\varepsilon+1)|^2 \varphi dxdt.$$

Hence, collecting these, (1.9) holds as desired.

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. In fact, we only need to combine Lemma 3.6 with Lemma 3.8.

4. Large-time behavior

In this section, we will investigate the large-time behavior of the generalized solution (u, v) determined in Theorem 1.2, under the additional assumptions (1.12)–(1.14). To achieve this, we begin with the following pointwise lower bound for the solution component v_ε, which will play a key role in the sequel.

Lemma 4.1. *Let* $(u_\varepsilon, v_\varepsilon)$ *come from Lemma 2.1, and let* (1.12) *be in force. Under the additional assumption that* Ω *is convex, then there exists* $c_1 > 0$*, independent of* t *and* ε*, fulfilling that*

$$v_\varepsilon(x,t) \ge c_1, \quad x \in \Omega, \ t > 0. \tag{4.1}$$

Proof. It immediately follows from [50, Corollary 3.1].

Let us state a straightforward consequence of Lemma 4.1.

Lemma 4.2. *Let all the assumptions in Lemma 4.1 be fulfilled. Then there exists a positive constant* c_2*, with the property that*

$$\int_\Omega (u_\varepsilon + v_\varepsilon^2)(\cdot,t)dx + \int_t^{t+1} \int_\Omega (u_\varepsilon v_\varepsilon + |\nabla v_\varepsilon|^2)(\cdot,s)dxds \le c_2, \quad t > 0 \text{ and } \varepsilon \in (0,1). \tag{4.2}$$

Proof. Invoking (2.7) and (4.1), we arrive at

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx + \frac{1}{2}\kappa\int_\Omega u_\varepsilon v_\varepsilon dx + \frac{1}{2}\kappa c_1\int_\Omega u_\varepsilon dx \le \int_\Omega h_1 dx \le \|h_1\|_{L^\infty(\Omega\times(0,\infty))}|\Omega|,\ t>0, \tag{4.3}$$

where c_1 is given in (4.1). By taking $\lambda := \frac{6}{\kappa}$, this, combined with (2.15), leads to

$$\frac{d}{dt}\int_\Omega \lambda u_\varepsilon + v_\varepsilon^2 dx + \int_\Omega 3c_1 u_\varepsilon + v_\varepsilon^2 dx + \int_\Omega u_\varepsilon v_\varepsilon + 2|\nabla v_\varepsilon|^2 dx \le C_1,\ t>0. \tag{4.4}$$

Setting $y(t) := \int_\Omega \lambda u_\varepsilon + v_\varepsilon^2 dx$, we get

$$y'(t) + \min\{3c_1\lambda^{-1}, 1\}y(t) \le C_1, \quad t>0,$$

which, employing a standard ODE argument, warrants that

$$\int_\Omega (\lambda u_\varepsilon + v_\varepsilon^2)(\cdot,t)dx \le C_2, \quad t>0. \tag{4.5}$$

Using this and integrating (4.4) over $[t, t+1]$, it follows that for any $t>0$

$$\int_\Omega (\lambda u_\varepsilon + v_\varepsilon^2)(\cdot,t+1)dx + \int_t^{t+1}\int_\Omega (u_\varepsilon v_\varepsilon + 2|\nabla v_\varepsilon|^2)(\cdot,s)dxds \le \int_\Omega (\lambda u_\varepsilon + v_\varepsilon^2)(\cdot,t)dx + C_1,$$

which, combined with (4.5), evidently ensures (4.2).

To prove the long-time behavior in Theorem 1.3, we shall consider the Helmholtz problem (1.16).

Lemma 4.3. *For given* $0 \not\equiv h_{2,\infty} \in C^1(\overline{\Omega})$*, the problem* (1.16) *possesses a unique classical solution* v_∞ *with the property that* $v_\infty \in C^{2+\theta}(\overline{\Omega})$ *for some* $\theta \in (0,1)$.

Proof. The assertion directly follows from [56].

We are also concerned with the decay in a linear differential inequality (see [50, Lemma 2.5]).

Lemma 4.4. *For* $\varepsilon \in (0,1)$*, let* $y_\varepsilon \in C^1([0,\infty))$ *be non-negative functions. If* $y_\varepsilon(0)$ *is dependent of* ε*, and there exist* $a>0$ *and the nonnegative function* $g(t) \in C([0,\infty)) \cap L^\infty([0,\infty))$ *which satisfies*

$$\lim_{t\to\infty}\int_t^{t+1} g(s)ds = 0$$

such that

$$y_\varepsilon'(t) + ay_\varepsilon(t) \le g(t) \quad \textit{for all} \quad t>0 \text{ and } \varepsilon\in(0,1),$$

then

$$y_\varepsilon(t) \to 0 \quad \textit{as} \quad t\to\infty \quad \textit{uniformly in} \quad \varepsilon.$$

As a consequence, under the additional assumptions (1.13)–(1.14), a stronger result than Lemma 4.2 can be shown as follows.

Lemma 4.5. *Let all the assumptions in Lemma 4.2 hold, and let* (1.13)–(1.14) *be in force. Then we have*

$$\int_\Omega |v_\varepsilon - v_\infty|^2(\cdot,t) + u_\varepsilon(\cdot,t)dx \to 0 \quad \text{as } t\to\infty \quad \text{uniformly in } \varepsilon, \tag{4.6}$$

$$\int_t^{t+1}\int_\Omega |\nabla(v_\varepsilon - v_\infty)|^2(\cdot,s)dxds \to 0 \quad \text{as } t\to\infty \quad \text{uniformly in } \varepsilon, \tag{4.7}$$

where v_∞ is a unique classical solution of (1.16).

Proof. Set $\widehat{v_\varepsilon} := v_\varepsilon - v_\infty$ for convenience. Lemmas 2.1 and 4.3 imply that for fixed u_ε from Lemma 2.1, the initial-boundary value problem

$$\begin{cases} \widehat{v_{\varepsilon t}} = \Delta\widehat{v_\varepsilon} - \widehat{v_\varepsilon} + u_\varepsilon + h_2 - h_{2,\infty}, & x\in\Omega,\ t>0,\\ \nabla\widehat{v_\varepsilon}\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ \widehat{v_\varepsilon}(x,0) = v_0(x) - v_\infty(x), & x\in\Omega, \end{cases} \tag{4.8}$$

admits a unique classical solution $\widehat{v_\varepsilon}$. We multiply the first equation in (4.8) by $\widehat{v_\varepsilon}$ to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \widehat{v_\varepsilon}^2dx + \int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx + \int_\Omega\widehat{v_\varepsilon}^2dx \le \int_\Omega u_\varepsilon v_\varepsilon - \int_\Omega u_\varepsilon v_\infty dx + \int_\Omega \widehat{v_\varepsilon}(h_2 - h_{2,\infty})dx,$$

and thereby obtain from Young's inequality that

$$\frac{d}{dt}\int_\Omega\widehat{v_\varepsilon}^2dx + 2\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx + \int_\Omega\widehat{v_\varepsilon}^2dx \le 2\int_\Omega u_\varepsilon v_\varepsilon + 2\|v_\infty\|_{L^\infty}\int_\Omega u_\varepsilon dx + \int_\Omega(h_2-h_{2,\infty})^2dx.$$

By taking $\lambda \ge \max\left\{\frac{4}{\kappa}, \frac{4\|v_\infty\|_{L^\infty}+2}{c_1\kappa}\right\}$, this, combined with (4.3), ensures

$$\frac{d}{dt}\int_\Omega\widehat{v_\varepsilon}^2 + \lambda u_\varepsilon dx + 2\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx + \int_\Omega\widehat{v_\varepsilon}^2 + u_\varepsilon dx \le \int_\Omega(h_2-h_{2,\infty})^2dx + \lambda\int_\Omega h_1dx.$$

Setting $g(t) := \int_\Omega(h_2-h_{2,\infty})^2dx + \lambda\int_\Omega h_1dx$ and $y_\varepsilon(t) := \int_\Omega\widehat{v_\varepsilon}^2 + \mu u_\varepsilon dx$, we have

$$y_\varepsilon'(t) + \min\{\lambda^{-1},1\}y_\varepsilon(t) + 2\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dx \le g(t). \tag{4.9}$$

By means of (1.13)–(1.14) and Lemma 4.4, the desired (4.6) holds. We now integrate (4.9) over $[t,t+1]$ to get

$$2\int_t^{t+1}\int_\Omega|\nabla\widehat{v_\varepsilon}|^2dxds \le \int_t^{t+1}g(s)ds + y_\varepsilon(t).$$

This, in view of (1.13), (1.14) and (4.6) again, ensures that (4.7) holds.

Our second result on the large-time behavior of generalized solutions featured in Theorem 1.3 is in fact a by-product of our previous analysis.

Proof of Theorem 1.3. In fact, Lemma 3.5, combining with the Fubini-Tonelli theorem, provides $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ and a null set $\mathcal{N}\subset(0,\infty)$ such that $\varepsilon_j\to0$ as $j\to\infty$ and

$$u_\varepsilon(\cdot,t)\to u(\cdot,t) \ \text{ and } \ v_\varepsilon(\cdot,t)\to(\cdot,t) \ \textit{a.e.} \text{ in } \Omega \text{ for all } t\in(0,\infty)\setminus\mathcal{N},$$

as $\varepsilon=\varepsilon_j\to0$. Based on this, Lemma 4.5, together with Fatou's lemma, presents the desired large-time behavior of the generalized solution in Theorem 1.3.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. N. Rodríguez, On the global well-posedness theory for a class of PDE models for criminal activity, *Phys. D Nonlinear Phenom.*, **260** (2013), 191–200. https://doi.org/10.1016/j.physd.2012.08.003
2. M. Short, M. D'Orsogna, V. Pasour, G. Tita, P. Brantingham, A. Bertozzi, et al., A statistical model of criminal behavior, *Math. Mod. Meth. Appl. Sci.*, **18** (2008), 1249–1267. https://doi.org/10.1142/S0218202508003029
3. M. Short, A. Bertozzi, P. Brantingham, G. Tita, Dissipation and displacement of hotspots in reaction-diffusion model of crime, *Proc. Natl. Acad. Sci. USA*, **107** (2010), 3961–3965. https://doi.org/0.1073/pnas.0910921107
4. H. Berestycki, J. Wei, M. Winter, Existence of symmetric and asymmetric spikes for a crime hotspot model, *SIAM J. Math. Anal.*, **46** (2014), 691–719. https://doi.org/10.1137/130922744
5. R. Cantrell, C. Cosner, R. Manásevich, Global bifurcation of solutions for crime modeling equations, *SIAM J. Math. Anal.*, **44** (2012), 1340–1358. https://doi.org/10.1137/110843356
6. Y. Gu, Q. Wang, G. Yi, Stationary patterns and their selection mechanism of urban crime models with heterogeneous near-repeat victimization effect, *Eur. J. Appl. Math.*, **28** (2017), 141–178. https://doi.org/10.1017/S0956792516000206
7. T. Kolokolnikov, M. Ward, J. Wei, The stability of steady-state hot-spot patterns for a reaction-diffusion model of urban crime, *Discrete Contin. Dyn. Syst. Ser. B*, **19** (2014), 1373–1410. https://doi.org/10.3934/dcdsb.2014.19.1373
8. D. Lloyd, H. O'Farrell, On localised hotspots of an urban crime model, *Phys. D Nonlinear Phenom.*, **253** (2013), 23–39. https://doi.org/10.1016/j.physd.2013.02.005
9. D. Lloyd, N. Santitissadeekorn, M. Short, Exploring data assimilation and forecasting issues for an urban crime model, *Eur. J. Appl. Math.*, **27** (2016), 451–478. https://doi.org/10.1017/S0956792515000625

10. L. Mei, J. Wei, The existence and stability of spike solutions for a chemotax is system modeling crime pattern formation, *Math. Models Methods Appl. Sci.*, **30** (2020), 1727–1764. https://doi.org/10.1142/S0218202520500359

11. M. Short, A. Bertozzi, P. Brantingham, Nonlinear patterns in urban crime: hotspots, bifurcations, and suppression, *SIAM J. Appl. Dyn. Syst.*, **9** (2010), 462–483. https://doi.org/10.1137/090759069

12. W. Tse, M. Ward, Hotspot formation and dynamics for a continuum model of urban crime, *Eur. J. Appl. Math.*, **27** (2016), 583–624. https://doi.org/10.1017/S0956792515000376

13. N. Rodríguez, A. Bertozzi, Local existence and uniqueness of solutions to a PDE model for criminal behavior, *Math. Models Methods Appl. Sci.*, **20** (2010), 1425–1457. https://doi.org/10.1142/S0218202510004696

14. N. Rodríguez, M. Winkler, On the global existence and qualitative behavior of one-dimensional solutions to a model for urban crime, *Eur. J. Appl. Math.*, **33** (2022), 919–959. https://doi.org/10.1017/S0956792521000279

15. Q. Wang, D. Wang, Y. Feng, Global well-posedness and uniform boundedness of urban crime models: One-dimensional case, *J. Differ. Equations*, **269** (2020), 6216–6235. https://doi.org/10.1016/j.jde.2020.04.035

16. M. Freitag, Global solutions to a higher-dimensional system related to crime modeling, *Math. Meth. Appl. Sci.*, **41** (2018), 6326–6335. https://doi.org/10.1002/mma.5141

17. J. Shen, B. Li, Mathematical analysis of a continuous version of statistical models for criminal behavior, *Math. Meth. Appl. Sci.*, **43** (2020), 409–426. https://doi.org/10.1002/mma.5898

18. J. Ahn, K. Kang, J. Lee, Global well-posedness of logarithmic Keller-Segel type systems, *J. Differ. Equations*, **287** (2021), 185–211. https://doi.org/10.1016/j.jde.2021.03.053

19. Y. Tao, M. Winkler, Global smooth solutions in a two-dimensional cross-diffusion system modeling propagation of urban crime, *Commun. Math. Sci.*, **19** (2021), 829–849. https://doi.org/10.4310/CMS.2021.v19.n3.a12

20. M. Winkler, Global solvability and stabilization in a two-dimensional cross-diffusion system modeling urban crime propagation, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, **36** (2019), 1747–1790. https://doi.org/10.1016/j.anihpc.2019.02.004

21. Y. Jiang, L. Yang, Global solvability and stabilization in a three-dimensional cross-diffusion system modeling urban crime propagation, *Acta Appl. Math.*, **178** (2022). https://doi.org/10.1007/s10440-022-00484-z

22. N. Rodríguez, M. Winkler, Relaxation by nonlinear diffusion enhancement in a two-dimensional cross-diffusion model for urban crime propagation, *Math. Models Methods Appl. Sci.*, **30** (2020), 2105–2137. https://doi.org/10.1142/S0218202520500396

23. F. Heihoff, Generalized solutions for a system of partial differential equations arising from urban crime modeling with a logistic source term, *Z. Für Angew. Math. Phys.*, **71** (2020). https://doi.org/10.1007/s00033-020-01304-w

24. B. Li, L. Xie, Generalized solution to a 2D parabolic-parabolic chemotaxis system for urban crime: Global existence and large time behavior, submitted for publication, 2022.

25. P. Jones, P. Brantingham, L. Chayes, Statistical models of criminal behavior: The effects of law enforcement actions, *Math. Models Methods Appl. Sci.*, **20** (2010), 1397–1423. https://doi.org/10.1142/S0218202510004647

26. A. Pitcher, Adding police to a mathematical model of burglary, *Eur. J. Appl. Math.*, **21** (2010), 401–419. https://doi.org/10.1017/S0956792510000112

27. J. Zipkin, M. Short, A. Bertozzi, Cops on the dots in a mathematical model of urban crime and police response, *Discrete Contin. Dyn. Syst. Ser. B*, **19** (2014), no. 5, 1479–1506. https://doi.org/10.3934/dcdsb.2014.19.1479

28. W. Tse, M. Ward, Asynchronous instabilities of crime hotspots for a 1-D reaction-diffusion model of urban crime with focused police patrol, *SIAM J. Appl. Dyn. Syst.*, **17** (2018), 2018–2075. https://doi.org/10.1137/17M1162585

29. A. Buttenschoen, T. Kolokolnikov, M. Ward, J. Wei, Cops-on-the-dots: the linear stability of crime hotspots for a 1-D reaction-diffusion model of urban crime, *Eur. J. Appl. Math.*, **31** (2020), 871–917. https://doi.org/10.1017/S0956792519000305

30. B. Li, L. Xie, Global large-data generalized solutions to a two-dimensional chemotaxis system stemming from crime modelling, *Discrete Contin. Dyn. Syst. Ser. B*, **2022** (2022). https://doi.org/10.3934/dcdsb.2022167

31. N. Rodríguez, Q. Wang, L. Zhang, Understanding the effects of on- and off-hotspot policing: Evidence of hotspot, oscillating, and chaotic activities, *SIAM J. Appl. Dyn. Syst.*, **20** (2021), 1882–1916. https://doi.org/10.1137/20M1359572

32. E. Keller, L. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.*, **26** (1970), 399–415. https://doi.org/10.1016/0022-5193(70)90092-5

33. K. Fujie, Boundedness in a fully parabolic chemotaxis system with strongly singular sensitivity, *J. Math. Anal. Appl.*, **424** (2015), 675–684. https://doi.org/10.1016/j.jmaa.2014.11.045

34. M. Winkler, Global solutions in a fully parabolic chemotaxis system with singular sensitivity, *Math. Methods Appl. Sci.*, **34** (2011), 176–190. https://doi.org/10.1002/mma.1346

35. J. Lankeit, A new approach toward boundedness in a two-dimensional parabolic chemotaxis system with singular sensitivity, *Math. Meth. Appl. Sci.*, **39** (2016), 394–404. https://doi.org/10.1002/mma.3489

36. C. Stinner, M. Winkler, Global weak solutions in a chemotaxis system with large singular sensitivity, *Nonliear Anal. Real Word Appl.*, **12** (2011), 3727–3740. https://doi.org/10.1016/j.nonrwa.2011.07.006

37. J. Lankeit, M. Winkler, A generalized solution concept for the Keller-Segel system with logarithmic sensitivity: global solvability for large nonradial data, *NoDEA-Nonlinear Differ. Equ. Appl.*, **24** (2017). https://doi.org/10.1007/s00030-017-0472-8

38. A. Zhigun, Generalised supersolutions with mass control for the Keller-Segel system with logarithmic sensitivity, *J. Math. Anal. Appl.*, **467** (2018), 1270–1286. https://doi.org/10.1016/j.jmaa.2018.08.001

39. M. Winkler, Unlimited growth in logarithmic Keller-Segel systems, *J. Differ. Equations*, **309** (2022), 74–97. https://doi.org/10.1016/j.jde.2021.11.026

40. M. Winkler, T. Yokota, Stabilization in the logarithmic Keller-Segel system, *Nonlinear Anal.*, **170** (2018), 123–141. https://doi.org/10.1016/j.na.2018.01.002

41. J. Ahn, Global well-posedness and asymptotic stabilization for chemotaxis system with signal-dependent sensitivity, *J. Differ. Equations*, **266** (2019), 6866–6904. https://doi.org/10.1016/j.jde.2018.11.015

42. Q. Hou, Z. Wang, K. Zhao, Boundary layer problem on a hyperbolic system arising from chemotaxis, *J. Differ. Equations*, **261** (2016), 5035–5070. https://doi.org/10.1016/j.jde.2016.07.018

43. H. Jin, J. Li, Z. Wang, Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity, *J. Differ. Equations*, **255** (2013), 193–219. doilinkhttps://doi.org/10.1016/j.jde.2013.04.002

44. H. Li, K. Zhao, Initial-boundary value problems for a system of hyperbolic balance laws arising from chemotaxis, *J. Differ. Equations*, **258** (2015), 302–308. https://doi.org/10.1016/j.jde.2014.09.014

45. J. Li, T. Li, Z. Wang, Stability of traveling waves of the Keller-Segel system with logarithmic sensitivity, *Math. Models Methods Appl. Sci.*, **24** (2014), 2819–2849. https://doi.org/10.1142/S0218202514500389

46. Y. Tao, L. Wang, Z. Wang, Large-time behavior of a parabolic-parabolic chemotaxis model with logarithmic sensitivity in one dimension, *Discrete Contin. Dyn. Syst. Ser. B*, **18** (2013), 821–845. https://doi.org/10.3934/dcdsb.2013.18.821

47. Z. Wang, Z. Xiang, P. Yu, Asymptotic dynamics in a singular chemotaxis system modeling onset of tumor angiogenesis, *J. Differ. Equations*, **260** (2016), 2225–2258. https://doi.org/10.1016/j.jde.2015.09.063

48. M. Winkler, Renormalized radial large-data solutions to the higher-dimensional Keller-Segel system with singular sensitivity and signal absorption, *J. Differ. Equations*, **264** (2018), 2310–2350. https://doi.org/10.1016/j.jde.2017.10.029

49. M. Winkler, The two-dimensional Keller-Segel system with singular sensitivity and signal absorption: global large-data solutions and their relaxation properties, *Math. Models Methods Appl. Sci.*, **26** (2016), 987–1024. https://doi.org/10.1142/S0218202516500238

50. B. Li, L. Xie, Generalized solution and its eventual smoothness to a logarithmic Keller-Segel system for criminal activities, submitted for publication, 2022.

51. M. Winkler, Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities, *SIAM J. Math. Anal.*, **47** (2015), 3092–3115. https://doi.org/dx.doi.org/10.1137/140979708

52. M. Aida, K. Osaka, T. Tsujikawa, M. Mimura, Chemotaxis and growth system with sigular sensitivity function, *Nonliear Anal. Real Word Appl.*, **6** (2005), 323–336. https://doi.org/10.1016/j.nonrwa.2004.08.011

53. X. Cao, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, *Discrete Contin. Dyn. Syst.*, **35** (2015), 1891–1904. https://doi.org/10.3934/dcds.2015.35.1891

54. M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differ. Equations*, **248** (2010), 2889–2905. https://doi.org/10.1016/j.jde.2010.02.008

55. J. Simon, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.*, **146** (1996), 65–96. https://doi.org/10.1007/BF01762360

56. O. Ladyzhenskaya, N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Academic Press, New York-London, 1968.

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