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Research article

Novel multiple criteria decision-making analysis under m-polar fuzzy aggregation operators with application

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Abstract: Aggregation is a very efficient indispensable tool in which several input values are transformed into a single output value that further supports dealing with different decision-making situations. Additionally, note that the theory of m-polar fuzzy (mF) sets is proposed to tackle multipolar information in decision-making problems. To date, several aggregation tools have been widely investigated to tackle multiple criteria decision-making (MCDM) problems in an m-polar fuzzy environment, including m-polar fuzzy Dombi and Hamacher aggregation operators (AOs). However, the aggregation tool to deal with m-polar information under Yager's operations (that is, Yager's t-norm and t-conorm) is missing in the literature. Due to these reasons, this study is devoted to investigating some novel averaging and geometric AOs in an mF information environment through the use of Yager's operations. Our proposed AOs are named as the mF Yager weighted averaging (mFYWA) operator, mF Yager ordered weighted averaging operator, mF Yager hybrid averaging operator, mF Yager weighted geometric (mFYWG) operator, mF Yager ordered weighted geometric operator and mF Yager hybrid geometric operator. The initiated averaging and geometric AOs are explained via illustrative examples and some of their basic properties, including boundedness, monotonicity, idempotency and commutativity are also studied. Further, to deal with different MCDM situations containing mF information, an innovative algorithm for MCDM is established under the under the condition of mFYWA and mFYWG operators. After that, a real-life application (that is, selecting a suitable site for an oil refinery) is explored under the conditions of developed AOs. Moreover, the initiated mF Yager AOs are compared with existing mF Hamacher and Dombi AOs through a numerical example. Finally, the effectiveness and reliability of the presented AOs are checked with the help of some existing validity tests.

Keywords: accuracy function; algorithm; m-polar fuzzy set; score function; Yager's t-norm; Yager's t-conorm

1. Introduction

Multi-criteria decision-making (MCDM) is an essential mathematical tool for solving various daily-life problems involving multiple parameters or attributes, and it is playing a vital role in several areas, including engineering, medical, economics, etc. Inspection of the past two decades show that the aggregation operator (AO) based MCDM methodologies are playing a significant role in solving several real-life problems by converting the raw data into a valuable piece of information. For the classification of alternatives in various daily-life scenarios, the experts or scientists used different types of traditional evaluation tools like crisp set theory. In different decision-making problems due to increasing uncertainties of datasets, it was difficult for the experts to tackle those situations with the help of exact numerical values. To remove this difficulty, Zadeh [1] originally launched the theory of fuzzy sets by proposing a membership function whose codomain is $[0, 1]$. Thus, crisp set theory is a particular case of fuzzy set theory. After that, many experts from all over the globe have been attracted to the powerful idea of fuzzy sets and solved different decision-making problems comprising vagueness and imprecision in their data-sets more accurately than crisp sets, e.g., [2–4]. Several AOs in a fuzzy information environment have been explored to deal with different decision-making situations. For instance, Song et al. [5] proposed some parameterized AOs under fuzzy information and studied their basic properties. Merigo and Gil-Lafuente [6] investigated fuzzy induced generalized AOs and applied them to solve decision-making problems.

As a direct extension of fuzzy sets, Atanassov [7] initiated the notion of intuitionistic fuzzy sets (IFSs) by adding a non-membership function with the membership function in the fuzzy set theory whose functional values sum should be bounded by 1. After the production of an IFS model, the experts moved their attraction to tackle decision-making situations using IFS theory. For example, Xu [8] investigated some IFS-based AOs, namely, IF weighted, ordered weighted, and hybrid weighted averaging AOs (see also [9, 10] for IFS-based power AOs and IFS-based ordered weighted distance AOs). In addition, Xu and Yager [11] proposed some IFS-based geometric AOs. Wei [12] introduced different induced geometric and generalized IFS-based AOs with the solution of a decision-making application. Tan et al. [13] proposed IFS-based generalized geometric AOs and studied their applications to MCDM. After the invention of Pythagorean fuzzy sets (PFSs) by Yager [14], Peng and Yang [15] studied some fundamental notions of PFS-based AOs in an interval-valued environment. Garg and Kumar [16] proposed some power geometric AOs based on the connection number in intuitionistic fuzzy format. Shahzadi et al. [17] developed some novel AOs under Pythagorean fuzzy Yager operations. Ali et al. [18] introduced some novel arithmetic and geometric AOs by using complex T-spherical fuzzy sets and studied their application in an investment problem. Ashraf et al. [19] submitted certain spherical fuzzy Dombi AOs and explored their application to multiple attribute group decision-making. In recent years, several studies have been completed which directly involve the aggregation of bipolar data with the help of existing operations, that is, Dombi and Hamacher t-norms and t-conorms. For example, Wei et al. [20] presented bipolar data-based Hamacher AOs with their MCDM applications. Afterwards, Jana et al. [21] proposed bipolar data-based Dombi AOs and solved a daily-life problem. In addition, Jana et al. [22] introduced bipolar fuzzy Dombi prioritized AOs.

Many daily life situations involve datasets from *m* different agents or sources ($m \geq 2$), which means the multipolar information emerges that cannot be portrayed mathematically through the traditional tools of crisp set theory, fuzzy set theory, IFS theory and PFS theory. The main goal of the work offered in this article is to tackle the shortcomings of mathematical tools considering multipolar, multi-attribute and multi-index information. These days, research scholars think that this world is nearing the concepts of multipolarity because multipolarity in information and data plays a substantial role in numerous disciplines ranging from arts to sciences. For example, a noisy communication channel may have different latency, bandwidth, radio frequency and network range. Concerning information technology, multipolar technology can be employed to analyze larger information systems. Concerning neurobiology, neurons in the brain collect data from other multiple neurons. Concerning a social network, the efficacy rate of distinct people may be distinct regarding trading relationships, proactiveness, and socialism. All of these multipolar scenarios contain fuzzy data. To deal with such multipolar situations, we need more innovative theoretical and mathematical models. In summary, the prevailing theories of fuzzy sets, IFSs and PFSs are very efficient mathematical tools to deal with vagueness and uncertainties; but they are inefficient in some scenarios, e.g., when the under-consideration datasets are multi-dimensional. To solve this difficulty in the implementation of fuzzy sets and their extensions, Chen et al. [23] generalized the theory of fuzzy sets and proposed the theory of *m*-polar fuzzy (*m*F) sets, which have the ability to deal with multipolarity in datasets of different domains of modern sciences. To date, some studies have focused on the aggregation of *m*F information by using different AOs. For example, Waseem et al. [24] launched *m*F Hamacher AOs and solved two MCDM problems. Khameneh and Kilicman [25] presented the ideas of *m*F soft weighted AOs and implemented them to solve MCDM problems. Additionally, Akram et al. [26] initiated the notions of *m*F Dombi AOs and explored some of their MCDM applications. Recently, Naz et al. [27] proposed some novel 2−tuple linguistic bipolar fuzzy Heronian mean AOs for group decision-making.

In the early 1980s, Yager proposed a t-norm (TN) and t-conorm (TCoN), which are more universal operators than the Lukasiewicz TN and TCoN, respectively. Recently, a number of researchers have been attracted toward these and introduced several new results in the area of MCDM. For example, Garg et al. [28] introduced Fermatean fuzzy Yager AOs and studied their application to COVID-19 testing facility. In addition, Liu et al. [29] presented some certain kinds of *q*-rung picture fuzzy Yager AOs for decision-making. Later, Akram et al. [30] launched the theory of complex Pythagorean fuzzy Yager AOs and illustrated their validity through an MCDM problem-solving method. All of these models do not consider the aggregation of *m*F information under Yager's TN and TCoN. We take Yager's operations due to their simple implementation compared to other TNs and TCoNs like Dombi, Hamacher and Frank. And, Yager's operations also consider a strong correlation between different estimated results compared to other operators. Therefore, in this article, we propose some other novel types of Yager AOs for the aggregation of *m*F information. For more related useful basic terminologies, the readers are referred to [31–42].

The following reasons motivate us to develop the *m*F Yager AOs.

1) The theory of *m*F sets being a generalized fruitful tool is playing a vital role in the execution

procedure of uncertain decision-making problems involving multipolar information.

2) The theory of fuzzy sets is only able to handle datasets in one dimension, and thus a loss of information may occur. This is because, in many practical situations, multiple attributes and all of their possible features can only be handled with *m*F set theory and its hybrid models.

3) Until today, several results on the aggregation of complex real-world problems involving *m*F datasets have been presented under different MCDM-AOs (i.e., *m*F Hamacher and Dombi AOs), but the aggregation of *m*F information with the help of Yager's operations (that is, Yager's TN and TCoN) has not been elucidated.

4) The *m*F Yager AOs provide an alternative approach for dealing with several MCDM problems like some existing *m*F AOs.

To sum up, from the aforementioned discussion, we notice that the work on the aggregation of *m*F information under Yager's operations is not present in the existing literature. Due to these shortcomings, in this article, we have presented *m*F Yager AOs and operated them to solve a practical MCDM problem. This article mainly contributes the following:

1) The concepts of some *m*F Yager arithmetic and geometric AOs are proposed along with their basic properties, including monotonicity, idempotency, boundedness and commutativity.

2) An algorithm is designed step-by-step for dealing with daily-life MCDM problems in an *m*F information environment.

3) A number of site selection problems have been explored in the literature via different fuzzy set-based hybrid models [43, 44]. Thus, to verify the applicability of the initiated *m*F Yager AOs in practical scenarios, an application is presented which deals with the selection of an appropriate site for an oil refinery.

4) To prove the feasibility and authenticity of the initiated *m*F Yager AOs, a comparison of these *m*F Yager AOs is investigated with existing *m*F Hamacher AOs [24], and *m*F Dombi AOs [26].

Table 1. Nomenclature of the research work.

Acronyms and Notations	Description
*m*F	*m*-polar fuzzy
*m*FYOWG	*m*F Yager ordered weighted geometric
*m*FDWA	*m*F Dombi weighted averaging
*m*FHWA	*m*F Hamacher weighted averaging
*m*FHWG	*m*F Hamacher weighted geometric
COVID-19	Corona-virus disease 2019
$\mathfrak{S}(\tilde{\eta})$	Accuracy function of *m*F number $\tilde{\eta}$
$\mathfrak{A}(\tilde{\eta})$	Score function of *m*F number $\tilde{\eta}$
$\tilde{\eta} = (\mathfrak{p}_1 \circ \eta, \ldots, \mathfrak{p}_m \circ \eta)$	*m*F number
$\Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n)^T$	weight-vector
$\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$	Universal set
$\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\}$	Universal set of parameters
$\tilde{\mathfrak{M}} = (\tilde{\mathfrak{d}}_{it})_{k\times n}$	*m*F decision matrix
$\tilde{\mathfrak{d}}_r$	Preference values

This article is structured as follows: Section 2 first reviews basic definitions and properties associated with mF numbers and then introduces the mF Yager weighted averaging (mFYWA) operator, mF Yager ordered weighted averaging (mFYOWA) operator, mF Yager hybrid averaging (mFYHA) operator, mF Yager weighted geometric (mFYWG) operator, mF Yager ordered weighted geometric operator (mFYOWG) and mF Yager hybrid geometric (mFYHG) operator. Section 3 first develops a MCDM method under initiated mF Yager AOs to solve real-life problems containing complicated mF information and then explores an MCDM application in which the selection of a suitable site for an oil refinery is investigated. Section 4 provides a comparison of the developed methodology for mF Yager AOs with mF Hamacher [24] and Dombi [26] AOs. Section 5 concludes our work by providing advantages, disadvantages and some further future directions.

The notations and abbreviations are provided in Table 1.

2. *m*F Yager AOs

This section first reviews the definition of mF sets and some operations of mF numbers; it then presents some essential Yager operations for mF numbers by using Yager's TCoN and Yager's TN and establishes mF Yager arithmetic and geometric AOs together with illustrative numerical examples.

Definition 2.1. [23] An mF set or mF set on a universal set $\mathcal{S}$ is a mapping $\eta : \mathcal{S} \to [0,1]^m$. The belongingness degree of each alternative is expressed as $\eta(\mathfrak{s}) = (\mathfrak{p}_1 \circ \eta(\mathfrak{s}), \mathfrak{p}_2 \circ \eta(\mathfrak{s}), \ldots, \mathfrak{p}_m \circ \eta(\mathfrak{s}))$ where $\mathfrak{s} \in \mathcal{S}$, and for $(j = 1, 2, \ldots, m)$, $\mathfrak{p}_j \circ \eta : [0,1]^m \to [0,1]$ is the j-th projection mapping.

For an mF number $\tilde{\eta} = (\mathfrak{p}_1 \circ \eta, \ldots, \mathfrak{p}_m \circ \eta)$, where $\mathfrak{p}_j \circ \eta \in [0,1]$, for all $j = 1, 2, \ldots, m$, the score and accuracy functions of mF number $\tilde{\eta}$ are respectively given as follows:

Definition 2.2. [24] For an mF number $\tilde{\eta} = (\mathfrak{p}_1 \circ \eta, \ldots, \mathfrak{p}_m \circ \eta)$, its score $\mathfrak{S}$ and accuracy $\mathfrak{A}$ functions are provided by

$$\mathfrak{S}(\tilde{\eta}) = \frac{1}{m}\Big(\sum_{t=1}^{m}(\mathfrak{p}_t \circ \eta)\Big), \quad \mathfrak{S}(\tilde{\eta}) \in [0,1],$$
$$\mathfrak{A}(\tilde{\eta}) = \frac{1}{m}\Big(\sum_{t=1}^{m}(-1)^{t+1}(\mathfrak{p}_t \circ \eta - 1)\Big), \quad \mathfrak{A}(\tilde{\eta}) \in [-1,1].$$

Clearly, the above Definition 2.2 provides us an ordered relation criterion for mF numbers, which is given as follows:

Definition 2.3. [24] For any two mF numbers $\tilde{\eta}_1 = (\mathfrak{p}_1 \circ \eta_1, \ldots, \mathfrak{p}_m \circ \eta_1)$, and $\tilde{\eta}_2 = (\mathfrak{p}_1 \circ \eta_2, \ldots, \mathfrak{p}_m \circ \eta_2)$, we have

1) $\tilde{\eta}_1 < \tilde{\eta}_2$, if $\mathfrak{S}(\tilde{\eta}_1) < \mathfrak{S}(\tilde{\eta}_2)$,
2) $\tilde{\eta}_1 > \tilde{\eta}_2$, if $\mathfrak{S}(\tilde{\eta}_1) > \mathfrak{S}(\tilde{\eta}_2)$,
3) If $\mathfrak{S}(\tilde{\eta}_1) = \mathfrak{S}(\tilde{\eta}_2)$ then
   - $\tilde{\eta}_1 < \tilde{\eta}_2$, if $\mathfrak{A}(\tilde{\eta}_1) < \mathfrak{A}(\tilde{\eta}_2)$,
   - $\tilde{\eta}_1 > \tilde{\eta}_2$, if $\mathfrak{A}(\tilde{\eta}_1) > \mathfrak{A}(\tilde{\eta}_2)$,
   - $\tilde{\eta}_1 = \tilde{\eta}_2$, if $\mathfrak{A}(\tilde{\eta}_1) = \mathfrak{A}(\tilde{\eta}_2)$.

Some useful fundamental properties of mF numbers are given as below [24]:

1) $\tilde{\eta}_1 \boxplus \tilde{\eta}_2 = \Big(\mathfrak{p}_1 \circ \eta_1 + \mathfrak{p}_1 \circ \eta_2 - \mathfrak{p}_1 \circ \eta_1.\mathfrak{p}_1 \circ \eta_2, \ldots, \mathfrak{p}_m \circ \eta_1 + \mathfrak{p}_m \circ \eta_2 - \mathfrak{p}_m \circ \eta_1.\mathfrak{p}_m \circ \eta_2\Big)$,
2) $\tilde{\eta}_1 \boxtimes \tilde{\eta}_2 = \Big(\mathfrak{p}_1 \circ \eta_1.\mathfrak{p}_1 \circ \eta_2, \ldots, \mathfrak{p}_m \circ \eta_1.\mathfrak{p}_m \circ \eta_2\Big)$,
3) $\varsigma\tilde{\eta} = (1 - (1 - \mathfrak{p}_1 \circ \eta)^{\varsigma}), \ldots, 1 - (1 - \mathfrak{p}_m \circ \eta)^{\varsigma}),\ \varsigma > 0$,
4) $(\tilde{\eta})^{\varsigma} = ((\mathfrak{p}_1 \circ \eta)^{\varsigma}, \ldots, (\mathfrak{p}_m \circ \eta)^{\varsigma}),\ \varsigma > 0$,
5) $\tilde{\eta}^c = (1 - \mathfrak{p}_1 \circ \eta, \ldots, 1 - \mathfrak{p}_m \circ \eta)$,
6) $\tilde{\eta}_1 \subseteq \tilde{\eta}_2$, if and only if $\mathfrak{p}_1 \circ \eta_1 \leq \mathfrak{p}_1 \circ \eta_2, \ldots, \mathfrak{p}_m \circ \eta_1 \leq \mathfrak{p}_m \circ \eta_2$,
7) $\tilde{\eta}_1 \cup \tilde{\eta}_2 = (\max(\mathfrak{p}_1 \circ \eta_1, \mathfrak{p}_1 \circ \eta_2), \ldots, \max(\mathfrak{p}_m \circ \eta_1, \mathfrak{p}_m \circ \eta_2))$,
8) $\tilde{\eta}_1 \cap \tilde{\eta}_2 = (\min(\mathfrak{p}_1 \circ \eta_1, \mathfrak{p}_1 \circ \eta_2), \ldots, \min(\mathfrak{p}_m \circ \eta_1, \mathfrak{p}_m \circ \eta_2))$.

Theorem 2.1. [24] Let $\tilde{\eta}_1 = (\mathfrak{p}_1 \circ \eta_1, \ldots, \mathfrak{p}_m \circ \eta_1)$ and $\tilde{\eta}_2 = (\mathfrak{p}_1 \circ \eta_2, \ldots, \mathfrak{p}_m \circ \eta_2)$ be mF numbers and $\varsigma, \varsigma_1, \varsigma_2 > 0$, then, we have

1) $\tilde{\eta}_1 \boxplus \tilde{\eta}_2 = \tilde{\eta}_2 \boxplus \tilde{\eta}_1$,
2) $\tilde{\eta}_1 \boxtimes \tilde{\eta}_2 = \tilde{\eta}_2 \boxtimes \tilde{\eta}_1$,
3) $\varsigma(\tilde{\eta}_1 \boxplus \tilde{\eta}_2) = \varsigma(\tilde{\eta}_1) \boxplus \varsigma(\tilde{\eta}_2)$,
4) $(\tilde{\eta}_1 \boxtimes \tilde{\eta}_2)^{\varsigma} = (\tilde{\eta}_1)^{\varsigma} \boxplus (\tilde{\eta}_2)^{\varsigma}$,
5) $\varsigma_1\tilde{\eta}_1 \boxplus \varsigma_2\tilde{\eta}_1 = (\varsigma_1 + \varsigma_2)\tilde{\eta}_1$,
6) $(\tilde{\eta}_1)^{\varsigma_1} \boxtimes (\tilde{\eta}_2)^{\varsigma_2} = (\tilde{\eta}_1)^{\varsigma_1+\varsigma_2}$,
7) $((\tilde{\eta}_1)^{\varsigma_1})^{\varsigma_2} = (\tilde{\eta}_1)^{\varsigma_1\varsigma_2}$.

Yager [41] initiated a useful TN (Yager product $\otimes$) and TCoN (Yager sum $\oplus$), which are respectively given by

$$\mathcal{Y}(\mathfrak{s}_1, \mathfrak{s}_2) = \mathfrak{s}_1 \otimes \mathfrak{s}_2 = 1 - \min\left(1, ((1 - \mathfrak{s}_1)^{\sigma} + (1 - \mathfrak{s}_2)^{\sigma})^{\frac{1}{\sigma}}\right), \tag{2.1}$$

$$\mathcal{Y}^*(\mathfrak{s}_1, \mathfrak{s}_2) = \mathfrak{s}_1 \oplus \mathfrak{s}_2 = \min\left(1, ((\mathfrak{s}_1)^{\sigma} + (\mathfrak{s}_2)^{\sigma})^{\frac{1}{\sigma}}\right), \tag{2.2}$$

where $\sigma \geq 0$ and $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathbb{R}$ (set of real numbers).

We are now ready to present some essential Yager operations for mF numbers by using Yager's TCoN and Yager's TN. For two mF numbers $\tilde{\eta}_1 = (\mathfrak{p}_1 \circ \eta_1, \ldots, \mathfrak{p}_m \circ \eta_1)$ *and* $\tilde{\eta}_2 = (\mathfrak{p}_1 \circ \eta_2, \ldots, \mathfrak{p}_m \circ \eta_2)$ and $\varsigma > 0$, we provide certain operations of mF numbers with Yager's TN and TCoN as below:

- $\tilde{\eta}_1 \oplus \tilde{\eta}_2 = \left(\sqrt{\min\left(1, ((\mathfrak{p}_1 \circ \eta_1)^{2\sigma} + (\mathfrak{p}_1 \circ \eta_2)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, ((\mathfrak{p}_m \circ \eta_1)^{2\sigma} + (\mathfrak{p}_m \circ \eta_2)^{2\sigma})^{\frac{1}{\sigma}}\right)}\right)$,
- $\tilde{\eta}_1 \otimes \tilde{\eta}_2 = \left(\sqrt{1 - \min\left(1, ((1 - (\mathfrak{p}_1 \circ \eta_1)^2)^{\sigma} + (1 - (\mathfrak{p}_1 \circ \eta_2)^2)^{\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{1 - \min\left(1, ((1 - (\mathfrak{p}_m \circ \eta_1)^2)^{\sigma} + (1 - (\mathfrak{p}_m \circ \eta_2)^2)^{\sigma})^{\frac{1}{\sigma}}\right)}\right)$,
- $\varsigma\tilde{\eta}_1 = \left(\sqrt{\min\left(1, (\varsigma(\mathfrak{p}_1 \circ \eta_1)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\varsigma(\mathfrak{p}_m \circ \eta_1)^{2\sigma})^{\frac{1}{\sigma}}\right)}\right)$,
- $(\tilde{\eta}_1)^{\varsigma} = \left(\sqrt{1 - \min\left(1, (\varsigma(1 - (\mathfrak{p}_1 \circ \eta_1)^2)^{\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{1 - \min\left(1, (\varsigma(1 - (\mathfrak{p}_m \circ \eta_1)^2)^{\sigma})^{\frac{1}{\sigma}}\right)}\right)$.

2.1. *mF Yager arithmetic AOs*

In this subsection, we introduce some novel mF Yager arithmetic AOs with their useful properties:

Definition 2.4. Let $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ with $t = 1, 2, \ldots, n$ be a finite set of mF numbers; then, a function $mFYWA_\Upsilon : \tilde{\eta}^n \to \tilde{\eta}$ is called an mF Yager weighted average operator, which is given as

$$mFYWA_\Upsilon(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \bigoplus_{t=1}^{n} (\Upsilon_t \tilde{\eta}_t), \tag{2.3}$$

where $\Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n)^T$ represents the weights for $\tilde{\eta}_t$, $\forall\, t = 1, \ldots, n$ and $\Upsilon_t > 0$ with $\sum_{t=1}^{n} \Upsilon_t = 1$.

Now we provide the main result to aggregate mF information with the proposed mF Yager operations.

Theorem 2.2. Let $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ be a finite collection of mF numbers, that is, $t = 1, 2, \ldots, n$; then, an aggregated value of these mF numbers using the mF Yager weighted average operator is given by

$$\begin{aligned} mFYWA_\Upsilon(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) &= \bigoplus_{t=1}^{n} (\Upsilon_t \tilde{\eta}_t), \\ &= \left(\sqrt{\min\left(1, \left(\sum_{t=1}^{n} \Upsilon_t (\mathfrak{p}_1 \circ \eta_t)^{2\sigma}\right)^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, \left(\sum_{t=1}^{n} \Upsilon_t (\mathfrak{p}_m \circ \eta_t)^{2\sigma}\right)^{\frac{1}{\sigma}}\right)} \right). \end{aligned} \tag{2.4}$$

The proof of this theorem is given in Appendix A.

Example 2.1. Let $\tilde{\eta}_1 = (0.5, 0.4, 0.7)$, $\tilde{\eta}_2 = (0.2, 0.4, 0.3)$, $\tilde{\eta}_3 = (0.8, 0.9, 0.6)$ and $\tilde{\eta}_4 = (0.7, 0.5, 0.3)$ be 3-polar fuzzy (3F) numbers and $\Upsilon = (0.3, 0.1, 0.4, 0.2)^T$ be weights associated with these 3F numbers. Then, for $\sigma = 5$,

$$\begin{aligned} mFYWA_\Upsilon(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4) &= \bigoplus_{t=1}^{4} (\Upsilon_t \tilde{\eta}_t) \\ &= \left(\sqrt{\min\left(1, \left(\sum_{t=1}^{4} \Upsilon_t (\mathfrak{p}_1 \circ \eta_t)^{2\sigma}\right)^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, \left(\sum_{t=1}^{4} \Upsilon_t (\mathfrak{p}_3 \circ \eta_t)^{2\sigma}\right)^{\frac{1}{\sigma}}\right)} \right), \\ &= \Bigg(\sqrt{\min\left(1, (0.3 \times (0.5)^{10} + 0.1 \times (0.2)^{10} + 0.4 \times (0.8)^{10} + 0.2 \times (0.7)^{10})^{1/5}\right)}, \\ &\quad \sqrt{\min\left(1, (0.3 \times (0.4)^{10} + 0.1 \times (0.4)^{10} + 0.4 \times (0.9)^{10} + 0.2 \times (0.5)^{10})^{1/5}\right)}, \\ &\quad \sqrt{\min\left(1, (0.3 \times (0.7)^{10} + 0.1 \times (0.3)^{10} + 0.4 \times (0.6)^{10} + 0.2 \times (0.3)^{10})^{1/5}\right)} \Bigg), \\ &= (0.7395, 0.8213, 0.6364). \end{aligned}$$

In what follows, some essential properties of mFYWA operators are explored.

Theorem 2.3. (Monotonicity) For two sets of mF numbers $\tilde{\eta}_t$ and $\tilde{\eta}'_t$, with $t \in \{1, 2, \ldots, n\}$, if each $\tilde{\eta}_t \leq \tilde{\eta}'_t$, then

$$mFYWA_\Upsilon(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) \leq mFYWA_\Upsilon(\tilde{\eta}'_1, \tilde{\eta}'_2, \ldots, \tilde{\eta}'_n). \tag{2.5}$$

Proof. It is straightforward by Definition 2.4 and Theorem 2.2. □

Theorem 2.4. (Idempotency) For a collection of mF numbers which are 'n' in number given as $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ such that $\tilde{\eta}_t = \tilde{\eta}$, we get

$$mFYWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \tilde{\eta}. \tag{2.6}$$

The proof of this theorem is provided in Appendix B.

Theorem 2.5. (Boundedness) For a set of 'n' mF numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$, if $\tilde{\eta}^l = \bigcap_{t=1}^{n}(\eta_t)$ and $\tilde{\eta}^u = \bigcup_{t=1}^{n}(\eta_t)$, then

$$\tilde{\eta}^l \leq mFYWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) \leq \tilde{\eta}^u. \tag{2.7}$$

Proof. Its proof is easily followed by Definition 2.4 and Theorem 2.2. □

Now we discuss the notion of mFYOWA operators with some basic results.

Definition 2.5. For a collection of mF numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$, $t = 1, 2, \ldots, n$, an mFYOWA operator is a function $m\text{FYOWA}_{\Upsilon} : \tilde{\eta}^n \to \tilde{\eta}$, which is given by

$$mFYOWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \bigoplus_{t=1}^{n}(\Upsilon_t \tilde{\eta}_{\varsigma(t)}), \tag{2.8}$$

where $\Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n)^T$ is the weight-vector and $\Upsilon_t \in (0, 1]$ with $\sum_{t=1}^{n} \Upsilon_t = 1$. $\varsigma(t)$, $(t = 1, 2, \ldots, n)$ represents the permutation, for which $\tilde{\eta}_{\varsigma(t-1)} \geq \tilde{\eta}_{\varsigma(t)}$.

Theorem 2.6. For a set of mF numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ with $t = 1, 2, \ldots, n$, an accumulated value of these mF numbers by utilizing the mFYOWA operator is provided by

$$\begin{aligned} mFYOWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) &= \bigoplus_{t=1}^{n}(\Upsilon_t \tilde{\eta}_{\varsigma(t)}) \\ &= \left(\sqrt{\min\Big(1, (\sum_{t=1}^{n} \Upsilon_t (\mathfrak{p}_1 \circ \eta_{\varsigma(t)})^{2\sigma})^{\frac{1}{\sigma}}\Big)}, \ldots, \sqrt{\min\Big(1, (\sum_{t=1}^{n} \Upsilon_t (\mathfrak{p}_m \circ \eta_{\varsigma(t)})^{2\sigma})^{\frac{1}{\sigma}}\Big)} \right). \end{aligned} \tag{2.9}$$

Proof. It is similar to the proof of Theorem 2.2. □

Example 2.2. Let $\tilde{\eta}_1 = (0.5, 0.4, 0.7, 0.6, 0.2)$, $\tilde{\eta}_2 = (0.1, 0.5, 0.4, 0.3, 0.6)$ and $\tilde{\eta}_3 = (0.6, 0.2, 0.4, 0.3, 0.7)$ be three 5-polar fuzzy numbers with weights $\Upsilon = (0.5, 0.3, 0.2)^T$. Then, for $\sigma = 4$, by Definition 2.2, we calculate the scores as below:

$$\mathfrak{S}(\tilde{\eta}_1) = \frac{0.5 + 0.4 + 0.7 + 0.6 + 0.2}{5} = 0.48, \qquad \mathfrak{S}(\tilde{\eta}_2) = \frac{0.1 + 0.5 + 0.4 + 0.3 + 0.6}{5} = 0.38,$$

$$\mathfrak{S}(\tilde{\eta}_3) = \frac{0.6 + 0.2 + 0.4 + 0.3 + 0.7}{5} = 0.44.$$

This implies $\mathfrak{S}(\tilde{\eta}_3) > \mathfrak{S}(\tilde{\eta}_1) > \mathfrak{S}(\tilde{\eta}_2)$; therefore,

$$\tilde{\eta}_{\varsigma(1)} = \tilde{\eta}_1 = (0.5, 0.4, 0.7, 0.6, 0.2), \qquad \tilde{\eta}_{\varsigma(2)} = \tilde{\eta}_3 = (0.6, 0.2, 0.4, 0.3, 0.7),$$
$$\tilde{\eta}_{\varsigma(3)} = \tilde{\eta}_2 = (0.1, 0.5, 0.4, 0.3, 0.6).$$

Then, from Definition 2.5,

$$\begin{aligned}
mFYOWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3) &= \bigoplus_{t=1}^{3} (\Upsilon_t \tilde{\eta}_{\varsigma(t)}), \\
&= \left(\sqrt{\min\left(1, (\sum_{t=1}^{3} \Upsilon_t (\mathfrak{p}_1 \circ \eta_{\varsigma(t)})^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\sum_{t=1}^{3} \Upsilon_t (\mathfrak{p}_5 \circ \eta_{\varsigma(t)})^{2\sigma})^{\frac{1}{\sigma}}\right)} \right), \\
&= \Bigg(\sqrt{\min\left(1, (0.5 \times (0.5)^8 + 0.3 \times (0.6)^8 + 0.2 \times (0.1)^8\right)}, \\
&\quad \sqrt{\min\left(1, (0.5 \times (0.4)^8 + 0.3 \times (0.2)^8 + 0.2 \times (0.5)^8\right)}, \\
&\quad \sqrt{\min\left(1, (0.5 \times (0.7)^8 + 0.3 \times (0.4)^8 + 0.2 \times (0.4)^8\right)}, \\
&\quad \sqrt{\min\left(1, (0.5 \times (0.6)^8 + 0.3 \times (0.3)^8 + 0.2 \times (0.3)^8\right)}, \\
&\quad \sqrt{\min\left(1, (0.5 \times (0.2)^8 + 0.3 \times (0.7)^8 + 0.2 \times (0.6)^8\right)} \Bigg), \\
&= (0.5377, 0.4272, 0.6428, 0.5505, 0.6157).
\end{aligned}$$

Remark 2.1. The mFYOWA operators verify different basic laws such as monotonicity, idempotency and boundedness as given by Theorems 2.3–2.5.

Theorem 2.7. (Abelian Property) For every two sets of mF numbers $\tilde{\eta}_t$ and $\tilde{\eta}'_t$ with $t \in \{1, 2, \ldots, n\}$, we have

$$mFYOWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = mFYOW\eta_{\Upsilon}(\tilde{\eta}'_1, \tilde{\eta}'_2, \ldots, \tilde{\eta}'_n); \tag{2.10}$$

here $\tilde{\eta}'_t$ serves as an arbitrary permutation of $\tilde{\eta}_t$.

Proof. Its proof is straightforward by Definition 2.5 and Theorem 2.6. □

From the above theory of arithmetic AOs (mFYWA and mFYOWA operators), we deduce that they efficiently aggregate mF numbers, but the first type of AOs do not consider ordering while the second type of AOs consider the ordering of mF numbers. In what follows, we provide a new kind of AOs, namely, the mFYHA operator, which keeps the characteristics of mFYWA and mFYOWA operators.

Definition 2.6. For a set of mF numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \mathfrak{p}_2 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ where $t \in \{1, 2, \ldots, n\}$, an mFYHA operator is provided by

$$mFYHA_{\Upsilon,\Omega}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \bigoplus_{t=1}^{n} (\Upsilon_t \tilde{\tilde{\eta}}_{\varsigma(t)}), \tag{2.11}$$

where $\Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n)^T$ is the weight-vector corresponding to the mF numbers $\tilde{\eta}_t$ with the following conditions: $\Upsilon_t \in (0, 1]$, $\sum_{t=1}^{n} \Upsilon_t = 1$ and $\tilde{\tilde{\eta}}_{\varsigma(t)}$ represents the jth biggest mF numbers such that $\tilde{\tilde{\eta}}_{\varsigma(t)} = (n\Omega_t)\tilde{\eta}_t, t \in \{1, 2, \ldots, n\}$, where $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^T$ is another weight-vector with $\Omega_t \in (0, 1]$, $\sum_{t=1}^{n} \Omega_t = 1$.

Notice that, when $\Upsilon = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, the mFYHA operator converts into the mFYWA operator. If $\Omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then the mFYHA operator becomes the mFYOWA operator. Thus, mFYHA operators investigate the mF degrees and ordering of mF numbers as an extension of both AOs, i.e., the mFYWA and mFYOWA operators.

Theorem 2.8. For a set of mF numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ with $t \in \{1, 2, \ldots, n\}$, an accumulated value of these mF numbers with the help of mFYHA operators is given by

$$mFYHA_{\Upsilon,\Omega}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \bigoplus_{t=1}^{n} (\Upsilon_t \tilde{\tilde{\eta}}_{\varsigma(t)})$$

$$= \left(\sqrt{\min\left(1, (\sum_{t=1}^{n} \Upsilon_t (\mathfrak{p}_1 \circ \tilde{\tilde{\eta}}_{\varsigma(t)})^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\sum_{t=1}^{n} \Upsilon_t (\mathfrak{p}_m \circ \tilde{\tilde{\eta}}_{\varsigma(t)})^{2\sigma})^{\frac{1}{\sigma}}\right)} \right). \tag{2.12}$$

Proof. It is similar to the proof of Theorem 2.2. □

Example 2.3. Let $\tilde{\eta}_1 = (0.4, 0.7, 0.3, 0.5)$, $\tilde{\eta}_2 = (0.3, 0.4, 0.2, 0.6)$, $\tilde{\eta}_3 = (0.7, 0.3, 0.4, 0.1)$ and $\tilde{\eta}_4 = (0.5, 0.6, 0.8, 0.7)$ be 4-polar fuzzy (4F) numbers with $\Upsilon = (0.4, 0.1, 0.2, 0.3)^T$, a weight-vector corresponding to these available 4F numbers and another weight-vector $\Omega = (0.2, 0.1, 0.3, 0.4)^T$. Then, using Definition 2.6, when $\sigma = 4$,

$$\begin{aligned}
\tilde{\tilde{\eta}}_1 &= \left(\sqrt{\min\left(1, (n\Omega_1(\mathfrak{p}_1 \circ \eta_1)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (n\Omega_1(\mathfrak{p}_4 \circ \eta_1)^{2\sigma})^{\frac{1}{\sigma}}\right)} \right), \\
&= \Bigg(\sqrt{\min\left(1, (4 \times 0.2 \times (0.4)^8)^{1/4}\right)}, \sqrt{\min\left(1, (4 \times 0.2 \times (0.7)^8)^{1/4}\right)}, \\
&\quad \sqrt{\min\left(1, (4 \times 0.2 \times (0.3)^8)^{1/4}\right)}, \sqrt{\min\left(1, (4 \times 0.2 \times (0.5)^8)^{1/4}\right)} \Bigg), \\
&= (0.3890, 0.6807, 0.2917, 0.4862).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{\tilde{\eta}}_2 &= \Bigg(\sqrt{\min\left(1, (4 \times 0.1 \times (0.3)^8)^{1/4}\right)}, \sqrt{\min\left(1, (4 \times 0.1 \times (0.4)^8)^{1/4}\right)}, \\
&\quad \sqrt{\min\left(1, (4 \times 0.1 \times (0.2)^8)^{1/4}\right)}, \sqrt{\min\left(1, (4 \times 0.1 \times (0.6)^8)^{1/4}\right)} \Bigg), \\
&= (0.2675, 0.3567, 0.1784, 0.5351),
\end{aligned}$$

$$\tilde{\tilde{\eta}}_3 = \Bigg(\sqrt{\min\left(1, (4 \times 0.3 \times (0.7)^8)^{1/4}\right)}, \sqrt{\min\left(1, (4 \times 0.3 \times (0.3)^8)^{1/4}\right)},$$
$$\sqrt{\min\left(1, (4 \times 0.3 \times (0.4)^8)^{1/4}\right)}, \sqrt{\min\left(1, (4 \times 0.3 \times (0.1)^8)^{1/4}\right)} \Bigg),$$
$$= (0.7161, 0.3069, 0.4092, 0.1023),$$

and

$$\tilde{\tilde{\eta}}_4 = \Bigg(\sqrt{\min\left(1, (4 \times 0.4 \times (0.5)^8)^{1/4}\right)}, \sqrt{\min\left(1, (4 \times 0.4 \times (0.6)^8)^{1/4}\right)},$$
$$\sqrt{\min\left(1, (4 \times 0.4 \times (0.8)^8)^{1/4}\right)}, \sqrt{\min\left(1, (4 \times 0.4 \times (0.7)^8)^{1/4}\right)} \Bigg),$$
$$= (0.5303, 0.6363, 0.8484, 0.7424).$$

Now the scores of mF numbers for $\sigma = 4$ are determined by

$$\mathfrak{S}(\tilde{\tilde{\eta}}_1) = \frac{0.3890 + 0.6807 + 0.2917 + 0.4862}{4} = 0.4619,$$
$$\mathfrak{S}(\tilde{\tilde{\eta}}_2) = \frac{0.2675 + 0.3567 + 0.1784 + 0.5351}{4} = 0.3344,$$
$$\mathfrak{S}(\tilde{\tilde{\eta}}_3) = \frac{0.7161 + 0.3069 + 0.4092 + 0.1023}{4} = 0.3836,$$
$$\mathfrak{S}(\tilde{\tilde{\eta}}_4) = \frac{0.5303 + 0.6363 + 0.8484 + 0.7424}{4} = 0.6893.$$

Since, $\mathfrak{S}(\tilde{\tilde{\eta}}_4) > \mathfrak{S}(\tilde{\tilde{\eta}}_1) > \mathfrak{S}(\tilde{\tilde{\eta}}_3) > \mathfrak{S}(\tilde{\tilde{\eta}}_2)$, thus

$$\tilde{\tilde{\eta}}_{\varsigma(1)} = \tilde{\tilde{\eta}}_4 = (0.5303, 0.6363, 0.8484, 0.7424), \qquad \tilde{\tilde{\eta}}_{\varsigma(2)} = \tilde{\tilde{\eta}}_1 = (0.3890, 0.6807, 0.2917, 0.4862),$$
$$\tilde{\tilde{\eta}}_{\varsigma(3)} = \tilde{\tilde{\eta}}_3 = (0.7161, 0.3069, 0.4092, 0.1023), \qquad \tilde{\tilde{\eta}}_{\varsigma(4)} = \tilde{\tilde{\eta}}_2 = (0.2675, 0.3567, 0.1784, 0.5351).$$

Then, from Theorem 2.8,

$$mFYHA_{\Upsilon,\Omega}(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4) = \bigoplus_{t=1}^{4} (\Upsilon_t \tilde{\tilde{\eta}}_{\varsigma(t)})$$
$$= \left(\sqrt{\min\left(1, (\sum_{t=1}^{4} \Upsilon_t (\mathfrak{p}_1 \circ \tilde{\tilde{\eta}}_{\varsigma(t)})^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\sum_{t=1}^{4} \Upsilon_t (\mathfrak{p}_4 \circ \tilde{\tilde{\eta}}_{\varsigma(t)})^{2\sigma})^{\frac{1}{\sigma}}\right)} \right),$$
$$= \Bigg(\sqrt{\min\left(1, (0.4 \times (0.5303)^8 + 0.1 \times (0.3890)^8 + 0.2 \times (0.7161)^8 + 0.3 \times (0.2675)^8)^{1/4}\right)},$$
$$\sqrt{\min\left(1, (0.4 \times (0.6363)^8 + 0.1 \times (0.6807)^8 + 0.2 \times (0.3069)^8 + 0.3 \times (0.3567)^8)^{1/4}\right)},$$
$$\sqrt{\min\left(1, (0.4 \times (0.8484)^8 + 0.1 \times (0.2917)^8 + 0.2 \times (0.4092)^8 + 0.3 \times (0.1784)^8)^{1/4}\right)},$$
$$\sqrt{\min\left(1, (0.4 \times (0.7424)^8 + 0.1 \times (0.4862)^8 + 0.2 \times (0.1023)^8 + 0.3 \times (0.5351)^8)^{1/4}\right)} \Bigg),$$
$$= (0.5982, 0.5938, 0.7567, 0.6671).$$

2.2. mF Yager geometric AOs

In what follows, some other kinds of geometric AOs are presented under the conditions of Yager's operations on *m*F information, and they are *m*FYWG, *m*FYOWG and *m*FYHG operators.

Definition 2.7. For a set of *m*F numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \mathfrak{p}_2 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$, $t = 1, 2, \ldots, n$, a mapping *m*FYWG : $\tilde{\eta}^n \to \tilde{\eta}$ is called the *m*FYWG operator, which is given by

$$mFYWG_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \bigotimes_{t=1}^{n} (\tilde{\eta}_j)^{\Upsilon_t}, \tag{2.13}$$

where $\Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n)^T$ is the weight-vector, with $\sum_{t=1}^{n} \Upsilon_t = 1,\ \Upsilon_t \in (0, 1]$.

Theorem 2.9. For a set of *m*F numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ with $t \in \{1, 2, \ldots, n\}$, an accumulated value of the given *m*F numbers with the help of *m*FYWG operators is provided by

$$mFYWG_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \bigotimes_{t=1}^{n} (\tilde{\eta}_t)^{\Upsilon_t},$$
$$= \left(\sqrt{1 - \min\left(1, \left(\sum_{t=1}^{n} (\Upsilon_t (1 - (\mathfrak{p}_1 \circ \eta_t)^2)^{\sigma})\right)^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{1 - \min\left(1, \left(\sum_{t=1}^{n} (\Upsilon_t (1 - (\mathfrak{p}_m \circ \eta_t)^2)^{\sigma})\right)^{\frac{1}{\sigma}}\right)} \right). \tag{2.14}$$

Proof. It is similar to the proof of Theorem 2.2. □

Example 2.4. Suppose that $\tilde{\eta}_1 = (0.3, 0.7, 0.5)$, $\tilde{\eta}_2 = (0.8, 0.9, 0.6)$, $\tilde{\eta}_3 = (0.4, 0.3, 0.1)$ and $\tilde{\eta}_4 = (0.5, 0.4, 0.8)$ be 3F numbers with the weight-vector $\Upsilon = (0.2, 0.4, 0.1, 0.3)^T$. For $\sigma = 4$, we get

$$mFYWG_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3) = \bigotimes_{t=1}^{4} (\tilde{\eta}_t)^{\Upsilon_t},$$
$$= \left(\sqrt{1 - \min\left(1, \left(\sum_{t=1}^{4} (\Upsilon_t (1 - (\mathfrak{p}_1 \circ \eta_t)^2)^{\sigma})\right)^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{1 - \min\left(1, \left(\sum_{t=1}^{4} (\Upsilon_t (1 - (\mathfrak{p}_3 \circ \eta_t)^2)^{\sigma})\right)^{\frac{1}{\sigma}}\right)} \right),$$
$$= \left(\sqrt{1 - \min\left(1, (0.2 \times (1 - (0.3)^2)^4 + 0.4 \times (1 - (0.8)^2)^4 + 0.1 \times (1 - (0.4)^2)^4 + 0.3 \times (1 - (0.5)^2)^4\right)^{1/4}}, \right.$$
$$\sqrt{1 - \min\left(1, (0.2 \times (1 - (0.7)^2)^4 + 0.4 \times (1 - (0.9)^2)^4 + 0.1 \times (1 - (0.3)^2)^4 + 0.3 \times (1 - (0.4)^2)^4\right)^{1/4}},$$
$$\left. \sqrt{1 - \min\left(1, (0.2 \times (1 - (0.5)^2)^4 + 0.4 \times (1 - (0.6)^2)^4 + 0.1 \times (1 - (0.1)^2)^4 + 0.3 \times (1 - (0.8)^2)^4\right)^{1/4}} \right),$$
$$= (0.5168, 0.5532, 0.5535).$$

One can easily prove from the above discussion that the *m*FYWG operators hold the following properties. So, we omit their proofs.

Theorem 2.10. (Idempotent Property) Let $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ be a set of '*n*' equal *m*F numbers, that is, $\tilde{\eta}_t = \tilde{\eta}$; then,

$$mFYWG_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \tilde{\eta}. \tag{2.15}$$

Theorem 2.11. (Bounded Property) Let $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ be a set of 'n' mF numbers, $\tilde{\eta}^l = \bigcap_{t=1}^{n}(\eta_t)$ and $\tilde{\eta}^u = \bigcup_{t=1}^{n}(\eta_t)$; then,

$$\tilde{\eta}^l \leq mFYWG_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) \leq \tilde{\eta}^u. \tag{2.16}$$

Theorem 2.12. (Monotone Property) For every two arbitrary sets of mF numbers $\tilde{\eta}_t$ and $\tilde{\eta}'_t$ with $t \in \{1, 2, \ldots, n\}$, if $\tilde{\eta}_t \leq \tilde{\eta}'_t$, then

$$mFYWG_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) \leq mFYWG_{\Upsilon}(\tilde{\eta}'_1, \tilde{\eta}'_2, \ldots, \tilde{\eta}'_n). \tag{2.17}$$

We now present some new mFYOWG operators as below:

Definition 2.8. For a set of mF numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \mathfrak{p}_2 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ with $t \in \{1, 2, \ldots, n\}$, an mFYOWG operator is a mapping mFYOWG : $\tilde{\eta}^n \to \tilde{\eta}$, which is given as:

$$mFYOWG_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \bigotimes_{t=1}^{n} (\tilde{\eta}_{\varsigma(t)})^{\Upsilon_t} \tag{2.18}$$

where $\Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n)^T$ is the weight-vector and $\Upsilon_t \in (0, 1]$ with $\sum_{t=1}^{n} \Upsilon_t = 1$. Here $\varsigma(t)$ with $(t = 1, 2, \ldots, n)$ serves as an arbitrary permutation which satisfies $\tilde{\eta}_{\varsigma(t-1)} \geq \tilde{\eta}_{\varsigma(t)}$.

Theorem 2.13. For a set of mF numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ with $t \in \{1, 2, \ldots, n\}$, an accumulated value of the given mF numbers with the help of mFYOWG operators is computed by

$$\begin{aligned} mFDOWG_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) &= \bigotimes_{t=1}^{n} (\tilde{\eta}_{\varsigma(t)})^{\Upsilon_t} \\ &= \left(\sqrt{1 - \min\left(1, \left(\sum_{t=1}^{n} (\Upsilon_t (1 - (\mathfrak{p}_1 \circ \eta_{\varsigma(t)})^2)^{\sigma}) \right)^{\frac{1}{\sigma}} \right)}, \ldots, \sqrt{1 - \min\left(1, \left(\sum_{t=1}^{n} (\Upsilon_t (1 - (\mathfrak{p}_m \circ \eta_{\varsigma(t)})^2)^{\sigma}) \right)^{\frac{1}{\sigma}} \right)} \right). \end{aligned} \tag{2.19}$$

Example 2.5. Let $\tilde{\eta}_1 = (0.4, 0.6, 0.2, 0.3)$, $\tilde{\eta}_2 = (0.4, 0.7, 0.2, 0.7)$, $\tilde{\eta}_3 = (0.5, 0.1, 0.6, 0.9)$ and $\tilde{\eta}_4 = (0.3, 0.9, 0.6, 0.4)$ be 4F numbers and $\Upsilon = (0.3, 0.4, 0.2, 0.1)^T$ be a weight-vector. Then, the score values of these 4F numbers for $\sigma = 5$ is calculated as:

$$S(\tilde{\eta}_1) = \frac{0.4 + 0.6 + 0.2 + 0.3}{4} = 0.375, \quad S(\tilde{\eta}_2) = \frac{0.4 + 0.7 + 0.2 + 0.7}{4} = 0.45,$$
$$S(\tilde{\eta}_3) = \frac{0.5 + 0.1 + 0.6, 0.9}{4} = 0.525, \quad S(\tilde{\eta}_4) = \frac{0.3 + 0.9 + 0.6 + 0.4}{4} = 0.55.$$

Since, $S(\tilde{\eta}_4) > S(\tilde{\eta}_3) > S(\tilde{\eta}_2) > S(\tilde{\eta}_1)$, thus

$$\tilde{\eta}_{\varsigma(1)} = \tilde{\eta}_4 = (0.3, 0.9, 0.6, 0.4), \quad \tilde{\eta}_{\varsigma(2)} = \tilde{\eta}_3 = (0.5, 0.1, 0.6, 0.9),$$
$$\tilde{\eta}_{\varsigma(3)} = \tilde{\eta}_2 = (0.4, 0.7, 0.2, 0.7), \quad \tilde{\eta}_{\varsigma(4)} = \tilde{\eta}_1 = (0.4, 0.6, 0.2, 0.3).$$

Then, from Definition 2.8,

$$mFYOWG_{\Upsilon}(\tilde{\eta}_1,\tilde{\eta}_2,\tilde{\eta}_3,\tilde{\eta}_4)=\bigotimes_{t=1}^{4}(\tilde{\eta}_{\varsigma(t)})^{\Upsilon_t},$$

$$=\left(\sqrt{1-\min\left(1,\left(\sum_{t=1}^{4}(\Upsilon_t(1-(\mathfrak{p}_1\circ\eta_{\varsigma(t)})^2)^{\sigma})\right)^{\frac{1}{\sigma}}\right)},\ldots,\sqrt{1-\min\left(1,\left(\sum_{t=1}^{4}(\Upsilon_t(1-(\mathfrak{p}_m\circ\eta_{\varsigma(t)})^2)^{\sigma})\right)^{\frac{1}{\sigma}}\right)}\right),$$

$$=\Bigg(\sqrt{1-\min\left(1,(0.3\times(1-(0.3)^2)^5+0.4\times(1-(0.5)^2)^5+0.2\times(1-(0.4)^2)^5+0.1\times(1-(0.4)^2)^5)^{1/5}\right)},$$

$$\sqrt{1-\min\left(1,(0.3\times(1-(0.9)^2)^5+0.4\times(1-(0.1)^2)^5+0.2\times(1-(0.7)^2)^5+0.1\times(1-(0.6)^2)^5)^{1/5}\right)},$$

$$\sqrt{1-\min\left(1,(0.3\times(1-(0.6)^2)^5+0.4\times(1-(0.6)^2)^5+0.2\times(1-(0.2)^2)^5+0.1\times(1-(0.2)^2)^5)^{1/5}\right)},$$

$$\sqrt{1-\min\left(1,(0.3\times(1-(0.4)^2)^5+0.4\times(1-(0.9)^2)^5+0.2\times(1-(0.7)^2)^5+0.1\times(1-(0.3)^2)^5)^{1/5}\right)}\Bigg),$$

$$=(0.4054,0.4102,0.4516,0.5282).$$

Remark 2.2. The *m*FYOWG operators verify different basic laws such as monotonicity, idempotency and boundedness as given by Theorems 2.10–2.12.

Theorem 2.14. (Commutativity Property) For every two arbitrary sets of *m*F numbers $\tilde{\eta}_t$ and $\tilde{\eta}'_t$ with $t\in\{1,2,\ldots,n\}$, if $\tilde{\eta}_t\leq\tilde{\eta}'_t$, then

$$mFYOWG_{\Upsilon}(\tilde{\eta}_1,\tilde{\eta}_2,\ldots,\tilde{\eta}_n)=mFYOWG_{\Upsilon}(\tilde{\eta}'_1,\tilde{\eta}'_2,\ldots,\tilde{\eta}'_n),\tag{2.20}$$

where $\tilde{\eta}'_t$ is any permutation of $\tilde{\eta}_t$.

Proof. Its proof is obvious by Definition 2.8 and Theorem 2.13. □

From Definitions 2.4 and 2.5, we conclude that *m*FYWG and *m*FYOWG operators are useful to efficiently aggregate *m*F numbers. The only difference is that *m*FYWG operators only aggregate *m*F information without considering the ordering of *m*F numbers while *m*FYOWG operators consider their ordering. We now present another general type of AOs called *m*FYHG operators, which keep the features of *m*FYWG and *m*FYOWG operators.

Definition 2.9. For a set of *m*F numbers $\tilde{\eta}_t=(\mathfrak{p}_1\circ\eta_t,\mathfrak{p}_2\circ\eta_t,\ldots,\mathfrak{p}_m\circ\eta_t)$ with $t\in\{1,2,\ldots,n\}$, an *m*FYHG operator is given as:

$$mFYHG_{\Upsilon,\Omega}(\tilde{\eta}_1,\tilde{\eta}_2,\ldots,\tilde{\eta}_n)=\bigotimes_{t=1}^{n}(\tilde{\tilde{\eta}}_{\varsigma(t)})^{\Upsilon_t},\tag{2.21}$$

where $\Upsilon=(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_n)^T$ denotes the weights associated with the *m*F numbers $\tilde{\eta}_t$, $t=1,2,\ldots,n$, $\Upsilon_t\in(0,1]$, $\sum_{t=1}^{n}\Upsilon_t=1$ and $\tilde{\tilde{\eta}}_{\varsigma(t)}$ represents the j-th largest *m*F numbers such that $\tilde{\tilde{\eta}}_{\varsigma(t)}=(n\Omega_t)\tilde{\eta}_t,(t=1,2,\ldots,n)$, $\Omega=(\Omega_1,\Omega_2,\ldots,\Omega_n)$ is a weight-vector with $\Omega_t\in(0,1]$, $\sum_{t=1}^{n}\Omega_t=1$.

Notice that, when $\Upsilon = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, the mFYHG operator becomes the mFYWG operator. When $\Omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, the mFYHG operators convert into mFYOWG operators. Thus, mFYHG operators are an extension of mFYWG and mFYOWG operators.

Theorem 2.15. For a set of mF numbers $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t)$ with $t \in \{1, 2, \ldots, n\}$, an accumulated value of the given mF numbers with the help of mFYHG operator is given by

$$mFYHG_{\Upsilon,\Omega}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \bigotimes_{t=1}^{n} (\tilde{\tilde{\eta}}_{\varsigma(t)})^{\Upsilon_t}$$
$$= \left(\sqrt{1 - \min\left(1, \left(\sum_{t=1}^{n} (\Upsilon_t (1 - (\mathfrak{p}_1 \circ \tilde{\tilde{\eta}}_{\varsigma(t)})^2)^{\sigma})\right)^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{1 - \min\left(1, \left(\sum_{t=1}^{n} (\Upsilon_t (1 - (\mathfrak{p}_m \circ \tilde{\tilde{\eta}}_{\varsigma(t)})^2)^{\sigma})\right)^{\frac{1}{\sigma}}\right)} \right). \quad (2.22)$$

Proof. It is similar to the proof of Theorem 2.2 via a mathematical induction method. □

Example 2.6. Let $\tilde{\eta}_1 = (0.7, 0.9, 0.8)$, $\tilde{\eta}_2 = (0.6, 0.5, 0.7)$, $\tilde{\eta}_3 = (0.9, 0.8, 0.4)$ and $\tilde{\eta}_4 = (0.5, 0.4, 0.5)$ be 3F numbers, and $\Upsilon = (0.1, 0.2, 0.4, 0.3)^T$ be an associated weight-vector and $\Omega = (0.2, 0.3, 0.4, 0.1)^T$ be another weight-vector. Then, using Definition 2.9, for $\sigma = 4$

$$\tilde{\tilde{\eta}}_1 = \left(\sqrt{1 - \min\left(1, (n\Omega_1 (1 - (\mathfrak{p}_1 \circ \eta_1)^2)^{\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{1 - \min\left(1, (n\Omega_1 (1 - (\mathfrak{p}_3 \circ \eta_1)^2)^{\sigma})^{\frac{1}{\sigma}}\right)} \right),$$
$$= \left(\sqrt{1 - \min\left(1, (4 \times 0.2 \times (1 - (0.7)^2)^4)^{1/4}\right)}, \sqrt{1 - \min\left(1, (4 \times 0.2 \times (1 - (0.9)^2)^4)^{1/4}\right)}, \sqrt{1 - \min\left(1, (4 \times 0.2 \times (1 - (0.8)^2)^4)^{1/4}\right)} \right),$$
$$= (0.7195, 0.9057, 0.8121).$$

Similarly,

$$\tilde{\tilde{\eta}}_2 = \left(\sqrt{1 - \min\left(1, (4 \times 0.3 \times (1 - (0.6)^2)^4)^{1/4}\right)}, \sqrt{1 - \min\left(1, (4 \times 0.3 \times (1 - (0.5)^2)^4)^{1/4}\right)}, \sqrt{1 - \min\left(1, (4 \times 0.3 \times (1 - (0.7)^2)^4)^{1/4}\right)} \right),$$
$$= (0.5746, 0.4637, 0.6828),$$

$$\tilde{\tilde{\eta}}_3 = \left(\sqrt{1 - \min\left(1, (4 \times 0.4 \times (1 - (0.9)^2)^4)^{1/4}\right)}, \sqrt{1 - \min\left(1, (4 \times 0.4 \times (1 - (0.8)^2)^4)^{1/4}\right)}, \sqrt{1 - \min\left(1, (4 \times 0.4 \times (1 - (0.4)^2)^4)^{1/4}\right)} \right),$$
$$= (0.8867, 0.7714, 0.2351),$$

and

$$\tilde{\tilde{\eta}}_4 = \Bigg(\sqrt{1 - \min\left(1, (4 \times 0.1 \times (1-(0.5)^2)^4)^{1/4}\right)}, \sqrt{1 - \min\left(1, (4 \times 0.1 \times (1-(0.4)^2)^4)^{1/4}\right)}, \sqrt{1 - \min\left(1, (4 \times 0.1 \times (1-(0.5)^2)^4)^{1/4}\right)}\Bigg), = (0.6353, 0.5762, 0.6353).$$

Now the scores of mF numbers for $\sigma = 3$ are computed as below:

$$S(\tilde{\tilde{\eta}}_1) = \frac{0.7195 + 0.9057 + 0.8121}{3} = 0.8124, \quad S(\tilde{\tilde{\eta}}_2) = \frac{0.5746 + 0.4637 + 0.6828}{3} = 0.5737,$$
$$S(\tilde{\tilde{\eta}}_3) = \frac{0.8867 + 0.7714 + 0.2351}{3} = 0.6311, \quad S(\tilde{\tilde{\eta}}_4) = \frac{0.6353 + 0.5762 + 0.6353}{3} = 0.6156.$$

Clearly, $S(\tilde{\tilde{\eta}}_1) > S(\tilde{\tilde{\eta}}_3) > S(\tilde{\tilde{\eta}}_4) > S(\tilde{\tilde{\eta}}_2)$; thus,

$$\tilde{\tilde{\eta}}_{\varsigma(1)} = \tilde{\eta}_1 = (0.7195, 0.9057, 0.8121), \qquad \tilde{\tilde{\eta}}_{\varsigma(2)} = \tilde{\eta}_3 = (0.8867, 0.7714, 0.2351),$$
$$\tilde{\tilde{\eta}}_{\varsigma(3)} = \tilde{\eta}_4 = (0.6353, 0.5762, 0.6353), \qquad \tilde{\tilde{\eta}}_{\varsigma(4)} = \tilde{\eta}_2 = (0.5746, 0.4637, 0.6828).$$

Now by Definition 2.8, we get

$$mFYHG_{\Upsilon,\Omega}(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4) = \bigotimes_{t=1}^{4} (\tilde{\eta}_{\varsigma(t)})^{\Upsilon_t}$$
$$= \left(\sqrt{1 - \min\left(1, \left(\sum_{t=1}^{4} (\Upsilon_t (1 - (\mathfrak{p}_1 \circ \tilde{\tilde{\eta}}_{\varsigma(t)})^2)^{\sigma}) \right)^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{1 - \min\left(1, \left(\sum_{t=1}^{4} (\Upsilon_t (1 - (\mathfrak{p}_3 \circ \tilde{\tilde{\eta}}_{\varsigma(t)})^2)^{\sigma}) \right)^{\frac{1}{\sigma}}\right)} \right),$$
$$= \Bigg(\sqrt{1 - \min\left(1, (0.1 \times (1-(0.7195)^2)^4 + 0.2 \times (1-(0.8867)^2)^4 + 0.4 \times (1-(0.6353)^2)^4 + 0.3 \times (1-(0.5746)^2)^4)^{1/4}\right)},$$
$$\sqrt{1 - \min\left(1, (0.1 \times (1-(0.9057)^2)^4 + 0.2 \times (1-(0.7714)^2)^4 + 0.4 \times (1-(0.5762)^2)^4 + 0.3 \times (1-(0.4637)^2)^4)^{1/4}\right)},$$
$$\sqrt{1 - \min\left(1, (0.1 \times (1-(0.8121)^2)^4 + 0.2 \times (1-(0.2351)^2)^4 + 0.4 \times (1-(0.6353)^2)^4 + 0.3 \times (1-(0.6828)^2)^4)^{1/4}\right)}\Bigg),$$
$$= (0.6445, 0.5763, 0.5507).$$

3. Application to MCDM

In this section, we present an MCDM methodology based on our initiated mF Yager AOs to tackle different real-world MCDM situations involving mF information. The terms used for this purpose are provided in the following subsection.

3.1. Methodology

Let $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ be a universe and $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\}$ be universal set of parameters. Let $\Upsilon = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n\}$ be a weight vector with $\sum_{t=1}^{n} \Upsilon_t = 1,\ \Upsilon_t \in (0, 1],\ \forall\ t \in \{1, 2, \ldots, n\}$. Suppose that an mF decision-matrix $\tilde{\mathfrak{M}} = (\tilde{\mathfrak{d}}_{it})_{k\times n} = (\mathfrak{p}_1 \circ \eta_{it}, \mathfrak{p}_2 \circ \eta_{it}, \ldots, \mathfrak{p}_m \circ \eta_{it})_{k\times n}$, which contains the experts' opinions in the form of membership degrees.

An algorithm is developed to tackle MCDM situations using mFYWA (or mFYWG) operators.

Algorithm: Selection of an appropriate alternative under *m*F Yager AOs

Step I: Input:

$\tilde{\mathfrak{M}}$, an mF decision matrix containing n attributes and k alternatives.

$\Upsilon = (\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n)^T$, the weight vector.

Step II: Utilize the mFYWA operators in the aggregation process of the given datasets in an mF decision matrix $\tilde{\mathfrak{M}}$ and determine the preference values $\tilde{\mathfrak{d}}_r$; here, the variation of 'r' is from 1 to k for given mF numbers η_t.

$$\tilde{\mathfrak{d}}_r = mFYWA_{\Upsilon}(\tilde{\eta}_{r1}, \tilde{\eta}_{r2}, \ldots, \tilde{\eta}_{rn}) = \bigoplus_{t=1}^{n}(\Upsilon_t\tilde{\eta}_{rt})$$
$$= \left(\sqrt{\min\Big(1, (\sum_{t=1}^{n}\Upsilon_t(\mathfrak{p}_1 \circ \tilde{\eta}_{rt})^{2\sigma})^{\frac{1}{\sigma}}\Big)}, \ldots, \sqrt{\min\Big(1, (\sum_{t=1}^{n}\Upsilon_t(\mathfrak{p}_m \circ \tilde{\eta}_{rt})^{2\sigma})^{\frac{1}{\sigma}}\Big)}\right).$$

When we use mFYWG operators, then

$$\tilde{\mathfrak{d}}_r = mFYWG_{\Upsilon}(\tilde{\eta}_{r1}, \tilde{\eta}_{r2}, \ldots, \tilde{\eta}_{rn}) = \bigotimes_{t=1}^{n}(\tilde{\eta}_{rt})^{\Upsilon_t},$$
$$= \left(\sqrt{1 - \min\Big(1, \Big(\sum_{t=1}^{n}(\Upsilon_t(1 - (\mathfrak{p}_1 \circ \tilde{\eta}_{rt})^2)^{\sigma})\Big)^{\frac{1}{\sigma}}\Big)}, \ldots, \sqrt{1 - \min\Big(1, \Big(\sum_{t=1}^{n}(\Upsilon_t(1 - (\mathfrak{p}_m \circ \tilde{\eta}_{rt})^2)^{\sigma})\Big)^{\frac{1}{\sigma}}\Big)}\right).$$

Step III: Determine the scores $\mathfrak{S}(\tilde{\mathfrak{d}}_r)$, where the variation of 'r' is from 1 to k.

Step IV: Write all of the alternatives $\mathcal{S}_r$, $(r = 1, 2, \ldots, k)$ in order in terms of their score values $\mathfrak{S}(\tilde{\mathfrak{d}}_r)$. In the case when the final score values of two alternatives are equal, one can use the accuracy function to find their exact ranking.

Output: An alternative with the highest score in the last step is the decision.

3.2. A case study: Site selection for a new refinery in Pakistan

An oil refinery or petroleum refinery is an industrial process plant where crude oil is processed and refined into more beneficial commodities like liquefied petroleum gas, kerosene, heating oil, asphalt base, petroleum naphtha, diesel fuel and gasoline. A petroleum refinery contains very sensitive and important substances, which is why making a suitable site selection is not an easy task due to the effects of different factors (parameters), including the availability of land, availability of raw water, resources/labor, effluent disposal, natural and geographic conditions of the site, conditions of society (humanities) and economy, conditions of traffic and transportation and conditions of utilities.

The government of Pakistan wants to build a new oil refinery and for this very important project, the first significant thing is site selection because the cost of this project is directly proportional to the site. Therefore, this crucial assignment is given to a team of experts of this domain from the eight areas proposed by the government officials. The proposed alternatives are $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_8$. After few meetings among the experts, they all agreed to evaluate the alternatives under their expertise with the following five common parameters:

$\mathcal{E}_1$ denotes the "Natural Conditions",
$\mathcal{E}_2$ denotes the "Traffic and Transportation Conditions",
$\mathcal{E}_3$ denotes the "Conditions of Utilities",
$\mathcal{E}_4$ denotes the "Cost",
$\mathcal{E}_5$ denotes the "Geographical Conditions".

Some other sub characteristics of these parameters are provided below to better understand the construction of 3F numbers.

- The parameter "Natural Conditions" in the site selection procedure includes temperature, humidity and wind.
- The parameter "Traffic and Transportation Conditions" in the site selection process includes by road, railway and sea.
- The parameter "Conditions of Utilities" includes power supply, availability of raw water and resources/labor.
- The "Cost" includes medium, high and very high.
- The parameter "Geographical Conditions" affects the site selection, and it includes hydro geology, soil type and rock exposure.

The final judgments of experts about the alternatives, as in terms of the favorable parameters, are presented in Table 2 in the form of 3F decision matrix.

Table 2. 3F decision matrix.

–	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$	$\mathcal{E}_5$
$\mathcal{S}_1$	(0.3, 0.5, 0.8)	(0.6, 0.9, 0.5)	(0.8, 0.5, 0.4)	(0.7, 0.4, 0.2)	(0.5, 0.7, 0.3)
$\mathcal{S}_2$	(0.5, 0.7, 0.8)	(0.4, 0.8, 0.7)	(0.6, 0.4, 0.2)	(0.7, 0.6, 0.9)	(0.6, 0.4, 0.7)
$\mathcal{S}_3$	(0.8, 0.4, 0.9)	(0.4, 0.8, 0.4)	(0.7, 0.8, 0.6)	(0.3, 0.5, 0.7)	(0.8, 0.6, 0.5)
$\mathcal{S}_4$	(0.7, 0.4, 0.5)	(0.9, 0.7, 0.6)	(0.8, 0.6, 0.5)	(0.7, 0.4, 0.6)	(0.6, 0.8, 0.5)
$\mathcal{S}_5$	(0.8, 0.5, 0.4)	(0.7, 0.2, 0.5)	(0.9, 0.4, 0.8)	(0.7, 0.9, 0.7)	(0.8, 0.3, 0.6)
$\mathcal{S}_6$	(0.5, 0.7, 0.4)	(0.6, 0.8, 0.7)	(0.8, 0.4, 0.6)	(0.1, 0.8, 0.9)	(0.6, 0.7, 0.3)
$\mathcal{S}_7$	(0.8, 0.3, 0.7)	(0.8, 0.7, 0.3)	(0.5, 0.9, 0.8)	(0.7, 0.6, 0.5)	(0.4, 0.5, 0.8)
$\mathcal{S}_8$	(0.7, 0.6, 0.2)	(0.5, 0.9, 0.1)	(0.8, 0.2, 0.5)	(0.6, 0.7, 0.9)	(0.9, 0.7, 0.5)

In view of the government officials, the team of experts assign a weight-vector to the set of parameters as follows:

$$\Upsilon_1 = 0.24,\ \Upsilon_2 = 0.35,\ \Upsilon_3 = 0.10,\ \Upsilon_4 = 0.25 \text{ and } \Upsilon_5 = 0.06.$$

Since $\sum_{t=1}^{5} \Upsilon_t = 1$. We now compute the most suitable ranking between the available sites for an oil refinery with the help of developed AOs, i.e., the: mFYWA and mFYWG operators:

Step I: When $\sigma = 4$, by implementing the mFYWA operator, we compute the values $\tilde{\mathfrak{d}}_i$ of the alternatives $\mathcal{S}_i,\ i = 1, 2, \ldots, 8$ in terms of the ranking of sites for an oil refinery.

$$\tilde{\mathfrak{d}}_1 = (0.6630, 0.7925, 0.6722), \quad \tilde{\mathfrak{d}}_2 = (0.6063, 0.7256, 0.8022),$$
$$\tilde{\mathfrak{d}}_3 = (0.6980, 0.7265, 0.7671), \quad \tilde{\mathfrak{d}}_4 = (0.8161, 0.6510, 0.5731),$$
$$\tilde{\mathfrak{d}}_5 = (0.7734, 0.7577, 0.6546), \quad \tilde{\mathfrak{d}}_6 = (0.6293, 0.7656, 0.7746),$$
$$\tilde{\mathfrak{d}}_7 = (0.7621, 0.7145, 0.6722), \quad \tilde{\mathfrak{d}}_8 = (0.7064, 0.8028, 0.7574).$$

Step II: Find the score values $\mathfrak{S}(\tilde{\mathfrak{d}}_i)$ of 3F numbers $\tilde{\mathfrak{d}}_i,\ (i = 1, 2, \ldots, 5)$ of the alternatives $\mathcal{S}_i$:

$$\mathfrak{S}(\tilde{\mathfrak{d}}_1) = 0.7092, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_2) = 0.7114, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_3) = 0.7305,$$
$$\mathfrak{S}(\tilde{\mathfrak{d}}_4) = 0.6800, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_5) = 0.7286, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_6) = 0.7232,$$
$$\mathfrak{S}(\tilde{\mathfrak{d}}_7) = 0.7162, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_8) = 0.7555.$$

Step III: Compute the ranking of alternatives using the scores obtained in the previous step: $\mathcal{S}_8 > \mathcal{S}_3 > \mathcal{S}_5 > \mathcal{S}_6 > \mathcal{S}_7 > \mathcal{S}_2 > \mathcal{S}_1 > \mathcal{S}_4$. **Step IV:** The alternative $\mathcal{S}_8$ has the highest score; thus, it is the most suitable suitable site for the construction of an oil refinery in Pakistan.

We now apply the mFYWG operator to compute a suitable option.

Step I: For $\sigma = 4$, by using the mFYWG operator, we find the values $\tilde{\mathfrak{d}}_i$ of the alternatives $\mathcal{S}_i,\ i = 1, 2, \ldots, 8$ in terms of the ranking of sites for an oil refinery.

$$\hat{\mathfrak{d}}_1 = (0.5342, 0.5501, 0.4426), \quad \tilde{\mathfrak{d}}_2 = (0.5135, 0.6199, 0.6443),$$
$$\tilde{\mathfrak{d}}_3 = (0.4762, 0.5640, 0.5564), \quad \tilde{\mathfrak{d}}_4 = (0.7338, 0.5187, 0.5564),$$
$$\tilde{\mathfrak{d}}_5 = (0.7331, 0.4177, 0.5354) \quad \tilde{\mathfrak{d}}_6 = (0.4599, 0.6840, 0.5745),$$
$$\tilde{\mathfrak{d}}_7 = (0.6744, 0.5416, 0.4873) \quad \tilde{\mathfrak{d}}_8 = (0.5977, 0.6174, 0.3510).$$

Step II: Find the score values $\mathfrak{S}(\tilde{\mathfrak{d}}_i)$ of 3F numbers $\tilde{\mathfrak{d}}_i,\ (i = 1, 2, \ldots, 8)$ of the alternatives $\mathcal{S}_i$:

$$\mathfrak{S}(\tilde{\mathfrak{d}}_1) = 0.5090, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_2) = 0.5926, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_3) = 0.5322,$$
$$\mathfrak{S}(\tilde{\mathfrak{d}}_4) = 0.6030, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_5) = 0.5621, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_6) = 0.5728,$$
$$\mathfrak{S}(\tilde{\mathfrak{d}}_7) = 0.5678, \quad \mathfrak{S}(\tilde{\mathfrak{d}}_8) = 0.5220.$$

Step III: Compute the ranking of alternatives using the scores $\mathfrak{S}(\tilde{\mathfrak{d}}_i),\ (i = 1, 2, \ldots, 8)$ determined in previous step: $\mathcal{S}_4 > \mathcal{S}_2 > \mathcal{S}_6 > \mathcal{S}_7 > \mathcal{S}_5 > \mathcal{S}_3 > \mathcal{S}_8 > \mathcal{S}_1$.

Step IV: The alternative $\mathcal{S}_4$ has the highest score; thus, it is the most suitable option for the construction of an oil refinery in Pakistan.

The method used to solve the above MCDM application is displayed in Figure 1.

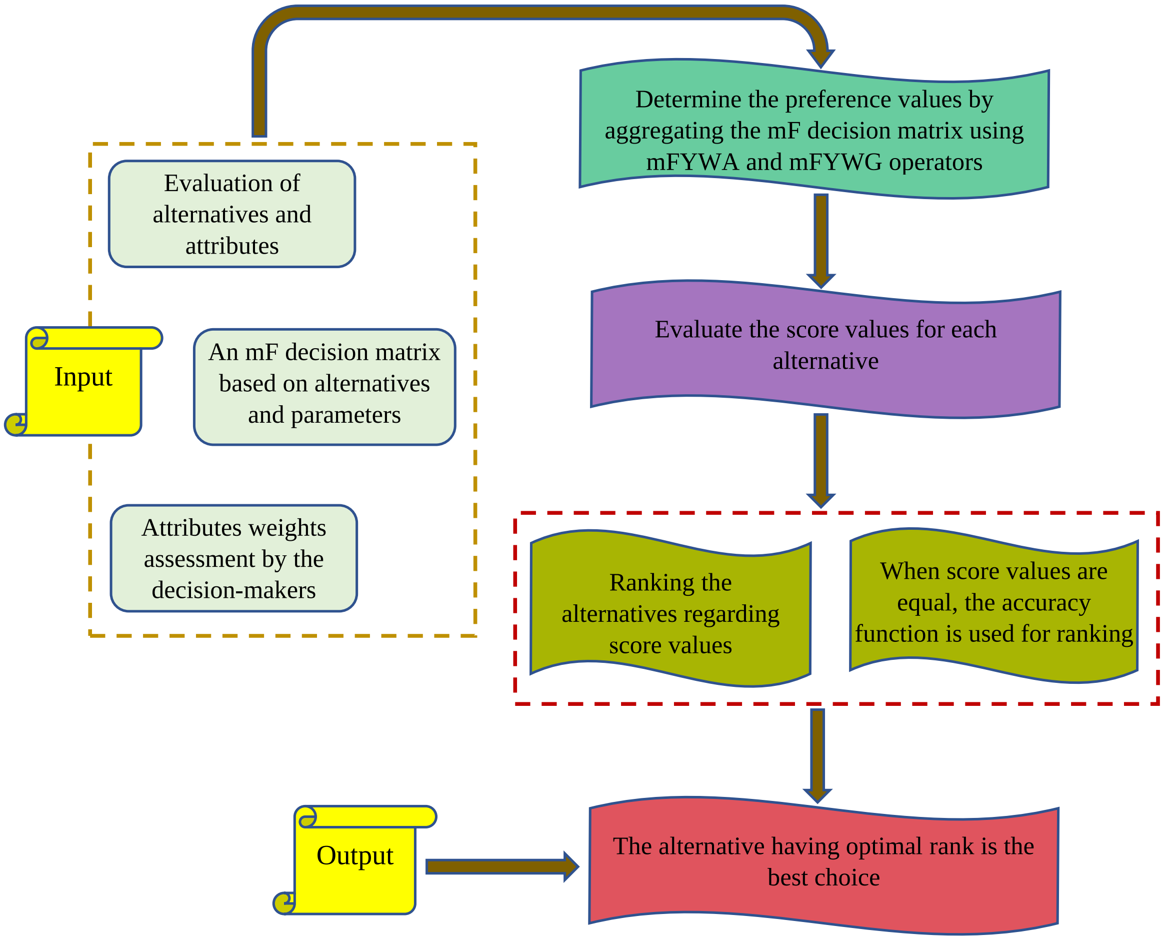

Figure 1. Flowchart diagram.

4. Comparison analysis and discussion

In this section, we give both qualitative and quantitative comparative analyses of the initiated *m*F Yager AOs with *m*F Dombi AOs [26], *m*F Hamacher AOs [24] and some Yager's operation-based AOs to prove their cogency and efficiency. Further, we discuss the validity of the proposed AOs through the use of three effectiveness tests which have been introduced by Wang and Triantaphyllou [45].

4.1. Comparative analysis

To effectively deal with *m*F information, the existing Yager's operation-based AOs, including Fermatean fuzzy Yager AOs [28], complex Pythagorean fuzzy Yager AOs [30] and *q*-rung picture fuzzy Yager AOs [29] are not useful; therefore, *m*F Yager AOs have been proposed in this study. In the literature, *m*F information is aggregated via Dombi and Hamacher TNs and TCoNs with very hard calculations. Yager's operations are simpler than Dombi and Hamacher TNs and TCoNs. This is another reason that has motivated us to select Yager's TN and TCoN in the current work.

We now discuss the comparison between the results of initiated *m*F Yager AOs and existing *m*F Dombi AOs [26] and *m*F Hamacher AOs [24]. For this, we applied these AOs to a daily-life scenario, and the computed results are provided in Tables 3 and 4 (for more detail see Figure 2). From Tables 3 and 4, we can easily see that the optimal object (i.e., $\mathcal{S}_2$) is the same by applying mFHWA and mFHWG [24] operators but it is not similar to the optimal object "$\mathcal{S}_4$" which is obtained by applying the proposed mFYWG operator. Besides, the optimal object (i.e., $\mathcal{S}_8$) is the same by applying existing mFDWA and mFDWG [26] operators and the proposed mFYWA operator. Thus, to deal with mF MCDM situations effectively, our proposed mF Yager AOs are much more versatile and generalized than certain existing MCDM tools, including mFDWA and mFDWG [26] operators, and mFHWA and mFHWG [24] AOs.

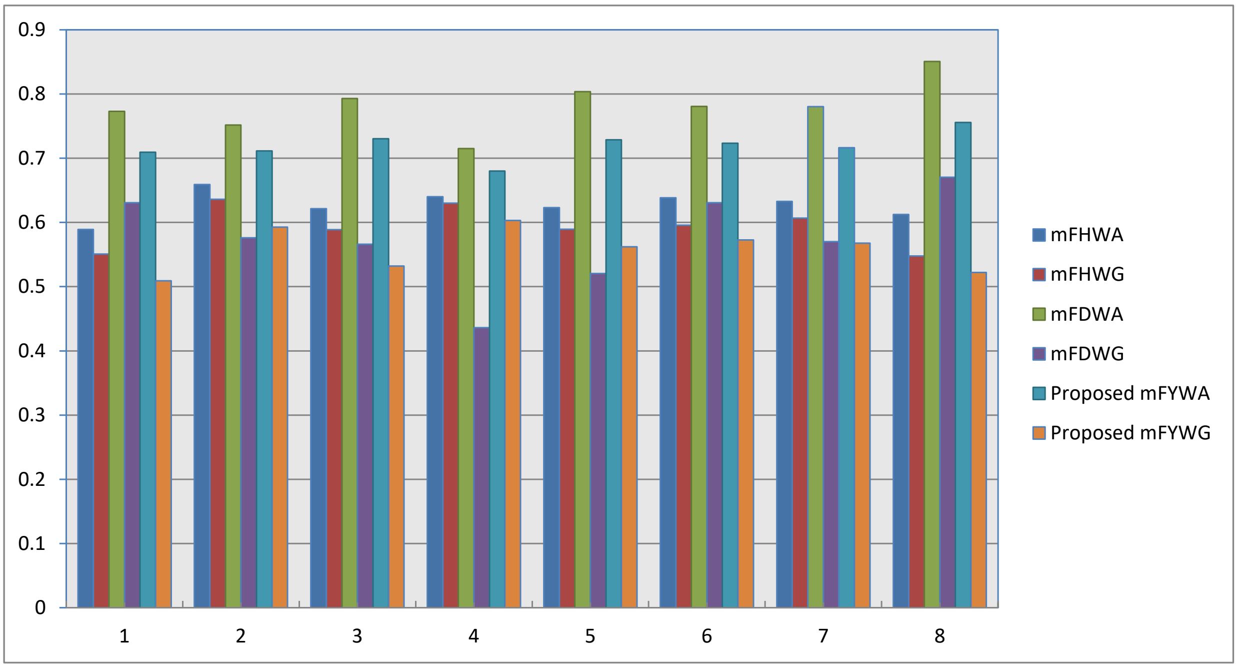

Figure 2. Comparison of mF Yager AOs with existing mF Dombi and Hamacher AOs on the application in Section 3.2.

Table 3. Comparison of mF Yager AOs with mF Dombi AOs [26] and mF Hamacher AOs [24].

Operators	$\mathfrak{S}(\tilde{\mathfrak{d}}_1)$	$\mathfrak{S}(\tilde{\mathfrak{d}}_2)$	$\mathfrak{S}(\tilde{\mathfrak{d}}_3)$	$\mathfrak{S}(\tilde{\mathfrak{d}}_4)$	$\mathfrak{S}(\tilde{\mathfrak{d}}_5)$	$\mathfrak{S}(\tilde{\mathfrak{d}}_6)$	$\mathfrak{S}(\tilde{\mathfrak{d}}_7)$	$\mathfrak{S}(\tilde{\mathfrak{d}}_8)$
mFHWA [24]	0.5892	0.6589	0.6214	0.6402	0.6231	0.6385	0.6328	0.6123
mFHWG [24]	0.5508	0.6361	0.5887	0.6300	0.5895	0.5957	0.6066	0.5479
mFDWA [26]	0.7731	0.7517	0.7930	0.7149	0.8035	0.7808	0.7803	0.8506
mFDWG [26]	0.6306	0.5761	0.5661	0.4361	0.5205	0.6307	0.5700	0.6704
Proposed mFYWA	0.7092	0.7114	0.7305	0.6800	0.7286	0.7232	0.7162	0.7555
Proposed mFYWG	0.5090	0.5926	0.5322	0.6030	0.5621	0.5728	0.5678	0.5220

Table 4. Comparison between the ranking results of mF Yager AOs and mF Dombi AOs [26] and mF Hamacher AOs [24].

Operators	Ranking Order	Best Option
mFHWA [24]	$\mathcal{S}_2 > \mathcal{S}_4 > \mathcal{S}_6 > \mathcal{S}_7 > \mathcal{S}_5 > \mathcal{S}_3 > \mathcal{S}_8 > \mathcal{S}_1$	$\mathcal{S}_2$
mFHWG [24]	$\mathcal{S}_2 > \mathcal{S}_4 > \mathcal{S}_7 > \mathcal{S}_6 > \mathcal{S}_5 > \mathcal{S}_3 > \mathcal{S}_1 > \mathcal{S}_1$	$\mathcal{S}_2$
mFDWA [26]	$\mathcal{S}_8 > \mathcal{S}_5 > \mathcal{S}_3 > \mathcal{S}_6 > \mathcal{S}_7 > \mathcal{S}_1 > \mathcal{S}_2 > \mathcal{S}_4$	$\mathcal{S}_8$
mFDWG [26]	$\mathcal{S}_8 > \mathcal{S}_6 > \mathcal{S}_1 > \mathcal{S}_2 > \mathcal{S}_7 > \mathcal{S}_3 > \mathcal{S}_5 > \mathcal{S}_4$	$\mathcal{S}_8$
Proposed mFYWA	$\mathcal{S}_8 > \mathcal{S}_3 > \mathcal{S}_5 > \mathcal{S}_6 > \mathcal{S}_7 > \mathcal{S}_2 > \mathcal{S}_1 > \mathcal{S}_4$	$\mathcal{S}_8$
Proposed mFYWG	$\mathcal{S}_4 > \mathcal{S}_2 > \mathcal{S}_6 > \mathcal{S}_7 > \mathcal{S}_5 > \mathcal{S}_3 > \mathcal{S}_8 > \mathcal{S}_1$	$\mathcal{S}_4$

4.2. Effectiveness tests

In the following, the feasibility and productiveness of the proposed algorithm based on mFYWA and mFYWG operators is justified via three tests criteria, which have been introduced by Wang and Triantaphyllou [45] to check the validity of MCDM methods.

- **Test-I:** When the belongingness degrees of a sub-optimal alternative are replaced with worse belongingness degrees without changing the criteria, then the decision object should be invariant.
- **Test-II:** The MCDM method should verify the transitive law.
- **Test-III:** If a given problem is resolved into different small portions by removing alternatives and the same MCDM approach is applied, then the ranking of alternatives should be the same as the original.

Now, we discuss the effectiveness of the proposed MCDM method under the conditions of mF Yager AOs by means of the above validity tests.

1) **Validity checking by Test I:** The proposed MCDM approach with Yager AOs verifies this test because when we replace the belongingness degrees of the object $\mathcal{S}_1$ with $\mathcal{S}_1^{'}$ and the object $\mathcal{S}_4$ with $\mathcal{S}_4^{'}$ in Table 2 (i.e., 3F decision matrix), then by applying the developed mFYWA operator to the new decision matrix, which is provided by Table 5, the scores of the alternatives $\mathcal{S}_1$ and $\mathcal{S}_6$ are $\mathfrak{S}(\mathfrak{d}_1) = 0.4383$ and $\mathfrak{S}(\mathfrak{d}_6) = 0.4858$, respectively. Clearly, $\mathcal{S}_8$ is again the best alternative, which is the same as the original decision object. In a similar manner, if we apply the mFYWG operator, the scores of the objects $\mathcal{S}_1$ and $\mathcal{S}_6$ are $\mathfrak{S}(\mathfrak{d}_1) = 0.3158$ and $\mathfrak{S}(\mathfrak{d}_6) = 0.4051$, respectively. Clearly, $\mathcal{S}_4$ is the best alternative which is the same as the original. Thus, the optimal alternatives were the same as that of the original ranking when we changed the sub-optimal alternatives belongingness values. Thus, the developed algorithm is reliable under the validity test criterion I.
2) **Validity checking by Tests II and III:** These test criteria also hold for our proposed MCDM approach with mF Yager AOs because when we remove some objects in the developed application (Section 3) and apply the developed mFYWA and mFYWG operators, we obtain similar ranking orders between the alternatives and the original. This is why, the overall ranking order of the alternatives will not be changed and the transitive property holds. Thus, the proposed algorithm is reliable under the validity checking tests II and III.

Table 5. 3F decision matrix.

–	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$	$\mathcal{E}_5$
$\mathcal{S}_1'$	(0.1, 0.3, 0.5)	(0.4, 0.6, 0.2)	(0.3, 0.4, 0.1)	(0.1, 0.3, 0.1)	(0.4, 0.6, 0.2)
$\mathcal{S}_2$	(0.5, 0.7, 0.8)	(0.4, 0.8, 0.7)	(0.6, 0.4, 0.2)	(0.7, 0.6, 0.9)	(0.6, 0.4, 0.7)
$\mathcal{S}_3$	(0.8, 0.4, 0.9)	(0.4, 0.8, 0.4)	(0.7, 0.8, 0.6)	(0.3, 0.5, 0.7)	(0.8, 0.6, 0.5)
$\mathcal{S}_4$	(0.6, 0.3, 0.2)	(0.7, 0.5, 0.3)	(0.4, 0.2, 0.4)	(0.5, 0.2, 0.4)	(0.2, 0.6, 0.3)
$\mathcal{S}_5$	(0.8, 0.5, 0.4)	(0.7, 0.2, 0.5)	(0.9, 0.4, 0.8)	(0.7, 0.9, 0.7)	(0.8, 0.3, 0.6)
$\mathcal{S}_6'$	(0.6, 0.3, 0.2)	(0.7, 0.5, 0.3)	(0.4, 0.2, 0.4)	(0.5, 0.2, 0.4)	(0.2, 0.6, 0.3)
$\mathcal{S}_7$	(0.8, 0.3, 0.7)	(0.8, 0.7, 0.3)	(0.5, 0.9, 0.8)	(0.7, 0.6, 0.5)	(0.4, 0.5, 0.8)
$\mathcal{S}_8$	(0.7, 0.6, 0.2)	(0.5, 0.9, 0.1)	(0.8, 0.2, 0.5)	(0.6, 0.7, 0.9)	(0.9, 0.7, 0.5)

5. Conclusions, limitations and future research

Nowadays, due to the existence of multipolar data and multiple attributes in several real-world problems, the fuzzification of multipolar information with AOs is emerging as a very popular mathematical topic for the unification of various inputs into a single useful output because traditional MCDM approaches fail to deal with complex decision-making problems. With the motivation to remove these issues of existing MCDM methods, and we have integrated *m*F numbers with Yager's TN and TCoN operations, and have presented some new Yager AOs in an *m*F environment, namely, *m*FYWA, *m*FYOWA, *m*FYHA, *m*FYWG, *m*FYOWG and *m*FYHG operators, which are respectively explained with illustrative numerical examples. Further, we have applied different results of the proposed AOs. To prove the feasibility and reliability of the developed *m*F AOs, we have implemented them to a daily-life problem, that is, the selection of an appropriate site for the construction of an oil refinery. Subsequently, we performed a comparative analysis of the initiated *m*F Yager AOs with existing *m*F Dombi [26] and *m*F Hamacher AOs [24]. From the comparative analysis (Tables 3 and 4), we have clearly observed that the optimal object (that is, $\mathcal{S}_8$) is the same by applying *m*FDWA and *m*FDWG [26] operators and the proposed *m*FYWA operator. On the other hand, the optimal object (that is, $\mathcal{S}_2$) is the same by applying *m*FHWA and *m*FHWG [24] operators, but it is not similar to the optimal object "$\mathcal{S}_4$" which is obtained by applying the proposed *m*FYWG operator. In the end, we have verified the effectiveness of the developed MCDM method by applying validity tests that were presented by Wang and Triantaphyllou [45].

The literature analysis revealed that the existing AOs have both pros and cons. Because of this, we noticed that our initiated AOs also have some limitations. The developed *m*F Yager AOs are not useful in the case of multipolar information from opposite sources because they only deal with multi-valued membership-based information. It may not be easy to compute final ranking results in the case of a big number of attributes without using mathematical software, including MAPAL, MATLAB, Mathematica, etc.

In the future, our presented work can be extended to the following

- *m*F Yager prioritized AOs,
- *m*F soft Yager AOs,
- Hesitant *m*F Yager AOs,

- mF Yager Bonferroni mean operators,
- Possibility mF Yager AOs,
- Rough mF Yager AOs.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix

A.1. Proof of Theorem 2.2

Proof. By utilizing the mathematical induction method we can easily prove it.

1). By putting $n = 1$ in Eq (2.4), we get

$$\begin{aligned} mFDWA_{\Upsilon}(\tilde{\eta}_1, , \tilde{\eta}_2, \ldots, \tilde{\eta}_n) &= \Upsilon_1 \tilde{\eta}_1 = \tilde{\eta}_1, \text{ (since } \Upsilon_1 = 1) \\ &= \left(\sqrt{\min\left(1, ((\mathfrak{p}_1 \circ \eta_1)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, ((\mathfrak{p}_m \circ \eta_1)^{2\sigma})^{\frac{1}{\sigma}}\right)} \right). \end{aligned}$$

Thus, Eq (2.4) holds for $n = 1$.

2). Now let us suppose that Eq (2.4) holds when $n = r$, where $r \in \mathbb{N}$ (set of natural numbers); then,

$$\begin{aligned} mFYWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_r) &= \bigoplus_{t=1}^{r} (\Upsilon_t \tilde{\eta}_t), \\ &= \left(\sqrt{\min\left(1, (\sum_{t=1}^{r} \Upsilon_t (\mathfrak{p}_1 \circ \eta_t)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\sum_{t=1}^{r} \Upsilon_t (\mathfrak{p}_m \circ \eta_t)^{2\sigma})^{\frac{1}{\sigma}}\right)} \right). \end{aligned} \tag{A.1}$$

For $n = r + 1$,

$$mFYWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_r, \tilde{\eta}_{r+1}) = \bigoplus_{t=1}^{r} (\Upsilon_t \tilde{\eta}_t) \oplus (\Upsilon_{r+1} \tilde{\eta}_{r+1}),$$

$$= \left(\sqrt{\min\left(1, (\sum_{t=1}^{r}\Upsilon_t(\mathfrak{p}_1 \circ \eta_t)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\sum_{t=1}^{r}\Upsilon_t(\mathfrak{p}_m \circ \eta_t)^{2\sigma})^{\frac{1}{\sigma}}\right)}\right) \oplus$$

$$\left(\sqrt{\min\left(1, (\Upsilon_{r+1}(\mathfrak{p}_1 \circ \eta_{r+1})^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\Upsilon_{r+1}(\mathfrak{p}_m \circ \eta_{r+1})^{2\sigma})^{\frac{1}{\sigma}}\right)}\right)$$

$$= \left(\sqrt{\min\left(1, (\sum_{t=1}^{r+1}\Upsilon_t(\mathfrak{p}_1 \circ \eta_t)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\sum_{t=1}^{r+1}\Upsilon_t(\mathfrak{p}_m \circ \eta_t)^{2\sigma})^{\frac{1}{\sigma}}\right)}\right).$$

Thus, Eq (2.4) holds for $n = r + 1$. Consequently, Eq (2.4) verifies for all $n \in \mathbb{N}$. □

A.2. Proof of Theorem 2.4

Proof. Since $\tilde{\eta}_t = (\mathfrak{p}_1 \circ \eta_t, \ldots, \mathfrak{p}_m \circ \eta_t) = \tilde{\eta}$, where $t = 1, \ldots, n$. Then, by Eq (2.4),

$$\begin{aligned}
mFYWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) &= \bigoplus_{t=1}^{n}(\Upsilon_t \tilde{\eta}_t), \\
&= \left(\sqrt{\min\left(1, (\sum_{t=1}^{n}\Upsilon_t(\mathfrak{p}_1 \circ \eta_t)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, (\sum_{t=1}^{n}\Upsilon_t(\mathfrak{p}_m \circ \eta_t)^{2\sigma})^{\frac{1}{\sigma}}\right)}\right), \\
&= \left(\sqrt{\min\left(1, ((\mathfrak{p}_1 \circ \eta)^{2\sigma})^{\frac{1}{\sigma}}\right)}, \ldots, \sqrt{\min\left(1, ((\mathfrak{p}_m \circ \eta)^{2\sigma})^{\frac{1}{\sigma}}\right)}\right), \\
&= (\mathfrak{p}_1 \circ \eta, \ldots, \mathfrak{p}_m \circ \eta), \text{ for } \sigma = 1 \\
&= \tilde{\eta}.
\end{aligned}$$

Hence, $mFYWA_{\Upsilon}(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n) = \tilde{\eta}$ holds if $\tilde{\eta}_t = \tilde{\eta}$, when '$t$' varies from 1 to n. □