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Research article

Existence, multiplicity and non-existence of solutions for modified Schrödinger-Poisson systems

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Abstract: We consider a class of modified Schrödinger-Poisson systems with general nonlinearity by variational methods. The existence and multiplicity of solutions are obtained. Besides, when $V(x) = 1$ and $f(x, u) = |u|^{p-2}u$, we obtain some existence and non-existence results for the modified Schrödinger-Poisson systems.

Keywords: modified Schrödinger-Poisson system; Nehari manifold; genus; Pohozaev identity

1. Introduction

In this paper, we consider the following modified Schrödinger-Poisson system, which is usually used to describe solitary waves for the nonlinear stationary Schrödinger equations interacting with an unknown electromagnetic field (refer to [1–4] for more physical background):

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u - \dfrac{1}{2}u\Delta(u^2) = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V(x) \in C(\mathbb{R}^3)$, $V_0 := \inf_{x\in\mathbb{R}^3} V(x) > 0$ and satisfies

(V): For any $M, r > 0$, there is a ball $B_r(y)$ centered at y with radius r such that

$$\mu(\{x \in B_r(y) : V(x) \le M\}) \to 0, \text{ as } |y| \to \infty.$$

In the past twenty years, there have been a lot of contributions about the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \ \lim_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

after the pioneering work in [2]. For example, Ruiz et al. gave some existence and nonexistence results for the case $V(x) = 1$ and $f(x, u) = |u|^{p-2}u$, $2 < p < 6$ in [5], while Azzollini [6] studied the existence of a ground state solution of Eq (1.2) with the same $f(x, u)$ but $3 < p < 6$. And [7–9] focused on the existence and multiplicity of nontrivial solutions with a superlinear and subcritical growth condition. Amongst them a global Ambrosetti-Rabinowitz type condition is given as follow

$$0 < F(x, u) := \int_0^u f(x, s)\mathrm{d}s \le \frac{1}{\gamma} u f(x, u), \quad \gamma > 4, \tag{A-R}$$

which is only valid for $f(x, u) = |u|^{p-2}u$ with $p > 4$. Then in [10], Liu, Wang and Zhang provided a supplement as $p \in (3, 4)$. And we discussed for a more general nonlinearity $f(x, u)$ without any growth restrictions at infinity in [11].

On the other hand, some researchers also considered a quasilinear Schrödinger equation defined by

$$-\Delta u + V(x)u - \Delta(u^2)u = f(x, u), \ x \in \mathbb{R}^N,$$

which arose in several models of physical phenomena, such as superfluid films in plasma physics (see e.g., [12–14]). And it has received considerable attention in mathematical analysis in the last twenty years (see [15–19]). Feng and Zhang in [20] added the quasilinear term $\Delta(u^2)u$ to Eq (1.2) and found that the new Eq (1.1) possesses at least one non-trivial solution by using perturbation method and mountain pass theorem based on the following assumptions:
(F_1): $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $|f(x, u)| \le C_1 \left(|u| + |u|^{p-1}\right)$ for some $C_1 > 0$ and $p \in (4, 6)$;
(F_2): $f(x, u) = o(u)$ uniformly in x as $u \to 0$;
(F_3): there exists $\mu > 4$ such that

$$0 < \mu F(x, s) = \mu \int_0^s f(x, t)\mathrm{d}t \le s f(x, s), \ s \in \mathbb{R} \setminus \{0\}, \ x \in \mathbb{R}^3;$$

(F_4): there exists $M > 0$ such that

$$\inf_{x \in \mathbb{R}^3, |u| \ge M} F(x, u) > 0.$$

After that, Chen L. et al. proved that Eq (1.1) possesses a sign-changing solution by a minimisation on a Nehari-type constraint for the corresponding Euler-Lagrange functional if f satisfies the following assumptions in [21]:
(F_1'): $f \in C^1(\mathbb{R}, \mathbb{R})$;
(F_2'): $\lim\limits_{s \to 0} \frac{f(s)}{s} = \lim\limits_{|s| \to \infty} \frac{f(s)}{|s|^{11}} = 0$;
(F_3'): there exists $\mu > 4$ such that

$$0 < \mu F(s) = \mu \int_0^s f(t)\mathrm{d}t \le s f(s), \ s \in \mathbb{R} \setminus \{0\};$$

(F_4'): $\frac{f(t)}{|t|^3}$ is increasing on $(-\infty, 0)$ and $(0, \infty)$, respectively.

Motivated by the above work, we discuss the existence of solutions for the modified Schrödinger-Poisson Eq (1.1) with coercive potential and more general assumptions on f but not need to be C^1. Concretely, let f satisfy

(f_1): $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $|f(x,u)| \le C_2\left(1 + |u|^{p-1}\right)$ for some $C_2 > 0$ and $p \in (4, 12)$;
(f_2): $f(x,u) = o(u)$ uniformly in x as $u \to 0$;
(f_3): $F(x,u)/u^4 \to \infty$ uniformly in x as $|u| \to \infty$, where $F(u) = \int_0^u f(s)\mathrm{d}s$;
(f_4): $u \to f(x,u)/u^3$ is positive for $u \neq 0$, strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$.

Clearly, (f_1) and (f_2) show that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1} \text{ for all } u \in \mathbb{R} \text{ and } x \in \mathbb{R}^3. \tag{1.3}$$

Using (f_2) and (f_4), we have

$$F(x,u) \ge 0 \text{ and } f(x,u)u > 4F(x,u) > 0 \text{ if } u \neq 0. \tag{1.4}$$

Remark 1.1. *(F_3) and (F_3') in [20, 21] can ensure the boundness of the Palais-Smale sequences of the corresponding Euler-Lagrange functional. Although they are quite natural, (F_3) and (F_3') are somewhat restrictive and eliminate many nonlinearities. For example, the function*

$$f(x,t) = |t|^2 t \log(1 + |t|)$$

does not satisfy (F_3) and (F_3') for any $\mu > 4$, but it satisfies $(f_1) - (f_4)$.

Now, we give our first main result.

Theorem 1.1. *Assume (V) and $(f_1) - (f_4)$ hold. Equation (1.1) has a nontrivial solution.*

For the proof of this theorem, we find that Eq (1.1) involves a second order derivative $\Delta(u^2)u$ and a nonlocal term ϕu, whose natural energy functional is not well defined in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ and variational methods cannot be used directly. In this case, we will make use of the perturbation method introduced in [22, 23]. Since f is not assumed to be differentiable, we do not know whether the Nehari manifold of the corresponding Euler-Lagrange functional is of class C^1 under our assumptions. Besides these, compared with [20,21], we do not assume f satisfying the Ambrosetti-Rabinowitz condition (see (F_3)), so the boundness of Palais-Smale sequence (or minimizing sequence) seems hard to prove.

Remark 1.2. *The condition (V) was firstly introduced by Bartsch, Pankov and Wang [24] to guarantee the compactness of embeddings of the work space. The limit of condition (V) can be replaced by one of the following simpler conditions:*
(V_1): $V(x) \in C(\mathbb{R}^3)$, $\mu(\{x \in \mathbb{R}^3 : V(x) \le M\}) < \infty$ for any $M > 0$ (see [25]);
(V_2): $V(x) \in C(\mathbb{R}^3)$, $V(x)$ is coercive, i.e., $\lim_{|x|\to\infty} V(x) = \infty$.

Next, we study the multiplicity of solutions of the Eq (1.1). Genus theory gives that Eq (1.1) has infinitely many high energy solutions.

Theorem 1.2. *Suppose that (V), $(f_1) - (f_4)$ are satisfied and f is odd in u. Equation (1.1) has infinitely many pairs of solutions.*

Remark 1.3. *Although the condition (V) plays a role in guaranteeing the compactness of the minimizing sequence for the energy functional I_λ, the existence result can also hold when V is a periodic potential because of the concentration-compactness principle.*

Suppose that Eq (1.1) has a periodic potential V and V satisfies
(V^*): $V(x) \in C(\mathbb{R}^3)$ and is 1-periodic in x_i, $1 \le i \le 3$, $V_0 := \inf_{x\in\mathbb{R}^3} V(x) > 0$
and f satisfies
(f_1'): $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, f is 1-periodic in x_i for $i = 1, 2, 3$ and $|f(x, u)| \le C_2 \left(1 + |u|^{p-1}\right)$ for some $C_2 > 0$ and $p \in (4, 12)$.

Our third main result is

Theorem 1.3. *Assume (V^*), (f_1') and $(f_2) - (f_4)$ hold. System (1.1) has a nontrivial solution.*

Remark 1.4. *When V is a periodic potential and f is odd in u, [26] proved the Schrödinger-Poisson has infinitely many solutions. But to the best of our knowledge, there is no result in the literature about the multiplicity of solutions of Eq (1.1) with the periodic potential. While we can still obtain that the perturbation functional of Eq (1.1) has infinitely many critical points by the method in [27]. But we can not make sure their critical values limit to be infinity, which is necessary to find distinct solutions of the Eq (1.1). For this reason, the multiplicity of solutions for the original problem with the periodic potential seems hard to obtain.*

Up to now, the functions f considered above is 4-superlinear at infinity (see (f_3)), specially $f(x, u) = |u|^{p-2}u$ with $4 < p < 12$. When $p = 4$, Nie and Wu [28] proved the existence of a non-trivial ground state solution and two non-trivial solutions for the Eq (1.1) with $f(x, u) = |u|^2 u + h(x)$. However, when $f(x, u) = |u|^{p-2}u$ with $1 \le p \le 4$, due to the effect of quasilinear term and the nonlocal Poisson term, it becomes quite complicated.

In the last part of our paper, we consider the following autonomous modified Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \phi(x)u - \frac{1}{2}u\Delta(u^2) = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $p \ge 1$.

We have the following result.

Theorem 1.4. *(1) If $1 \le p \le 3$ or $p \ge 12$, problem (1.5) does not admit any nontrivial solution.*
(2) If $p \in (4, 12)$, problem (1.5) has a radial solution.

This paper is organized as follows. In Section 2, we describe the related mathematical tools. And Sections 3–6 give the proofs of Theorems 1.1–1.4, respectively.

In what follows, C and C_i always denote positive generic constants.

2. Preliminaries and functional setting

We first recall some definitions and known facts which will be used. Let $L^p(\mathbb{R}^3)$ be the usual Lebesgue space with the norm

$$\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p \mathrm{d}x\right)^{1/p}.$$

And the Sobolev space $W^{1,p}(\mathbb{R}^N)$ is the space endowed with the follow norm

$$\|u\|_{W^{1,p}} = \left(\int_{\mathbb{R}^3} (|\nabla u|^p + u^p)\, \mathrm{d}x\right)^{1/p}.$$

Moreover, $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm

$$\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^{1/2}.$$

Due to the existence of quasilinear term $u\Delta(u^2)$, we consider a family of the perturbation functional I_λ (see Eq (2.3)), which is well-defined in

$$E = W^{1,4}(\mathbb{R}^3) \cap H_V^1(\mathbb{R}^3),$$

where

$$H_V^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < +\infty\}$$

and $V(x)$ satisfies (V), which is a Hilbert space endowed with the following norm

$$\|u\|_{H_V} = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x\right)^{1/2}$$

and $W^{1,4}(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_W = \left(\int_{\mathbb{R}^3} (|\nabla u|^4 + u^4) \mathrm{d}x\right)^{1/4}.$$

The norm of E is defined by

$$\|u\| = \left(\|u\|_W^2 + \|u\|_{H_V}^2\right)^{1/2}.$$

Notice that the embedding from $H_V^1(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ is compact (see [25]). Thus, by applying the interpolation inequality, and so the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $2 \leq s \leq 12$.

Next, the Lax-Milgram theorem (see [5]) shows that, for every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = u^2$$

and

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v \mathrm{d}x = \int_{\mathbb{R}^3} u^2 v \mathrm{d}x, \text{ for all } v \in D^{1,2}(\mathbb{R}^3).$$

The following lemma gives some properties of ϕ_u. See [5].

Lemma 2.1. *For any $u \in E \subset H^1(\mathbb{R}^3)$, the following conclusions are true*
(1) $\phi_u \geq 0$;
(2) $\phi_{tu} = t^2\phi_u$, $\forall t \in \mathbb{R}$;
(3) $\|\phi_u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} \phi_u u^2 dx \leq C_3\|u\|_{12/5}^4 \leq C_4\|u\|_{H_V}^4$, *where* $C_3, C_4 > 0$ *are constants;*
(4) If $u_n \rightharpoonup u$ *weakly in* $H^1(\mathbb{R}^3)$, *then up to a subsequence,* $\phi_{u_n} \to \phi_u$ *strongly in* $D^{1,2}(\mathbb{R}^3)$ *and*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

We look for a weak solution $u \in H_V^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ of system (1.1), such that for all $\varphi \in C^\infty(\mathbb{R}^3)$, satisfying

$$\begin{aligned}\int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(x)u\varphi)\,\mathrm{d}x + \int_{\mathbb{R}^3} u^2 \nabla u \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u|^2 u\varphi \mathrm{d}x \\ + \int_{\mathbb{R}^3} \phi_u u\varphi \mathrm{d}x - \int_{\mathbb{R}^3} f(x,u)\varphi \mathrm{d}x = 0,\end{aligned} \tag{2.1}$$

which is formally associated to the energy functional given by

$$\begin{aligned}I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 + u^2|\nabla u|^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \\ - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x, \text{ for } u \in H_V^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3),\end{aligned} \tag{2.2}$$

where $F(x,u) = \int_0^u f(x,s)\mathrm{d}s$.

Due to $\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x$ is not well-defined in $H_V^1(\mathbb{R}^3)$, we take a perturbation functional of Eq (2.2) given by

$$I_\lambda(u) = \frac{\lambda}{4}\int_{\mathbb{R}^3} \left(|\nabla u|^4 + u^4\right)\mathrm{d}x + I(u). \tag{2.3}$$

It follows from conditions (V), Eqs (1.3) and (1.4), that $I_\lambda \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle I_\lambda'(u),\varphi\rangle = \lambda\int_{\mathbb{R}^3} \left(|\nabla u|^2\nabla u\nabla\varphi + u^3\varphi\right)\mathrm{d}x + \int_{\mathbb{R}^3} (\nabla u\nabla\varphi + V(x)u\varphi)\,\mathrm{d}x + \int_{\mathbb{R}^3} u^2\nabla u\nabla\varphi \mathrm{d}x \\ + \int_{\mathbb{R}^3} |\nabla u|^2 u\varphi \mathrm{d}x + \int_{\mathbb{R}^3} \phi_u u\varphi \mathrm{d}x - \int_{\mathbb{R}^3} f(x,u)\varphi \mathrm{d}x, \text{ for all } \varphi \in E.\end{aligned} \tag{2.4}$$

For a proof, we refer to the Lemma 2.1 in [29].

3. Proof of Theorem 1.1

First of all, we discuss some properties of the perturbation functional I_λ on $\mathcal{N}_\lambda$ which are useful to apply the general Nehari theory.

Lemma 3.1. *Assume (V) and $(f_1)-(f_4)$ hold and $\lambda \in (0,1]$.*
(1) For $u \in E \setminus \{0\}$, there exists a unique $t_u = t(u) > 0$ such that $m(u) := t_u u \in \mathcal{N}_\lambda$ and

$$I_\lambda(m(u)) = \max_{t\in\mathbb{R}^+} I_\lambda(tu);$$

(2) For all $u \in \mathcal{N}_\lambda$, there exists $\alpha_0 > 0$ such that $\|u\|_W \geq \alpha_0$;
(3) There exists $\rho > 0$ such that $c := \inf_{\mathcal{N}_\lambda} I_\lambda \geq \inf_{S_\rho} I_\lambda > 0$, where $S_\rho := \{u \in E : \|u\| = \rho\}$;
(4) If $\mathcal{V} \subset E \setminus \{0\}$ is a compact subset, there exists $R > 0$ such that $I_\lambda \leq 0$ on $\mathbb{R}^+\mathcal{V} \setminus B_R(0)$.

Proof. (1) For any $u \in E \setminus \{0\}$, we consider $h_u(t) = I_\lambda(tu)$ for $t \in (0,\infty)$,

$$\begin{aligned}h_u(t) = \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} \left(|\nabla u|^4 + u^4\right)\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x + \frac{t^4}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x \\ + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,tu)\mathrm{d}x.\end{aligned} \tag{3.1}$$

Combining Eq (1.3) and the Sobolev embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ for $s \in [2, 12]$ is continuous, for $\varepsilon > 0$ small enough, we obtain

$$\begin{aligned} h_u(t) &\geq \frac{\lambda t^4}{4}\|u\|_W^4 + \frac{t^2}{2}\|u\|_{H_V}^2 + \frac{t^4}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \\ &\quad - \frac{\varepsilon t^2}{2}\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x - \frac{C_\varepsilon t^p}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x \\ &\geq \frac{\lambda t^4}{4}\|u\|_W^4 + \frac{t^2}{4}\|u\|_{H_V}^2 - C_5 t^p \|u\|_p^p. \end{aligned}$$

Hence for $t > 0$ small enough, we have $h_u(t) > 0$.

On the other hand, using Lemma 2.1-(3), (f_3) and Fatou's lemma, it is easy to say that

$$\begin{aligned} h_u(t) \leq &\frac{\lambda t^4}{4}\|u\|_W^4 + \frac{t^2}{2}\|u\|_{H_V}^2 + C_6 t^4 \|u\|_{H_V}^4 + C_7 t^4 \|u\|_W^4 \\ &- t^4 \int_{\mathbb{R}^3} \frac{F(x, tu)}{|tu|^4} u^4 \mathrm{d}x \to -\infty, \text{as } t \to \infty. \end{aligned}$$

Hence, there exists a $t_u = t(u) > 0$ such that $h_u'(t_u) = 0$, $h_u(t_u)$ is a positive maximum and $t_u u \in \mathcal{N}_\lambda$. Next, we prove the uniqueness of t_u. Otherwise, there exists $t_u^* > 0$ with $t_u^* \neq t_u$ such that $h_u'(t_u^*) = 0$. Then we obtain

$$\lambda\|u\|_W^4 + \frac{\|u\|_{H_V}^2}{(t_u^*)^2} + \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x + 2\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x = \int_{\mathbb{R}^3} \frac{f(x, t_u^* u)}{(t_u^* u)^3} u^4 \mathrm{d}x,$$

and

$$\lambda\|u\|_W^4 + \frac{\|u\|_{H_V}^2}{(t_u)^2} + \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x + 2\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x = \int_{\mathbb{R}^3} \frac{f(x, t_u u)}{(t_u u)^3} u^4 \mathrm{d}x.$$

Then

$$\left(\frac{1}{(t_u^*)^2} - \frac{1}{(t_u)^2}\right)\|u\|_{H_V}^2 = \int_{\mathbb{R}^3} \left(\frac{f(x, t_u^* u)}{(t_u^* u)^3} - \frac{f(x, t_u u)}{(t_u u)^3}\right) u^4 \mathrm{d}x,$$

which contradicts with (f_4).

(2) From Eq (1.3) and $u \in \mathcal{N}_\lambda$, we see that for $\varepsilon > 0$ sufficiently small, there has

$$\begin{aligned} 0 &= \lambda\|u\|_W^4 + \|u\|_{H_V}^2 + \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x + 2\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^3} f(x, u)u \mathrm{d}x \\ &\geq \lambda\|u\|_W^4 + \|u\|_{H_V}^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x - \frac{C_\varepsilon}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x \\ &\geq \lambda\|u\|_W^4 + \frac{1}{2}\|u\|_{H_V}^2 - C_8\|u\|_W^p \\ &\geq \lambda\|u\|_W^4 - C_8\|u\|_W^p. \end{aligned}$$

The above result means that for any $u \in \mathcal{N}_\lambda$, there exists a constant $\alpha_0 > 0$ such that $\|u\|_W \geq \alpha_0 > 0$.

(3) For any $\rho > 0$, let $u \in E \setminus \{0\}$ with $\|u\| \leq \rho$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x \leq C\rho^4.$$

By (V), (f_1), (f_2) and the Sobolev inequality, without losing generality, if we choose $\rho < 1$ small enough and $\varepsilon = \frac{1}{2}$, then for any $\|u\| \le \rho$, we have

$$\begin{aligned} I_\lambda(u) \ge & \frac{\lambda}{4}\|u\|_W^4 + \frac{1}{2}\|u\|_{H_V}^2 + \frac{1}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \\ & - \varepsilon \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x - C_\varepsilon \int_{\mathbb{R}^3} |u|^{12} \mathrm{d}x \\ \ge & \frac{\lambda}{4}\|u\|_W^4 + \frac{1}{4}\|u\|_{H_V}^2 + \frac{1}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x - C_9\left(\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x\right)^3 \\ \ge & \frac{\lambda}{4}\|u\|_W^4 + \frac{1}{4}\|u\|_{H_V}^2 \\ \ge & \frac{\lambda}{8}\|u\|^4. \end{aligned} \tag{3.2}$$

Owing to Lemma 3.1-(1), for any $u \in \mathcal{N}_\lambda$, we arrive at

$$I_\lambda(u) = \max_{t \in \mathbb{R}^+} I_\lambda(tu). \tag{3.3}$$

Taking $s > 0$ with $su \in S_\rho$. From Eqs (3.2) and (3.3), we get

$$I_\lambda(u) \ge I_\lambda(su) \ge \inf_{v \in S_\rho} I_\lambda(v) \ge \frac{\lambda}{8}\rho^4 > 0.$$

Therefore

$$c := \inf_{\mathcal{N}_\lambda} I_\lambda \ge \inf_{S_\rho} I_\lambda > 0.$$

(4) Suppose this is not true, there must exist $u_n \in \mathcal{V}$ and $v_n = t_n u_n$ such that $I_\lambda(v_n) \ge 0$ for all n and $t_n \to \infty$ as $n \to \infty$. Without losing generality, we assume that $\|u_n\| = 1$ for every $u_n \in \mathcal{V}$. Passing to subsequence, there exists $u \in E$ with $\|u\| = 1$ and $u_n \to u$ strongly in E. For $u(x) \ne 0$, we have $|v_n(x)| \to \infty$, it follows from (f_3) and Fatou's lemma that

$$\int_{\mathbb{R}^3} \frac{F(x, v_n)}{v_n^4} u_n^4 \mathrm{d}x \to \infty, \text{ as } n \to \infty,$$

which, jointly with Lemma 2.1-(3), one has

$$\begin{aligned} 0 \le & \frac{I_\lambda(v_n)}{\|v_n\|^4} \\ = & \frac{1}{\|v_n\|^4}\left(\frac{\lambda}{4}\|v_n\|_W^4 + \frac{1}{2}\|v_n\|_{H_V}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} v_n^2|\nabla v_n|^2 \mathrm{d}x\right) - \int_{\mathbb{R}^3} \frac{F(x, v_n)}{v_n^4} u_n^4 \mathrm{d}x \\ \le & C_{10} - \int_{\mathbb{R}^3} \frac{F(x, v_n)}{v_n^4} u_n^4 \mathrm{d}x \to -\infty \text{ as } n \to \infty. \end{aligned}$$

This is a contradiction.

Now we are in a position to study the minimizing sequence for I_λ on $\mathcal{N}_\lambda$.

Lemma 3.2. *For fixed $\lambda \in (0, 1]$, let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence for I_λ. Then $\{u_n\}$ is bounded in E. In addition, passing to a subsequence there exists $u \in E$ such that $u_n \rightharpoonup u \neq 0$ and $u_n \to u$ in E.*

Proof. Let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of I_λ. i.e.

$$I_\lambda(u_n) \to c := \inf_{\mathcal{N}_\lambda} I_\lambda \text{ and } \langle I'_\lambda(u_n), u_n\rangle = 0. \tag{3.4}$$

By Eq (3.4), one sees that

$$\begin{aligned} c + o(1) &= I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2_{H_V} + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x, u_n)u_n - F(x, u_n)\right)\mathrm{d}x. \end{aligned}$$

Thus, we deduce $\|u_n\|_{H_V}$ is bounded, which in turn means that $\|u_n\|_W$ is bounded.
If $\{u_n\}$ is unbounded in $W^{1,4}(\mathbb{R}^3)$, set $\omega_n = \|u_n\|^{-1}_W u_n$, we have

$$\omega_n \rightharpoonup \omega \text{ weakly in } W^{1,4}(\mathbb{R}^3),\ \omega_n \to \omega \text{ strongly in } L^p(\mathbb{R}^3),\ \omega_n \to \omega\ a.e. \text{ on } x \in \mathbb{R}^3.$$

This proof can be split into two steps.
Step 1: If $\omega = 0$, it follows from Lemma 3.1-(1) that

$$I_\lambda(u_n) = \max_{t\in\mathbb{R}^+} I_\lambda(tu_n).$$

For any $m > 0$ and set $v_n = (8m)^{1/4}\omega_n$, since $v_n \to 0$ strongly in $L^p(\mathbb{R}^3)$, we infer from Eq (1.3) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} F(x, v_n)\mathrm{d}x = 0. \tag{3.5}$$

So for n large enough, $(8m)^{1/4}\|u_n\|^{-1}_W \in (0, 1)$, and

$$\begin{aligned} I_\lambda(u_n) &\geq I_\lambda(v_n) \\ &= 2\lambda m + (2m)^{1/2}\frac{\|u_n\|^2_{H_V}}{\|u_n\|^2_W} + 2m\frac{\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x}{\|u_n\|^4_W} \\ &\quad + 4m\frac{\int_{\mathbb{R}^3}u_n^2|\nabla u_n|^2\mathrm{d}x}{\|u_n\|^4_W} - \int_{\mathbb{R}^3}F(x, v_n)\mathrm{d}x \\ &\geq \lambda m. \end{aligned}$$

Thus, for fixed $\lambda > 0$, together with the arbitrariness of m, we can obtain that $I_\lambda(u_n) \to \infty$. This contradicts with $I_\lambda(u_n) \to c > 0$.

Step 2: If $\omega \neq 0$, the set $\Theta = \{x \in \mathbb{R}^3 : \omega(x) \neq 0\}$ has a positive Lebesgue measure. For $x \in \Theta$ and $|u_n(x)| \to \infty$, this together with condition (f_3), implies

$$\frac{F(x, u_n(x))}{|u(x)|^4}|\omega_n(x)|^4 \to \infty \text{ as } n \to \infty.$$

According to $I_\lambda(u_n) \to c$, Lemma 2.1-(3), (f_3), Sobolev inequality and Fatou's Lemma, there holds that

$$\begin{aligned}\frac{c+o(1)}{\|u_n\|_W^4} &= \frac{\lambda}{4} + \frac{\|u_n\|_{H_V}^2}{\|u_n\|_W^4} + \frac{\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x}{4\|u_n\|_W^4} + \frac{\int_{\mathbb{R}^3}u_n^2|\nabla u_n|^2\mathrm{d}x}{2\|u_n\|_W^4} - \frac{\int_{\mathbb{R}^3}F(x,u_n)\mathrm{d}x}{\|u_n\|_W^4}\\ &\le \frac{\lambda}{4} + C_{11} - \left(\int_{\omega\neq 0} + \int_{\omega=0}\right)\frac{F(x,u_n(x))}{|u_n(x)|^4}|\omega_n(x)|^4\mathrm{d}x + o(1)\\ &\le \frac{\lambda}{4} + C_{11} - \int_{\omega\neq 0}\frac{F(x,u_n(x))}{|u_n(x)|^4}|\omega_n(x)|^4\mathrm{d}x \to -\infty, \text{ as } n\to\infty,\end{aligned}$$

where C_{11} is a constant independent of n. This is impossible.

In any case, we obtain a contradiction. It follows that $\{u_n\}$ is bounded in $W^{1,4}(\mathbb{R}^3)$. Therefore, $\{u_n\}$ is bounded in E, so $u_n \rightharpoonup u$ weakly in E after passing to a subsequence. If $u = 0$, since $u_n \in \mathcal{N}_\lambda$, for n sufficiently large, we see as in Eq (3.5) that

$$c + 1 \ge I_\lambda(u_n) \ge I_\lambda(su_n) \ge C_{12}s^4 - \int_{\mathbb{R}^3}F(x,su_n)\mathrm{d}x \to C_{12}s^4$$

for all $s > 0$, where $C_{12} = \frac{\lambda}{4}\left(\inf\limits_{u\in\mathcal{N}_\lambda}\|u\|_W\right)^4 > 0$, it is a contradiction. Hence $u \neq 0$.

Owing to the fact that embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is compact, similar to Lemma 3.1 in [20], it is easily to obtain $u_n \to u$ strongly in E.

Lemma 3.3. *For fixed* $\lambda \in (0,1]$, *there exits* $u \in \mathcal{N}_\lambda$ *such that* $I_\lambda(u) = \inf\limits_{\mathcal{N}_\lambda} I_\lambda$.

Proof. Let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of I_λ, then by Lemma 3.2 we have $\{u_n\} \subset E$ is bounded. Thus, passing to a subsequence we have $u_n \rightharpoonup u \neq 0$ weakly in E, as is known to all $\langle I'_\lambda(u), u\rangle = 0$. It follows that $u \in \mathcal{N}_\lambda$. Thus, $I_\lambda(u) \ge c > 0$. To complete the proof, we just need to prove $I_\lambda(u) \le c$. Indeed, by Eq (1.4), Fatou's lemma and the weakly lower semi-continuity of norm, it is clear that

$$\begin{aligned}c + o(1) &= I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle\\ &= \frac{1}{4}\|u_n\|_{H_V}^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right)\mathrm{d}x\\ &\ge \frac{1}{4}\|u\|_{H_V}^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,u)u - F(x,u)\right)\mathrm{d}x + o(1)\\ &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle + o(1)\\ &= I_\lambda(u) + o(1).\end{aligned}$$

The proof is completed.

Let S be the unit sphere in E. Define a mapping $m(\omega) : S \to \mathcal{N}_\lambda$ and a functional $J_\lambda(\omega) : S \to \mathbb{R}$ by

$$m(\omega) = t_\omega\omega \text{ and } J_\lambda(\omega) := I_\lambda(m(\omega)),$$

where t_ω is as in Lemma 3.1-(1). As Proposition 2.9 and Corollary 2.10 in [27], the following proposition is a consequence of Lemma 3.1 and the above observation.

Proposition 3.1. *Assume* (V) *and* $(f_1)-(f_4)$. *For fixed* $\lambda\in(0,1]$, *then*
(1) $J_\lambda\in C^1(S,\mathbb{R})$, *and*

$$J'_\lambda(\omega)z=\|m(\omega)\|\langle I'_\lambda(m(\omega)),z\rangle$$

for any $z\in T_\omega S=\{v\in E:\langle v,\omega\rangle=0\}$;
(2) $\{\omega_n\}$ *is a Palais-Smale sequence for* J_λ *if and only if* $\{m(\omega_n)\}$ *is a Palais-Smale sequence for* I_λ;
(3) $\omega\in S$ *is a critical point of* J_λ *if and only if* $m(\omega)\in\mathcal{N}$ *is a critical point of* I_λ. *Moreover, the corresponding critical values of* J_λ *and* I_λ *coincide and* $c=\inf\limits_S J_\lambda=\inf\limits_{\mathcal{N}_\lambda}I_\lambda$.

Finally, the proof of Theorem 1.1 is based on the following convergence result for the modified functional I_λ.

Lemma 3.4. *Let* $\lambda_n\to0$ *and* $\{u_n\}\subset E$ *be a sequence of critical points of* I_{λ_n} *satisfying* $I'_{\lambda_n}(u_n)=0$ *and* $I_{\lambda_n}(u_n)\le C$ *for some C independent of n. Then up to a subsequence* $u_n\rightharpoonup\widetilde{u}$ *weakly in* $H^1_V(\mathbb{R}^3)$ *as* $n\to\infty$ *and* $\widetilde{u}$ *is a critical point of I.*

Proof. The proof is similar to many existing literature (see [20, 21]).

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $\{\omega_n\}\subset S$ be a minimizing sequence for J_λ. By Ekeland's variational principle we may assume $J'_\lambda(\omega_n)\to0$ and $J_\lambda(\omega_n)\to c$ as mentioned above. From Proposition 3.1-(2), we have $I_\lambda(u_n)\to c$ and $I'_\lambda(u_n)\to0$, where $u_n=m(\omega_n)$. Therefore, $\{u_n\}$ is a minimizing sequence for I_λ on $\mathcal{N}_\lambda$, by using Lemma 3.3, it is clear that there exists a minimizer u of $I_\lambda|_{\mathcal{N}_\lambda}$. Therefore $m^{-1}(u)\in S$ is a minimizer of J_λ and also a critical point of J_λ, we can obtain that u is a critical point of I_λ by Proposition 3.1-(3).

Choose a sequence $\lambda_i\to0$. Let $\{u_i\}\subset E$ be a sequence of critical points of I_{λ_i} with $I_{\lambda_i}(u_i)=c_{\lambda_i}\le C$. From Lemma 3.4, there exists a critical point $\widetilde{u}$ of I such that $\widetilde{u}\in H^1_V(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Next, we need prove that $\widetilde{u}$ is a non-trivial critical point of I. Considering $\langle I'_{\lambda_i}(u_i),u_i\rangle=0$, it follows from Sobolev inequality and Young's inequality that

$$\begin{aligned}0&=\lambda_i\|u_i\|_W^4+\|u_i\|_{H_V}^2+\int_{\mathbb{R}^3}\phi_{u_i}u_i^2\mathrm{d}x+2\int_{\mathbb{R}^3}u_i^2|\nabla u_i|^2\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u_i)u_i\mathrm{d}x\\&\ge\|u_i\|_{H_V}^2+2\int_{\mathbb{R}^3}u_i^2|\nabla u_i|^2\mathrm{d}x-\frac{\varepsilon}{2}\int_{\mathbb{R}^3}|u_i|^2\mathrm{d}x-\frac{C_\varepsilon}{p}\int_{\mathbb{R}^3}|u_i|^{12}\mathrm{d}x\\&\ge\frac12\|u_i\|_{H_V}^2+C_{13}\|u_i\|_{12}^4-C_{14}\|u_i\|_{12}^{12}\\&\ge C_{13}\|u_i\|_{12}^4-C_{14}\|u_i\|_{12}^{12},\end{aligned}$$

which implies $\|u_i\|_{12}\ge(\frac{C_{13}}{C_{14}})^{1/8}$. Recall that $u_i\to\widetilde{u}$ strongly in $L^{12}(\mathbb{R}^3)$. Therefore, it is clear that $\widetilde{u}\neq0$.

4. Proof of Theorem 1.2

To prove Theorem 1.2, we need recall some concepts. Denote

$$\Gamma=\{A\subset E\setminus\{0\}:\ A\text{ is closed},-A=A\}.$$

For $A \in \Gamma$, we define the Z_2 genus of A as follows

$$\gamma(A) = \min\{n \in \mathbb{N} : \text{ there exists a odd, continuous } \phi : A \to \mathbb{R}^n \setminus \{0\}\},$$

if the minimum does not exist, we let $\gamma(A) = +\infty$. In addition, set $\gamma(\emptyset) = 0$.

Next, we notice that the nonlinearity f no longer meets Ambrosetti-Rabinowitz condition, so it seems difficult to prove the boundedness of Palais-Smale sequences. Which is why, in what follows, we need to illustrate the functional I_λ satisfies the Cerami condition. We say I_λ satisfies the Cerami condition, if any $(C)_c$-sequence has a convergent subsequence in E. We know the $(C)_c$-sequence $\{u_n\}$ in E at the level c means,

$$I_\lambda(u_n) \to c \text{ and } (1 + \|u_n\|)I'_\lambda(u_n) \to 0.$$

Lemma 4.1. *For all $\lambda \in (0, 1)$, there exist $\alpha_i < \beta_i$ independent of λ such that $\alpha_i \to \infty$ as $i \to \infty$ and the functional I_λ has sequence of critical points $\{u_i(\lambda)\}$ with $I_\lambda(u_i(\lambda)) \in [\alpha_i, \beta_i]$.*

Proof. We split the proof into two steps.

Step 1. For $0 < \lambda < 1$. Let $\{u_n\} \subset E$ be any Cerami sequence of I_λ, i.e.

$$I_\lambda(u_n) \to c \text{ and } (1 + \|u_n\|)\|I'_\lambda(u_n)\| \to 0 \text{ as } n \to \infty. \tag{4.1}$$

From Eq (4.1), we can obtain that

$$\begin{aligned} c + o(1) &= I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2_{H_V} + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x, u_n)u_n - F(x, u_n)\right)\mathrm{d}x. \end{aligned}$$

Thus, $\|u_n\|_{H_V}$ is bounded and then $\|u_n\|_W$ is bounded. Otherwise, if $\{u_n\}$ is unbounded in $W^{1,4}(\mathbb{R}^3)$, $u_n \neq 0$ for all n. For $t \in \mathbb{R}^+$, Lemma 3.1 implies that

$$\alpha(t) := I_\lambda(tu_n)$$

has a positive maximum. Take $t_n \in [0, 1]$,

$$I_\lambda(t_nu_n) = \max_{t\in[0,1]} I_\lambda(tu).$$

We show that $\{I_\lambda(t_nu_n)\}$ is bounded. Indeed, if $t_n = 0$ or $t_n = 1$, it is obvious. Assume $t_n \in (0, 1)$ for all $n \in \mathbb{N}$. It follows from the condition (f_4) and Eq (4.1) that

$$\begin{aligned} I_\lambda(t_nu_n) &= I_\lambda(t_nu_n) - \frac{1}{4}\langle I'_\lambda(t_nu_n), t_nu_n\rangle \\ &= \frac{1}{4}\|t_nu_n\|^2_{H_V} + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x, t_nu_n)u_n - F(x, t_nu_n)\right)\mathrm{d}x \\ &\leq \frac{1}{4}\|u_n\|^2_{H_V} + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x, u_n)u_n - F(x, u_n)\right)\mathrm{d}x \\ &= I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &= c + o_n(1). \end{aligned}$$

Then, set $\omega_n = \|u_n\|_W^{-1} u_n$, up to a subsequence we have

$$\omega_n \rightharpoonup \omega \text{ weakly in } W^{1,4}(\mathbb{R}^3),\ \omega_n \to \omega \text{ strongly in } L^p(\mathbb{R}^3),\ \omega_n \to \omega\ a.e. \text{ on } x \in \mathbb{R}^3.$$

Case 1. $\omega = 0$. For any $M > 0$ and n sufficiently large, from $\|u_n\|_W \to +\infty$, we obtain

$$I_\lambda(t_n u_n) = \max_{t\in[0,1]} I_\lambda(tu_n) \geq I_\lambda(M\omega_n) \geq \frac{1}{4}M^4 - \int_{\mathbb{R}^3} F(x, M\omega_n)\mathrm{d}x. \tag{4.2}$$

By Eq (1.3), we have

$$\left|\int_{\mathbb{R}^3} F(x, M\omega_u)\mathrm{d}x\right| \leq \varepsilon M^2 \int_{\mathbb{R}^3} \omega_n^2 \mathrm{d}x + C_\varepsilon M^p \int_{\mathbb{R}^3} |\omega_n|^p \mathrm{d}x.$$

As $n \to \infty$, by the arbitrariness of ε, we have $\left|\int_{\mathbb{R}^3} F(x, M\omega_u)\mathrm{d}x\right| \to 0$. It follows from Eq (4.2) that

$$\liminf_{n\to\infty} I_\lambda(t_n u_n) \geq \frac{M^4}{4}, \text{ for all } M > 0.$$

From the arbitrariness of M and the boundedness of $\{I_\lambda(t_n u_n)\}$, we have a contradiction.

Case 2. $\omega \neq 0$. The set $\Theta = \{x \in \mathbb{R}^3 : \omega(x) \neq 0\}$ has positive Lebesgue measure. For $x \in \Theta$ that $|u_n(x)| \to \infty$, which together with condition (f_3) implies

$$\frac{F(x, u_n(x))}{|u(x)|^4}|\omega_n(x)|^4 \to \infty \text{ as } n \to \infty.$$

From $I_\lambda(u_n) \to c$, Lemma 2.1-(3), (f_3), Sobolev inequality and Fatou's Lemma, we obtain

$$\begin{aligned}\frac{c + o(1)}{\|u_n\|_W^4} &= \frac{\lambda}{4} + \frac{\|u_n\|_{H_V}^2}{\|u_n\|_W^4} + \frac{\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x}{4\|u_n\|_W^4} + \frac{\int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 \mathrm{d}x}{2\|u_n\|_W^4} - \frac{\int_{\mathbb{R}^3} F(x, u_n)\mathrm{d}x}{\|u_n\|_W^4}\\ &\leq \frac{\lambda}{4} + C_{15} - \int_{\omega\neq 0} \frac{F(x, u_n(x))}{|u_n(x)|^4}|\omega_n(x)|^4 \mathrm{d}x \to -\infty, \text{ as } n \to \infty,\end{aligned}$$

where C_{15} is a constant independent on n. This is impossible.

Summing up the aforementioned arguments, we know that the Cerami sequence of I_λ is bounded. Since the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is compact, similar to Lemma 3.1 in [20], it is easy to check that the sequence u_n possesses a convergent subsequence in E.

Step 2. For all $0 < \lambda < 1$, we show that the functional I_λ has a sequence of critical points $\{u_i(\lambda)\}$ with $I_\lambda(u_i(\lambda)) \in [\alpha_i, \beta_i]$, and $\alpha_i \to \infty$ as $i \to \infty$.

Consider the eigenvalue problem

$$\int_{\mathbb{R}^3} (\nabla u \cdot \nabla\phi + V(x)u\phi)\, \mathrm{d}x = \mu \int_{\mathbb{R}^3} u\phi \mathrm{d}x, \text{ for all } \phi \in H_V^1(\mathbb{R}^3),$$

where μ is a eigenvalue of the operator $L = -\Delta + V$. From the compactness of the embedding $H_V^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$, we obtain that the spectrum $\sigma(L) = \{\mu_1, \mu_2, \cdots, \mu_n, \cdots\}$ of L with

$$\mu_1 < \mu_2 \leq \cdots \leq \mu_n \leq \cdots$$

and $\mu_n \to +\infty$ as $n \to \infty$. Let $e_n \in E$ be corresponding orthogonal eigenfunctions of μ_n. Denote by $E_n = \{e_1, e_2, \cdots, e_n\}$. Then the space E is decomposed as $E = E_n \oplus W_n$ for $n = 1, 2, \cdots$, where W_n is an orthogonal complement to E_n in $H^1_V(\mathbb{R}^3)$.

Hereafter, we use the following notations:

$$Q_\rho = \{u \in E : \ \|u\|^2_{H_V} + \int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x \le \rho^2\},$$

where $\rho > 0$;

$$D_n = E_n \cap Q_{r_n},$$

where r_n is chosen in the following Claim 1; and

$$G_n = \{\phi \in C(D_n, E) : \ \phi \text{ is odd and } \phi|_{\partial Q_{r_n} \cap E_n} = id\}.$$

$$\Gamma_i = \{\phi(\overline{D_n \setminus A}) : \ \phi \in G_n, n \ge i, A = -A \subset E_n \cap Q_{r_n} \text{ is closed and } \gamma(A) \le n - i\},$$

where $\gamma(\cdot)$ is the genus;

$$c_i(\lambda) = \inf_{B \in \Gamma_i} \sup_{u \in B} I_\lambda(u), \ i = 1, 2, \cdots$$

Claim 1. For n dimensional subspace E_n, there exists $r_n > 0$ such that

$$I_\lambda < 0 \text{ on } \overline{E_n \setminus Q_{r_n}}. \tag{4.3}$$

It suffices to prove that for $u \in E_n$,

$$I_\lambda(u) \to -\infty, \text{ as } \|u\|^2_{H_V} + \textstyle\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x \to \infty.$$

For any $\{u_j\} \subset E_n$ with $\|u_j\|^2_{H_V} + \int_{\mathbb{R}^3} u_j^2|\nabla u_j|^2 \mathrm{d}x \to \infty$ as $j \to \infty$, then $\|u_j\| \to \infty$. Set

$$a_j = \frac{u_j}{\|u_j\|}.$$

Clearly $\{a_j\}$ is bounded in E_n, and there exists $a \in E_n \setminus \{0\}$ such that

$$a_j \to a \text{ strongly in } E_n,$$
$$a_j \to a \ a.e. \text{ on } \mathbb{R}^3.$$

For $x \in \{a \neq 0\}$, we obtain

$$|u_j(x)| \to \infty. \tag{4.4}$$

Therefore, from condition (f_3) and Eq (4.4), there is

$$\frac{F(x, u_j)}{\|u_j\|^4} = \frac{F(x, u_j(x))}{|u_j(x)|^4}|a_j(x)|^4 \to \infty \text{ as } j \to \infty.$$

By Fatou's Lemma, we have

$$\textstyle\int_{\mathbb{R}^3} \frac{F(x,u_j)}{\|u_j\|^4} \mathrm{d}x \ge \int_{\{a \neq 0\}} \frac{F(x,u_j)}{\|u_j\|^4} \mathrm{d}x \to +\infty, \text{ as } j \to \infty,$$

and

$$I_\lambda(u_j) = \|u_j\|^4 \left(\frac{\lambda \|u_j\|_W^4}{4\|u_j\|^4} + \frac{\|u_j\|_{H_V}^2}{\|u_j\|^4} + \frac{\int_{\mathbb{R}^3} \phi_{u_j} u_j^2 \mathrm{d}x}{4\|u_j\|^4} + \frac{\int_{\mathbb{R}^3} u_j^2 |\nabla u_j|^2 \mathrm{d}x}{2\|u_j\|^4} - \frac{\int_{\mathbb{R}^3} F(x,u_j)\mathrm{d}x}{\|u_j\|^4} \right)$$
$$\leq \frac{\lambda}{4} + C_{16} - \left(\int_{a\neq 0} + \int_{\{a\neq 0\}} \right) \frac{F(x,u_j)}{\|u_j\|^4}\mathrm{d}x \to -\infty, \text{ as } n \to \infty.$$

Claim 2. For each $B \in \Gamma_i$, if $0 < \rho < r_n$ for all $n \geq i$, $B \cap \partial Q_\rho \cap W_{i-1} \neq \emptyset$.

For a rigorous proof of this Claim, readers can refer to the Lemma 2.4 in [22].

Claim 3. There exist constants $\alpha_i < \beta_i$ such that $c_i(\lambda) \in [\alpha_i, \beta_i]$ and $\alpha_i \to +\infty$ as $i \to \infty$.

Indeed, when $\rho < r_n$ for all $n \geq i$, from Claim 2 and the definition of $c_i(\lambda)$, we obtain

$$c_i(\lambda) \geq \inf_{u\in\partial Q_\rho \cap W_{i-1}} I_\lambda(u) \geq \inf_{u\in\partial Q_\rho \cap W_{i-1}} I(u).$$

For small $\varepsilon > 0$ and $u \in \partial Q_\rho \cap W_{i-1}$, by (V), (f_1), (f_2), the Sobolev inequality and interpolation inequality, there is

$$\begin{aligned}
I_\lambda(u) &\geq I(u) \\
&\geq \frac{1}{2}\|u\|_{H_V}^2 + \frac{1}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \\
&\quad - \varepsilon \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x - C_\varepsilon \int_{\mathbb{R}^3} |u|^p \mathrm{d}x \\
&\geq \frac{1}{4}\|u\|_{H_V}^2 + \frac{1}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 \mathrm{d}x - C_\varepsilon \int_{\mathbb{R}^3} |u|^p \mathrm{d}x \\
&\geq \frac{\rho^2}{4} - C_\varepsilon \|u\|_2^{(1-t)p}\|u\|_{12}^{tp} \\
&\geq \frac{\rho^2}{4} - C_\varepsilon \mu_i^{-\frac{(1-t)p}{2}} \rho^{(1-t)p+\frac{tp}{2}} \\
&= \rho^2 \left(\frac{1}{4} - C_\varepsilon \mu_i^{-\frac{(1-t)p}{2}} \rho^{(1-t)p+\frac{tp}{2}-2} \right),
\end{aligned}$$

where $t \in (0,1)$ satisfies $\frac{1}{p} = \frac{t}{12} + \frac{1-t}{2}$. Take $\rho = \rho_i$ satisfying $\rho_i^{(1-t)p+\frac{tp}{2}-2} = \frac{1}{8C_\varepsilon}\mu_i^{\frac{(1-t)p}{2}}$ and choose $r_n > \rho_n$. Since $\mu_i \to +\infty$ as $i \to \infty$, $I_\lambda(u) \geq \frac{\rho_i^2}{8} := \alpha_i \to +\infty$ as $i \to \infty$. By $c_i(\lambda) \leq c_i(1) := \beta_i$, we have completed this claim.

Claim 4. $c_i(\lambda), i = 1, 2, \cdots$, are critical values of I_λ.

In fact, the Deformation Lemma still holds under the Cerami condition [30]. If $c_i(\lambda)$ is not a critical value of I_λ, by Theorem A.4 in [31], for $0 < \overline{\varepsilon} < \min\{\alpha_i : i = 1, 2, \cdots\}$, there exist $\varepsilon \in (0, \overline{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

(a) $\eta(t,u) = u$ for all $t \in [0,1]$ if $I_\lambda(u) \notin [c_i(\lambda) - \overline{\varepsilon}, c_i(\lambda) + \overline{\varepsilon}]$;

(b) $\eta(1, I_\lambda^{c_i(\lambda)-\varepsilon})$, where $I_\lambda^{c_i(\lambda)-\varepsilon} = \{u \in E : I_\lambda(u) \leq c_i(\lambda) - \varepsilon\}$;

(c) $\eta(t,u)$ is odd in u.

Set $\phi = \eta(1,\cdot)$, Eq (4.3) gives that $\phi = id$ on $\partial Q_{r_n} \cap E_n$ for all n. According to the definition of $c_i(\lambda)$, there exists $B \in \Gamma_i$ such that

$$\sup_{u\in B} I_\lambda(u) \leq c_i(\lambda) + \varepsilon,$$

which means $A = \phi(B) \in \Gamma_i$. By (b), we have

$$c_i(\lambda) \leq \sup_{u \in A} I_\lambda(u) \leq c_i(\lambda) - \varepsilon.$$

This is a contradiction.

Obviously, from previous discussions, for all $0 < \lambda < 1$, the functional I_λ has sequence of critical points $\{u_i(\lambda)\}$ with $I_\lambda(u_i(\lambda)) \in [\alpha_i, \beta_i]$, and $\alpha_i \to \infty$ as $i \to \infty$.

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.1, for each λ and $i \geq 2$, we have a sequence of critical values $c_i(\lambda)$ for which there are critical points $u_i(\lambda)$ of I_λ. Since $c_i(\lambda) \in [\alpha_i, \beta_i]$ for all λ, by using Lemma 3.4, as $\lambda_n \to 0$, we obtain a critical point u_i of I with the critical value in $[\alpha_i, \beta_i]$. Because $\alpha_i \to +\infty$, infinitely many pairs of geometrically distinct solutions $\pm u_i$ have been obtained.

5. Proof of Theorem 1.3

The proof of Theorem 1.3 is analogous to that in Section 3. From Lemma 3.1 and 3.2, we know the functional I_λ on $\mathcal{N}_\lambda$ has a bounded minimizing sequence $\{u_n\}$. The question arises whether the minimizing sequence is convergent or not. In this section, let $E^* := W^{1,4}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, which endowed with the norm

$$\|u\| = \left(\|u\|_W^2 + \|u\|_H^2\right)^{1/2}.$$

Therefore, it is necessary to research some compact properties of the minimizing sequence for I_λ on $\mathcal{N}_\lambda$. Firstly, we can get the following result from P. L. Lions (see [32, 33]),

Lemma 5.1. *Let $r > 0$. If $\{u_n\}$ is bounded in E^* and*

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^3} \int_{B_r(y)} |u_n|^2 dx = 0,$$

we have $u_n \to 0$ strongly in $L^s(\mathbb{R}^3)$ for any $s \in (2, 12)$.

Next we are in a position to study the minimizing sequence for I_λ on $\mathcal{N}_\lambda$.

Lemma 5.2. *Let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence for I_λ. $\{u_n\}$ is bounded in E^*. Furthermore, after a suitable $\mathbb{Z}^3$-translation, passing to a subsequence there exists $u \in \mathcal{N}_\lambda$ such that $u_n \rightharpoonup u$ and $I_\lambda(u) = \inf_{\mathcal{N}_\lambda} I_\lambda$.*

Proof. Let $c = \inf_{\mathcal{N}_\lambda} I_\lambda$. Notice that $\{u_n\}$ is bounded from Lemma 3.2, $u_n \rightharpoonup u$ weakly in E^* after passing to subsequence. If

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^3} \int_{B_r(y)} |u_n|^2 \mathrm{d}x = 0,$$

due to Lemma 5.1, one has $u_n \to 0$ strongly in $L^s(\mathbb{R}^3)$ for any $s \in (2, 12)$. From the above fact and Eq (1.3), we have

$$\int_{\mathbb{R}^3} f(x, u_n) u_n \mathrm{d}x = o(\|u_n\|_W).$$

Therefore,

$$\begin{aligned}0 &= \langle I'_\lambda(u_n), u_n\rangle \\ &= \lambda\|u_n\|_W^4 + \int_{\mathbb{R}^3}\left(|\nabla u_n|^2 + V(x)u_n^2\right)\mathrm{d}x + 2\int_{\mathbb{R}^3} u_n^2|\nabla u_n|^2\mathrm{d}x + \int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} f(x,u_n)u_n\mathrm{d}x \\ &\geq \lambda\|u_n\|_W^4 - o(\|u_n\|_W),\end{aligned}$$

which implies $\|u_n\|_W \to 0$. This is contradictory to Lemma 3.1-(2). Therefore, there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty}\int_{B_r(y_n)} |u_n|^2\mathrm{d}x \geq \delta > 0,$$

where we suppose $y_n \in \mathbb{Z}^3$. Owing to the invariance of I_λ and $\mathcal{N}_\lambda$ under translations, $\{y_n\}$ is bounded in $\mathbb{Z}^3$. Hence, passing to a subsequence we imply $u_n \rightharpoonup u \neq 0$ weakly in E and $\langle I'_\lambda(u), u\rangle = 0$. It follows that $u \in \mathcal{N}_\lambda$. Thus, $I_\lambda(u) \geq c > 0$.

It follows from Eq (1.4), Fatou's lemma and the weakly lower semi-continuity that

$$\begin{aligned}c + o(1) &= I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|_{H_V}^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right)\mathrm{d}x \\ &\geq \frac{1}{4}\|u\|_{H_V}^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,u)u - F(x,u)\right)\mathrm{d}x + o(1) \\ &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle + o(1) \\ &= I_\lambda(u) + o(1),\end{aligned}$$

which implies $I_\lambda(u) \leq c$.

Proof of Theorem 1.3 Using the similar methods of proving Theorem 1.1, by Lemma 5.2, we can prove Theorem 1.3.

6. Proof of Theorem 1.4

Consider the problem

$$\begin{cases} -\Delta u + u + \phi(x)u - \frac{1}{2}u\Delta(u^2) = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

for $p \in [1,\infty)$ and $(u,\phi) \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$. In order to prove nonexistence results, we need to build a related Pohozaev equality for Eq (1.5). With this equality, we can prove that there does not exist nontrivial solutions of Eq (1.5) for $1 \leq p \leq 2$ or $p \geq 12$. For $p \in (2,3]$, we make use of the trick introduced in [5] for the Schrödinger-Poisson system.

Proof of Theorem 1.4 We divide this proof into three steps. First of all, we give the following two nonexistence results. Finally, we prove that there exists a radial solution of Eq (1.5) for $p \in (4, 12)$.

Step 1: When $p \in (2, 3]$.

Suppose that $(u, \phi) \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of Eq (1.5). Multiply the first equation of Eq (1.5) by u and integrate, there is

$$\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 + \phi u^2 + 2u^2|\nabla u|^2\right) \mathrm{d}x - \int_{\mathbb{R}^3} |u|^p \mathrm{d}x = 0. \tag{6.1}$$

By the definition of ϕ, we have that

$$\int_{\mathbb{R}^3} \phi u^2 \mathrm{d}x = \int_{\mathbb{R}^3} |\nabla \phi|^2 \mathrm{d}x.$$

On the other hand,

$$\int_{\mathbb{R}^3} |u|^3 \mathrm{d}x = \int_{\mathbb{R}^3} \langle \nabla \phi, \nabla |u| \rangle \mathrm{d}x.$$

As in [32], we can easily conclude

$$\int_{\mathbb{R}^3} |u|^3 \mathrm{d}x \le \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 \mathrm{d}x.$$

Inserting this inequality into Eq (6.1), we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 + |\nabla \phi|^2 + 2u^2|\nabla u|^2\right) \mathrm{d}x - \int_{\mathbb{R}^3} |u|^p \mathrm{d}x \\ &\ge \int_{\mathbb{R}^3} \left(u^2 + |u|^3 - |u|^p\right) \mathrm{d}x. \end{aligned}$$

It is easy to check that, if $p \in (2, 3]$, the function $l(u) = u^2 + |u|^3 - |u|^p$ is nonnegative and vanishes only at zero. Therefore, u must be equal to zero.

Step 2: When $1 \le p \le 2$ or $p \ge 12$.

For the general case, recall that $(u, \phi) \in H^2_{loc}(\mathbb{R}^3) \times H^2_{loc}(\mathbb{R}^3)$ is a solution of Eq (1.5). Multiply the first equation of the Eq (1.5) by $x \cdot \nabla u$ and integrate by parts on a ball B_R, we deduce

$$\begin{aligned} &-\frac{1}{2}\int_{B_R} |\nabla u|^2 \mathrm{d}x - \frac{3}{2}\int_{B_R} u^2 \mathrm{d}x - \frac{1}{2}\int_{B_R} u^2 (x \cdot \nabla \phi) \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 u^2 \mathrm{d}x \\ &\quad - \frac{3}{2}\int_{B_R} \phi u^2 \mathrm{d}x + \frac{3}{p}\int_{B_R} |u|^p \mathrm{d}x \\ &= \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla u|^2 \mathrm{d}\sigma - \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 \mathrm{d}\sigma - \frac{R}{2}\int_{\partial B_R} u^2 \mathrm{d}\sigma \\ &\quad + \frac{1}{R}\int_{\partial B_R} u^2 |x \cdot \nabla u|^2 \mathrm{d}x - \frac{R}{2}\int_{\partial B_R} u^2 |\nabla u|^2 \mathrm{d}x \\ &\quad - \frac{R}{2}\int_{\partial B_R} \phi u^2 \mathrm{d}\sigma + \frac{R}{p}\int_{\partial B_R} |u|^p \mathrm{d}\sigma. \end{aligned} \tag{6.2}$$

Multiply the Possion equation by $x \cdot \nabla\phi$ and integrate on B_R, we obtain

$$\begin{aligned}\int_{B_R} u^2(x\cdot\nabla\phi)\mathrm{d}x &= \int_{B_R} -\Delta\phi(x\cdot\nabla\phi)\mathrm{d}x \\ &= -\frac{1}{2}\int_{B_R}|\nabla\phi|^2\mathrm{d}x - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla\phi|^2\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R}|\nabla\phi|^2\mathrm{d}\sigma.\end{aligned} \tag{6.3}$$

This together with Eq (6.2) implies

$$\begin{aligned}&-\frac{1}{2}\int_{B_R}\left(|\nabla u|^2 - \frac{1}{2}|\nabla\phi|^2\right)\mathrm{d}x - \frac{3}{2}\int_{B_R}u^2\mathrm{d}x - \frac{1}{2}\int_{B_R}|\nabla u|^2u^2\mathrm{d}x - \frac{3}{2}\int_{B_R}\phi u^2\mathrm{d}x \\ &+\frac{3}{p}\int_{B_R}|u|^p\mathrm{d}x = \frac{1}{2R}\int_{\partial B_R}|x\cdot\nabla u|^2\mathrm{d}\sigma - \frac{R}{2}\int_{\partial B_R}\left(|\nabla u|^2 - \frac{1}{2}|\nabla\phi|^2\right)\mathrm{d}\sigma \\ &-\frac{R}{2}\int_{\partial B_R}u^2\mathrm{d}\sigma + \frac{1}{R}\int_{\partial B_R}u^2|x\cdot\nabla u|^2\mathrm{d}x - \frac{R}{2}\int_{\partial B_R}u^2|\nabla u|^2\mathrm{d}x \\ &-\frac{R}{2}\int_{\partial B_R}\phi u^2\mathrm{d}\sigma + \frac{R}{p}\int_{\partial B_R}|u|^p\mathrm{d}\sigma.\end{aligned} \tag{6.4}$$

A similar method used in [34] can show the existence of a sequence $R_n \to +\infty$ such that the right hand side of Eq (6.4) vanishing. Hence

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(|\nabla u|^2 - \frac{1}{2}|\nabla\phi|^2\right)\mathrm{d}x + 3\int_{\mathbb{R}^3}u^2\mathrm{d}x + \int_{\mathbb{R}^3}|\nabla u|^2u^2\mathrm{d}x \\ &\quad + 3\int_{\mathbb{R}^3}\phi u^2\mathrm{d}x - \frac{6}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x = 0.\end{aligned} \tag{6.5}$$

By

$$\int_{\mathbb{R}^3}|\nabla\phi|^2\mathrm{d}x = \int_{\mathbb{R}^3}\phi u^2\mathrm{d}x,$$

and Eq (6.5) we get

$$-2\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x - 6\int_{\mathbb{R}^3}u^2\mathrm{d}x - 2\int_{\mathbb{R}^3}|\nabla u|^2u^2\mathrm{d}x - 5\int_{\mathbb{R}^3}\phi u^2\mathrm{d}x + \frac{12}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x = 0. \tag{6.6}$$

On the other hand, because $(u, \phi) \in H^2_{loc}(\mathbb{R}^3) \times H^2_{loc}(\mathbb{R}^3)$ is a solution of Eq (1.5),

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^3}u^2\mathrm{d}x + 2\int_{\mathbb{R}^3}|\nabla u|^2u^2\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\phi u^2\mathrm{d}x - \int_{\mathbb{R}^3}|u|^p\mathrm{d}x = 0.\end{aligned} \tag{6.7}$$

Isolate the third term in Eq (6.7) and substitute it in Eq (6.6), we have

$$-\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x - 5\int_{\mathbb{R}^3}u^2\mathrm{d}x - 4\int_{\mathbb{R}^3}\phi u^2\mathrm{d}x + (\frac{12}{p} - 1)\int_{\mathbb{R}^3}|u|^p\mathrm{d}x = 0, \tag{6.8}$$

which indicates that $\int_{\mathbb{R}^3}\phi u^2\mathrm{d}x \geq 0$. Hence, if $p \geq 12$, $u = 0$ is valid from Eq (6.8).

Then we isolate the second term in Eq (6.7) and substitute it in Eq (6.6),

$$4\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x + 10\int_{\mathbb{R}^3}|\nabla u|^2u^2\mathrm{d}x + \int_{\mathbb{R}^3}\phi u^2\mathrm{d}x + (\frac{12}{p}-6)\int_{\mathbb{R}^3}|u|^p\mathrm{d}x = 0. \tag{6.9}$$

If $1 \leq p \leq 2$, Eq (6.9) combining with Lemma 2.1 implies $u = 0$.

Step 3: When $4 < p < 12$.

Denote by

$$H^1_r(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}$$

and consider the problem

$$\begin{cases} -\Delta u + u + \phi(x)u - \frac{1}{2}u\Delta(u^2) = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2,\ u \in H^1_r(\mathbb{R}^3) \cap W^{1,4}(\mathbb{R}^3), & \text{in } \mathbb{R}^3. \end{cases}$$

It is well known that the embedding $H^1_r(\mathbb{R}^3) \cap W^{1,4}(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ is compact for $2 \leq p < 12$. Owing to the symmetric critical principle and the perturbation method, the existence result of Theorem 1.4 is as same as Theorem 1.1. However, now we have the fact that Nemyskii operator corresponding to the nonlinearity f is still compact and the nonlinearity $f(x, u) = |u|^{p-2}u$ ($p \in (4, 12)$) satisfies Ambrosetti-Rabinowitz condition, so some parts of the proofs become simpler than Theorem 1.1. The details are omitted.

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Conflict of interest

The authors declare no conflict of interest.

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