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Research article

Non-smooth dynamics of a SIR model with nonlinear state-dependent impulsive control

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Abstract: The classic SIR model is often used to evaluate the effectiveness of controlling infectious diseases. Moreover, when adopting strategies such as isolation and vaccination based on changes in the size of susceptible populations and other states, it is necessary to develop a non-smooth SIR infectious disease model. To do this, we first add a non-linear term to the classical SIR model to describe the impact of limited medical resources or treatment capacity on infectious disease transmission, and then involve the state-dependent impulsive feedback control, which is determined by the convex combinations of the size of the susceptible population and its growth rates, into the model. Further, the analytical methods have been developed to address the existence of non-trivial periodic solutions, the existence and stability of a disease-free periodic solution (DFPS) and its bifurcation. Based on the properties of the established Poincaré map, we conclude that DFPS exists, which is stable under certain conditions. In particular, we show that the non-trivial order-1 periodic solutions may exist and a non-trivial order-k ($k \geq 1$) periodic solution in some special cases may not exist. Moreover, the transcritical bifurcations around the DFPS with respect to the parameters p and AT have been investigated by employing the bifurcation theorems of discrete maps.

Keywords: SIR model; state-dependent feedback control; disease-free periodic solution; transcritical bifurcation; Poincaré map

1. Introduction

The mathematical model of infectious diseases has always played an important role in the prevention and control of infectious diseases, for example the classical SIR model is usually used to describe the transmission dynamics of infectious diseases among humans such as measles, chickenpox, whooping cough, mumps, etc [1–5]. It not only provides important dynamic descriptions for the evolution of infectious diseases, but also provides important quantitative tools for evaluating the effectiveness of various prevention and control strategies. The above important roles have been more important in the three years of the COVID-19 outbreak. Cui et al. consider the impact of limited medical resources or treatment capacity on infectious disease transmission to develop the SIR model [6–8] with nonlinear term, as shown in Figure 1, which illustrates the relationships between S, I and R, where S, I, R represent the density of susceptible, infected and recovered population, respectively. The corresponding model is as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - \beta S(t)I(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu + d + v)I(t) - \frac{cI(t)}{b + I(t)}, \\ \frac{dR(t)}{dt} = vI(t) + \frac{cI(t)}{b + I(t)} - \mu R(t), \end{cases} \tag{1.1}$$

where

- Λ is the constant recruitment rate of susceptible population, μ is natural death rate, β is the transmission rate;
- d represents the death rate caused by disease. v is the recovery rate without hospital treatment. The funtion $h(I) = cI/(b+I)$ is the recovered population with hospital treatment, where c gives the maximum recovery rate and b is the infected size at which there is 50% saturation ($h(b) = c/2$).

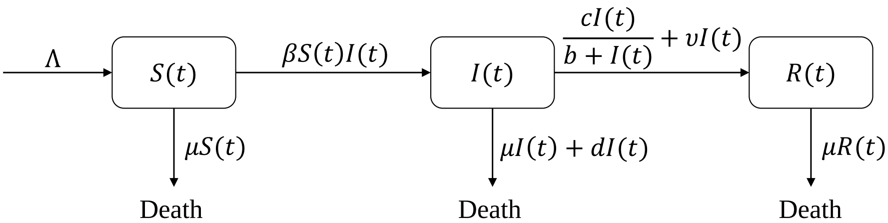


Figure 1. Diagram of the SIR model adopted in the study for simulating certain infectious diseases.

Due to the first two equations being independent of the third one, we can only study the following reduced model:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S(t) - \beta S(t)I(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \frac{cI(t)}{b + I(t)}, \end{cases} \tag{1.2}$$

where $\gamma = \mu + d + v$.

Infectious disease prevention and control has been a hot topic in recent years, usually by controlling the source of infection, cutting off the transmission route and protecting susceptible people. In practice, we often respond to infectious diseases by vaccinating susceptible populations and isolating infected individuals for treatment, which often result in the right function of the dynamical system to be unsmooth or even discontinuous. However the continuous dynamical systems like the above cannot describe the aforementioned situation. By establishing appropriate mathematical model, we can quantify these control measures and analyze the effectiveness of the measures qualitatively or quantitatively, and the state-dependent impulsive model is one type of mathematical model that can well characterize infectious disease control [9–11]. The model assumes that no control measures are implemented when the number of susceptible population is within a certain range, and that comprehensive measures including immunization of the susceptible population and treatment of infected persons are taken when the size of the susceptible population reaches or exceeds the control threshold. Subsequently, numerous scholars have developed a large number of state-dependent impulsive models for different types of infectious disease characteristics and discussed the impact of state-dependent impulsive control strategies on the dynamic behaviors, such as disease elimination and epidemics.

Zhang et al. [12] and Cheng et al. [13] qualitatively analyzed the state-dependent impulsive models under different control measures. There have been relatively systematic studies on single-threshold state-dependent impulsive models [14–20], but it is of concern that if only the size of the susceptible population is used as the basis for control, the following two situations may occur: one in which the size of the susceptible population is small but the growth rate is large, and the other in which there is a larger susceptible population but its growth rate is relatively small. Then in the first case, due to the delay effect, it may not be possible to achieve the desired prevention and control goal; similarly, in the second case, if the growth rate is small, there is no need for infectious disease prevention and control [21–24]. Therefore, this paper integrates the case of threshold control for a convex combination of the size of the susceptible population and its growth rates, which is more complex but closer to reality. Thus, based on model (1.2), we propose the following state-dependent feedback control SIR model:

$$\begin{cases} \left.\begin{aligned} \frac{dS(t)}{dt} &= \Lambda - \mu S(t) - \beta S(t)I(t), \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - \gamma I(t) - \frac{cI(t)}{b+I(t)}, \end{aligned}\right\} & \alpha_1 S(t) + \alpha_2 \frac{dS(t)}{dt} < AT, \\ \left.\begin{aligned} S(t^+) &= (1-p)S(t), \\ I(t^+) &= (1-q)I(t), \end{aligned}\right\} & \alpha_1 S(t) + \alpha_2 \frac{dS(t)}{dt} = AT. \end{cases} \tag{1.3}$$

Here, non-negative constant AT represents the action threshold and non-negative constants α_1 and α_2 are the weight parameter in the threshold condition and satisfies $\alpha_1 + \alpha_2 = 1$. $p \in [0, 1]$ denotes the vaccination rate of the susceptible population and $q \in [0, 1]$ the isolated ratio of the infected population. Considering the practical significance, we assume that the initial value (S_0, I_0) of system (1.3) comes from the domain Ω, where

$$\Omega := \{(S, I) \mid \alpha_1 S_0 + \alpha_2(\Lambda - \mu S_0 - \beta S_0 I_0) < AT, S_0 \geq 0, I_0 \geq 0\}. \tag{1.4}$$

Otherwise, the initial values are taken after an integrated control strategy application [25, 26].

The main purpose of this study is to investigate the dynamics of the proposed state-dependent impulsive model, including but not limited to the periodic solutions and the bifurcations. The organization of the rest part of the paper is as follows: In Section 2, we first give the basic definitions of the state-dependent impulsive model and some lemmas on the stability of the disease-free periodic solution (DFPS). The main properties of the ordinary differential equation (ODE) are also introduced in this section. In Section 3, we address the existence and stability of the periodic solution including DFPS and the non-trivial periodic solution by analyzing the properties of ODE and the Poincaré map.

In Section 4, the transcritical bifurcations have been investigated with respect to two interesting parameters [27–39]. Finally, we summarize the whole work and give some discussions in the last section.

2. Preliminaries and notations

2.1. Planar impulsive semi-dynamic systems and preliminaries

In this section, we will give a brief summary about the main results used in the following section. Consider the following generalized planar impulsive semi-dynamic system

$$\begin{cases} \dfrac{dx}{dt} = F_1(x,y), \quad \dfrac{dy}{dt} = F_2(x,y), \quad if \quad \phi(x,y) \neq 0, \\ \Delta x = \bar{\alpha}(x,y), \quad \Delta y = \bar{\beta}(x,y), \quad if \quad \phi(x,y) = 0. \end{cases} \tag{2.1}$$

Here, $(x,y) \in R_+^2 = \{(x,y)|x \geq 0, y \geq 0\}$, $\triangle x = x^+ - x$ and $\triangle y = y^+ - y$. $F_1,\ F_2,\ \bar{\alpha},\ \bar{\beta}$ are continuous functions from R_+^2 into R. The impulsive function $H : R_+^2 \to R_+^2$ is defined as

$$H(x,y) = (H_1(x,y), H_2(x,y)) = (x + \bar{\alpha}(x,y), y + \bar{\beta}(x,y))$$

and $N^+ = (x^+, y^+)$ is called an impulsive point of $M = (x,y)$. We can define the planar impulsive semi-dynamic system and an order-k periodic solution of model (2.1) in the following based on the notation and definition presented in literatures [17,40,41].

Definition 2.1. *A solution $(x(t),y(t))$ of an impulsive system is said to be an order-k periodic solution with period T, if T is the smallest positive number satisfying $(x(t+kT), y(t+kT)) = (x(t), y(t))$ for all $k \geq 0$ and $t \geq 0$, and the trajectory $(x(t), y(t))$ pulses k times within period T.*

Further, the following analogue of Poincaré criterion can be used to analyze the local stability of an order-k periodic solution.

Lemma 2.1. *([42]) The solution $(x(t),y(t)) = (\xi(t),\eta(t))$ with T-periodic of the system (2.1) is orbitally asymptotically stable if the Floquet multiplier μ_2 satisfies the condition $|\ \mu_2\ | < 1$, where*

$$\mu_2 = \prod_{k=1}^{q} \triangle_k \exp\left[\int_0^T \left(\frac{\partial F_1}{\partial x}(\xi(t),\eta(t)) + \frac{\partial F_2}{\partial y}(\xi(t),\eta(t))\right) dt\right], \tag{2.2}$$

with

$$\Delta_k = \frac{F_1^+\left(\frac{\partial\bar{\beta}}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\bar{\beta}}{\partial x}\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial x}\right) + F_2^+\left(\frac{\partial\bar{\alpha}}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial\bar{\alpha}}{\partial y}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right)}{F_1\frac{\partial\phi}{\partial x} + F_2\frac{\partial\phi}{\partial y}}, \tag{2.3}$$

and F_1, F_2, $\frac{\partial\bar{\alpha}}{\partial x}$, $\frac{\partial\bar{\alpha}}{\partial y}$, $\frac{\partial\bar{\beta}}{\partial x}$, $\frac{\partial\bar{\beta}}{\partial y}$, $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial y}$ can be calculated at the point $(\xi(\tau_k),\eta(\tau_k))$, and $F_1^+ = F_1(\xi(\tau_k^+),\eta(\tau_k^+))$, $F_2^+ = F_2(\xi(\tau_k^+),\eta(\tau_k^+))$. Here $\phi(x,y)$ is sufficiently smooth such that grad $\phi(x,y) \neq 0$, and $\tau_k (k \in N)$ is the moment of the k-th impulse effect.

In order to discuss the bifurcation of the Poincaré map, we introduce the following lemma:

Lemma 2.2. *(Transcritical bifurcation [43]) Let* : $U \times \Theta \longrightarrow \mathbb{R}$ *define a one-parameter family of maps* $\mathcal{P}(I, \alpha)$*, where* $\mathcal{P}$ *is* C^r *with* $r \geq 2$*, and* U*,* Θ *are open intervals of the real line containing 0. Assume that*

$$\mathcal{P}(0, \alpha) = 0 \text{ for all } \alpha, \frac{\partial \mathcal{P}}{\partial I}(0, 0) = 1, \ \frac{\partial^2 \mathcal{P}}{\partial I \partial \alpha}(0, 0) > 0, \ \frac{\partial^2 \mathcal{P}}{\partial I^2}(0, 0) > 0.$$

Then there are $\alpha_1 < 0 < \alpha_2$ *and* $\varepsilon > 0$ *such that:*

(i) If $\alpha_1 < \alpha < 0$*,* $\mathcal{P}_\alpha$ *has two fixed points, 0 and* $I_{1\alpha} > 0$ *in* $(-\varepsilon, \varepsilon)$*. The origin is asymptotically stable, the other fixed point is unstable.*

(ii) If $0 < \alpha < \alpha_2$*,* $\mathcal{P}_\alpha$ *has two fixed points, 0 and* $I_{1\alpha} < 0$ *in* $(-\varepsilon, \varepsilon)$*. The origin is unstable, the other fixed point is asymptotically stable.*

Here again, the case $\frac{\partial^2 \mathcal{P}}{\partial I \partial \alpha}(0, 0) < 0$ is handled by making the change of parameter $\alpha \to -\alpha$.

2.2. The main properties of ODE system

Firstly, we focus on the ODE system:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu S(t) - \beta S(t) I(t) \\ \dfrac{dI(t)}{dt} = \beta S(t) I(t) - \gamma I(t) - \dfrac{c I(t)}{b + I(t)}. \end{cases} \tag{2.4}$$

It's easy to know that system (2.4) has one disease-free equilibrium at $E_0 = (\frac{\Lambda}{\mu}, 0)$. By calculating the Jacobian matrix at E_0, we have the following lemma.

Lemma 2.3. E_0 *is locally asymptotically stable if* $R_0 := \frac{\Lambda b \beta}{\mu(b\gamma + c)} \leq 1$ *and unstable if* $R_0 > 1$.

Then we focus on the existence and stability of the endemic equilibrium. Let

$$\begin{cases} \Lambda - \mu S - \beta S I = 0 \\ \beta S I - \gamma I - \frac{cI}{b+I} = 0, \end{cases} \tag{2.5}$$

then

$$\begin{cases} S = \dfrac{1}{\beta}(\gamma + \dfrac{c}{b + I}) \\ I^2 + a_1 I + a_2 = 0, \end{cases} \tag{2.6}$$

where $a_1 = \frac{\beta\gamma b + \mu\gamma + c\beta - \Lambda\beta}{\beta\gamma}, a_2 = \frac{\mu\gamma b + c\mu - \Lambda\beta b}{\beta\gamma} = \frac{\Lambda b}{\gamma}(\frac{1}{R_0} - 1)$. The positive root of $I^2 + a_1 I + a_2 = 0$ implies the existence of the endemic equilibrium. So it's easy to prove the following lemma:

Lemma 2.4. *For the existence of the endemic equilibrium, we have the following conclusion:*

1. *If* $R_0 > 1$*, then there exists a unique endemic equilibrium.*
2. *If* $R_0 = 1$*, then there is no endemic equilibrium when* $a_1 \geq 0$ *and there exists a unique endemic equilibrium when* $a_1 < 0$.
3. *If* $R_0 < 1$ *and* $a_1 \geq 0$ *, then there is no endemic equilibrium.*

4. *If $R_0 < 1$ and $a_1 < 0$, then we have $R_0^* = (\frac{a_1^2\gamma}{4\Lambda b} + 1)^{-1}$ and there is no endemic equilibrium when $R_0 < R_0^*$, otherwise there exists a unique endemic equilibrium of multiplicity two or two endemic equilibria when $R_0 = R_0^*$ or $R_0 > R_0^*$, respectively.*

If $R_0^* < R_0 < 1$ and $a_1 < 0$, $E_1^*(S_1^*, I_1^*)$ and $E_2^*(S_2^*, I_2^*)$ are the corresponding equilibria, where

$$S_1^* = \frac{1}{\beta}(\gamma + \frac{c}{b + I_1^*}), \quad S_2^* = \frac{1}{\beta}(\gamma + \frac{c}{b + I_2^*}),$$

$$I_1^* = \frac{-a_1 + \sqrt{\Delta}}{2}, \quad I_2^* = \frac{-a_1 - \sqrt{\Delta}}{2},$$

and $\Delta = a_1^2 - 4a_2$.

The Jacobian of the system (2.4) at $E^*(S^*, I^*)$ is

$$\begin{pmatrix} -\mu - \beta I^* & -\beta S^* \\ \beta I^* & \frac{cI^*}{(b+I^*)^2} \end{pmatrix}$$

and the characteristic equation is given by

$$\lambda^2 + H(I^*)\lambda + \beta I^* G(I^*) = 0, \tag{2.7}$$

where

$$H(I^*) = \mu + \beta I^* - \frac{cI^*}{(b + I^*)^2}, G(I^*) = \gamma + \frac{c}{(b + I^*)^2}(b - \frac{\mu}{\beta}). \tag{2.8}$$

Therefore, we have the following lemmas about the stability of the endemic equilibrium:

Lemma 2.5 ([7]). *If $R_0 > 1, b > \frac{\mu}{\beta}$, then the endemic equilibrium E^* is a stable node or focus when $H(I^*) > 0$; E^* is an unstable node or focus when $H(I^*) < 0$; E^* is a center when $H(I^*) = 0$.*

Lemma 2.6 ([7]). *If $R_0^* < R_0 < 1, b > \frac{\mu}{\beta}$ and $a_1 < 0$, then the endemic equilibrium E_2^* is a saddle; E_1^* is a stable node or focus when $H(I^*) > 0$; E_1^* is an unstable node or focus when $H(I^*) < 0$; E_1^* is a center when $H(I^*) = 0$.*

2.3. Notations

We firstly denote some essential curves for the further study in the next section.

The vertical and horizontal isoclines of system (1.2) are shown below,

$$L_1 : I = \frac{\Lambda}{\beta S} - \frac{\mu}{\beta} \text{ and } L_2 : I = \frac{c}{\beta S - \gamma} - b,$$

which intersect with S-axis at $(\frac{\Lambda}{\mu}, 0)$ and $(\frac{1}{\beta}(\frac{c}{b} + \gamma), 0)$, respectively. The positional relationship of the two points is determined by R_0, that is, $\frac{\Lambda}{\mu} > \frac{1}{\beta}(\frac{c}{b} + \gamma)$ when $R_0 > 1$ and $\frac{\Lambda}{\mu} < \frac{1}{\beta}(\frac{c}{b} + \gamma)$ when $R_0 < 1$. In addition, the intersection point of L_1 and L_2 in the first quadrant is the endemic equilibrium of system (1.2).

In the phase plane, we can define the impulsive curve L_M and the phase curve L_N, which once the trajectory intersects with L_M, it will impulse to L_N. And we can get the expression of L_M and L_N by $\alpha_1 S(t) + \alpha_2 \frac{dS(t)}{dt} = AT$ and impulse functions [21–24].

When $\alpha_1 = 1$, the impulsive curve L_M and the phase curve L_N of system (1.3) are straight lines $S = AT$ and $S = (1-p)AT$, respectively.

When $\alpha_1 \in [0, 1)$, the impulsive curve L_M of system (1.3) is a curve $I = L_M(S)$, where

$$L_M(S) = -\frac{AT - \alpha_2\Lambda}{\alpha_2\beta S} + \frac{(\alpha_1 - \alpha_2\mu)}{\alpha_2\beta}, \tag{2.9}$$

which follows from $\alpha_1 S(t) + \alpha_2 \frac{dS(t)}{dt} = AT$. L_M intersects with S-axis and L_1 at $M_0(S_v, 0)$ and $M_1(S_L, L_1(S_{l1}))$, where

$$S_v = \frac{AT - \alpha_2\Lambda}{\alpha_1 - \alpha_2\mu} \text{ and } S_{l1} = \frac{AT}{\alpha_1}, \tag{2.10}$$

respectively. Based on the biological significance of system (1.3), here

$$AT - \alpha_2\Lambda > 0 \text{ and } \alpha_1 - \alpha_2\mu > 0. \tag{2.11}$$

And thus function $I = L_M(S)$ monotonically increases with respect to S. The phase curve L_N is represented by an increasing function $I = L_N(S)$ correspondingly, where

$$L_N(S) = (1-q)L_M\left(\frac{S}{1-p}\right). \tag{2.12}$$

By solving $L_M(S) = L_N(S)$, we conclude that curves L_M and L_N have a unique intersection point, denoted as $(S_{mn}, L_M(S_{mn}))$, where $S_{mn} = \frac{1-(1-p)(1-q)}{q} S_v > S_v$.

We define some auxiliary functions as follows (see more details in [44, 45]),

$$P(S,I) := L_M(S) - I = -\frac{AT - \alpha_2\Lambda}{\alpha_2\beta S} + \frac{(\alpha_1 - \alpha_2\mu)}{\alpha_2\beta} - I,$$

$$Q(S,I) := L_N(S) - I = -(1-p)(1-q)\frac{AT - \alpha_2\Lambda}{\alpha_2\beta S} + (1-q)\left(\frac{(\alpha_1 - \alpha_2\mu)}{\alpha_2\beta} - I\right).$$

Thus,

$$(P_S, P_I) = (\frac{AT - \alpha_2\Lambda}{\alpha_2\beta S^2}, -1) \text{ and } (Q_S, Q_I) = ((1-p)(1-q)\frac{AT - \alpha_2\Lambda}{\alpha_2\beta S^2}, -(1-q))$$

represent the normal vectors of the impulsive curve L_M and the phase curve L_N, respectively. Let $(\frac{dS}{dt}, \frac{dI}{dt})$ denote the tangent vector of the phase orbit of system (1.2). Denote

$$\sigma_M(S,I) = (P_S, P_I) \cdot \left(\frac{dS}{dt}, \frac{dI}{dt}\right) \text{ and } \sigma_N(S,I) = (Q_S, Q_I) \cdot \left(\frac{dS}{dt}, \frac{dI}{dt}\right), \tag{2.13}$$

which are useful in section 3. Note that for a point (S, I) satisfying $P(S, I) = 0$,

- if $\sigma_M(S, I) > 0$, a trajectory pulses at point (S, I) and pulses to point $((1-p)S, (1-q)I)$;
- if $\sigma_M(S, I) = 0$, a trajectory is tangent to the impulsive curve L_M and pulses to point $((1-p)S, (1-q)I)$;
- if $\sigma_M(S, I) < 0$, a trajectory passes through point (S, I) and no pulse occurs at this point.

For a point (S, I) satisfying $Q(S, I) = 0$,

- if $\sigma_N(S, I) > 0$, a trajectory passes through point (S, I) from above curve L_N to below;
- if $\sigma_N(S, I) = 0$, a trajectory is tangent to the phase curve L_N at (S, I);
- if $\sigma_N(S, I) < 0$, a trajectory passes through point (S, I) from below curve L_N to above.

3. Existence and stability of periodic solution

3.1. Disease-free periodic solution

Let $I(0) = 0$ in system (1.3), then $I(t) \equiv 0$ and $S(t)$ satisfies the following equations:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu S(t), & S(t) < S_v, \\ S(t^+) = (1-p)S(t), & S(t) = S_v, \end{cases} \tag{3.1}$$

where $S_v = \frac{AT-\alpha_2\Lambda}{\alpha_1-\alpha_2\mu} > 0$ since both $\alpha_1 - \alpha_2\mu$ and $AT - \alpha_2\Lambda$ are positive in this article.

Solving the first equation of system (3.1) with the initial condition $S(0) = (1-p)S_v$, we have

$$S(t) = \frac{\Lambda}{\mu} - \frac{\Lambda - \mu(1-p)S_v}{\mu} e^{-\mu t}.$$

Obviously, if $S_v < \frac{\Lambda}{\mu}$, then the model (3.1) has a periodic solution $\tilde{S}(t)$ with period T, here

$$T = -\frac{1}{\mu} \ln \frac{\Lambda - \mu S_v}{\Lambda - \mu(1-p)S_v} > 0 \tag{3.2}$$

and

$$\tilde{S}(t) = \frac{\Lambda}{\mu} - \frac{\Lambda - \mu(1-p)S_v}{\mu} e^{-\mu(t-(n-1)T)}, \;\; t \in ((n-1)T, nT], \;\; n \in \mathbb{N}. \tag{3.3}$$

It follows from the relationship of system (1.3) and system (3.1) that the following theorem holds true naturally.

Theorem 3.1. *If $S_v < \frac{\Lambda}{\mu}$, then system* (1.3) *has a DFPS* $(\tilde{S}(t), 0)$ *with period T.*

Using Lemma 2.1, the stabilities of the DFPS are shown in the following theorems.

Theorem 3.2. *If $R_0 \leq 1$ and $S_v < \frac{\Lambda}{\mu}$, then the DFPS* $(\tilde{S}(t), 0)$ *of system* (1.3) *is locally orbitally asymptotically stable.*

Proof. Under the assumptions $R_0 \leq 1$ and $S_v < \frac{\Lambda}{\mu}$, we claim that the DFPS $(\tilde{S}(t), 0)$ of system (1.3) is orbitally asymptotically stable. For system (1.3), there are $F_1(S,I) = \Lambda - \mu S - \beta SI$, $F_2(S,I) = \beta SI - \gamma I - \frac{cI}{b+I}$, $\bar{\alpha}(S,I) = -pS$, $\bar{\beta}(S,I) = -qI$, $\phi(S,I) = \alpha_1 S + \alpha_2(\Lambda - \mu S - \beta SI) - AT$, $(\tilde{S}(T), 0) = (S_v, 0)$, and $(\tilde{S}(T^+), 0) = ((1-p)S_v, 0)$. Thereby,

$$\begin{gathered} \tfrac{\partial F_1}{\partial S} = -\mu - \beta I, \quad \tfrac{\partial F_2}{\partial I} = \beta S - \gamma - \tfrac{bc}{(b+I)^2}, \quad \tfrac{\partial \bar{\alpha}}{\partial S} = -p, \quad \tfrac{\partial \bar{\beta}}{\partial I} = -q, \\ \tfrac{\partial \bar{\alpha}}{\partial I} = \tfrac{\partial \bar{\beta}}{\partial S} = 0, \quad \tfrac{\partial \phi}{\partial S} = \alpha_1 - \alpha_2\mu - \alpha_2\beta I, \quad \tfrac{\partial \phi}{\partial I} = -\alpha_2\beta S, \end{gathered} \tag{3.4}$$

$$\begin{aligned} \triangle_1 &= \frac{F_1^+\left(\frac{\partial \bar{\beta}}{\partial I}\frac{\partial \phi}{\partial S} - \frac{\partial \bar{\beta}}{\partial S}\frac{\partial \phi}{\partial I} + \frac{\partial \phi}{\partial S}\right) + F_2^+\left(\frac{\partial \bar{\alpha}}{\partial S}\frac{\partial \phi}{\partial I} - \frac{\partial \bar{\alpha}}{\partial I}\frac{\partial \phi}{\partial S} + \frac{\partial \phi}{\partial I}\right)}{F_1\frac{\partial \phi}{\partial S} + F_2\frac{\partial \phi}{\partial I}} \\ &= \frac{(1-q)F_1^+}{F_1} = \frac{(1-q)F_1(\tilde{S}(T^+), 0)}{F_1(\tilde{S}(T), 0)} = (1-q)\frac{\Lambda - \mu(1-p)S_v}{\Lambda - \mu S_v}, \end{aligned}$$

and

$$\exp\left(\int_0^T\left(\frac{\partial F_1}{\partial S}(\tilde{S}(t),0)+\frac{\partial F_2}{\partial I}(\tilde{S}(t),0)\right)dt\right)$$
$$=\exp\left(\int_0^T\left(-\mu-\gamma-\frac{c}{b}+\beta\tilde{S}(t)\right)dt\right)$$
$$=\left(\frac{\Lambda-\mu S_v}{\Lambda-\mu(1-p)S_v}\right)^{\frac{b\mu^2+b\mu\gamma+c\mu-\Lambda b\beta}{b\mu^2}}\exp\left(-\frac{p\beta S_v}{\mu}\right).$$

Taking the above two equations into formula (2.2) yields

$$\begin{aligned}\mu_2=&\triangle_1\exp\left(\int_0^T\left(\frac{\partial F_1}{\partial S}(\tilde{S}(t),0)+\frac{\partial F_2}{\partial I}(\tilde{S}(t),0)\right)dt\right)\\=&(1-q)\left(\frac{\Lambda-\mu S_v}{\Lambda-\mu(1-p)S_v}\right)^{\frac{b\mu\gamma+c\mu-\Lambda b\beta}{b\mu^2}}\exp\left(-\frac{p\beta S_v}{\mu}\right),\end{aligned}\tag{3.5}$$

where $(1-q)\exp\left(-\frac{p\beta S_v}{\mu}\right)\in[0,1)$. Moreover, It follows from $S_v<\frac{\Lambda}{\mu}$ and $R_0=\frac{\Lambda b\beta}{\mu(b\gamma+c)}\leq 1$ that $\left(\frac{\Lambda-\mu S_v}{\Lambda-\mu(1-p)S_v}\right)^{\frac{b\mu\gamma+c\mu-\Lambda b\beta}{b\mu^2}}\in(0,1]$. Therefore, when $S_v<\frac{\Lambda}{\mu}$ and $R_0\leq 1$ there is $|\mu_2|<1$, i.e., $(\tilde{S}(t),0)$ is orbitally asymptotically stable. □

The attraction domain of DFPS when $R_0\leq 1$ and $S_v<\frac{\Lambda}{\mu}$ will be introduced in the next subsection (see Theorem 3.4). With regard to $R_0>1$, we have the following conclusion:

Theorem 3.3. *If $R_0>1$ and $S_v\leq(\gamma+\frac{c}{b})/\beta$ hold true, then the DFPS is locally orbitally asymptotically stable.*

Proof. It follows from Eq (3.5) that

$$\begin{aligned}\mu_2|_{q=0}=&\left(\frac{\Lambda-\mu S_v}{\Lambda-\mu(1-p)S_v}\right)^{\frac{b\mu\gamma+c\mu-\Lambda b\beta}{b\mu^2}}\exp\left(-\frac{p\beta S_v}{\mu}\right)\\=&\exp\left(\int_{(1-p)S_v}^{S_v}\frac{\beta s-\gamma-\frac{c}{b}}{\Lambda-\mu s}ds\right)>0.\end{aligned}\tag{3.6}$$

Let

$$f(s):=\frac{\beta s-\gamma-\frac{c}{b}}{\Lambda-\mu s}.$$

It follows from $R_0>1$ that

$$\frac{df(s)}{ds}=\frac{\Lambda b\beta-b\mu\gamma-c\mu}{b(\Lambda-\mu s)^2}>0,$$

which indicates that $f(s)$ is increasing and $f(\frac{1}{\beta}(\gamma+\frac{c}{b}))=0$. Thus, if $S_v\leq\frac{1}{\beta}(\gamma+\frac{c}{b})$ then $f(s)\leq 0$ for all $s\in[(1-p)S_v,S_v]$ and $0<\mu_2|_{q=0}\leq 1$. Moreover, it follows from the formula of μ_2 that we have

$$\frac{\partial\mu_2}{\partial q}=-\mu_2|_{q=0}<0,\ \mu_2|_{q=1}=0.$$

Therefore, if $R_0>1$ and $S_v\leq\frac{1}{\beta}(\gamma+\frac{c}{b})$, i.e., $0<\mu_2|_{q=0}\leq 1$, then $|\mu_2|=(1-q)(\mu_2|_{q=0})<1$ holds for all $q\in(0,1]$, which means that the DFPS is locally orbitally asymptotically stable. □

Corollary 3.1. *If $R_0 > 1$, $(\gamma + \frac{c}{b})/\beta \leq (1-p)S_v < S_v < \frac{\Lambda}{\mu}$ and $q = 0$ hold true, then the DFPS is unstable.*

Remark 3.1. *When $R_0 > 1$, $(1-p)S_v < (\gamma + \frac{c}{b})/\beta < S_v < \frac{\Lambda}{\mu}$ and $q = 0$ hold true, $\mu_2 = 1$ may occur, which means that the bifurcation phenomenon may exist near the DFPS with respect to the critical parameters (see Section 4).*

3.2. Non-trivial periodic solution

In this section, we first define the Poincaré map of system (1.3), and then discuss its main properties, which help us to discuss the non-trivial periodic solution of the system.

3.2.1. Formation of Poincaré map $\mathcal{P}_{\mathcal{M}}$

Suppose $\mathcal{N}_0$ is the phase set and $\mathcal{M}_0$ is the impulsive set. If the solution starting from $N_n^+(S_n^+, I_n^+) \in \mathcal{N}_0 \subset L_N$ will arrive at the threshold line L_M for the first time after a finite time, then the intersection point can be marked as $M_{n+1}(S_{n+1}, I_{n+1})$, as shown in Figure 2. Point $M_{n+1} \in \mathcal{M}_0$ and it will pulse to point $N_{n+1}^+(S_{n+1}^+, I_{n+1}^+) \in \mathcal{N}_0$. The relation between N_n^+ and N_{n+1}^+ is determined by the solution of the ODE system. Thus we define

$$S_{n+1} := P_M(S_n^+, I_n^+), \quad I_{n+1} := P_N(S_n^+, I_n^+),$$

where

$$I_n^+ = L_N(S_n^+), \quad I_{n+1} = L_M(S_{n+1}).$$

Therefore we have the following difference equations:

$$\begin{cases} S_{n+1} = P_M(S_n^+, I_n^+), \\ I_{n+1} = L_M(P_M(S_n^+, I_n^+)), \end{cases} \tag{3.7}$$

i.e., we have

$$\begin{cases} S_{n+1}^+ = (1-p)P_M(S_n^+, I_n^+), \\ I_{n+1}^+ = (1-q)L_M(P_M(S_n^+, I_n^+)). \end{cases} \tag{3.8}$$

Then a Poincaré map can be defined as follows:

$$\mathcal{P}_{\mathcal{M}}(S_n^+) := S_{n+1}^+ = (1-p)P_M(S_n^+, L_N(S_n^+)). \tag{3.9}$$

According to the existence of the endemic equilibrium, we discuss the existence and stability of system (1.3) by analyzing the properties of Poincaré map $\mathcal{P}_{\mathcal{M}}$ in the following cases. It is worth reiterating that we assume that the initial value (S_0, I_0) comes from the domain Ω where

$$\Omega := \{(S, I) \mid I > L_M(S), S \geq 0, I \geq 0\}. \tag{3.10}$$

3.2.2. The system has no endemic equilibrium

Suppose $R_0 \leq 1$ and $a_1 \geq 0$ or $R_0 < 1$ and $R_0 < R_0^*$ in this subsection. Lemma 2.4 shows that under the above conditions system (1.2) has a unique disease-free equilibrium $(\frac{\Lambda}{\mu}, 0)$, which is globally stable. Combining it with $(1-p)S_v < \frac{\Lambda}{\mu}$, we can conclude that any trajectory of system (1.3) undergoes finitely times impulses or is free from pulse effect, and then tends to $(\frac{\Lambda}{\mu}, 0)$ when $\frac{\Lambda}{\mu} \leq S_v < \frac{1}{\beta}(\frac{c}{b} + \gamma)$. When $S_v \geq \frac{1}{\beta}(\frac{c}{b} + \gamma)$, by calculating the function $\sigma_M(S, L_M(S))$ defined in (2.13), we can easily get that $\sigma_M(S, L_M(S)) < 0$ for all $S \geq S_v$, that is, system (1.3) has no impulsive effect in this case. Thus, system (1.3) may experience infinitely many impulsive effects only if $S_v < \frac{\Lambda}{\mu}$. So we discuss the dynamics of system (1.3) under the condition $S_v < \frac{\Lambda}{\mu}$ in the following text.

Firstly, the curves mentioned in Section 2.3, L_1, L_2, L_M, L_N, divide the plane into several parts. For example, if the relative positions of the impulsive curve and the isoclinic lines are shown in the Figure 2, we marked the parts as $V_1, V_2, ..., V_9$ (see Figure 2).

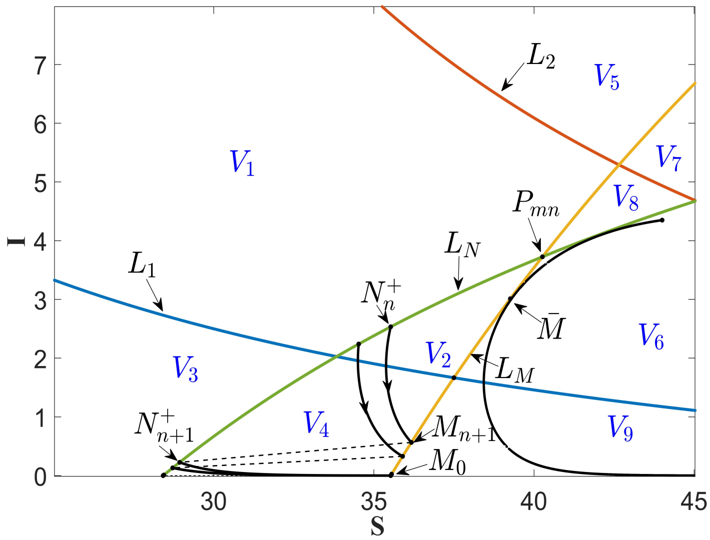


Figure 2. Phase diagram of system (1.3) when $R_0 \leq 1$, $a_1 \geq 0$ or $R_0 < R_0^*$, and $S_{mn} > (1-p)S_\sigma$. The parameters are $\Lambda = 15, \mu = 0.2, \beta = 0.12, \gamma = 1.5, b = 3, c = 30, AT = 30, p = 0.2, q = 0.6, \alpha_1 = 0.8, \alpha_2 = 0.2$.

What's more, by calculating the function $\sigma_M(S, L_M(S))$, we can get that there exists at least one point $\bar{M}(S_\sigma, L_M(S_\sigma))$ satisfying $\sigma_M(\bar{M}) = 0$. Therefore, the precise impulsive set is $\mathcal{M}_0 = \{(S, L_M(S))|S \in \mathbb{D})\}$ and the precise phase set is $\mathcal{N}_0 = \{(S, L_N(S))|S \in (1-p)\mathbb{D}\}$, where

$$\mathbb{D} := [S_v, S_d], \quad S_d := \min\left\{S_\sigma, \frac{S_{mn}}{1-p}\right\}, \tag{3.11}$$

$S_\sigma \in (S_v, +\infty)$ is the root of $\sigma_M(S, L_M(S)) = 0$, and S_{mn} is the unique root of $L_M(S) - L_N(S) = 0$.

We first discuss the following in the situation that $S_{mn} > (1-p)S_\sigma$ (see Figure 2). Any trajectory from initial point in part V_1, part V_2 ,part V_3 and part V_5 will stay in part V_4 after a finite number of impulsive effects. However, we have $\frac{dI}{dt} < 0$ in part V_4, and the impulsive effect will decrease S and I. Thus, the trajectory will decreasingly converge to the disease free periodic solution and will not have a non-trivial periodic solution (Figure 2). The domain of attraction of DFPS $(\tilde{S}(t), 0)$ is Ω. Then if $S_{mn} < (1-p)S_\sigma$, we denote the trajectory passing through the intersection of L_M and L_N, P_{mn}, as Γ_{mn}, which divides Ω as Ω_1(above) and Ω_2(below). Apparently, any trajectory from an initial point in Ω_1 will converge to DFPS $(\tilde{S}(t), 0)$ with the impulsive effect and any trajectory from an initial point in Ω_2 undergoes at most one pulse and then tends to the boundary equilibrium $(\frac{\Lambda}{\mu}, 0)$. So we have the following theorem:

Theorem 3.4. *Suppose $R_0 \le 1$ and $a_1 \ge 0$ or $R_0 < 1$ and $R_0 < R_0^*$. When $S_v \ge \frac{\Lambda}{\mu}$ then $(\frac{\Lambda}{\mu}, 0)$ is globally stable on Ω. When $S_v < \frac{\Lambda}{\mu}$ then $(\frac{\Lambda}{\mu}, 0)$ and DFPS $(\tilde{S}(t), 0)$ are bistable and their domains of attraction are Ω_1 and Ω_2, respectively.*

3.2.3. The system has one endemic equilibrium

Based on the analysis in Section 2.2 (Lemma 2.4) and the relative position between the isoclinic lines and the impulsive curve, we will discuss the following cases:

- case (A): $R_0 > 1$
  - case (a_1): $S_v > \frac{\Lambda}{\mu}$
  - case (a_2): $\frac{1}{\beta}(\gamma + \frac{c}{b}) < S_v < \frac{\Lambda}{\mu}$
  - case (a_3): $S_v < \frac{1}{\beta}(\gamma + \frac{c}{b})$
- case (B): $R_0 < 1, a_1 < 0, R_0 = R_0^*$

In case (a_1), the trajectory of system (1.3) has no impulsive effect since $\sigma_M(S, L_M(S)) < 0$ for all $S \ge S_v$, that is, the solution of system (1.3) is determined by the ODE system (2.4) completely.

In case (a_2), it's easy to calculate that $\sigma_M(M_0) > 0$ and $\sigma_M(M_1) < 0$, where $M_0(S_v, 0)$ and $M_1(S_{l1}, L_1(S_{l1}))$ are the intersections of pulse curve L_M and S-axis or L_1, respectively (see Figure 3(a)). So there exists at least one point $\bar{M}$ satisfying $\sigma_M(\bar{M}) = 0$. Assume that $\bar{M}(S_\sigma, L_M(S_\sigma))$ is the only point within the first quadrant that satisfies $\sigma_M(\bar{M}) = 0$. Here, $S_\sigma \in (S_v, S_{l1})$. Under this assumption, the precise impulsive set is $\mathcal{M}_0 = \{(S, L_M(S))|S \in \mathbb{D})\}$ and the precise phase set is $\mathcal{N}_0 = \{(S, L_N(S))|S \in (1-p)\mathbb{D}\}$, where $\mathbb{D}$ is shown in Eq (3.11).

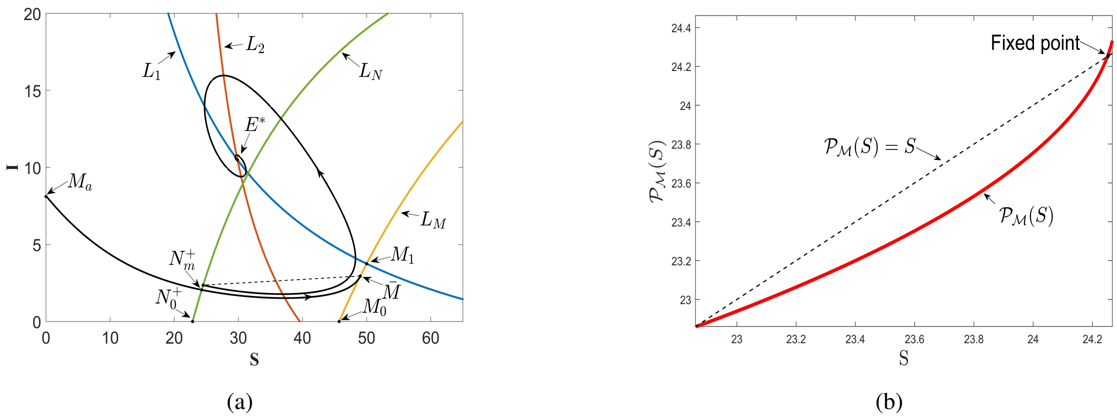


Figure 3. (a) is the phase diagram of system (1.3) when $R_0 > 1$ and $\frac{1}{\beta}(\gamma + \frac{c}{b}) < S_v < \frac{\Lambda}{\mu}$, and (b) is the corresponding Poincaré map $\mathcal{P}_\mathcal{M}$. The parameters are $\Lambda = 40, \mu = 0.5, \beta = 0.08, \gamma = 1.5, b = 12, c = 20, AT = 40, p = 0.5, q = 0.2, \alpha_1 = 0.8, \alpha_2 = 0.2$.

Assume $b > \frac{\mu}{\beta}$ and $H(I^*) > 0$ in case (a_2), which ensures that E^* is a globally stable focus or node of system (1.2) (see Lemma 2.5). Then there is a unique point M_a either on the I-axis or curve L_M and the trajectory of system (1.2) starting from M_a first reaches L_M after a finite time $t > 0$ and intersects at point $\bar{M}$ when $S_\sigma < \frac{S_{mn}}{1-p}$. The trajectory $\widehat{M_a\bar{M}}$ divides Ω as Ω_3 and Ω_4, where Ω_3 (resp. Ω_4) at the left (resp. right) of trajectory $\widehat{M_a\bar{M}}$. Any trajectory starting from Ω_3 will be free from impulsive effects and tend to E^*. Any trajectory starting from Ω_4 will pulse finite (infinite) times. Then by discussing the properties of the Poincaré map $\mathcal{P}_\mathcal{M}$ defined in Eq (3.9), we have the following main results:

Theorem 3.5. *Suppose $R_0 > 1, \frac{1}{\beta}(\gamma + \frac{c}{b}) < S_v < \frac{\Lambda}{\mu}$, $b > \frac{\mu}{\beta}$, $H(I^*) > 0$, S_σ is the unique root of $\sigma_M(S, L_M(S)) = 0$ on interval (S_v, S_{l1}) and $S_\sigma < \frac{S_{mn}}{1-p}$.*

(*i*) *When $\widehat{M_a\bar{M}}$ and L_N have at least one intersection point, denoting the smaller one is $N_m^+(S_m, L_N(S_m))$, then the domain of map $\mathcal{P}_\mathcal{M}$ is $\mathbb{D}_1 = [(1-p)S_v, S_m]$ and $\mathcal{P}_\mathcal{M}$ are increasing and continuous, that is, system* (1.3) *does not have order-k ($k \geq 2$) periodic solution. Moreover, if $S_m < (1-p)S_\sigma$, map $\mathcal{P}_\mathcal{M}(S)$ has at least one fixed point on interval $\mathbb{D}_1$ (see Figure 3(b)), that is, system* (1.3) *has at least one order-*1 *non-trivial periodic solution.*

(*ii*) *When $\widehat{M_a\bar{M}}$ and L_N have no intersection point, then the domain of map $\mathcal{P}_\mathcal{M}$ is $\mathbb{D}_2 = [(1-p)S_v, (1-p)S_\sigma]$ and $\mathcal{P}_\mathcal{M}(S)$ are increasing and continuous, that is, system* (1.3) *does not have order-k ($k \geq 2$) periodic solution.*

Proof. Firstly, under the assumptions of Theorem 3.5, it follows from Lemma 2.5 that E^* is a globally stable focus or node of system (1.2).

It follows from the vector field of system (1.2), the Poincaré map $\mathcal{P}_\mathcal{M}$ is well defined on interval $\mathbb{D}_1$ for (*i*) and on interval $\mathbb{D}_2$ for (*ii*). Moreover, as previously analyzed, when $S_\sigma < \frac{S_{mn}}{1-p}$, the precise impulsive set of system (1.3) is $[S_v, S_\sigma]$. Therefore, the domain of map $\mathcal{P}_\mathcal{M}(S)$ as shown in Theorem (3.5). And to discuss the existence of the periodic solution of system (1.3), we only need to discuss the properties of the Poincaré map $\mathcal{P}_\mathcal{M}$ defined on interval $\mathbb{D}_1$ (resp. $\mathbb{D}_2$) for (*i*) (rsep. for (*ii*)).

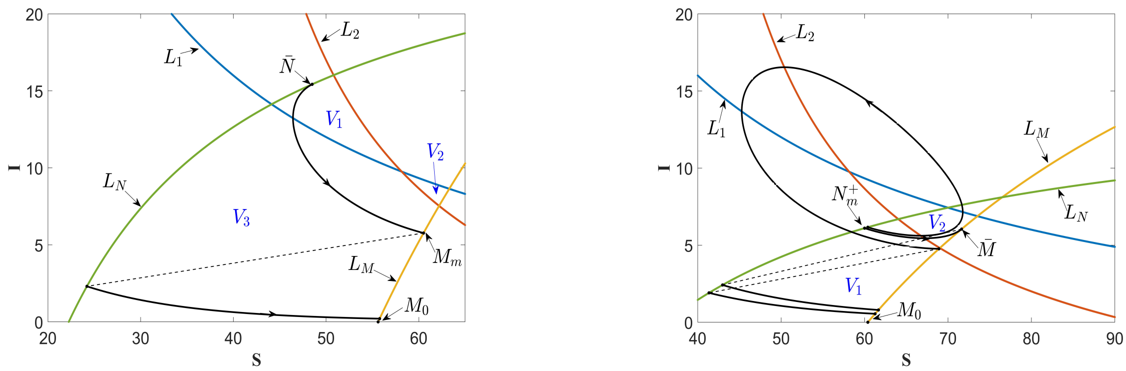
According to the vector field and the uniqueness of the solution of ODE (1.2), we can conclude that map $\mathcal{P}_\mathcal{M}$ is increasing on its domain. And the continuity of $\mathcal{P}_\mathcal{M}$ can be confirmed by using the theorem of continuity of the solution of an ODE with respect to its initial value. For a one-dimensional monotonically increasing discrete map, it is evident that there is no k-periodic point ($k \geq 2$), which indicates that system (1.3) does not have order-k ($k \geq 2$) periodic solution.

Finally, we prove that the last part of (*i*) holds. Theorem 3.3 shows that the DFPS is locally stable, that is, $(1-p)S_v = \mathcal{P}_\mathcal{M}((1-p)S_v)$ and $\frac{d\mathcal{P}_\mathcal{M}(S_0^+)}{dS_0^+}\Big|_{S_0^+=(1-p)S_v} = \mu_2 < 1$. Combining the above with condition $S_m < (1-p)S_\sigma = \mathcal{P}_\mathcal{M}(S_m)$ for increasing map $\mathcal{P}_\mathcal{M}$, we can conclude that $\mathcal{P}_\mathcal{M}$ has at least one fixed point on interval $((1-p)S_v, S_m)$, as shown in Figure 3(b), which indicates that system (1.3) has at least one order-1 non-trivial periodic solution. □

Remark 3.2. *When the unique endemic equilibrium E^* is an unstable focus or unstable node or center, the domain and continuity of map $\mathcal{P}_\mathcal{M}$ are relatively complex. Thus for this case, the properties of map $\mathcal{P}_\mathcal{M}$ and the existence of periodic solutions of system* (1.3) *are not discussed in detail in this work, but it is worth further consideration in the future.*

In case (a_3), we consider two special cases that the trajectory is tangent to L_N and L_M, respectively. Firstly, there must be a point $\bar{N} \in L_N$ satisfying $\sigma_N(\bar{N}) = 0$ because $\sigma_N(N_1) \cdot \sigma_N(N_2) < 0$. The same exists for $\bar{M} \in L_M$ so that $\sigma_M(\bar{M}) = 0$.

Assume that $\bar{N}$ is the unique point in first quadrant satisfying $\sigma_N(S, L_N(S)) = 0$. If the trajectory from the initial point $\bar{N}$ can reach the impulsive curve L_M, we mark the intersection as M_m. Then the precise impulsive set is $\mathcal{M}_0 = \{(S, L_M(S)) | S \in [S_v, S_{M_m}]\}$, where S_{M_m} is the horizontal coordinate of M_m (Figure 4(a)). Further, we mark the parts that are worth discussing as V_1, V_2, V_3 if the relative positions of the impulsive curve and the isoclinic lines are shown in the Figure 4(a). Based on the analysis of the ODE system, $\frac{dI}{dt} < 0$ holds for V_1 and V_3 while $\frac{dI}{dt} > 0$ for V_2. And the impulsive effect will always decrease the S and I, so the periodic solution will not appear only in part V_3. Once the orbit crosses from V_3 to V_2, there may be an order-1 periodic solution.



(a) $\bar{N}$ exists (b) $\bar{M}$ exists

Figure 4. The phase diagram of system (1.3) when $R_0 > 1, S_v < \frac{1}{\beta}(\gamma + \frac{c}{b})$. The parameters are $\Lambda = 40, \mu = 0.2, \beta = 0.05, \gamma = 1.5, b = 8, c = 25, AT = 50, p = 0.6, q = 0.6, \alpha_1 = 0.79, \alpha_2 = 0.21$ in (a) and $\Lambda = 40, \mu = 0.2, \beta = 0.05, \gamma = 1.5, b = 8, c = 25, AT = 50, p = 0.4, q = 0.6, \alpha_1 = 0.68, \alpha_2 = 0.32$ in (b).

Assume that $\bar{M}$ is the unique point in first quadrant satisfying $\sigma_M(S, L_M(S)) = 0$. Moreover, assuming a reverse trajectory starting from $\bar{M}$ intersects with L_N and first intersects at point $N_m^+(S_m, L_N(S_m))$ (see Figure 4(b)). Thus, the precise impulsive set is $\mathcal{M}_0 = \{(S, L_M(S)) | S \in [S_v, S_\sigma]\}$, where S_σ is the horizontal coordinate of $\bar{M}$. Further, we mark the parts that are worth discussing as V_1, V_2 if the relative positions of the impulsive curve and the isoclinic lines are shown in the Figure 4(b). Based on the analysis of the ODE system, $\frac{dI}{dt} < 0$ holds for V_1 while $\frac{dI}{dt} > 0$ for V_2. And the impulsive effect will always decrease the S and I, so if the orbit crosses from V_1 to V_2, there may be an order-1 periodic solution.

Theorem 3.6. *Suppose* $R_0 > 1$ *and* $S_v < \frac{1}{\beta}(\gamma + \frac{c}{b})$, *the system* (1.3) *may have an order-1 periodic solution.*

In case (B), L_M and L_2 have a unique intersection point, denoted as M_{L_2}, where $S_{M_{L_2}}$ is the horizontal coordinate of M_{L_2}. It follows from the vector field of system (1.2) that $\sigma_M(S, L_M S) < 0$ for all $S \geq S_{M_{L_2}}$. Thus, if a trajectory starting from Ω experiences pulse effect, it must pulse at point $(S_{w1}, L_M(S_{w1}))$, where $S_{w1} < S_{M_{L_2}}$, and then pulse to point $((1-p)S_{w1}, (1-q)L_M(S_{w1}))$, which in the part $\frac{dI}{dt} < 0$. Thus, after a finite time, this trajectory will decrease and reach point $(S_{w2}, L_M(S_{w2}))$ on the pulse curve L_M. Here, $L_M(S_{w2}) < (1-q)L_M(S_{w1}) < L_M(S_{w1})$. From this, it can be concluded that the system (1.3) does not have order-1 non-trivial periodic solution, which also indicates that the system (1.3) does not have an order-k ($k \geq 1$) periodic solution except for the DFPS. The specific conclusion is as follows.

Theorem 3.7. *Suppose $R_0 < 1, a_1 < 0, R_0 = R_0^*$, the system (1.3) will not have an order-k periodic solution ($k \geq 1$) except for the DFPS.*

3.2.4. The system has two endemic equilibria

Same as case (B), it's easy to know that the non-trivial periodic solutions are only possible when $S_1^* < \frac{AT}{\alpha_1} < S_2^*$, where S_1^* and S_2^* are the horizontal coordinates of E_1^* and E_2^*, respectively. So we just focus on this situation (see Figure 5).

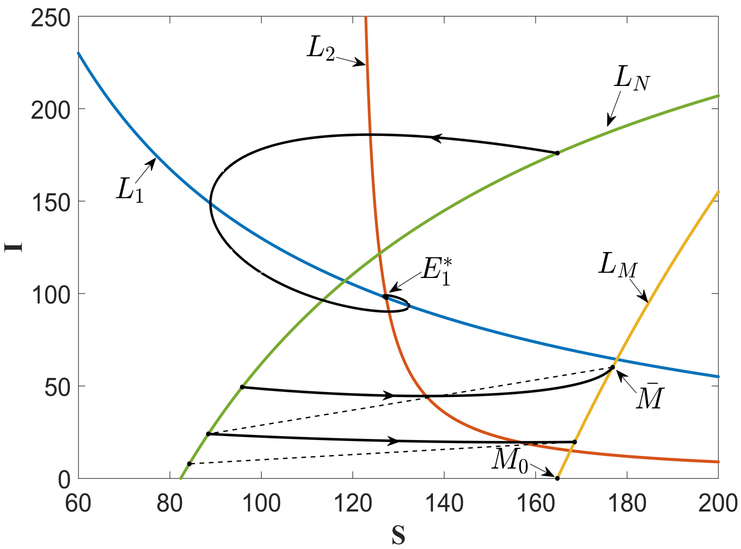


Figure 5. The phase diagram of system (1.3) when $R_0 < 1, a_1 < 0, R_0 > R_0^*, S_1^* < S_v < S_2^*$. The parameters are $\Lambda = 150, \mu = 0.2, \beta = 0.01, \gamma = 1.2, b = 0.01, c = 7.16, AT = 160, p = 0.5, q = 0.6, \alpha_1 = 0.9, \alpha_2 = 0.1$.

Similar to the above analysis, there must be an $\bar{M}$ satisfying $\sigma_M(\bar{M}) = 0$. So the precise impulsive set is $\mathcal{M}_0 = \{(S, L_M(S))|S \in [S_v, S_\sigma]\}$, where S_σ is the horizontal coordinate of $\bar{M}$. What's more, when the trajectory crosses from the part satisfying $\frac{dI}{dt} < 0$ to the part $\frac{dI}{dt} > 0$, there may be an order-1 periodic solution.

Theorem 3.8. *Suppose $R_0 < 1$, $a_1 < 0$, $R_0 > R_0^*$ and $S_1^* < \frac{AT}{\alpha_1} < S_2^*$, where S_1^* and S_2^* are the horizontal coordinates of E_1^* and E_2^*, respectively. The system (1.3) may have an order-1 periodic solution.*

Corollary 3.2. *Replace the inequalities $R_0 > 1$ and $\frac{1}{\beta}(\gamma + \frac{c}{b}) < S_v < \frac{\Lambda}{\mu}$ in Theorem 3.5 by $R_0 < 1$, $a_1 < 0$, $R_0 > R_0^*$ and $S_1^* < \frac{AT}{\alpha_1} < S_2^*$. Then the conclusion of Theorem 3.5 still holds.*

4. Bifurcations when $q = 0$

Based on Remark 3.1, in order to discuss the bifurcation near the DFPS, we assume that $R_0 > 1$, $(1-p)S_v < \frac{1}{\beta}(r + \frac{c}{b}) < S_v < \frac{\Lambda}{\mu}$ and $q = 0$ always hold true in this section.

On the phase space $\{(S, I)|0 \le I < L_1(S), 0 < S < \frac{\Lambda}{\mu}\}$, considering the following scalar differential equation

$$\frac{dI}{dS} = \frac{\beta SI - \gamma I - \frac{cI}{b+I}}{\Lambda - \mu S - \beta SI} := h(S, I), (S_0, I_0) := (S_n^+, I_n^+), \tag{4.1}$$

where $I_n^+ \in [0, \epsilon)$, $S_n^+ = L_N^{-1}(I_n^+)$ and ϵ is small enough. We can solve I with respect to S as the following:

$$I(S; S_n^+, I_n^+) = I_n^+ + \int_{S_n^+(I_n^+)}^{S} h(s, I(s; S_n^+, I_n^+))ds. \tag{4.2}$$

Thus, when $q = 0$ and the definition of point $(S_{n+1}, I_{n+1}) \in L_M$ is shown in Subsection 3.2.1, the Poincaré map can be also represented as

$$\mathcal{P}(I_n^+, \alpha) := I_{n+1}^+ = I_{n+1} = I(S_{n+1}; S_n^+, I_n^+). \tag{4.3}$$

Here, $I_n^+ \in [0, \epsilon)$ is the variable and $\alpha \in \Theta$ is the parameter of map $\mathcal{P}(I_n^+, \alpha)$. Thus, $\mathcal{P}$ is defined as a one-parameter-family of maps from $[0, \epsilon) \times \Theta$ to $\mathbb{R}$.

Referring to Lemma 2.2, for the purpose of bifurcation analysis about map $\mathcal{P}(I_n^+, \alpha)$, we first calculate that

$$\frac{\partial I(S; S_n^+, I_n^+)}{\partial I_n^+} = 1 + \int_{S_n^+}^{S} \frac{\partial h(s, I(s; S_n^+, I_n^+))}{\partial I} \frac{\partial I(s; S_n^+, I_n^+)}{\partial I_n^+} ds - \left(\frac{dS_n^+}{dI_n^+}\Big|_{I_n^+ = L_N(S_n^+)} \right) h(S_n^+, I_n^+). \tag{4.4}$$

Let

$$W(I_n^+) = 1 - \left(\frac{dS_n^+}{dI_n^+}\Big|_{I_n^+ = L_N(S_n^+)} \right) h(S_n^+, I_n^+).$$

Utilizing the variable formula, we have

$$\frac{\partial I(S; S_n^+, I_n^+)}{\partial I_n^+} = W(I_n^+) \exp\left(\int_{S_n^+}^{S} \frac{\partial h(s, I(s; S_n^+, I_n^+))}{\partial I} ds \right). \tag{4.5}$$

Further, we denote

$$I_{n+1} = I(S_{n+1}; S_n^+, I_n^+) = I_n^+ + \int_{S_n^+}^{S_{n+1}} h(s, I(s; S_n^+, I_n^+))ds, \tag{4.6}$$

where $(S_n^+, I_n^+) \in L_N$ and $(S_{n+1}, I_{n+1}) \in L_M$. Then we calculate its first-order derivative and second-order derivative with respect to I_n^+ as follows:

$$\begin{aligned} \frac{\partial I_{n+1}}{\partial I_n^+} =& 1 + \int_{S_n^+}^{S_{n+1}} \frac{\partial h(s, I(s; S_n^+, I_n^+))}{\partial I} \frac{\partial I(s; S_n^+, I_n^+)}{\partial I_n^+} ds - \left(\frac{dS_n^+}{dI_n^+}\Big|_{I_n^+ = L_N(S_n^+)} \right) h(S_n^+, I_n^+) + \frac{dS_{n+1}}{dI_n^+} h(S_{n+1}, I_{n+1}) \\ =& \frac{\partial I(S; S_n^+, I_n^+)}{\partial I_n^+}\Big|_{S = S_{n+1}} + \frac{dS_{n+1}}{dI_n^+} h(S_{n+1}, I_{n+1}) \\ =& \frac{\partial I(S; S_n^+, I_n^+)}{\partial I_n^+}\Big|_{S = S_{n+1}} + \left(\frac{dS_{n+1}}{dI_{n+1}}\Big|_{I_{n+1} = L_M(S_{n+1})} \right) \frac{\partial I_{n+1}}{\partial I_n^+} h(S_{n+1}, I_{n+1}), \end{aligned} \tag{4.7}$$

$$\begin{aligned}\frac{\partial^2 I_{n+1}}{\partial (I_n^+)^2} =&\frac{\partial}{\partial I_n^+}\left(W(I_n^+)\exp\left(\int_{S_n^+}^{S_{n+1}}\frac{\partial h(s,I(s;S_n^+,I_n^+))}{\partial I}ds\right)\right)+\frac{dS_{n+1}}{dI_{n+1}}\frac{\partial^2 I_{n+1}}{\partial (I_n^+)^2}h(S_{n+1},I_{n+1})\\&+\frac{dS_{n+1}}{dI_{n+1}}\frac{\partial I_{n+1}}{\partial I_n^+}\frac{\partial h(S_{n+1},I_{n+1})}{\partial I_n^+}+\frac{d^2 S_{n+1}}{d(I_{n+1})^2}\left(\frac{\partial I_{n+1}}{\partial I_n^+}\right)^2 h(S_{n+1},I_{n+1})\\=&\exp\left(\int_{S_n^+}^{S_{n+1}}\frac{\partial h(s,I(s;S_n^+,I_n^+))}{\partial I}ds\right)\left[\frac{\partial W(I_n^+)}{\partial I_n^+}+W(I_n^+)\left(\frac{dS_{n+1}}{dI_{n+1}}\frac{\partial I_{n+1}}{\partial I_n^+}\frac{\partial h(S_{n+1},I_{n+1})}{\partial I}\right.\right.\\&\left.\left.-\frac{dS_n^+}{dI_n^+}\frac{\partial h(S_n^+,I_n^+)}{\partial I}+\int_{S_n^+}^{S_{n+1}}\frac{\partial^2 h(s,I(s;S_n^+,I_n^+))}{\partial I^2}\frac{\partial I(s;S_n^+,I_n^+)}{\partial I_n^+}ds\right)\right]\\&+\frac{dS_{n+1}}{dI_{n+1}}\frac{\partial^2 I_{n+1}}{\partial (I_n^+)^2}h(S_{n+1},I_{n+1})+\frac{dS_{n+1}}{dI_{n+1}}\frac{\partial I_{n+1}}{\partial I_n^+}\frac{\partial h(S_{n+1},I_{n+1})}{\partial I_n^+}\\&+\frac{d^2 S_{n+1}}{d(I_{n+1})^2}\left(\frac{\partial I_{n+1}}{\partial I_n^+}\right)^2 h(S_{n+1},I_{n+1}),\end{aligned}\tag{4.8}$$

where

$$\frac{\partial W(I_n^+)}{\partial I_n^+}=-\frac{d^2 S_n^+}{d(I_n^+)^2}h(S_n^+,I_n^+)-\frac{dS_n^+}{dI_n^+}\frac{\partial h(S_n^+,I_n^+)}{\partial I_n^+},\tag{4.9}$$

$$\frac{\partial h(S_n^+,I_n^+)}{\partial I_n^+}=\frac{\partial h(S_n^+,I_n^+)}{\partial I}+\frac{\partial h(S_n^+,I_n^+)}{\partial S}\frac{dS_n^+}{dI_n^+},\tag{4.10}$$

and

$$\frac{\partial h(S_{n+1},I_{n+1})}{\partial I_n^+}=\frac{\partial h(S_{n+1},I_{n+1})}{\partial I}\frac{\partial I_{n+1}}{\partial I_n^+}+\frac{\partial h(S_{n+1},I_{n+1})}{\partial S}\frac{\partial S_{n+1}}{\partial I_n^+}.\tag{4.11}$$

Because $I_{n+1}=L_M(S_{n+1})$ and $I_n^+=L_M(S_n^+)$, we have

$$\frac{dS_{n+1}}{dI_{n+1}}=\left(\left.\frac{dL_M(S)}{dS}\right|_{S=S_{n+1}}\right)^{-1}=\frac{\alpha_2\beta(S_{n+1})^2}{AT-\alpha_2\Lambda}\tag{4.12}$$

and

$$\frac{dS_n^+}{dI_n^+}=\left(\left.\frac{dL_N(S)}{dS}\right|_{S=S_n^+}\right)^{-1}=\frac{\alpha_2\beta(S_n^+)^2}{(1-p)(AT-\alpha_2\Lambda)}.\tag{4.13}$$

When $I_n^+=0$, then $S_n^+=(1-p)S_v$, $I_{n+1}=0$, $S_{n+1}=S_v=\frac{AT-\alpha_2\Lambda}{\alpha_1-\alpha_2\Lambda}$, $h(S_n^+,I_n^+)=h((1-p)S_v,0)=0$, $h(S_{n+1},I_{n+1})=h(S_v,0)=0$, and thus

$$\left.\frac{\partial I_{n+1}}{\partial I_n^+}\right|_{I_n^+=0}=\exp\left(\int_{(1-p)S_v}^{S_v}\frac{\partial h(s,0)}{\partial I}ds\right)=\exp\left(\int_{(1-p)S_v}^{S_v}\frac{\beta s-\gamma-\frac{c}{b}}{\Lambda-\mu s}ds\right)=\mu_2,\tag{4.14}$$

$$\begin{aligned}\left.\frac{\partial^2 I_{n+1}}{\partial (I_n^+)^2}\right|_{I_n^+=0}=&\mu_2\Big(-\frac{2\alpha_2\beta(1-p)S_v^2}{(AT-\alpha_2\Lambda)}f\big((1-p)S_v\big)+\frac{2\alpha_2\beta S_v^2}{AT-\alpha_2\Lambda}f(S_v)\mu_2\\&+\int_{(1-p)S_v}^{S_v}\frac{\partial^2 h(s,I(s;(1-p)S_v,0))}{\partial I^2}\frac{\partial I(s;(1-p)S_v,0)}{\partial I_n^+}ds\Big).\end{aligned}\tag{4.15}$$

Denote

$$\theta=\int_{(1-p)S_v}^{S_v}\frac{\partial^2 h(s,I(s;(1-p)S_v,0))}{\partial I^2}\frac{\partial I(s;(1-p)S_v,0)}{\partial I_n^+}ds$$

and

$$f(s)=\frac{\partial h(s,0)}{\partial I}=\frac{\beta s-\gamma-\frac{c}{b}}{\Lambda-\mu s}.$$

Then

$$\begin{aligned}\theta &= \int_{(1-p)S_v}^{S_v}\left(\frac{2c}{b^2(\Lambda-\mu s)}+\frac{2\beta s}{\Lambda-\mu s}f(s)\right)\left(\exp\int_{(1-p)S_v}^{S}f(s)ds\right)ds\\ &= \int_{(1-p)S_v}^{S_v}\left(\frac{2c}{b^2(\Lambda-\mu s)}\right)\left(\exp\int_{(1-p)S_v}^{S}f(s)ds\right)ds+\int_{(1-p)S_v}^{S_v}\frac{2\beta s}{\Lambda-\mu s}d\left(\exp\int_{(1-p)S_v}^{S}f(s)ds\right)\\ &= \int_{(1-p)S_v}^{S_v}\left(\frac{2c}{b^2(\Lambda-\mu s)}\right)\left(\exp\int_{(1-p)S_v}^{S}f(s)ds\right)ds+\left[\frac{2\beta s}{\Lambda-\mu s}\left(\exp\int_{(1-p)S_v}^{S}f(s)ds\right)\right]\Bigg|_{(1-p)S_v}^{S_v}\\ &\quad-\int_{(1-p)S_v}^{S_v}\left(\exp\int_{(1-p)S_v}^{S}f(s)ds\right)d\left(\frac{2\beta s}{\Lambda-\mu s}\right),\end{aligned} \tag{4.16}$$

$f(s)=0$ has a unique root $s=\frac{1}{\beta}(\gamma+\frac{c}{b})$ and $f'(s)=(\Lambda\beta-\mu(\gamma+\frac{c}{b}))/(\Lambda-\mu s)^2>0$ when $R_0>1$.

Therefore, when $(1-p)S_v<\frac{1}{\beta}(\gamma+\frac{c}{b})<S_v<\frac{\Lambda}{\mu}$, there are $f\big((1-p)S_v\big)<0<f(S_v)$ and

$$0<\exp\left(\int_{(1-p)S_v}^{S}f(s)ds\right)<\max\left\{\exp\left(\int_{(1-p)S_v}^{S_v}f(s)ds\right),1\right\}=\max\{\mu_2,1\} \tag{4.17}$$

for all $S\in((1-p)S_v,S_v)$. Moreover, if $\mu_2=1$ then $0<\exp\left(\int_{(1-p)S_v}^{S}f(s)ds\right)<1$ and $\theta>0$. Thus when $R_0>1$, $(1-p)S_v<\frac{1}{\beta}(\gamma+\frac{c}{b})<S_v<\frac{\Lambda}{\mu}$ and $q=0$, there is

$$\frac{\partial^2 I_{n+1}}{\partial(I_n^+)^2}\bigg|_{I_n^+=0,\ \mu_2=1}>0. \tag{4.18}$$

4.1. Bifurcations with respect to p

Now, under the assumptions of $q=0$ and $\frac{1}{\beta}(\gamma+\frac{c}{b})<S_v<\frac{\Lambda}{\mu}$, we consider the bifurcations near the DFPS $(\tilde{S}(t),0)$ with respect to parameter p.

Taking μ_2 as a function of p with $p\in[0,1]$ and taking the derivative of μ_2 with respect to p yields

$$\frac{d\mu_2(p)}{dp}=\frac{\mu_2 S_v}{\mu}\left(\frac{\Lambda b\beta-b\mu\gamma-c\mu}{b\big(\Lambda-\mu(1-p)S_v\big)}-\beta\right). \tag{4.19}$$

Thus $p=\tilde{p}$ with $\tilde{p}=1-\frac{1}{\beta S_v}(\gamma+\frac{c}{b})\in(0,1)$ is the unique root of $\frac{d\mu_2(p)}{dp}=0$ and $\frac{d\mu_2(p)}{dp}>0$ for $p\in(0,\tilde{p})$, $\frac{d\mu_2(p)}{dp}<0$ for $p\in(\tilde{p},1)$. Furthermore, we have

$$\mu_2\big|_{p=0}=1 \ \text{ and } \ \mu_2\big|_{p=1}=\left(\frac{\Lambda-\mu S_v}{\Lambda}\right)^{\frac{-\Lambda b\beta+b\mu\gamma+c\mu}{b\mu^2}}e^{\frac{\beta S_v}{\mu}}>0.$$

Therefore,

- if $\mu_2\big|_{p=1}\geq 1$, then there is no $p^*\in(0,1)$ satisfying $\mu_2\big|_{p=p^*}=1$.

- if $\mu_2\big|_{p=1} < 1$, then there is a unique $p^* \in (\tilde{p}, 1)$ satisfying $\mu_2\big|_{p=p^*} = 1$ and $\frac{d\mu_2}{dp}\big|_{p=p*} < 0$.

Theorem 4.1. *When $q = 0$, $\frac{1}{\beta}(\gamma + \frac{c}{b}) < S_v < \frac{\Lambda}{\mu}$ and $\mu_2\big|_{p=1} < 1$, map $\mathcal{P}$ undergoes the transcritical bifurcation at $p = p^*$, where p^* is the unique root of $\mu_2(p) = 1$. Moreover, an unstable non-trivial fixed point (non-trivial order-1 periodic solution of system (1.3)) appears if $p \in (p^*, p^* + \epsilon)$ with $\epsilon > 0$ is small enough.*

Proof. According to Eqs (4.14) and (4.18), we already have

$$\frac{\partial \mathcal{P}}{\partial I_n^+}(0, p^*) = \frac{\partial I_{n+1}}{\partial I_n^+}\Bigg|_{I_n^+=0}^{p=p^*} = 1 \ \text{ and } \ \frac{\partial^2 \mathcal{P}}{\partial (I_n^+)^2}(0, p^*) = \frac{\partial^2 I_{n+1}}{\partial (I_n^+)^2}\Bigg|_{I_n^+=0}^{p=p^*} > 0.$$

What's more, it's easy to know that $\mathcal{P}(0, p) = 0$ for all $p \in [0, 1]$. Thus, to prove the Theorem 4.1 by using Lemma 2.2, we just need to verify the sign of $\frac{\partial^2 \mathcal{P}}{\partial I_n^+ \partial p}(0, p^*)$, where $\frac{\partial^2 \mathcal{P}}{\partial I_n^+ \partial p}(0, p^*) = \frac{\partial^2 I_{n+1}}{\partial I_n^+ \partial p}\big|_{I_n^+=0}^{p=p^*}$.

From Eq (4.7) it follows that

$$\begin{aligned}
\frac{\partial^2 I_{n+1}}{\partial I_n^+ \partial p} =& \frac{\partial}{\partial p}\left(W(I_n^+) \exp \int_{S_n^+}^{S_{n+1}} \frac{\partial h(s, I(s; S_n^+, I_n^+))}{\partial I} ds \right) + \frac{\partial}{\partial p}\left(\frac{dS_{n+1}}{dI_{n+1}}\right)\frac{\partial I_{n+1}}{\partial I_n^+} h(S_{n+1}, I_{n+1}) \\
&+ \frac{dS_{n+1}}{dI_{n+1}} \frac{\partial}{\partial p}\left(\frac{\partial I_{n+1}}{\partial I_n^+}\right) h(S_{n+1}, I_{n+1}) + \frac{dS_{n+1}}{dI_{n+1}} \frac{\partial I_{n+1}}{\partial I_n^+} \frac{\partial}{\partial p}\big(h(S_{n+1}, I_{n+1})\big) \\
=& \Big(\exp \int_{S_n^+}^{S_{n+1}} \frac{\partial h(s, I(s; S_n^+, I_n^+))}{\partial I} ds \Big) \Big[\frac{\partial W(I_n^+)}{\partial p} + W(I_n^+) \Big(\frac{\partial S_{n+1}}{\partial p} \frac{\partial h(S_{n+1}, I_{n+1})}{\partial I} \\
&- \frac{\partial S_n^+}{\partial p} \frac{\partial h(S_n^+, I_n^+)}{\partial I} + \int_{S_n^+}^{S_{n+1}} \frac{\partial^2 h(s, I(s; S_n^+, I_n^+))}{\partial I \partial p} ds \Big) \Big] \\
&+ \frac{\partial}{\partial p}\left(\frac{dS_{n+1}}{dI_{n+1}}\right) \frac{\partial I_{n+1}}{\partial I_n^+} h(S_{n+1}, I_{n+1}) + \frac{dS_{n+1}}{dI_{n+1}} \frac{\partial}{\partial p}\left(\frac{\partial I_{n+1}}{\partial I_n^+}\right) h(S_{n+1}, I_{n+1}) \\
&+ \frac{dS_{n+1}}{dI_{n+1}} \frac{\partial I_{n+1}}{\partial I_n^+} \frac{\partial}{\partial p}\big(h(S_{n+1}, I_{n+1})\big),
\end{aligned} \tag{4.20}$$

where $W(I_n^+ = 0) = 1$ and

$$\begin{aligned}
\frac{\partial W(I_n^+)}{\partial p}\bigg|_{I_n^+=0} =& \Big(- \frac{dS_n^+}{dI_n^+} \frac{h(S_n^+, I_n^+)}{\partial p} - \frac{\partial}{\partial p}\left(\frac{dS_n^+}{dI_n^+}\right) h(S_n^+, I_n^+) \Big)\bigg|_{I_n^+=0} \\
=& - \frac{\alpha_2 \beta (1-p) S_v^2}{(AT - \alpha_2 \Lambda)} \Big(\frac{\partial h(S_n^+, I_n^+)}{\partial S} \frac{\partial S_n^+}{\partial p} + \frac{\partial h(S_n^+, I_n^+)}{\partial I} \frac{\partial I_n^+}{\partial p} \Big)\bigg|_{I_n^+=0} \\
=& 0.
\end{aligned} \tag{4.21}$$

We have $\frac{\partial I_n^+}{\partial p} = 0$ because I_n^+ and p are independent. Thus, we have the following equation

$$0 = \frac{\partial I_n^+}{\partial p} = \frac{\partial L_M\left(\frac{S_n^+}{1-p}\right)}{\partial\left(\frac{S_n^+}{1-p}\right)} \frac{\partial\left(\frac{S_n^+}{1-p}\right)}{p} = L_M^{'}\left(\frac{S_n^+}{1-p}\right) \frac{\frac{\partial S_n^+}{\partial p}(1-p) + S_n^+}{(1-p)^2}. \tag{4.22}$$

By solving Eq (4.22), we have

$$\frac{\partial S_n^+}{\partial p} = -\frac{S_n^+}{1-p} L_M^{'}\left(\frac{S_n^+}{1-p}\right). \tag{4.23}$$

In order to calculate $\frac{\partial I_{n+1}}{\partial S_n^+}$, we first calculate $\frac{\partial I(S;S_n^+,I_n^+)}{\partial S_n^+}$. That is

$$\frac{\partial I(S;S_n^+,I_n^+)}{\partial S_n^+} = \frac{dI_n^+}{dS_n^+} - h(S_n^+,I_n^+) + \int_{S_n^+}^{S} \frac{\partial h(s,I(s;S_n^+,I_n^+))}{\partial I} \frac{\partial I(s;S_n^+,I_n^+)}{\partial S_n^+} ds. \tag{4.24}$$

Using the variational formula we have

$$\frac{\partial I(S;S_n^+,I_n^+)}{\partial S_n^+} = \left(\frac{dI_n^+}{dS_n^+} - h(S_n^+,I_n^+)\right) \exp \int_{S_n^+}^{S} \frac{\partial h(s,I(s;S_n^+,I_n^+))}{\partial I} ds. \tag{4.25}$$

Thus

$$\begin{aligned} \frac{\partial I_{n+1}}{\partial S_n^+} =& \frac{dI_n^+}{dS_n^+} - h(S_n^+,I_n^+) + \int_{S_n^+}^{S_{n+1}} \frac{\partial h(s,I(s;S_n^+,I_n^+))}{\partial I} \frac{\partial I(s;S_n^+,I_n^+)}{\partial S_n^+} ds + \frac{dS_{n+1}}{dS_n^+} h(S_{n+1},I_{n+1}) \\ =& \left.\frac{\partial I(S;S_n^+,I_n^+)}{\partial S_n^+}\right|_{S=S_{n+1}} + \frac{dS_{n+1}}{dS_n^+} h(S_{n+1},I_{n+1}). \end{aligned} \tag{4.26}$$

Then we can verify the symbol of the following equations

$$\begin{aligned} & \left.\int_{S_n^+}^{S_{n+1}} \frac{\partial^2 h(s,I(s;S_n^+,I_n^+))}{\partial I \partial p} ds\right|_{I_n^+=0} \\ =& \int_{(1-p)S_v}^{S_v} \frac{\partial^2 h(s,I(s;(1-p)S_v,0))}{\partial I^2} \frac{\partial I(s,I(s;(1-p)S_v,0))}{\partial p} ds \\ =& \int_{(1-p)S_v}^{S_v} \frac{\partial^2 h(s,I(s;(1-p)S_v,0))}{\partial I^2} \left(\frac{\partial I(s,I(s;(1-p)S_v,0))}{\partial I_n^+} \frac{\partial I_n^+}{\partial p} \right. \\ & \left.\left. + \frac{\partial I(s,I(s;(1-p)S_v,0))}{\partial S_n^+} \frac{\partial S_n^+}{\partial p}\right)\right|_{I_n^+=0} ds \\ =& \int_{(1-p)S_v}^{S_v} \frac{\partial^2 h(s,I(s;(1-p)S_v,0))}{\partial I^2} \left.\left(\frac{\partial I(s,I(s;(1-p)S_v,0))}{\partial S_n^+} \frac{\partial S_n^+}{\partial p}\right)\right|_{I_n^+=0} ds \\ =& -\theta S_v \left(\left.\frac{dL_M(S)}{dS}\right|_{S=S_v}\right)\left(\left.\frac{dL_N(S)}{dS}\right|_{S=(1-p)S_v}\right) < 0 \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} \left.\frac{\partial h(S_{n+1},I_{n+1})}{\partial p}\right|_{I_n^+=0} =& \left.\left(\frac{\partial h(S_{n+1},I_{n+1})}{\partial S} \frac{\partial S_{n+1}}{\partial p} + \frac{\partial h(S_{n+1},I_{n+1})}{\partial I} \frac{\partial I_{n+1}}{\partial p}\right)\right|_{I_n^+=0} \\ =& -\frac{(AT-\alpha_2\Lambda)^2}{(1-p)(\alpha_2)^2\beta^2(S_v)^3} f(S_v) < 0. \end{aligned} \tag{4.28}$$

Therefore, we can conclude that $(1-p^*)S_v < \frac{1}{\beta}(\gamma + \frac{c}{b}) < S_v < \frac{\Lambda}{\mu}$, $f((1-p^*)S_v) < 0 < f(S_v)$ and

$$\left.\frac{\partial^2 I_{n+1}}{\partial (I_n^+) \partial p}\right|_{I_n^+=0,\ p=p*} < 0. \tag{4.29}$$

Combining Eq (4.18) and Lemma 2.2, we complete the proof. □

4.2. Bifurcations with respect to AT

When $R_0 > 1$ and $q = 0$, we consider the bifurcations near the DFPS $(\tilde{S}(t), 0)$ with respect to parameter AT, where $AT \in (\alpha_2\Lambda, \alpha_1\frac{\Lambda}{\mu})$ since $0 < S_v = \frac{AT-\alpha_2\Lambda}{\alpha_1-\alpha_2\mu} < \frac{\Lambda}{\mu}$.

Taking μ_2 as a function of AT, we have

$$\frac{d\mu_2(AT)}{dAT} = \frac{\partial\mu_2}{\partial S_v}\frac{\partial S_v}{\partial AT} = \frac{p\mu_2}{\mu}\frac{g(S_v)}{\alpha_1 - \alpha_2\mu}, \tag{4.30}$$

where

$$g(S_v) = \frac{\Lambda(\Lambda b\beta - b\mu\gamma - c\mu)}{b(\Lambda - \mu S_v)(\Lambda - \mu(1-p)S_v)} - \beta.$$

The sign of $\frac{d\mu_2(AT)}{dAT}$ is determined by $g(S_v)$. Let $g(S_v) = 0$, we have

$$S_v^2 + \frac{\Lambda}{\mu}\frac{p-2}{1-p}S_v + \frac{\Lambda}{\mu}\frac{b\gamma + c}{b\beta(1-p)} = 0$$

with

$$\Delta = \left(\frac{\Lambda}{\mu}\frac{p-2}{1-p}\right)^2 - 4\frac{\Lambda}{\mu}\frac{b\gamma+c}{b\beta(1-p)} = \frac{\Lambda^2}{\mu^2(1-p)R_0}\left[\frac{(p-2)^2}{1-p}R_0 - 4\right].$$

Let $D(p) = \frac{(p-2)^2}{1-p}$, then we have $D(0) = 4$ and $D'(p) > 0$ for all $p \in (0,1)$.Thus $D(p) > 4$ holds for all $p \in (0,1)$. This means $\Delta > 0$ when $R_0 > 1$. Hence $g(S_v) = 0$ has two roots, S_{v1} and S_{v2}, which satisfy $0 < S_{v1} < \frac{\Lambda}{\mu} < S_{v2}$. Thus $g(S_v)$ is decreasing on $(0, S_{v1})$ and increasing on $(S_{v1}, \frac{\Lambda}{\mu})$, which indicates that $\mu_2(AT)$ is decreasing on $(\alpha_2\Lambda, (\alpha_1 - \alpha_2\mu)S_{v1} + \alpha_2\Lambda)$ and increasing on $((\alpha_1 - \alpha_2\mu)S_{v1} + \alpha_2\Lambda, \alpha_1\frac{\Lambda}{\mu})$.

Furthermore, there are

$$\mu_2\big|_{S_v=0 \text{ or } AT=\alpha_2\Lambda} = 1 \quad \text{and} \quad \mu_2\big|_{S_v=\frac{\Lambda}{\mu} \text{ or } AT=\alpha_1\frac{\Lambda}{\mu}} = +\infty.$$

Therefore, there is a unique $S_v^* \in (S_{v_1}, \frac{\Lambda}{\mu})$ such that $\mu_2\big|_{S_v=S_v^*} = 1$. That is to say, there is a unique $AT^* = (\alpha_1 - \alpha_2\mu)S_v^* + \alpha_2\Lambda \in ((\alpha_1 - \alpha_2\mu)S_{v1} + \alpha_2\Lambda, \alpha_1\frac{\Lambda}{\mu}) \subset (\alpha_2\Lambda, \alpha_1\frac{\Lambda}{\mu})$ satisfying $\mu_2\big|_{AT=AT^*} = 1$. Similarly, we have the following theorem:

Theorem 4.2. *When $R_0 > 1$ and $q = 0$, map $\mathcal{P}$ undergoes the transcritical bifurcation at $AT = AT^*$, where $AT^* \in (\alpha_2\Lambda, \alpha_1\Lambda/\mu)$ is the unique root of $\mu_2(AT) = 1$. Moreover, an unstable non-trivial fixed point (non-trivial order-1 periodic solution of system* (1.3)*) appears if $AT \in (AT^* - \epsilon, AT^*)$ with $\epsilon > 0$ small enough.*

Proof. Same as the proof of Theorem 4.1, we already have

$$\frac{\partial\mathcal{P}}{\partial I_n^+}(0, AT^*) = \frac{\partial I_{n+1}}{\partial I_n^+}\Bigg|_{I_n^+=0}^{AT=AT^*} = 1 \text{ and } \frac{\partial^2\mathcal{P}}{\partial(I_n^+)^2}(0, AT^*) = \frac{\partial^2 I_{n+1}}{\partial(I_n^+)^2}\Bigg|_{I_n^+=0}^{AT=AT^*} > 0.$$

And it's easy to know that $\mathcal{P}(0, AT) = 0$ for all $AT \in (\alpha_2\Lambda, \alpha_1\Lambda/\mu)$. Then we just need to verify the sign of $\frac{\partial^2 \mathcal{P}}{\partial I_n^+ \partial AT}(0, AT^*)$, where $\frac{\partial^2 \mathcal{P}}{\partial I_n^+ \partial AT}(0, AT^*) = \frac{\partial^2 I_{n+1}}{\partial I_n^+ \partial AT}\Big|_{I_n^+=0}^{AT=AT^*}$. Firstly, we can calculate that

$$\begin{aligned} \frac{\partial^2 I_{n+1}}{\partial I_n^+ \partial AT}\bigg|_{I_n^+=0} =& \bigg[\frac{\partial}{\partial AT}\left(W(I_n^+)\Big(\exp\int_{S_n^+}^{S_{n+1}} \frac{\partial h(s, I(s; S_n^+, I_n^+))}{\partial I} ds\Big)\right) \\ &+ \frac{dS_{n+1}}{dI_{n+1}}\frac{\partial I_{n+1}}{\partial I_n^+}\frac{\partial}{\partial AT}\big(h(S_{n+1}, I_{n+1})\big)\bigg]\bigg|_{I_n^+=0} \\ =& \exp\bigg(\int_{S_n^+}^{S_{n+1}} \frac{\partial h(s, I(s; S_n^+, I_n^+))}{\partial I} ds\bigg)\bigg[\frac{\partial W(I_n^+)}{\partial AT} + W(I_n^+)\Big(\frac{\partial S_{n+1}}{\partial AT}\frac{\partial h(S_{n+1}, I_{n+1})}{\partial I} \\ &- \frac{\partial S_n^+}{\partial AT}\frac{\partial h(S_n^+, I_n^+)}{\partial I} + \int_{S_n^+}^{S_{n+1}} \frac{\partial^2 h(s, I(s; S_n^+, I_n^+))}{\partial I \partial AT} ds\Big)\bigg]\bigg|_{I_n^+=0} \\ &+ \bigg[\frac{dS_{n+1}}{dI_{n+1}}\frac{\partial I_{n+1}}{\partial I_n^+}\frac{\partial}{\partial AT}\big(h(S_{n+1}, I_{n+1})\big)\bigg]\bigg|_{I_n^+=0}, \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} \frac{\partial W(I_n^+)}{\partial AT}\bigg|_{I_n^+=0} =& \Big(-\frac{dS_n^+}{dI_n^+}\frac{\partial h(S_n^+, I_n^+)}{\partial AT} - \frac{\partial}{\partial AT}\Big(\frac{dS_n^+}{dI_n^+}\Big)h(S_n^+, I_n^+)\Big)\bigg|_{I_n^+=0} \\ =& \Big(-\frac{dS_n^+}{dI_n^+}\Big(\frac{\partial h(S_n^+, I_n^+)}{\partial S}\frac{\partial S_n^+}{\partial AT} + \frac{\partial h(S_n^+, I_n^+)}{\partial I}\frac{\partial I_n^+}{\partial AT}\Big)\Big)\bigg|_{I_n^+=0} = 0. \end{aligned} \tag{4.32}$$

Similarly, we have the following equation

$$\begin{aligned} 0 = \frac{\partial I_n^+}{\partial AT} = \frac{\partial L_N(S_n^+)}{\partial AT} &= \frac{\partial}{\partial AT}\Big(\frac{(1-p)(\alpha_2\Lambda - AT)}{\alpha_2\beta S_n^+} + \frac{1}{\beta}\big(\frac{\alpha_1}{\alpha_2} - \mu\big)\Big) \\ &= \frac{(p-1)S_n^+ - (1-p)(\alpha_2\Lambda - AT)\frac{\partial S_n^+}{\partial AT}}{\alpha_2\beta(S_n^+)^2}. \end{aligned} \tag{4.33}$$

Therefore,

$$\frac{\partial S_n^+}{\partial AT}\bigg|_{I_n^+=0} = \frac{S_n^+}{AT - \alpha_2\Lambda}\bigg|_{I_n^+=0} = \frac{(1-p)S_v}{AT - \alpha_2\Lambda} \text{ and } \frac{\partial I_{n+1}}{\partial AT} = \frac{\partial I_{n+1}}{\partial S_n^+}\frac{\partial S_n^+}{\partial AT}.$$

Then we figure out the symbol of the following equations

$$\begin{aligned} \frac{\partial h(S_{n+1}, I_{n+1})}{\partial AT}\bigg|_{I_n^+=0} =& \Big(\frac{\partial h(S_{n+1}, I_{n+1})}{\partial S}\frac{\partial S_{n+1}}{\partial AT} + \frac{\partial h(S_{n+1}, I_{n+1})}{\partial I}\frac{\partial I_{n+1}}{\partial AT}\Big)\bigg|_{I_n^+=0} \\ =& \mu_2 f(S_v)\Big(\frac{dL_N(S)}{dS}\bigg|_{S=(1-p)S_v}\Big)\frac{(1-p)S_v}{AT - \alpha_2\Lambda} > 0 \end{aligned} \tag{4.34}$$

and

$$\begin{aligned}
&\int_{S_n^+}^{S_{n+1}} \frac{\partial^2 h(s, I(s; S_n^+, I_n^+))}{\partial I \partial AT} ds \bigg|_{I_n^+=0} \\
=& \int_{(1-p)S_v}^{S_v} \frac{\partial^2 h(s, I(s; (1-p)S_v, 0))}{\partial I^2} \frac{\partial I(s, I(s; (1-p)S_v, 0))}{\partial AT} ds \\
=& \int_{(1-p)S_v}^{S_v} \frac{\partial^2 h(s, I(s; (1-p)S_v, 0))}{\partial I^2} \bigg(\frac{\partial I(s, I(s; (1-p)S_v, 0))}{\partial I_n^+} \frac{\partial I_n^+}{\partial AT} \\
&+ \frac{\partial I(s, I(s; (1-p)S_v, 0))}{\partial S_n^+} \frac{\partial S_n^+}{\partial AT} \bigg) \bigg|_{I_n^+=0} ds \\
=& \int_{(1-p)S_v}^{S_v} \frac{\partial^2 h(s, I(s; (1-p)S_v, 0))}{\partial I^2} \bigg(\frac{\partial I(s, I(s; (1-p)S_v, 0))}{\partial S_n^+} \frac{\partial S_n^+}{\partial AT} \bigg) \bigg|_{I_n^+=0} ds \\
=& \frac{(1-p)S_v}{AT - \alpha_2 \Lambda} \bigg(\frac{dL_N(S)}{dS} \bigg|_{S=(1-p)S_v} \bigg) \theta > 0.
\end{aligned} \tag{4.35}$$

According to equations (4.31) to (4.35), it can be concluded that

$$\frac{\partial^2 I_{n+1}}{\partial (I_n^+) \partial AT} \bigg|_{I_n^+=0}^{AT=AT^*} > 0. \tag{4.36}$$

□

5. Discussion and conclusions

In previous state-dependent feedback control SIR models [12, 13], the timing of implementing control measures was only related to whether the number of susceptible individuals reached the threshold, ignoring the impact of growth rates on control thresholds. This will inevitably increase the risk of effective control of infectious diseases. In this work, we propose a SIR model with nonlinear state-dependent feedback control, in which the control measures, such as isolation and vaccination, are adopted when the convex combinations of the size of the susceptible population and their growth rates reach the action threshold. The control form adopted is more in line with the development laws of the population. And we add a non-linear term to the classical SIR model to describe the impact of limited medical resources or treatment capacity on infectious disease transmission. To analyze the dynamical behavior of the proposed model, we first analyze the relevant properties of its ODE system (2.4), including the existence and stability of the equilibria. And then the analytical methods are developed to address the existence of order-k ($k \geq 1$) non-trivial periodic solutions, the existence and stability of a DFPS and its bifurcations.

In Section 3, we first prove the existence and the asymptotical stability of DFPS in Section 3.1. Then we analyse the direction of the phase plane solution trajectories for different positional relationships between the impulsive curves and the isoclinic lines, and using the defined sigma function, we find the maximum set of impulsive effect in each case, thus giving the domain of definition of the Poincaré map. In Section 3.2.2, we discuss the precise impulsive set when the system has no endemic equilibrium and figure out the attractive domain of the DFPS, which is not mentioned in Section 3.1. In Sections 3.2.3 and 3.2.4, we analyse the existence of the non-trivial periodic solutions in some special cases. Moreover, we give the sufficient conditions for the existence of the order-1 non-trivial periodic solution and the non-existence of the order-k ($k \geq 1$) periodic solution when the endemic equilibrium E^* is a stable focus or node. However, as shown in Remark 3.2, when endemic equilibrium E^* is an unstable focus or node, the domain, continuity and convexity of map $\mathcal{P}_{\mathcal{M}}$ are relatively complex. Thus, discussing the existence of non-trivial periodic solutions of system (1.3) in this situation is challenging, which is worth further consideration in the future.

In Section 4, we calculate the transcritical bifurcation with respect to the parameters p and AT in the case $q = 0$. From a biological perspective, $q = 0$ means that when the threshold condition is reached, we only vaccinate susceptible individuals without isolating infected individuals. Mathematically, the reason for the calculation at $q = 0$ is that when calculating the relevant derivatives according to the Lemma 2.2, we can only get the inequality (4.17) at $q = 0$, which means that $q = 0$ is a sufficient condition. Then we conclude that the transcritical bifurcation around the DFPS exists with respect to the parameter p or AT (see Theorem 4.1, 4.2).

When the endemic equilibrium of the system (2.4) is unstable, the dynamic behavior of the pulse system (1.3) is complex and rich, and thus it is well worth following up. The analytical techniques developed here not only can be applied to analyze epidemic models with nonlinear state-dependent impulsive control, but also can be extended to other fields including integrated pest management and tumor control.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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