Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(10): 18592–18629.
DOI: 10.3934/mbe.2023825
Received: 16 July 2023
Revised: 06 September 2023
Accepted: 17 September 2023
Published: 27 September 2023

Research article

Dynamic analysis of a Leslie-Gower predator-prey model with the fear effect and nonlinear harvesting

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Abstract: In this paper, we investigate the stability and bifurcation of a Leslie-Gower predator-prey model with a fear effect and nonlinear harvesting. We discuss the existence and stability of equilibria, and show that the unique equilibrium is a cusp of codimension three. Moreover, we show that saddle-node bifurcation and Bogdanov-Takens bifurcation can occur. Also, the system undergoes a degenerate Hopf bifurcation and has two limit cycles (i.e., the inner one is stable and the outer is unstable), which implies the bistable phenomenon. We conclude that the large amount of fear and prey harvesting are detrimental to the survival of the prey and predator.

Keywords: nonlinear harvesting; fear effect; Leslie-Gower; Hopf bifurcation; Bogdanov-Takens bifurcation

1. Introduction

Fishing is a method used in the industry to acquire fish products from natural or artificial bodies of water. With the development of fisheries, fishing has become more common. We see that harvesting for economic gain is a relatively regular occurrence in nature and that it significantly affects both the ecological balance and system dynamics. It is crucial to develop biological resources at their maximum sustainable yield while preserving the survival of all interacting populations, both ecologically and economically. However, if a species is overharvested, it can lead to ecological problems, as some people may prioritize profit over protecting the environment. Thus, the authors of [1, 2] built mathematical models to analyze these problems, whose dynamical behaviors have attracted the interest of many scholars. There are three forms of harvesting: 1) constant harvesting, $h(x) = h$; 2) linear harvesting, $h(x) = qEx$; 3) nonlinear harvesting, $h(x) = \frac{qEx}{m_1E+m_2x}$, which is also called Michaelis-Menten-type harvesting.

Leslie and Gower studied the predator-prey relationship between two species and developed the famous Leslie-Gower predator-prey model [3], which has been widely discussed [4–6]. For example, the Leslie-Gower predator-prey model with the Allee effect and a generalist predator was studied in [7], where the authors found that the system exhibits a multi-stability phenomenon and undergoes various bifurcations. Huang et al. [8] studied the Leslie-Gower-type predator-prey model with constant-yield harvesting, and they found that the dynamical behavior of the model is very sensitive to the constant yield harvest of the predator.

Gupta et al. [9] considered the following Leslie-Gower predator-prey model with Michaelis-Menten-type prey-harvesting:

$$\begin{aligned}
&\dot{x} = rx\left(1 - \frac{x}{K}\right) - \alpha xy - \frac{qEx}{m_1 E + m_2 x},\\
&\dot{y} = sy\left(1 - \frac{y}{nx}\right), \ \text{ if } (x, y) \neq (0, 0),\\
&\dot{y} = 0, \ \text{ if } (x, y) = (0, 0),
\end{aligned} \tag{1.1}$$

where x and y are the prey and predator population densities, respectively. They studied the stability and bifurcation (saddle-node bifurcation and Hopf bifurcation) of system (1.1). Also, the existence of bionomic equilibria and optimal singular control were investigated. Based on system (1.1), Gupta and Chandra [10] introduced the Holling type II functional response and obtained the bistable situation. The model exhibits several local bifurcations (saddle-node, Hopf, homoclinic and Bogdanov-Takens) which are ecologically important. Considering the group defense and nonlinear harvesting in prey, Kumar and Kharbanda [11] obtained that the density of the predator increases as the harvest rate of the predator decreases. Caraballo Garrido et al. [12] investigated the predator prey model with nonlinear harvesting with both constant and distributed delay by varying parameters. Some scholars [13–17] have combined other functional responses and harvesting to obtain more complex dynamical behaviors.

As with direct killing, indirect killing also has a significant impact on the dynamic behaviors of the system. The studies mentioned above, however, solely take into account the predator's direct killing. Predation danger may drive the prey to engage in anti-predation behaviors, such as habitat modifications or foraging, which may lower the prey's birth rate. Hence, Wang et al. [18] incorporated the fear effect into the reproduction of prey animals and obtained the following prey-predator model:

$$\begin{aligned}
&\dot{x} = r_0 x f(k, y) - dx - ax^2 - pxy,\\
&\dot{y} = cpxy - my,
\end{aligned} \tag{1.2}$$

where $f(k, y) = \frac{1}{1+k_0 y}$ accounts for the cost of anti-predator defense due to fear. They studied a model with a linear functional response or Holling type II functional response. It was found that the fear effect has no impact on the dynamic behaviors of model (1.2). However, the dynamic behavior of model (1.2) with Holling type II functional responses can be affected by the fear effect. Chen et al. [19] considered the influence of the fear effect and Leslie-Gower function on the dynamic behavior of the predator-prey model, and they demonstrated that there are many types of bifurcation phenomena, including transcritical bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation. Zhang et al. [20] studied a delayed diffusive predator-prey model with spatial memory and a nonlocal fear effect, and they investigated the stability, Hopf bifurcation and Turing-Hopf bifurcation of the system. Many scholars [21–25] have studied the prey-predator model with a fear effect.

In this paper, we incorporate the fear effect into system (1.1) and obtain the following model:

$$\begin{aligned}
\dot{x} &= x\left(\frac{r_0}{1+k_0y} - d_0 - ax - c_0y\right) - \frac{q_0Ex}{m_1E + m_2x},\\
\dot{y} &= s_0y\left(1 - \frac{y}{hx}\right), \quad \text{if } (x,y) \neq (0,0),\\
\dot{y} &= 0, \quad \text{if } (x,y) = (0,0),
\end{aligned} \tag{1.3}$$

where r_0 is the birth rate of the prey and d_0 is the natural death rate of the prey. In the ecological sense, it is clear that $r_0 > d_0$. a represents the intra-species competition, c_0 is the maximum predation rate, s_0 is the intrinsic growth rates of the predators, h is a measure of the quality of the prey as food for the predator, k_0 is the fear parameter, q_0 is the catchability coefficient, E is the effort applied to harvest the prey species and m_1, m_2 are suitable constants. For simplicity, letting

$$\begin{aligned}
&\bar{x} = \frac{m_2}{m_1E}x, \quad \bar{y} = \frac{m_2}{m_1Eh}y, \quad \bar{t} = \frac{am_1^2E^2}{m_2^2x}t,\\
&\bar{r} = \frac{r_0m_2}{am_1E}, \quad \bar{k} = \frac{k_0m_1Eh}{m_2}, \quad \bar{d} = \frac{d_0m_2}{am_1E},\\
&\bar{c} = \frac{c_0h}{a}, \quad \bar{q} = \frac{q_0m_2}{am_1^2E}, \quad \bar{s} = \frac{s_0m_2}{am_1E},
\end{aligned}$$

for x, y be positive, and dropping the bars, system (1.3) becomes

$$\begin{aligned}
\dot{x} &= x^2\left(\frac{r}{1+ky} - d - x - cy\right) - \frac{qx^2}{1+x},\\
\dot{y} &= sy\left(x - y\right),
\end{aligned} \tag{1.4}$$

where $r > d$, and r, k, d, c, q and s are positive constants.

The key aim of this study on prey-predator models is to discuss the impacts of prey fear and prey harvesting on system dynamics. The bifurcation phenomenon that distinguishes system (1.4) from system (1.1) deserves further discussion. In addition, by analyzing the observed bifurcation phenomena, we can elucidate the benefits and drawbacks of prey harvesting on both populations.

The structure of the article is as follows. In Section 2, we obtain the boundedness of solutions and analyze the dynamical behaviors of origin. In Section 3, we discuss the existence of boundary equilibria and positive equilibria. In Section 4, we analyze the stability of equilibria. In Section 5, we show that system (1.4) undergoes saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation. In Section 6, we have a summary of the article.

2. Preliminaries

We show that the positive solutions of system (1.4) are ultimately bounded.

Theorem 2.1. *All solutions of system* (1.4) *are bounded for all* $t \geq 0$.

Proof. Since system (1.3) is equivalent to system (1.4), we now prove that the solution of system (1.3) is bounded. From the first equation of system (1.3), we have

$$\dot{x} \leq x(r_0 - d_0 - ax),$$

for $t \geq 0$, which immediately implies that $\limsup_{t\to\infty} x(t) \leq \frac{r_0 - d_0}{a} \triangleq M$. Then from the second equation of system (1.3), it follows that

$$\dot{y} \leq s_0 y\left(1 - \frac{y}{hM}\right)$$

for $t \geq 0$, that is, $\limsup_{t\to\infty} y(t) \leq hM$. Hence, $x(t)$ and $y(t)$ are bounded. The proof is completed.

Denote

$$q_0 = (d + q)k + c + 1.$$

Next, we show the dynamic behaviors of the origin of system (1.4).

Lemma 2.1. *The types of origin in system* (1.4) *are as follows:*

1) if $r - d < q$ or $r - d = q \leq q_0$, the origin of system (1.4) *is a non-hyperbolic attractor;*

2) if $r - d > q$ or $r - d = q > q_0$, the origin of system (1.4) *is a non-hyperbolic repeller.*

Proof. The Jacobian matrix of system (1.4) at the origin is degenerate; then, we apply the blow-up method to analyze the type of origin. Notice that when $x = 0$, we have that $\dot{x} = 0$ and $\dot{y} = -sy^2 < 0$, which means that system (1.4) has the invariant line $x = 0$. Using the horizontal blow-up

$$(x, y) = (u, uv) \quad \text{and} \quad \mathrm{d}\tau = u\mathrm{d}t,$$

system (1.4) can be rewritten as

$$\begin{aligned} \dot{u} &= u\left(\frac{r}{1 + kuv} - d - u - cuv - \frac{q}{1 + u}\right), \\ \dot{v} &= sv(1 - v) - v\left(\frac{r}{1 + kuv} - d - u - cuv - \frac{q}{1 + u}\right). \end{aligned} \tag{2.1}$$

(i) The equilibria of system (2.1) in $u = 0$ are $A(0, 0)$ and $B\left(0, 1 - \frac{r-d-q}{s}\right)$ when $r - d < q + s$. The Jacobian matrix at the equilibria A and B are, respectively

$$J_A = \begin{pmatrix} r - d - q & 0 \\ 0 & q + s - (r - d) \end{pmatrix}$$

and

$$J_B = \begin{pmatrix} r - d - q & 0 \\ \dfrac{(s - r + d + q)R}{s^2} & r - d - (q + s) \end{pmatrix},$$

where

$$R = -kr^2 + (ks - c + (d + q)k)r + (c - q + 1)s + c(d + q).$$

The eigenvalues of matrix J_A are $\lambda_{J_{A1}} = r - d - q$ and $\lambda_{J_{A2}} = q + s - (r - d) > 0$, and the eigenvalues of matrix J_B are $\lambda_{J_{B1}} = r - d - q$ and $\lambda_{J_{B2}} = r - d - (q + s) < 0$. If $r - d < q$, that is, if $\lambda_{J_{A1}} < 0$ and $\lambda_{J_{B1}} < 0$, A is a saddle and B is a stable node (see Figure 1(a)). If $q < r - d < q + s$, that is, if $\lambda_{J_{A1}} > 0$ and $\lambda_{J_{B1}} > 0$, A is an unstable node and B is a saddle (see Figure 1(c)).

If $r - d = q$, that is, if $\lambda_{J_{A1}} = 0$ and $\lambda_{J_{B1}} = 0$, both A and B are degenerate equilibria. First, we consider the degenerate equilibrium A. Taking the time variable

$$d\tau = s\,dt,$$

we can obtain the Taylor expansion of system (2.1) as follows:

$$\begin{cases} \dot{u} = \dfrac{q-1}{s}u^2 - \dfrac{q}{s}u^3 - \dfrac{dk+kq+c}{s}u^2v + o(|u,v|^2), \\ \dot{v} = v - \dfrac{q-1}{s}uv - v^2 + \dfrac{dk+kq+c}{s}uv^2 + \dfrac{q}{s}u^2v + o(|u,v|^2). \end{cases}$$

By Theorem 7.1 in [26], the degenerate equilibrium A is a saddle-node if $q \neq 1$. If $q = 1$, we have that the equilibrium A is a degenerate saddle from $-\dfrac{q}{s} < 0$.

Next, for the degenerate equilibrium B, make the following transformation:

$$(u, v) = (X, Y + 1).$$

System (2.1) becomes

$$\begin{cases} \dot{X} = -[(k-1)q + dk + c + 1]X^2 + [(k^2-1)q + dk^2]X^3 + o(|X,Y|^2), \\ \dot{Y} = [(k-1)q + dk + c + 1]X - sY - [(k^2-1)q + dk^2]X^2 \\ \qquad +[(2k-1)q + 2dk + 2c + 1]XY - sY^2 + o(|X,Y|^2). \end{cases} \tag{2.2}$$

Using $(X, Y) = (sX_1, [(k-1)q + dk + c + 1]X_1 + Y_1)$ and $d\tau = -s\,dt$, system (2.2) becomes

$$\begin{cases} \dot{X}_1 = \alpha_{20}X_1^2 + \alpha_{30}X_1^3 + \alpha_{21}X_1^2Y_1 + o(|X_1, Y_1|^2), \\ \dot{Y}_1 = Y_1 + \beta_{20}X_1^2 + \beta_{11}X_1Y_1 + \beta_{02}Y_1^2 + o(|X_1, Y_1|^2), \end{cases} \tag{2.3}$$

where

$$\begin{aligned} &\alpha_{20} = q_0 - q, \quad \alpha_{21} = (d+q)k + c, \\ &\alpha_{30} = (k^2-k)q^2 + (2dk^2 - k^2s + 2ck - dk - c + k + s)q + d^2k^2 - dk^2s + 2cdk \\ &\qquad +c^2 + dk + c, \\ &\beta_{20} = \left(-2k^2 + 3k - 1\right)q^2 + \left(-4dk^2 + k^2s - 4ck + 3dk + 3c - 3k - s + 2\right)q - 2d^2k^2 \\ &\qquad +dk^2s - 4cdk - 2c^2 - 3dk - 3c - 1, \quad \beta_{11} = 1 - q, \quad \beta_{02} = 1. \end{aligned}$$

If $\alpha_{20} > 0$ (or $\alpha_{20} < 0$), that is, if $q < q_0$ (or $q > q_0$), B a saddle-node with a stable parabolic sector on the right (or left). If $\alpha_{20} = 0$, which implies that $k < 1$, we get that $q = \frac{dk+c+1}{1-k}$. Next, substituting $q = \frac{dk+c+1}{1-k}$ into the coefficients of the X^3 term of system (2.3), we have

$$\alpha_{30} = s[dk + (1+c)(1+k)] > 0.$$

Explicitly, from Theorem 7.1 in [26] we know that the degenerate equilibrium B is a stable node if $q = q_0$.

In summary, when $r - d = q \leq 1$, A and B are as shown in Figure 1(a). When $1 < r - d = q \leq q_0$, A and B are as shown in Figure 1(b). When $r - d = q > q_0$, A and B are as shown in Figure 1(c).

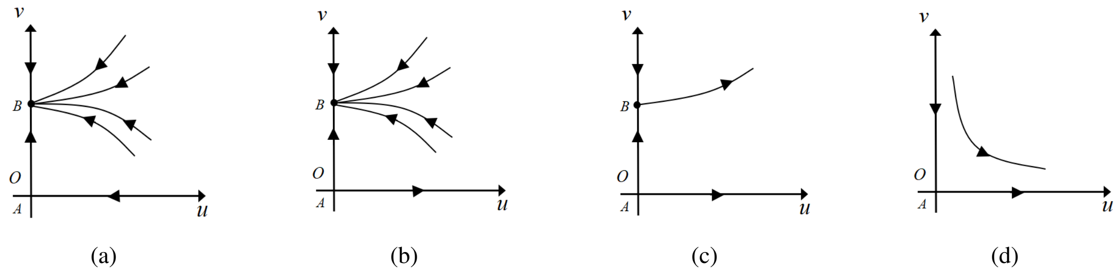


Figure 1. Phase portrait around the origin of system (2.1). (a) $r-d<q$ or $r-d=q\leq 1$. (b) $1<r-d=q\leq q_0$. (c) $r-d=q>q_0$ or $q<r-d<q+s$. (d) $r-d\geq q+s$.

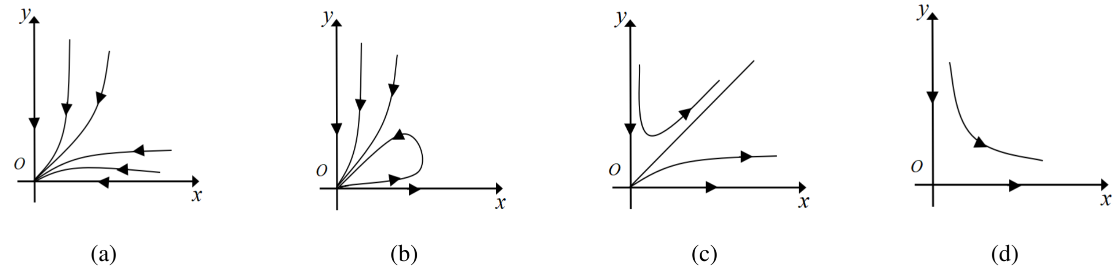


Figure 2. (a) The origin of system (1.4) is an attractor when $r-d<q$ or $r-d=q\leq 1$. (b) The origin of system (1.4) is an attractor when $1<r-d=q\leq q_0$. (c) The origin of system (1.4) is a repeller when $r-d=q>q_0$ or $q<r-d<q+s$. (d) The origin of system (1.4) is a repeller when $r-d\geq q+s$.

After a blow-down, the origin in system (1.4) is an attractor when $r-d<q$, $r-d=q\leq 1$ (see Figure 2(a)) or $1<r-d=q\leq q_0$ (see Figure 2(b)). The origin is a repeller when $r-d=q>q_0$ or $q<r-d<q+s$ (see Figure 2(c)).

(ii) When $r-d=q+s$, system (2.1) has only one equilibrium, $A(0,0)$ at $u=0$, whose Jacobian matrix is

$$J_A=\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}.$$

Expanding system (2.1) in a Taylor series and taking a time variable $\mathrm{d}\tau=s\,\mathrm{d}t$, we have

$$\begin{cases} \dot{u}=u+\dfrac{q-1}{s}u^2+o(|u,v|^2), \\ \dot{v}=\dfrac{1-q}{s}uv-v^2+o(|u,v|^2). \end{cases}$$

The coefficient of v^2 is $-1<0$; from Theorem 7.1 in [26], we know that the equilibrium A is a saddle-node (see Figure 1(d)). After a blow-down, we see that the origin is a repeller for system (1.4) (see Figure 2(d)).

(iii) System (2.1) has a unique equilibrium $A(0,0)$ when $r-d>q+s$. The Jacobian matrix at the equilibrium A is

$$J_A = \begin{pmatrix} r-d-q & 0 \\ 0 & d-r+q+s \end{pmatrix}.$$

Obviously, the two eigenvalues are $r-d-q > 0$ and $d-r+q+s < 0$. It shows that the equilibrium A is a saddle (see Figure 1(d)). Similarly, the origin is a repeller for system (1.4) (see Figure 2(d)). The proof is completed.

3. Existence of equilibria

In this section, we will discuss the existence of the boundary equilibria and the positive equilibria of system (1.4).

First, we analyze the existence of boundary equilibria. When $y = 0$, the first equation of system (1.4) can be simplified into

$$\dot{x} \;=\; x^2(r-d-x) - \frac{qx^2}{1+x}.$$

We have

$$f(x) = x^2 - (r-d-1)\,x - (r-d-q),$$

and the discriminant of $f(x)$ is as follows:

$$\Delta_1 = 4(q^* - q),$$

where

$$q^* = \frac{(r-d+1)^2}{4}.$$

The two roots of $f(x) = 0$ can be expressed as

$$x_1 = \frac{r-d-1-\sqrt{\Delta_1}}{2}, \quad x_2 = \frac{r-d-1+\sqrt{\Delta_1}}{2}.$$

If $r-d > q$, $f(x) = 0$ has only one positive root x_2. If $r-d = q$, $f(x) = 0$ has only one positive root $q-1$ if $q > 1$.

Assume that $r-d < q$. When $r-d \le 1$, obviously, $f(x) = 0$ has no positive roots. When $r-d > 1$, obviously, $q^* > r-d$. If $q > q^*$, then $f(x) = 0$ has no positive roots. If $q = q^*$, then $f(x) = 0$ has only one positive root $\frac{r-d-1}{2}$. If $q < q^*$, then $f(x) = 0$ has two positive roots x_1 and x_2.

To summarize, we have the following lemma.

Lemma 3.1. *The following claims regarding the existence of the boundary equilibria of system* (1.4) *are true.*

1) If $r-d > q$, system (1.4) *has a unique boundary equilibrium $E_2(x_2, 0)$.*

2) If $1 < r-d = q$, system (1.4) *has a unique boundary equilibrium $E_2(q-1, 0)$.*

3) If $1 < r-d < q$, we have the following three cases:

(a) if $1 < r-d < q^* < q$, system (1.4) *has no boundary equilibrium;*

(b) if $1 < r-d < q = q^*$, system (1.4) *has a unique boundary equilibrium $\overline{E}\left(\frac{r-d-1}{2}, 0\right)$;*

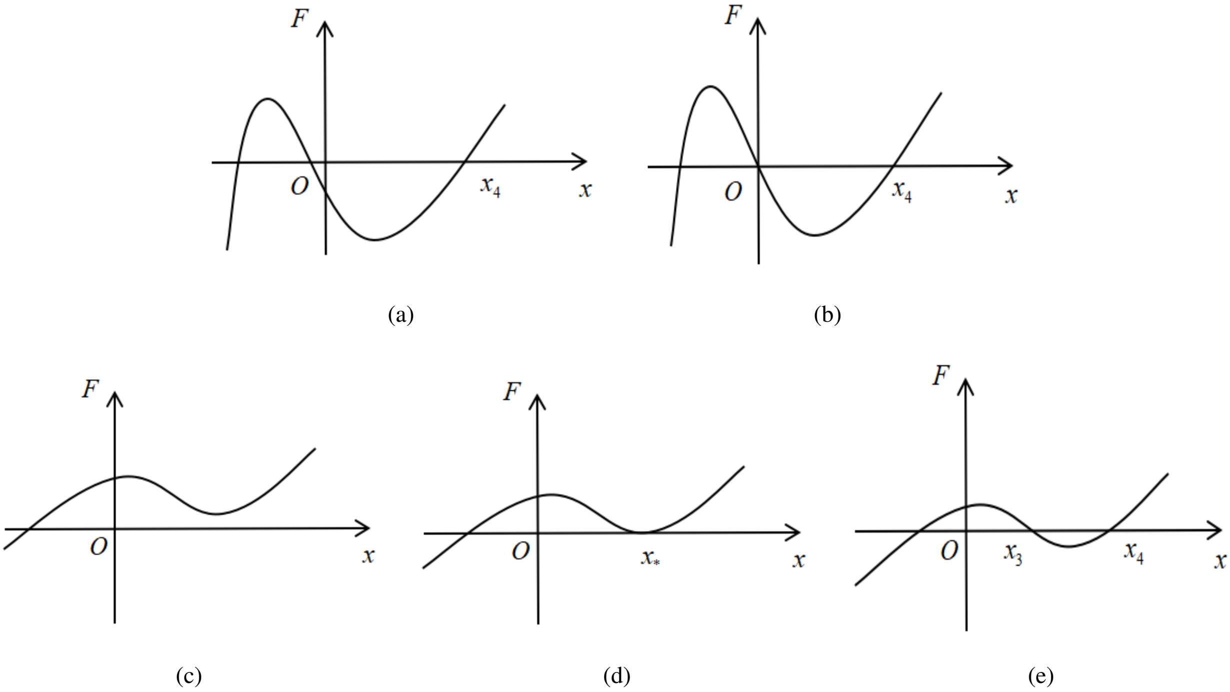


Figure 3. (a) $F(x) = 0$ has a unique positive root x_4 when $r - d > q$. (b) $F(x) = 0$ has a unique positive root x_4 when $q_0 < r - d = q$. (c) When $q_0 < r - d < q$ and $F(x_*) > 0$, $F(x) = 0$ has no positive root. (d) When $q_0 < r - d < q$ and $F(x_*) = 0$, $F(x) = 0$ has a unique positive root x_*. (e) When $q_0 < r - d < q$ and $F(x_*) < 0$, $F(x) = 0$ has two different positive roots x_3 and x_4.

(c) if $1 < r - d < q < q^*$, *system* (1.4) *has two boundary equilibria* $E_1(x_1, 0)$, $E_2(x_2, 0)$.

Next, we will discuss the positive equilibria $E(x, y)$ of system (1.4). Letting $\dot{x} = \dot{y} = 0$ in system (1.4), we have

$$\begin{cases} \dfrac{r}{1+ky} - d - x - cy - \dfrac{q}{1+x} = 0, \\ x - y = 0. \end{cases}$$

We denote

$$F(x) = k(c+1)x^3 + ((k+1)(c+1) + dk)x^2 + (q_0 - (r-d))x - (r - d - q)$$

and

$$F'(x) = 3k(c+1)x^2 + 2((k+1)(c+1) + dk)x + q_0 - (r-d),$$

where the discriminant of $F'(x)$ is

$$\Delta_2 = 4((k+1)(c+1) + dk)^2 - 12k(c+1)(q_0 - (r-d)).$$

Define

$$x_* = \frac{-2((k+1)(c+1) + dk) + \sqrt{\Delta_2}}{6k(1+c)}, \quad y_* = x_*.$$

From $F(x) = 0$, we have

$$q = -\frac{(1+x)[k(c+1)x^2 + (dk+c+1)x + d - r]}{kx+1}. \tag{3.1}$$

The Jacobian matrix of system (1.4) at $E(x, y)$ is

$$J_E = \begin{pmatrix} \dfrac{x^2[q-(x+1)^2]}{(1+x)^2} & -\dfrac{x^2[rk + c(1+kx)^2]}{(1+kx)^2} \\ sx & -sx \end{pmatrix},$$

and

$$DetJ_E = \left[\frac{x^2[rk + c(1+kx)^2]}{(1+kx)^2} - \frac{x^2[q-(x+1)^2]}{(1+x)^2}\right] sx,$$

$$TrJ_E = \frac{x^2[q-(x+1)^2]}{(1+x)^2} - sx.$$

Substituting (3.1) into $DetJ_E$ and $F'(x)$, we have

$$DetJ_E = \frac{(x+s)(1+x)^2}{x} F'(x). \tag{3.2}$$

When $r - d > q$, it is easy to get that the equation $F(x) = 0$ has a unique positive root x_4 (see Figure 3(a)).

When $q_0 < r - d = q$, $\Delta_2 > 0$. We obtain that $F(x) = 0$ has only one positive root x_4 (see Figure 3(b)). When $q_0 \geq r - d = q$, we find that $F(x) = 0$ has no positive roots.

When $r - d < q$, obviously, $F(x) = 0$ has no positive roots if $q_0 \geq r - d$. If $q_0 < r - d$, we obtain that $F(x) = 0$ has no positive roots when $F(x_*) > 0$ (see Figure 3(c)). When $F(x_*) = 0$, $F(x) = 0$ has only one positive root x_* (see Figure 3(d)). When $F(x_*) < 0$, $F(x) = 0$ has two positive roots x_3 and x_4 (see Figure 3(e)).

To summarize, we have the following lemma.

Lemma 3.2. *The following claims regarding the existence of the boundary equilibria of system* (1.4) *are true.*

1) If $r - d > q$, system (1.4) *has a unique positive equilibrium $E_4(x_4, y_4)$.*

2) If $q_0 < r - d = q$, system (1.4) *has a unique positive equilibrium $E_4(x_4, y_4)$.*

3) If $q_0 < r - d < q$, we obtain the following results:

(a) if $F(x_*) > 0$, system (1.4) *has no positive equilibrium;*

(b) if $F(x_*) = 0$, system (1.4) *has a unique positive equilibrium $E_*(x_*, y_*)$;*

(c) if $F(x_*) < 0$, system (1.4) *has two positive equilibria $E_3(x_3, y_3)$ and $E_4(x_4, y_4)$.*

4. Stability of equilibria

In this section, we will discuss the stability of the equilibria.

4.1. Stability of the boundary equilibria E_1 and E_2

Theorem 4.1. *1) If $r-d>q$, the unique boundary equilibrium E_2 is a saddle.*
2) If $1<r-d=q$, the unique boundary equilibrium E_2 is a saddle.
3) If $1<r-d<q=q^*$, the unique boundary equilibrium $\overline{E}$ is a saddle-node.
4) If $1<r-d<q<q^*$, E_1 is unstable and E_2 is a saddle.

Proof. 1) The Jacobian matrix of system (1.4) at E_2 is

$$J_{E_2}=\begin{pmatrix} \dfrac{-\sqrt{\Delta_1}x_2^2}{1+x_2} & -(rk+c)x_2^2 \\ 0 & sx_2 \end{pmatrix}.$$

Obviously, the equilibrium E_2 is a saddle.
2) The Jacobian matrix of system (1.4) at $E_2(q-1,0)$ is

$$J_{E_2(q-1,0)}=\begin{pmatrix} -\dfrac{(q-1)^3}{q} & -(rk+c)(q-1)^2 \\ 0 & s(q-1) \end{pmatrix},$$

which implies that E_2 is a saddle.
3) The Jacobian matrix of system (1.4) at $\overline{E}$ is

$$J_{\overline{E}}=\begin{pmatrix} 0 & -\dfrac{(rk+c)(r-d-1)^2}{4} \\ 0 & \dfrac{s(r-d-1)}{2} \end{pmatrix},$$

which means that $\overline{E}$ is a degenerate equilibrium. First, we transform $\overline{E}$ to the origin by letting $X=x-\dfrac{r-d-1}{2}$, $Y=y$. Then, system (1.4) can be rewritten as

$$\begin{cases} \dot{X}=a_{01}Y+a_{20}X^2+a_{11}XY+a_{02}Y^2+o(|X,Y|^2), \\ \dot{Y}=b_{01}Y+b_{11}XY+b_{02}Y^2+o(|X,Y|^2), \end{cases} \tag{4.1}$$

where

$$a_{01}=-\frac{(r-d-1)^2(kr+c)}{4},\quad a_{20}=\frac{(r-d-1)^2}{2(-r+d-1)},\quad a_{11}=-(r-d-1)(kr+c),$$
$$a_{02}=\frac{(r-d-1)^2rk^2}{4},\quad b_{01}=\frac{s(r-d-1)}{2},\quad b_{11}=s,\quad b_{02}=-s.$$

Next, applying the following transformation:

$$X=u+v,\quad Y=-\frac{2s}{(kr+c)(r-d-1)}v,\quad \mathrm{d}\tau=\frac{r-d-1}{2}\mathrm{d}t,$$

system (4.1) becomes

$$\begin{cases} \dot{u}=c_{20}u^2+c_{11}uv+c_{02}v^2+o(|u,v|^2), \\ \dot{v}=v+d_{11}uv+d_{02}v^2+o(|u,v|^2), \end{cases}$$

where

$$
\begin{aligned}
c_{20} &= -\frac{r-d-1}{s(r-d+1)}, \quad c_{11} = \frac{2[(r-d-1)^2 - s(r-d+1)]}{(r-d+1)(r-d-1)s},\\
c_{02} &= \frac{k^2r^5 + (-3dk^2 - 2k^2s + 2ck - 3k^2)r^4 + h_1r^3 + h_2r^2 + h_3r + h_4}{(r-d+1)(kr+c)^2(r-d-1)^2s},\\
d_{11} &= -\frac{2}{r-d-1}, \quad d_{02} = -\frac{2[-kr^2 + (dk-c+k)r + cd + c - 2s]}{(r-d-1)^2(kr+c)},\\
h_1 &= 3d^2k^2 + 4dk^2s - 2k^2s^2 - 6cdk - 4cks + 6dk^2 + c^2 - 6ck + 3k^2,\\
h_2 &= (-d^3 - 2sd^2 + 4s^2d - 3d^2 - 3d + 2s - 1)k^2 + (6cd^2 + 8cds + 12cd + 4s^2\\
&\quad +6c)k - 3c^2d - 2c^2s - 3c^2,\\
h_3 &= (3d^2 + 4sd + 6d + 3)c^2 + (-2d^3k - 4sd^2k - 6d^2k - 6dk + 4sk + 4s^2 - 2k)c\\
&\quad -2d^2k^2s^2 - 4dks^2 + 2k^2s^2 + 4ks^2,\\
h_4 &= (-d^3 - 2sd^2 - 3d^2 - 3d + 2s - 1)c^2 + (-4ds^2 + 4s^2)c.
\end{aligned}
$$

Since $c_{20} < 0$, we get that $\overline{E}$ is a saddle-node from Theorem 7.1 in [26].
4) The Jacobian matrices of system (1.4) at $E_1\,(x_1, 0)$ and $E_2\,(x_2, 0)$, respectively, are

$$
J_{E_1} = \begin{pmatrix} \dfrac{x_1^2\sqrt{\Delta_1}}{1+x_1^2} & -x_1^2(rk+c) \\ 0 & sx_1 \end{pmatrix}
$$

and

$$
J_{E_2} = \begin{pmatrix} -\dfrac{x_2^2\sqrt{\Delta_1}}{1+x_2^2} & -x_2^2(rk+c) \\ 0 & sx_2 \end{pmatrix}.
$$

This proves that E_1 is unstable and E_2 is a saddle point. The proof is completed.

4.2. Stability of the interior equilibrium when $r - d \geq q$

If E_3 exists, from $F'(x_3) < 0$ and (3.2), we obtain that $DetJ_{E_3} < 0$. That is, E_3 is a saddle if it exists. Hence, in the following discussion, we only study the stability of E_4.

Define

$$
s^* = \frac{x_4[q - (1+x_4)^2]}{(1+x_4)^2}.
$$

Theorem 4.2. *When $r - d > q$ or $r - d = q > q_0$, system* (1.4) *has a positive equilibrium E_4. In addition, the following statements are true.*

1) If $s^* \leq 0$ or $0 < s^* < s$, E_4 is locally asymptotically stable.

2) If $s < s^*$, E_4 is an unstable node or focus.

3) If $s = s^* > 0$, E_4 is a center or weak focus.

Proof. The Jacobian matrix at the equilibrium $E_4(x_4, y_4)$ is

$$
J_{E_4} = \begin{pmatrix} \dfrac{x_4^2[q-(1+x_4)^2]}{(1+x_4)^2} & -\dfrac{x_4^2[rk + c(1+kx_4)^2]}{(1+kx_4)^2} \\ sx_4 & -sx_4 \end{pmatrix}.
$$

Notice that $F'(x_4) > 0$; so, from (3.2), we know that $DetJ_{E_4} > 0$.

Obviously,

$$TrJ_{E_4} = x_4(s^* - s).$$

Hence, it is easy to see that E_4 is locally asymptotically stable if $s^* \leq 0$ or $0 < s^* < s$, an unstable node or focus if $s < s^*$ and a center or weak focus if $s = s^* > 0$. The proof is completed.

Define

$$q_1 = -\frac{1}{8}s^2 + \frac{5}{2}s + 1 + \frac{1}{8}\sqrt{s(s+8)^3}.$$

Theorem 4.3. *If E_4 is locally asymptotically stable and $0 < q < q_1$, then E_4 is globally asymptotically stable.*

Proof. Noting that $r - d > q$ or $r - d = q > q_0$, and according to Lemmas 3.1 and 3.2, system (1.4) has a boundary equilibrium E_2 and positive equilibrium E_4. By Lemma 2.1 and Theorem 4.1, both the origin and E_2 are unstable. We assume that E_4 is locally asymptotically stable. Now, we want to show that there is no limit cycle around E_4. Hence, taking the Dulac function $\Phi(x, y) = \frac{1}{x^2y^2}$, we have

$$\frac{\partial(\Phi F)}{\partial x} + \frac{\partial(\Phi G)}{\partial y} = -\frac{x^3 + (s+2)x^2 + (2s+1-q)x + s}{x(1+x)^2y^2},$$

where $F = x^2\left(\frac{r}{1+ky} - d - x - cy\right) - \frac{qx^2}{1+x}$ and $G = sy(x - y)$.

Define

$$H = x^3 + (s+2)x^2 + (2s+1-q)x + s.$$

Obviously, $H > 0$ for $q \leq 1 + 2s$. In what follows, we only consider that $q > 1 + 2s$. By calculation, the discriminant of H is

$$\Delta_3 = -\frac{q[4q^2 + (s^2 - 20s - 8)q - 4(s-1)^3]}{108}.$$

By calculation, we obtain that $q_1 > 1 + 2s$ and

$$\Delta_3\Big|_{q=1+2s} = \frac{s(1+2s)(2s^2 + 11s + 32)}{108} > 0.$$

Thus, we have that $\Delta_3 > 0$ for $1 + 2s < q < q_1$, which means that $H = 0$ has no positive roots. Then, $H > 0$.

To sum up, $H > 0$ when $0 < q < q_1$. Then, we have

$$\frac{\partial(\Phi F)}{\partial x} + \frac{\partial(\Phi G)}{\partial y} < 0,$$

which implies that E_4 is globally asymptotically stable. The proof is completed.

Remark 4.1. *If $q < min\{1, r - d\}$, from Theorem 4.2, E_4 is locally asymptotically stable. Therefore, by Theorem 4.3, E_4 is globally asymptotically stable. That is, when the intrinsic growth rate of the prey is high and the catchability coefficient for prey is low, the prey and predator will reach a steady state. Hence, a small catchability coefficient for prey will not lead to the extinction of prey and predator.*

Remark 4.2. *Assume that $r - d = q \leq q_0$. From Lemma 2.1, the origin is a attractor. By Lemma 3.1 and Theorem 4.1, E_2 is unstable if it exists. From Lemma 3.2, system* (1.4) *has no positive equilibrium. Note that system* (1.4) *has no limit cycle. Therefore, the origin is globally asymptotically stable. That is, when the intrinsic growth rate of the prey and the catchability coefficient for prey are low, the prey and predator will become extinct.*

4.3. Stability of the interior equilibrium when $r - d < q$

Lemma 4.1 ([27]). *The system*

$$\begin{cases} \dot{x} &= y + Ax^2 + Bxy + Cy^2 + o(|x, y|^2), \\ \dot{y} &= Dx^2 + Exy + Fy^2 + o(|x, y|^2), \end{cases}$$

is equivalent to the system

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= Dx^2 + (E + 2A)xy + o(|x, y|^2) \end{cases}$$

in some small neighborhood of $(0, 0)$ *after changes to the coordinates.*

Lemma 4.2 ([27]). *The system given by*

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = x^2 + a_{30}x^3 + a_{40}x^4 + y(a_{21}x^2 + a_{31}x^3) + y^2(a_{12}x + a_{22}x^2) + o(|x, y|^4), \end{aligned}$$

is equivalent to the system given by

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = x^2 + Gx^3y + o(|x, y|^4) \end{aligned}$$

by some nonsingular transformations in the neighborhood of $(0, 0)$*, where $G = a_{31} - a_{30}a_{21}$.*

By computation, from $F(x_*) = F'(x_*) = 0$, we can express k and r in terms of x_*, c, d, s and q, as follows:

$$\begin{aligned} k &= \frac{q - (1 + x_*)^2(1 + c)}{(2c + 2)x_*^3 + (4c + d + 4)x_*^2 + (2c + 2d + 2)x_* + d + q}, \\ r &= \frac{\left[(c + 1)x_*^2 + (c + d + 1)x_* + d + q\right]^2}{(2c + 2)x_*^3 + (4c + d + 4)x_*^2 + (2c + 2d + 2)x_* + d + q}, \end{aligned} \tag{4.2}$$

where $q > (1 + x_*)^2(1 + c)$ because k and r are positive.
Notice that

$$\begin{aligned} q - (r - d) &= \frac{x_*^2\left[(2(x_* + 1)(c + 1) + d)q - (x_* + 1)^2(c + 1)^2\right]}{(x_* + 1)^2(2cx_* + d + 2x_*) + q}, \\ (r - d) - q_0 &= \frac{x_*\left[((3x_* + 4)(c + 1) + 2d)q + (x_* - 2)(x_* + 1)^2(c + 1)^2\right]}{(x_* + 1)^2(2cx_* + d + 2x_*) + q}; \end{aligned}$$

clearly, $q - (r - d) > 0$ and $(r - d) - q_0 > 0$ when $q > (1 + x_*)^2(1 + c)$. Thus, $q_0 < r - d < q$.
By computation, we have

$$(1 + x_*)^2(1 + c) - \frac{(x_* + s)(1 + x_*)^2}{x_*} = \frac{(1 + x_*)^2(cx_* - s)}{x_*}.$$

From the above discussions, we can obtain the following theorem.

Theorem 4.4. *Assume that* (4.2) *and* $q > (1+x_*)^2(1+c)$ *hold; system* (1.4) *has a unique positive equilibrium* $E_*(x_*, y_*)$.

1) *If* $s \le cx_*$ *and* $q > (1+x_*)^2(1+c)$, *or if* $s > cx_*$ *and* $q > \dfrac{(x_*+s)(1+x_*)^2}{x_*}$, *then* E_* *is a saddle-node with an unstable parabolic sector.*
2) *If* $s > cx_*$ *and* $(1+x_*)^2(1+c) < q < \dfrac{(x_*+s)(1+x_*)^2}{x_*}$, *then* E_* *is a saddle-node with a stable parabolic sector.*

Proof. Obviously, we have that $DetJ_{E_*} = 0$ by (3.2). Then, the type of E_* depends on the sign of TrJ_{E_*}, as follows:

$$TrJ_{E_*} = \frac{x_*^2}{(1+x_*)^2}\left(q - \frac{(x_*+s)(1+x_*)^2}{x_*}\right).$$

First, moving $E_*(x_*, y_*)$ to the origin by the transformation $(x, y) = (X + x_*, Y + y_*)$, it follows that system (1.4) becomes

$$\begin{cases} \dot{X} = \hat{a}_{10}X + \hat{a}_{01}Y + \hat{a}_{20}X^2 + \hat{a}_{11}XY + \hat{a}_{02}Y^2 + o(|X,Y|^2), \\ \dot{Y} = \hat{b}_{10}X + \hat{b}_{01}Y + \hat{b}_{20}X^2 + \hat{b}_{11}XY + \hat{b}_{02}Y^2 + o(|X,Y|^2), \end{cases} \tag{4.3}$$

where

$$\begin{aligned} &\hat{a}_{10} = \frac{x_*^2[q-(1+x_*)^2]}{(1+x_*)^2}, \quad \hat{a}_{01} = -\frac{x_*^2[q-(1+x_*)^2]}{(1+x_*)^2}, \quad \hat{a}_{11} = -\frac{2x_*[q-(1+x_*)^2]}{(1+x_*)^2}, \\ &\hat{a}_{20} = \frac{x_*[q(2+x_*)-2(1+x_*)^3]}{(1+x_*)^3}, \quad \hat{a}_{02} = \frac{x_*^2[(1+c)(1+x_*)^2-q]^2}{[(1+x_*)(cx_*+d+x_*)+q](1+x_*)^3}, \\ &\hat{b}_{10} = sx_*, \quad \hat{b}_{01} = -sx_*, \quad \hat{b}_{20} = 0, \quad \hat{b}_{11} = s, \quad \hat{b}_{02} = -s. \end{aligned}$$

The eigenvalues of the Jacobian matrix at point E_* are $\lambda_1 = 0$ and $\lambda_2 = \hat{a}_{10} + \hat{b}_{01}$. If $q \neq \dfrac{(x_*+s)(1+x_*)^2}{x_*}$, then $\lambda_2 \neq 0$.

Next, taking the transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \hat{a}_{01} & \hat{a}_{10} \\ -\hat{a}_{10} & \hat{b}_{10} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

and, by introducing the new time variable

$$\mathrm{d}\tau = TrJ_{E_*}\mathrm{d}t,$$

system (4.3) is rewritten as

$$\begin{cases} \dot{\hat{u}} = \hat{c}_{20}\hat{u}^2 + \hat{c}_{11}\hat{u}\hat{v} + \hat{c}_{02}\hat{v}^2 + o(|\hat{u},\hat{v}|^2), \\ \dot{\hat{v}} = \hat{v} + \hat{d}_{20}\hat{u}^2 + \hat{d}_{11}\hat{u}\hat{v} + \hat{d}_{02}\hat{v}^2 + o(|\hat{u},\hat{v}|^2), \end{cases}$$

where

$$\hat{c}_{20} = \frac{s(\hat{a}_{01}^2\hat{a}_{20}x_* - \hat{a}_{01}\hat{a}_{10}\hat{a}_{11}x_* + \hat{a}_{02}\,\hat{a}_{10}^2x_*)}{(\hat{a}_{01}sx_* + \hat{a}_{10}^2)TrJ_{E_*}},$$
$$\hat{c}_{11} = \frac{s[(-s^3x_*^2 + \hat{a}_{10}\,s^2x_*)\hat{a}_{01} + \hat{a}_{02}\hat{a}_{10}\,s^2x_*^2 + \hat{a}_{10}^2\hat{a}_{11}sx_* + \hat{a}_{10}^3\hat{a}_{20}]}{(\hat{a}_{01}sx_* + \hat{a}_{10}^2)TrJ_{E_*}},$$
$$\hat{c}_{02} = \frac{sx_*[(\hat{a}_{20} - s)\hat{a}_{10}^2 + (\hat{a}_{11}sx_* + s^2x_*)\hat{a}_{10} + \hat{a}_{02}\,s^2x_*^2]}{(\hat{a}_{01}sx_* + \hat{a}_{10}^2)TrJ_{E_*}},$$
$$\hat{d}_{20} = \frac{\hat{a}_{10}(\hat{a}_{01}^2\hat{a}_{20} - \hat{a}_{01}\hat{a}_{10}\hat{a}_{11} + \hat{a}_{02}\,\hat{a}_{10}^2)}{(\hat{a}_{01}sx_* + \hat{a}_{10}^2)TrJ_{E_*}},$$
$$\hat{d}_{11} = \frac{\hat{a}_{01}^2s^2x_* + (\hat{a}_{10}\hat{a}_{11}sx_* + 2\hat{a}_{10}\,s^2x_* + 2\hat{a}_{10}^2\hat{a}_{20} - \hat{a}_{10}^2s)\hat{a}_{01} - 2\hat{a}_{02}\,\hat{a}_{10}^2sx_* - \hat{a}_{10}^3\hat{a}_{11}}{(\hat{a}_{01}sx_* + \hat{a}_{10}^2)TrJ_{E_*}},$$
$$\hat{d}_{02} = \frac{-\hat{a}_{01}\,s^3x_*^2 + \hat{a}_{02}\hat{a}_{10}\,s^2x_*^2 + \hat{a}_{01}\hat{a}_{10}\,s^2x_* + \hat{a}_{10}^2\hat{a}_{11}sx_* + \hat{a}_{10}^3\hat{a}_{20}}{(\hat{a}_{01}sx_* + \hat{a}_{10}^2)TrJ_{E_*}}.$$

By a simple calculation, we get

$$\hat{c}_{20} = \frac{sx_*^4[q - (1 + x_*)^2]M}{(1 + x_*)^2TrJ_{E_*}[(1 + c)x_*^2 + (c + d + 1)x_* + d + q]},$$

where

$$M = -[(3x_* + 2)(1 + c) + d]q + (1 + c)^2(1 + x_*)^3.$$

Note that $q > (1 + x_*)^2(1 + c)$; then, $q > (1 + x_*)^2$. By computation, we can obtain

$$M\Big|_{q=(1+x_*)^2(1+c)} = -((2x_* + 1)(c + 1) + d)(1 + x_*)^2(1 + c),$$

which implies that $M < 0$ for $q > (1 + x_*)^2(1 + c)$. Therefore, the sign of $\hat{c}_{20}$ is determined by TrJ_{E_*}. Considering the time transformation, and by using Theorem 7.1 in [26], if $s > cx_*$ and $(1+x_*)^2(1+c) < q < \frac{(x_*+s)(1+x_*)^2}{x_*}$, that is, if $TrJ_{E_*} < 0$, then E_* is a saddle-node with a stable parabolic sector (see Figure 4(a)). If $s \le cx_*$ and $q > (1 + x_*)^2(1 + c)$, or if $s > cx_*$ and $q > \frac{(x_*+s)(1+x_*)^2}{x_*}$, that is, if $TrJ_{E_*} > 0$, then E_* is a saddle-node with an unstable parabolic sector (see Figure 4(b)). The proof is completed.

From $F(x_*) = F'(x_*) = TrJ_{E_*} = 0$, we can express k, r and q in terms of x_*, c, d and s, as follows:

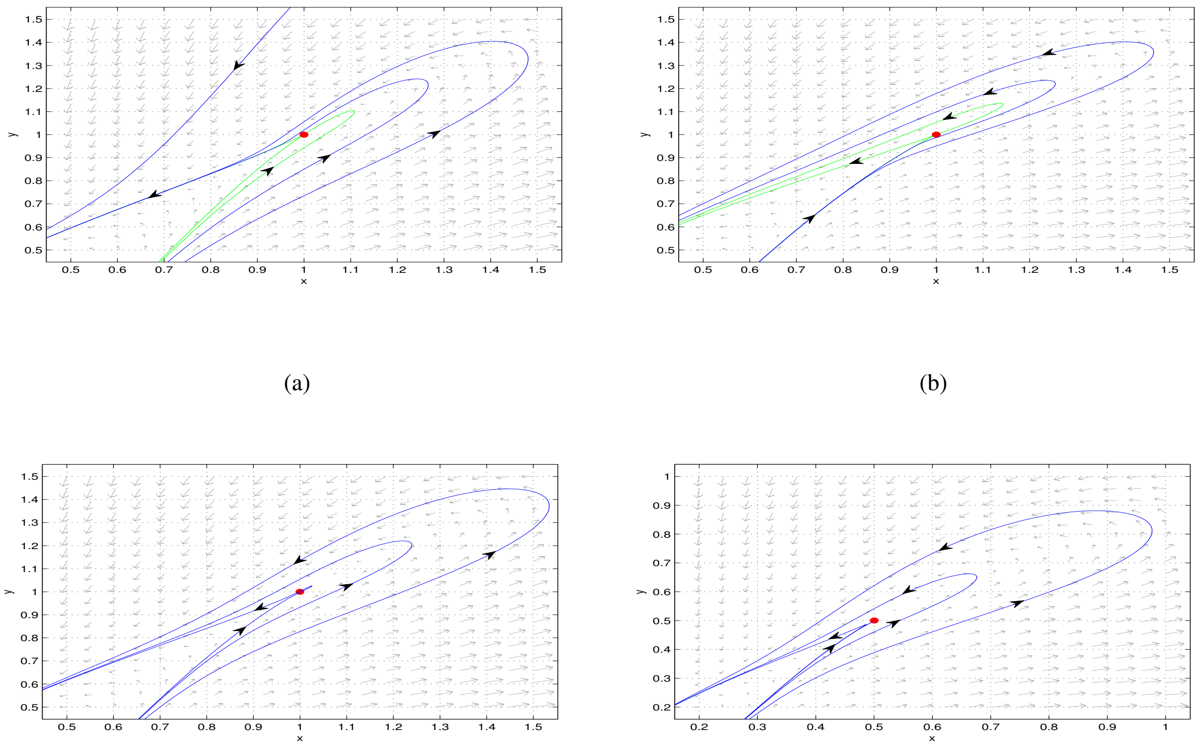
$$\begin{aligned} k &= \frac{s - cx_*}{(2c + 2)x_*^2 + (d + 1)x_* + s}, \\ r &= \frac{[(c + 2)x_*^2 + (d + s + 1)x_* + s]^2}{x_*[(2c + 2)x_*^2 + (d + 1)x_* + s]}, \\ q &= \frac{(x_* + s)(1 + x_*)^2}{x_*}, \end{aligned} \tag{4.4}$$

where $s > cx_*$.

Theorem 4.5. *Assume that* (4.4) *and* $s > cx_*$ *hold.*

1) If one of the following conditions holds: (1.1) $x_* \ge 1$*; (1.2)* $0 < x_* \le \frac{c}{c+2}$*; (1.3)* $\frac{c}{c+2} < x_* < 1$, $s \ne \frac{2x_*^2}{1-x_*}$, E_* *is a cusp of codimension two.*

2) If $\frac{c}{c+2} < x_* < 1$ *and* $s = \frac{2x_*^2}{1-x_*}$ *hold,* E_* *is a cusp of codimension three.*



(a) (b) (c) (d)

Figure 4. Phase portraits of system (1.4). Let $s = 2$, $c = 1$ and $d = 2$. (a) E_* is a saddle-node with an attracting parabolic sector when $k = \frac{1}{17}$, $q = 10$ and $r = \frac{162}{17}$. (b) E_* is a saddle-node with a repelling parabolic sector when $k = \frac{7}{39}$, $q = 15$ and $r = \frac{529}{39}$. (c) E_* is a cusp of codimension two when $k = \frac{1}{9}$, $q = 12$ and $r = \frac{100}{9}$. (d) E_* is a cusp of codimension three when $s = 1$, $c = 1$, $d = 1$, $k = \frac{1}{6}$, $q = \frac{27}{4}$ and $r = \frac{169}{24}$.

Proof. 1) Let $X = x - x_*$ and $Y = y - y_*$; then, system (1.4) can be rewritten as follows:

$$\begin{cases} \dot{X} = sx_*X - sx_*Y + \dfrac{sx_* - x_*^2 + 2s}{x_* + 1}X^2 - 2sXY + \dfrac{x_*(cx_* - s)^2}{(c+2)x_*^2 + (d+s+1)x_* + s}Y^2 + o(|X, Y|^2), \\ \dot{Y} = sx_*X - sx_*Y + sXY - sY^2 + o(|X, Y|^2). \end{cases} \tag{4.5}$$

Applying the transformation $(u, v) = (-\frac{1}{sx_*}X, -X + Y)$, system (4.5) becomes

$$\begin{cases} \dot{u} = v + e_{20}u^2 + e_{11}uv + e_{02}v^2 + o(|u, v|^2), \\ \dot{v} = f_{20}u^2 + f_{11}uv + f_{02}v^2 + o(|u, v|^2), \end{cases} \tag{4.6}$$

where

$$
\begin{aligned}
&e_{20} = -\frac{sx_*^3 T_1}{T_2(1+x_*)},\ e_{11} = \frac{2[c^2x_*^3 - s(3c+2)x_*^2 - s(d+1)x_* - s^2]}{T_2},\ e_{02} = -\frac{(cx_* - s)^2}{T_2},\\
&f_{20} = -\frac{s^2x_*^4 T_1}{T_2(1+x_*)},\ f_{11} = \frac{sx_*[2c^2x_*^3 - s(5c+2)x_*^2 + s(s-1-d)x_* - s^2]}{T_2},\\
&f_{02} = -\frac{c^2x_*^3 + s(2-c)x_*^2 + s(d+2s+1)x_* + s^2}{T_2},\\
&T_1 = -(3cx_* + 2c + d + 3x_* + 2)s + c^2x_*^2 + c^2x_* - cx_*^2 - dx_* - 2x_*^2 - x_*,\\
&T_2 = (c+2)x_*^2 + (d+s+1)x_* + s.
\end{aligned}
$$

By Lemma 4.1, system (4.6) is equivalent to the following:

$$
\begin{cases} \dot{x} = y,\\ \dot{y} = D_{20}x^2 + D_{11}xy + o(|x,y|^2), \end{cases}
$$

where

$$
D_{20} = f_{20},\ D_{11} = f_{11} + 2e_{20} = \frac{sx_*(2x_*^2 - s(1-x_*))}{1+x_*}.
$$

Substituting $s = cx_*$ into T_1, we get

$$
T_1\Big|_{s=cx_*} = -x_*(c+1)(2cx_* + c + d + 2x_* + 1) < 0.
$$

So, $T_1 < 0$ for $s > cx_*$, that is, $D_{20} \neq 0$.

Obviously, if $x_* \geq 1$, we have that $D_{11} > 0$, that is, E_* is a cusp of codimension two by the result in [28] (see Figure 4(c)). When $x_* < 1$, from $D_{11} = 0$, we have that $s = \frac{2x_*^2}{1-x_*}$. Noting that $s > cx_*$, we have

$$
s - cx_*\Big|_{s=\frac{2x_*^2}{1-x_*}} = x_*((c+2)x_* - c).
$$

Therefore, if $0 < x_* \leq \frac{c}{c+2}$ or $\frac{c}{c+2} < x_* < 1,\ s \neq \frac{2x_*^2}{1-x_*}$ holds and E_* is a cusp of codimension two.

2) If $\frac{c}{c+2} < x_* < 1$ and $s = \frac{2x_*^2}{1-x_*}$ hold, that is, $D_{11} = 0$; we will show that E_* is a cusp of codimension three. When $\frac{c}{c+2} < x_* < 1,\ s = \frac{2x_*^2}{1-x_*}$ and (4.4) reduces to the following:

$$
\begin{aligned}
k &= -\frac{cx_* - c + 2x_*}{2(c+1)x_*^2 + (-2c+d-3)x_* - d - 1},\\
r &= \frac{(cx_*^2 + (-c+d-3)x_* - d - 1)^2}{[2(c+1)x_*^2 + (-2c+d-3)x_* - d - 1](x_* - 1)},\\
q &= \frac{(1+x_*)^3}{1-x_*}.
\end{aligned} \tag{4.7}
$$

Note that $\frac{c}{c+2} < x_* < 1$; then, k, r, q in (4.7) are positive.

Then, system (4.5) becomes

$$
\begin{cases} \dot{x}_1 = g_{10}x_1 + g_{01}y_1 + g_{20}x_1^2 + g_{11}x_1y_1 + g_{02}y_1^2 + g_{30}x_1^3 + g_{21}x_1^2y_1 + g_{12}x_1y_1^2\\ \qquad + g_{03}y_1^3 + g_{40}x_1^4 + g_{22}x_1^2y_1^2 + g_{13}x_1y_1^3 + g_{04}y_1^4 + o(|x_1,y_1|^4),\\ \dot{y}_1 = h_{10}x_1 + h_{01}y_1 + h_{11}x_1y_1 + h_{02}y_1^2 + o(|x_1,y_1|^4), \end{cases} \tag{4.8}
$$

where

$$
\begin{aligned}
&g_{10} = -\frac{2x_*^3}{x_* - 1}, \quad g_{01} = \frac{2x_*^3}{x_* - 1}, \quad g_{20} = -\frac{3x_*^2}{x_* - 1}, \quad g_{11} = \frac{4x_*^2}{x_* - 1}, \quad g_{21} = \frac{2x_*}{x_* - 1},\\
&g_{02} = \frac{(cx_* - c + 2x_*)^2 x_*^2}{(cx_*^2 + (-c + d - 3)x_* - d - 1)(x_* - 1)}, \quad g_{30} = -\frac{x_*^2}{(x_* - 1)(1 + x_*)},\\
&g_{12} = \frac{2(cx_* - c + 2x_*)^2 x_*}{(cx_*^2 + (-c + d - 3)x_* - d - 1)(x_* - 1)}, \quad g_{40} = \frac{1}{(x_* - 1)(1 + x_*)^2},\\
&g_{03} = \frac{(cx_* - c + 2x_*)^3 x_*^2}{(cx_*^2 + (-c + d - 3)x_* - d - 1)^2(x_* - 1)}, \quad h_{10} = -\frac{2x_*^3}{x_* - 1},\\
&g_{22} = \frac{(cx_* - c + 2x_*)^2}{(cx_*^2 + (-c + d - 3)x_* - d - 1)(x_* - 1)}, \quad h_{01} = \frac{2x_*^3}{x_* - 1},\\
&g_{13} = \frac{2(cx_* - c + 2x_*)^3 x_*}{(cx_*^2 + (-c + d - 3)x_* - d - 1)^2(x_* - 1)}, \quad h_{11} = -\frac{2x_*^2}{x_* - 1},\\
&g_{04} = \frac{(cx_* - c + 2x_*)^4 x_*^2}{(cx_*^2 + (-c + d - 3)x_* - d - 1)^3(x_* - 1)}, \quad h_{02} = \frac{2x_*^2}{x_* - 1}.
\end{aligned}
$$

Let $x_2 = y_1$ and $y_2 = \dot{y}_1$; then, system (4.8) becomes

$$
\begin{cases}
\dot{x}_2 = y_2,\\
\dot{y}_2 = i_{20}x_2^2 + i_{02}y_2^2 + i_{30}x_2^3 + i_{21}x_2^2y_2 + i_{12}x_2y_2^2 + i_{03}y_2^3 + i_{40}x_2^4 + i_{31}x_2^3y_2\\
\qquad +i_{22}x_2^2y_2^2 + i_{13}x_2y_2^3 + i_{04}y_2^4 + o(|x_2, y_2|^4),
\end{cases}
\tag{4.9}
$$

where

$$
\begin{aligned}
&i_{20} = -\frac{2((c+4)(c+1)x_* - c^2 + d + 1)x_*^5}{(cx_*^2 + (-c + d - 3)x_* - d - 1)(x_* - 1)}, \quad i_{02} = \frac{5}{2x_*},\\
&i_{30} = -\frac{2x_*^4 Q_1}{(cx_*^2 + (-c + d - 3)x_* - d - 1)^2(1 + x_*)(x_* - 1)},\\
&i_{21} = \frac{x_* Q_2}{(cx_*^2 + (-c + d - 3)x_* - d - 1)(1 + x_*)(x_* - 1)}, \quad i_{12} = -\frac{7 + 4x_*}{2x_*^2(1 + x_*)},\\
&i_{03} = -\frac{x_* - 1}{4x_*^4(1 + x_*)}, \quad i_{40} = -\frac{2x_*^3 Q_3}{(cx_*^2 + (-c + d - 3)x_* - d - 1)^3(x_* - 1)(1 + x_*)^2},\\
&i_{31} = \frac{2Q_4}{(cx_*^2 + (-c + d - 3)x_* - d - 1)^2(1 + x_*)^2}, \quad i_{13} = \frac{(x_*^2 + x_* + 2)(x_* - 1)}{2x_*^6(1 + x_*)^2},\\
&i_{22} = -\frac{Q_5}{2x_*^3(cx_*^2 + (-c + d - 3)x_* - d - 1)(1 + x_*)^2}, \quad i_{04} = -\frac{(x_* - 1)^2}{8x_*^9(1 + x_*)^2},\\
&Q_1 = (4c^3 + 20c^2 + 24c + 8)x_*^4 + (-4c^3 + 3c^2d - 8c^2 + 16cd - 24c + 12d - 20)x_*^3\\
&\qquad +(-4c^3 - 3c^2d - 24c^2 + 2cd + 2d^2 - 70c - 50)x_*^2 + (4c^3 - 3c^2d + 9c^2\\
&\qquad -18cd + d^2 - 18c - 22d - 23)x_* + 3c^2d + 3c^2 - 3d^2 - 6d - 3,\\
&Q_2 = (2c^2 + 9c + 8)x_*^3 + (-2c^2 + 3c + d + 5)x_*^2 + (-2c^2 - 12c + 3d - 13)x_* + 2c^2 - 4d - 4,
\end{aligned}
$$

$$\begin{aligned}
Q_3 = {} & (7c^4 + 39c^3 + 72c^2 + 56c + 16)x_*^7 + (-7c^4 + 9c^3 d - 26c^3 + 45c^2 d - 63c^2 + 60cd - 68c \\
& +24d - 24)x_*^6 + (-14c^4 - 9c^3 d + 3c^2 d^2 - 87c^3 - 15c^2 d + 15cd^2 - 234c^2 - 30cd + 12d^2 \\
& -229c - 24d - 68)x_*^5 + (14c^4 - 18c^3 d - 3c^2 d^2 + 51c^3 - 96c^2 d + 3c\,d^2 + d^3 + 51c^2 \\
& -198cd + 3d^2 + 87c - 117d + 73)x_*^4 + (7c^4 + 18c^3 d - 6c^2 d^2 + 59c^3 + 27c^2 d - 30c\,d^2 \\
& +d^3 + 189c^2 - 24cd - 33d^2 + 342c - 21d + 205)x_*^3 + (-7c^4 + 9c^3 d + 6c^2 d^2 - 27c^3 \\
& +63c^2 d - 9c\,d^2 - 2d^3 + 9c^2 + 150cd - 18d^2 + 159c + 114d + 130)x_*^2 + (-9c^3 d + 3c^2 d^2 \\
& -9c^3 - 18c^2 d + 21c\,d^2 - 3d^3 - 21c^2 + 42cd + 27d^2 + 21c + 63d + 33)x_* - 3c^2 d^2 \\
& -6c^2 d + 3d^3 - 3c^2 + 9d^2 + 9d + 3, \\
Q_4 = {} & (2c^3 + 10c^2 + 16c + 8)x_*^5 + (c^2 d + 7c^2 + 4cd + 20c + 4d + 12)x_*^4 - (4c^3 + 12c^2 - 4cd \\
& +12c - 8d + 8)x_*^3 + (-2c^2 d - 10c^2 - 32c + 4d - 28)x_*^2 + (2c^3 + 4c^2 - 8cd + 2d^2 - 8c \\
& -12d - 14)x_* + c^2 d + c^2 - 2d^2 - 4d - 2, \\
Q_5 = {} & (c^2 + 4)x_*^4 + (-3c - 4d + 20)x_*^3 + (-2c^2 + 6c - 7d + 41)x_*^2 + (-3c + 10d + 14)x_* \\
& +c^2 + d + 1.
\end{aligned}$$

Let $x_3 = x_2$ and $y_3 = (1 - i_{02}x_2)y_2$; then, system (4.9) becomes

$$\begin{cases} \dot{x}_3 = y_3, \\ \dot{y}_3 = j_{20}x_3^2 + j_{30}x_3^3 + j_{21}x_3^2 y_3 + j_{12}x_3 y_3^2 + j_{03}y_3^3 + j_{40}x_3^4 + j_{31}x_3^3 y_3 \\ \qquad + j_{22}x_3^2 y_3^2 + j_{13}x_3 y_3^3 + j_{04}y_3^4 + o(|x_3, y_3|^4), \end{cases} \tag{4.10}$$

where

$$\begin{aligned}
& j_{20} = i_{20}, \quad j_{30} = 2i_{02}i_{20} + i_{30}, \quad j_{21} = i_{21}, \quad j_{12} = -i_{02}^2 + i_{12}, \\
& j_{03} = i_{03}, \quad j_{40} = i_{02}^2 i_{20} - 2i_{02}i_{30}i_{40}, \quad j_{31} = i_{21}i_{02} + i_{31}, \\
& j_{22} = -i_{02}^3 + i_{22}, \quad j_{13} = i_{03}i_{02} + i_{13}, \quad j_{04} = i_{04}.
\end{aligned}$$

To delete the y_3^3-term, $x_3y_3^3$-term and y_3^4-term in system (4.10), we do the following two transformations:

$$x_3 = x_4 + \frac{j_{03}}{2}x_4^2 y_4 + \frac{j_{13}}{6}x_4^3 y_4 + \frac{j_{04}}{2}x_4^2 y_4^2, \quad y_3 = (1 + j_{03}x_4 y_4 + \frac{j_{13}}{2}x_4^2 y_4 + j_{04}x_4 y_4^2)y_4;$$

$$x_4 = x_5, \quad y_4 = y_5 + \frac{1}{2}j_{03}j_{20}x_5^4.$$

Hence, system (4.10) becomes

$$\begin{cases} \dot{x}_5 = y_5, \\ \dot{y}_5 = m_{20}x_5^2 + m_{30}x_5^3 + m_{21}x_5^2 y_5 + m_{12}x_5 y_5^2 + m_{40}x_5^4 + m_{31}x_5^3 y_5 + m_{22}x_5^2 y_5^2 + o(|x_5, y_5|^4), \end{cases} \tag{4.11}$$

where

$$m_{20} = j_{20}, \quad m_{30} = j_{30}, \quad m_{21} = j_{21}, \quad m_{12} = j_{12}, \quad m_{40} = j_{40}, \quad m_{31} = j_{31} - 3j_{20}j_{03}, \quad m_{22} = j_{22}.$$

Using $\frac{c}{c+2} < x_* < 1$, we have that $m_{20} = -\dfrac{2((c+4)(c+1)x_* - c^2 + d + 1)x_*^5}{(cx_*^2 + (-c + d - 3)x_* - d - 1)(x_* - 1)} < 0$. Letting

$$x_6 = -x_5, \quad y_6 = \frac{y_5}{-\sqrt{-m_{20}}}, \quad \tau = \sqrt{-m_{20}}t,$$

system (4.11) becomes (still denoting τ by t)

$$\begin{cases} \dot{x}_6 = y_6, \\ \dot{y}_6 = x_6^2 + n_{30}x_6^3 + n_{21}x_6^2y_5 + n_{12}x_6y_6^2 + n_{40}x_6^4 + n_{31}x_6^3y_6 + n_{22}x_6^2y_6^2 + o(|x_6, y_6|^4), \end{cases} \tag{4.12}$$

where

$$n_{30} = \frac{m_{30}}{m_{20}}, \quad n_{21} = -\frac{m_{21}}{\sqrt{-m_{20}}}, \quad n_{12} = m_{12}, \quad n_{40} = \frac{m_{40}}{m_{20}}, \quad n_{31} = \frac{m_{31}}{\sqrt{-m_{20}}}, \quad n_{22} = -m_{22}.$$

By Lemma 4.2, system (4.12) is equivalent to the following system:

$$\begin{cases} \dot{X} = Y, \\ \dot{Y} = X^2 + GX^3Y + o(|X, Y|^4), \end{cases}$$

where

$$G = \frac{\sqrt{\dfrac{2((c+4)(c+1)x_* - c^2 + d + 1)x_*}{(cx_*^2 + (-c+d-3)x_* - d - 1)(x_* - 1)}}\delta(x_*)}{4((c+4)(c+1)x_* - c^2 + d + 1)^2(cx_*^2 + (-c+d-3)x_* - d - 1)^2(1+x_*)^2x_*^3}$$

and

$$\delta(x_*) = \sum_{i=0}^{9} P_i x_*^i;$$

here, the coefficients of $P_i, i = 0, \cdots, 9$ are given in Appendix A.

Using $\frac{c}{c+2} < x_* < 1$, the sign of G is determined by $\delta(x_*)$. By computation, we have that $\delta(\frac{c}{c+2}) = \frac{(16(29c^2+80c+56))(3c^2+cd+5c+2d+2)^4}{(c+2)^9} > 0$ and $\delta(1) = 384c + 128d + 384 > 0$. Using Lemma 3.1 in [29], the number of roots for $\delta(x_*)$ in $\frac{c}{c+2} < x_* < 1$ is equal to that of positive roots for

$$\mu(x_*) = (1+x_*)^9\delta\left(\frac{cx_* + c + 2}{(c+2)(1+x_*)}\right) = \frac{16}{(c+2)^9}\sum_{i=0}^{9} M_i x_*^i$$

in $\frac{c}{c+2} < x_* < 1$, and the coefficients of $M_i, i = 0, \cdots, 9$ are given in Appendix A. Obviously, $M_i, i = 0, \cdots, 9$ are positive. Hence, $\mu(x_*) > 0$ in $\frac{c}{c+2} < x_* < 1$, which implies that $\delta(x_*)$ has no positive zeros in $\frac{c}{c+2} < x_* < 1$. Then, $\delta(x_*) \neq 0$, that is $G \neq 0$ in $\frac{c}{c+2} < x_* < 1$, which means that E_* is a cusp of codimension three (see Figure 4(d)). The proof is completed.

Theorem 4.6. *Assume that $q_0 < r - d < q$ and $F(x_*) < 0$; system* (1.4) *has two positive equilibria $E_3(x_3, y_3)$ and $E_4(x_4, y_4)$, where E_3 is always a saddle point. Moveover,*

1) if $q < \dfrac{(x_4+s)(1+x_4)^2}{x_4}$, E_4 is a stable node or focus;

2) if $q > \dfrac{(x_4+s)(1+x_4)^2}{x_4}$, E_4 is an unstable node or focus;

3) if $q = \dfrac{(x_4+s)(1+x_4)^2}{x_4}$, E_4 is a center or weak focus.

Proof. It is clear that $F'(x_3) < 0$ and $F'(x_4) > 0$ (see Figure 3(e)). Combining (3.2) again, we have that $DetJ_{E_3} < 0$ and $DetJ_{E_4} > 0$. Thus, E_3 is a saddle point.
In what follows, we consider that

$$TrJ_{E_4} = \frac{x_4^2[q - (x_4 + 1)^2]}{(1 + x_4)^2} - sx_4.$$

When $TrJ_{E_4} < 0$, E_4 is a stable node or focus; when $TrJ_{E_4} > 0$, E_4 is an unstable node or focus; when $TrJ_{E_4} = 0$, E_4 is a center or weak focus. The proof is completed.

5. Bifurcation

We will analyze the bifurcations of system (1.4) in this section, including saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation.

5.1. Saddle-node bifurcation

From Lemma 3.1, when $1 < r - d < q < q^*$, system (1.4) has two boundary equilibria $E_1(x_1, 0)$ and $E_2(x_2, 0)$. However, when $q = q_{SN} = q^*$, only the boundary equilibrium $\overline{E}$ exists. Therefore, according to Sotomayor's theorem [28], system (1.4) will produce a saddle-node bifurcation at $\overline{E}$.

Theorem 5.1. *Assume that* $1 < r - d < q$, *with* $q = q_{SN}$ *being the bifurcation parameter; then, system* (1.4) *will undergo saddle-node bifurcation at* $\overline{E}$.

Proof. The eigenvalues of $J_{\overline{E}}$ are $\lambda_1 = 0$ and $\lambda_2 = \dfrac{s(r - d - 1)}{2}$. Denote the eigenvectors of $J_{\overline{E}}$ and $J_{\overline{E}}^T$ as

$$V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$W = \begin{pmatrix} 1 \\ \dfrac{(rk + c)(r - d - 1)}{2} \end{pmatrix},$$

respectively.
Denote

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} = \begin{pmatrix} x^2\left(\dfrac{\mathrm{r}}{1 + ky} - d - x - cy\right) - \dfrac{qx^2}{1 + x} \\ sy(x - y) \end{pmatrix}.$$

Then,

$$F_q(\overline{E}; q_{SN}) = \begin{pmatrix} -\dfrac{(r - d - 1)^2}{2(r - d + 1)} \\ 0 \end{pmatrix},$$

$$D^2F(\overline{E}; q_{SN})(V, V) = \begin{pmatrix} -\dfrac{(r - d - 1)^2}{r - d + 1} \\ 0 \end{pmatrix}.$$

It is easy to know that

$$W^T F_q(\overline{E}; q_{SN}) = -\frac{(r-d-1)^2}{2(r-d+1)} \neq 0,$$
$$W^T[D^2F(\overline{E}; q_{SN})(V,V)] = -\frac{(r-d-1)^2}{r-d+1} \neq 0.$$

This proves the transversality conditions, which means that system (1.4) will undergo saddle-node bifurcation at $\overline{E}$. The proof is completed.

Similarly, it follows from Lemma 3.2 that a saddle-node bifurcation occurs at the positive equilibrium E_*.

Theorem 5.2. *Assume that $q_0 < r - d < q$ and $F(x_*) = 0$; system* (1.4) *will undergo saddle-node bifurcation at E_*.*

5.2. Hopf bifurcation

It follows from Theorem 4.6 that, if $q = \frac{(x_4+s)(1+x_4)^2}{x_4}$, then $TrJ_{E_4} = 0$. Noting that $DetJ_{E_4} > 0$, the Jacobian matrix of E_4 has a pair of purely imaginary eigenvalues. Thus, system (1.4) may undergo Hopf bifurcation at E_4.

For simplicity, similar to the analyses of Dai et al. [30] and Lu et al. [31], we prove the Hopf bifurcation. Letting

$$\tilde{x} = \frac{x}{x_4}, \quad \tilde{y} = \frac{y}{y_4}, \quad \tilde{t} = x_4^2 t, \quad \tilde{r} = \frac{r}{x_4}, \quad \tilde{k} = k\, x_4,$$
$$\tilde{d} = \frac{d}{x_4}, \quad \tilde{c} = c, \quad \tilde{\alpha} = \frac{1}{x_4}, \quad \tilde{q} = \frac{q}{x_4^2}, \quad \tilde{s} = \frac{s}{x_4},$$

and by dropping the tilde, system (1.4) becomes

$$\begin{aligned} \dot{x} &= x^2\left(\frac{r}{1+ky} - d - x - cy\right) - \frac{qx^2}{\alpha + x}, \\ \dot{y} &= sy\,(x-y)\,, \end{aligned} \tag{5.1}$$

where $r > d$ and the other parameters are positive.

Clearly, $\tilde{E}_4(1,1)$ is an equilibrium of system (5.1), which implies that

$$r = \frac{[(c+d+1)(\alpha+1)+q](k+1)}{\alpha+1}.$$

Define

$$\tilde{q}_0 = \frac{\alpha[(k+1)(c+1)+dk](\alpha+1)}{(1-\alpha k)},$$
$$\tilde{q}_1 = \frac{(\alpha+1)^2[(2k+1)(c+1)+dk]}{(1-\alpha k)}.$$

Assume that system (5.1) has another positive equilibrium $\tilde{E}_3(\tilde{x}_3, \tilde{y}_3)$. By computation, $\tilde{x}_3$ satisfies the following equation:

$$(x-1)\Phi(x) = 0,$$

where

$$\Phi(x) \quad = \quad k(c+1)(\alpha+1)x^2 + (\alpha+1)[(c+1)(\alpha k+k+1)+dk]x + (1-\alpha k)(\tilde{q}_0 - q).$$

Note that $\tilde{x}_3 < 1$ is a unique positive root of $\Phi(x)$, which implies that $\alpha k < 1$ and $\tilde{q}_0 < q$. Also, substituting $x = 1$ into $\Phi(x)$, we have

$$\Phi(1) = (1 - \alpha k)(\tilde{q}_1 - q) > 0,$$

that is, $\alpha k < 1$ and $q < \tilde{q}_1$.

The Jacobian matrix of system (5.1) at $\tilde{E}_4$ is

$$J_{\tilde{E}_4} = \begin{pmatrix} -1 + \dfrac{q}{(\alpha+1)^2} & \dfrac{[(-2c-d-1)\alpha - 2c - d - q - 1]k - c(\alpha+1)}{(k+1)(\alpha+1)} \\ s & -s \end{pmatrix},$$

and

$$DetJ_{\tilde{E}_4} = \frac{s(1-\alpha k)(\tilde{q}_1 - q)}{(\alpha+1)^2(k+1)}, \quad TrJ_{\tilde{E}_4} = \frac{q - \tilde{q}}{(\alpha+1)^2},$$

where

$$\tilde{q} = (s+1)(\alpha+1)^2.$$

We have the following results.

Theorem 5.3. *Assuming that $\alpha k < 1$ and $\tilde{q}_0 < q < \tilde{q}_1$, system (5.1) has the equilibrium $\tilde{E}_4(1,1)$. Moveover,*

1) $\tilde{E}_4(1,1)$ is a stable hyperbolic node or focus if $q < \tilde{q}$;

2) $\tilde{E}_4(1,1)$ is an unstable hyperbolic node or focus if $q > \tilde{q}$;

3) $\tilde{E}_4(1,1)$ is a fine focus or center if $q = \tilde{q}$.

Now, we will study the Hopf bifurcation around $\tilde{E}_4$ in system (5.1). Obviously, the transversality condition

$$\left.\frac{\mathrm{d}TrJ_{\tilde{E}_4}}{\mathrm{d}q}\right|_{q=\tilde{q}} = \frac{1}{(\alpha+1)^2} \neq 0$$

holds. Then, we can determine the stability of the limit cycle around $\tilde{E}_4$ by calculating the first Lyapunov number. First, using the transformation $(\tilde{x}, \tilde{y}) = (x-1, y-1)$, the Taylor expansion of system (5.1) at the origin takes the following form:

$$\begin{cases} \dot{\tilde{x}} = \tilde{a}_{10}\tilde{x} + \tilde{a}_{01}\tilde{y} + \tilde{a}_{20}\tilde{x}^2 + \tilde{a}_{11}\tilde{x}\tilde{y} + \tilde{a}_{02}\tilde{y}^2 + \tilde{a}_{30}\tilde{x}^3 + \tilde{a}_{21}\tilde{x}^2\tilde{y} + \tilde{a}_{12}\tilde{x}\tilde{y}^2 + \tilde{a}_{03}\tilde{y}^3 + o(|\tilde{x},\tilde{y}|^2), \\ \dot{\tilde{y}} = \tilde{b}_{10}\tilde{x} + \tilde{b}_{01}\tilde{y} + \tilde{b}_{20}\tilde{x}^2 + \tilde{b}_{11}\tilde{x}\tilde{y} + \tilde{b}_{02}\tilde{y}^2, \end{cases} \tag{5.2}$$

where

$$\begin{aligned}
&\tilde{a}_{10} = s, \quad \tilde{a}_{01} = \frac{[(-s-1)\alpha - 2c - d - s - 2]k - c}{k+1}, \quad \tilde{a}_{20} = \frac{2s\alpha + s - 1}{\alpha+1}, \\
&\tilde{a}_{11} = \frac{[(-2s-2)\alpha - 4c - 2d - 2s - 4]k - 2c}{k+1}, \quad \tilde{a}_{02} = \frac{(s\alpha + \alpha + c + d + s + 2)k^2}{(k+1)^2} \\
&\tilde{a}_{30} = \frac{\alpha^2 s - 2\alpha - 1}{(\alpha+1)^2}, \quad \tilde{a}_{21} = \frac{[(-s-1)\alpha - 2c - d - s - 2]k - c}{k+1}, \\
&\tilde{a}_{12} = \frac{2(s\alpha + \alpha + c + d + s + 2)k^2}{(k+1)^2}, \quad \tilde{a}_{03} = -\frac{(s\alpha + \alpha + c + d + s + 2)k^3}{(k+1)^3}, \\
&\tilde{b}_{10} = s, \quad \tilde{b}_{01} = -s, \quad \tilde{b}_{20} = 0, \quad \tilde{b}_{11} = s, \quad \tilde{b}_{02} = -s.
\end{aligned}$$

Next, using the transformation $(\tilde{u},\tilde{v}) = \left(-\tilde{x}, \dfrac{\tilde{a}_{10}\tilde{x}+\tilde{a}_{01}\tilde{y}}{\sqrt{D}}\right)$, where $D = \tilde{a}_{10}\tilde{b}_{01} - \tilde{a}_{01}\tilde{b}_{10} = DetJ_{\tilde{E}_4} > 0$, system (5.2) becomes

$$\begin{cases} \dot{\tilde{u}} = -\sqrt{D}\tilde{v} + \tilde{c}_{20}\tilde{u}^2 + \tilde{c}_{11}\tilde{u}\tilde{v} + \tilde{c}_{02}\tilde{v}^2 + \tilde{c}_{30}\tilde{u}^3 + \tilde{c}_{21}\tilde{u}^2\tilde{v} + \tilde{c}_{12}\tilde{u}\tilde{v}^2 + \tilde{c}_{03}\tilde{v}^3 + o(|\tilde{u},\tilde{v}|^3), \\ \dot{\tilde{v}} = \sqrt{D}\tilde{u} + \tilde{d}_{20}\tilde{u}^2 + \tilde{d}_{11}\tilde{u}\tilde{v} + \tilde{d}_{02}\tilde{v}^2 + \tilde{d}_{30}\tilde{u}^3 + \tilde{d}_{21}\tilde{u}^2\tilde{v} + \tilde{d}_{12}\tilde{u}\tilde{v}^2 + \tilde{d}_{03}\tilde{v}^3 + o(|\tilde{u},\tilde{v}|^3), \end{cases} \tag{5.3}$$

where the coefficients are given in Appendix B.

According to the results of [26], the first-order Lyapunov number can be written as

$$l_1 = \frac{\gamma_1 k^2 + \gamma_2 k + \gamma_3}{8[(\alpha s + \alpha + 2c + d + s + 2)k + c](\alpha+1)^2(k+1)D},$$

where

$$\begin{aligned} \gamma_1 = {} & (s^3 + 2s^2 + s)\alpha^4 + (4c\, s^2 + 2d\, s^2 + 4cs + 2ds - 2s^2 - 6s - 4)\alpha^3 \\ & +(4c^2 s + 4cds - 5c\, s^2 + d^2 s - 2d\, s^2 - 2s^3 - 16cs - 8ds - 15s^2 \\ & -16c - 8d - 29s - 17)\alpha^2 + (-8c^2 s - 8cds - 6c\, s^2 - 2d^2 s - 3d\, s^2 \\ & -s^3 - 16c^2 - 16cd - 32cs - 4d^2 - 17ds - 9s^2 - 36c - 18d - 24s - 18)\alpha \\ & +c\, s^2 - 4c^2 - 4cd + 2cs - d^2 + s^2 - 4c - 2d + 2s, \\ \gamma_2 = {} & (2c\, s^2 - s^3 + 2cs - s^2)\alpha^3 + (4c^2 s + 2cds - 5c\, s^2 - d\, s^2 - 8cs + 2s^2 - 8c + 3s)\alpha^2 \\ & +(-8c^2 s - 4cds - c\, s^2 + d\, s^2 + 2s^3 - 16c^2 - 8cd - 9cs + 3ds + 10s^2 - 18c + 12s \\ & +2)\alpha + 3c\, s^2 + d\, s^2 + s^3 - 4c^2 - 2cd + 10cs + 4ds + 6s^2 + 2c + 2d + 10s + 4, \\ \gamma_3 = {} & (c^2 s - c\, s^2)\alpha^2 + (-2c^2 s + c\, s^2 - 4c^2 + 3cs)\alpha + c\, s^2 - c^2 + 4cs + 2c. \end{aligned}$$

Thus, we can obtain the following theorem about the Hopf bifurcation.

Theorem 5.4. *If $\alpha k < 1$, $\tilde{q}_0 < q < \tilde{q}_1$ and $q = \tilde{q}$, then the following statements hold.*

1) If $l_1 > 0$, then system (5.1) *undergoes subcritical Hopf bifurcation and an unstable limit cycle comes out around $\tilde{E}_4$.*

2) If $l_1 < 0$, then system (5.1) *undergoes supercritical Hopf bifurcation and a stable limit cycle appears around $\tilde{E}_4$.*

3) If $l_1 = 0$, then system (5.1) *undergoes a degenerate Hopf bifurcation and multiple limit cycles may appear around $\tilde{E}_4$.*

By numerical simulation, we show the existence of limit cycles. Letting $k = 0.1, \alpha = 1, d = 1, c = 1, s = 1, q = 8$ and $r = 7.7$, we have that $l_1 = 0.001984126984$. We perturb q to $q = 8 - 0.005$; then, there exists an unstable limit cycle around $\tilde{E}_4$ (see Figure 5(a),(b)). On the other hand, letting $k = 0.1, \alpha = 1, d = 1, c = 1, s = 0.7, q = 6.8$ and $r = 7.04$, we obtain that $l_1 = -0.06095323795$. We perturb q to $q = 6.8 + 0.03$; then, there exists a stable limit cycle around $\tilde{E}_4$ (see Figure 5(c),(d)).

Now, we give an example to illustrate the existence of two limit cycles. The parameters are given as follows:

$$d = 1, c = 1, s = 1, \alpha = \frac{1}{2}, k = \frac{7}{116} + \frac{3\sqrt{57}}{116}, r = \frac{369}{58} + \frac{9\sqrt{57}}{58}, q = \frac{9}{2},$$

where $l_1 = 0$. We perturb k and q to $k = \frac{7}{116} + \frac{3\sqrt{57}}{116} + 0.03$ and $q = \frac{9}{2} + 0.01$. Hence, system (5.1) undergoes a degenerate Hopf bifurcation and has two limit cycles (the inner one is stable and the outer is unstable) around $\tilde{E}_4$ (Figure 5(e),(f)).

Remark 5.1. *In Figure 5(a),(b), the origin is a stable node and the boundary equilibria are unstable. In addition, system* (5.1) *has two positive equilibria, where $\tilde{E}_3$ is a saddle point and $\tilde{E}_4$ is a stable point, and an unstable limit cycle appears around $\tilde{E}_4$. The orbits of the phase portraits reveal that the prey and predator tend to coexistent in steady states only when the initial values of system* (5.1) *lie inside the unstable limit cycle; otherwise, the prey and predator become extinct.*

In Figure 5(c),(d), in addition to the origin being stable, the other two boundary equilibria and the two positive equilibria are unstable. System (5.1) *has a stable limit cycle that appears around $\tilde{E}_4$. When the initial values lie to the right of the two stable invariant manifolds of the saddle, the prey and predator tend to coexist in periodic orbits. In addition, when the initial values lie to the left of the two stable invariant manifolds of the saddle, the prey and predator tend to go extinct.*

Figure 5(e),(f) show that system (5.1) *undergoes a degenerate Hopf bifurcation and has two limit cycles (the inner one is stable and the outer is unstable) around $\tilde{E}_4$. Prey and predator will oscillate and coexist if the initial values lie inside of the unstable limit cycle, while the prey and predator will become extinct if the initial values lie outside of the unstable limit cycle.*

5.3. Bogdanov-Takens bifurcation

From Theorem 4.5(1), the unique positive equilibrium E_* of system (1.4) is a cusp of codimension two, which means that a Bogdanov-Takens bifurcation of codimension two may occur. Hence, using q and s as the bifurcation parameters, system (1.4) becomes

$$
\begin{aligned}
\dot{x} &= x^2\left(\frac{r}{1+ky} - d - x - cy\right) - \frac{(q+\lambda_1)x^2}{1+x}, \\
\dot{y} &= (s+\lambda_2)y\,(x-y),
\end{aligned}
\tag{5.4}
$$

where $\lambda = (\lambda_1, \lambda_2)$ is a parameter vector in a small neighborhood of the origin.

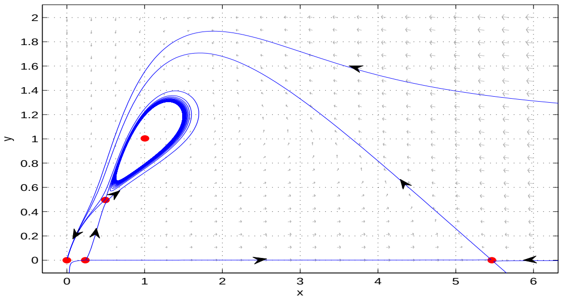
Theorem 5.5. *Assuming that the conditions of Theorem 4.5 (1) hold, system* (1.4) *undergoes a Bogdanov-Takens bifurcation of codimension two around E_*.*

Proof. First, by initiating the transformation $x_1 = x - x_*$ and $y_1 = y - y_*$ to move the positive equilibrium E_* to the origin, system (5.4) becomes

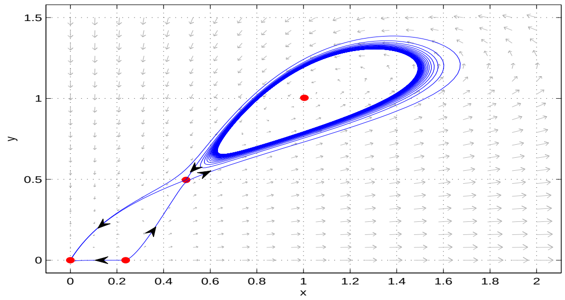
$$
\begin{cases}
\dot{x}_1 = g_{00} + g_{10}x_1 + g_{01}y_1 + g_{20}x_1^2 + g_{11}x_1y_1 + g_{02}y_2^2 + o(|x_1, y_1|^2), \\
\dot{y}_1 = h_{00} + h_{10}x_1 + h_{01}y_1 + h_{20}x_1^2 + h_{11}x_1y_1 + h_{02}y_1^2 + o(|x_1, y_1|^2),
\end{cases}
\tag{5.5}
$$

where

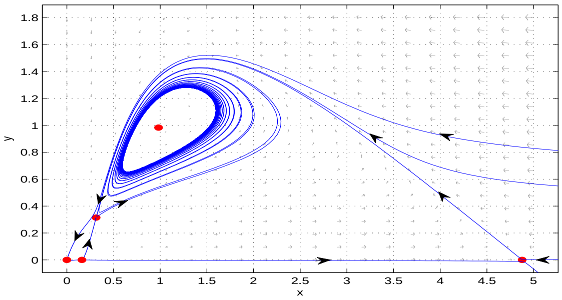
$$
\begin{aligned}
&g_{00} = -\frac{x_*^2\lambda_1}{1+x_*}, \quad g_{10} = \frac{[sx_*^2 + (2s-\lambda_1)x_* + s - 2\lambda_1]x_*}{(1+x_*)^2}, \quad g_{01} = sx_*, \\
&g_{20} = \frac{-x_*^4 + (s-2)x_*^3 + (4s-1)x_*^2 + 5sx_* + 2s - \lambda_1}{(1+x_*)^3}, g_{11} = -2s, \\
&g_{02} = \frac{(cx_* - s)^2 x_*}{[(c+2)x_*^2 + (d+s+1)x_* + s]}, \quad h_{00} = 0, \quad h_{10} = (s+\lambda_2)x_*, \\
&h_{01} = -(s+\lambda_2)x_*, \quad h_{20} = 0, \quad h_{11} = s + \lambda_2, \quad h_{02} = -s - \lambda_2.
\end{aligned}
$$



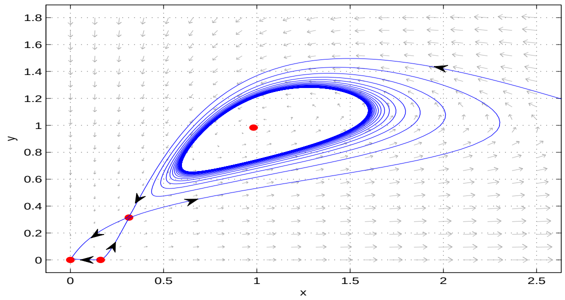
(a)



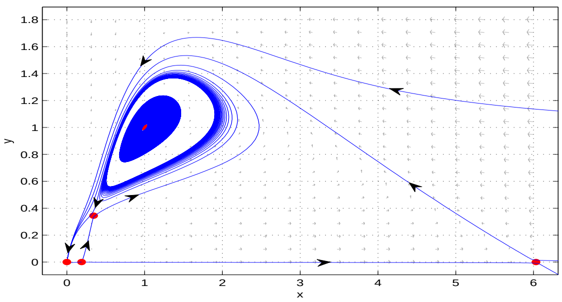
(b)



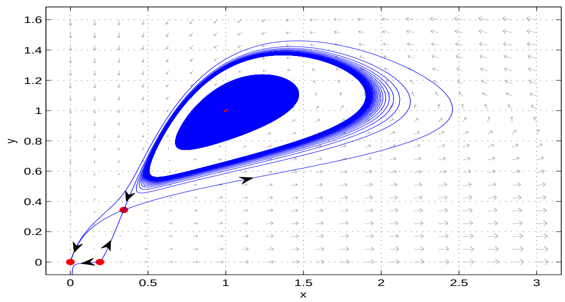
(c)



(d)



(e)



(f)

Figure 5. (a) An unstable limit cycle appears in system (5.1) with $k = 0.1, \alpha = 1, d = 1, c = 1, s = 1, q = 8-0.005, r = 7.7$. (b) The local amplified phase portrait of (a). (c) A stable limit cycle appears in system (5.1) with $k = 0.1, \alpha = 1, d = 1, c = 1, s = 0.7, q = 6.8 + 0.03, r = 7.04$. (d) The local amplified phase portrait of (c). (e) Two limit cycles (the inner one is stable and the outer is unstable) appear in system (5.1) with $d = 1, c = 1, s = 1, \alpha = \frac{1}{2}, k = \frac{7}{116} + \frac{3\sqrt{57}}{116} + 0.03, r = \frac{369}{58} + \frac{9\sqrt{57}}{58}, q = \frac{9}{2} + 0.01$. (f) The local amplified phase portrait of (e).

Second, letting

$$x_2 = y_1,$$
$$y_2 = h_{10}x_1 + h_{01}y_1 + h_{20}x_1^2 + h_{11}x_1y_1 + h_{02}y_1^2,$$

system (5.5) can be written as follows:

$$\begin{cases} \dot{x}_2 = y_2, \\ \dot{y}_2 = j_{00} + j_{10}x_2 + j_{01}y_2 + j_{20}x_2^2 + j_{11}x_2y_2 + j_{02}y_2^2 + o(|x_2, y_2|^2), \end{cases} \tag{5.6}$$

where

$$j_{00} = g_{00}h_{00}, \quad j_{10} = g_{00}h_{11} + g_{01}h_{10} - g_{10}h_{01}, \quad j_{01} = g_{10}h_{01},$$
$$j_{20} = \frac{g_{01}h_{10}h_{11} + g_{02}h_{10}^2 - g_{10}h_{02}h_{10} - g_{11}h_{01}h_{10} + g_{20}h_{01}^2}{h_{10}},$$
$$j_{11} = \frac{g_{11}h_{10} - 2g_{20}h_{01} - h_{11}h_{01} + 2h_{10}h_{02}}{h_{10}}, \quad j_{02} = \frac{g_{20} + h_{11}}{h_{10}}.$$

Taking a new time variable τ with $\mathrm{d}t = (1 - j_{02}x_2)\mathrm{d}\tau$ and $x_3 = x_2, y_3 = (1 - j_{02}x_2)y_2$, system (5.6) becomes

$$\begin{cases} \dot{x}_3 = y_3, \\ \dot{y}_3 = k_{00} + k_{10}x_3 + k_{01}y_3 + k_{20}x_3^2 + k_{11}x_3y_3 + k_{02}y_3^2 + o(|x_3, y_3|^2), \end{cases} \tag{5.7}$$

where

$$k_{00} = j_{00}, \quad k_{10} = -2j_{00}j_{02} + j_{10}, \quad k_{01} = j_{01},$$
$$k_{20} = j_{00}j_{02}^2 - 2j_{02}j_{10} + j_{20}, \quad k_{11} = -j_{11}, \quad k_{02} = \frac{g_{20} + h_{11}}{h_{10}}.$$

From the proof of Theorem 4.4, we have

$$k_{20}\Big|_{\lambda_1=\lambda_2=0} = \frac{sx_*^3T_1}{[(c+2)x_*^2 + (d+s+1)x_* + s](1 + x_*)} < 0,$$

where T_1 is defined in Theorem 4.4. Letting

$$x_4 = x_3, \qquad y_4 = \frac{y_3}{\sqrt{-k_{20}}}, \qquad \tau = \sqrt{-k_{20}}t,$$

system (5.7) becomes

$$\begin{cases} \dot{x}_4 = y_4, \\ \dot{y}_4 = m_{00} + m_{10}x_4 + m_{01}y_4 - x_4^2 + m_{11}x_4y_4 + o(|x_4, y_4|^2), \end{cases} \tag{5.8}$$

where

$$m_{00} = -\frac{k_{00}}{k_{20}}, \qquad m_{10} = -\frac{k_{10}}{k_{20}}, \qquad m_{01} = \frac{k_{01}}{\sqrt{-k_{20}}}, \qquad m_{11} = \frac{k_{11}}{\sqrt{-k_{20}}}.$$

Next, letting $x_5 = x_4 - \dfrac{m_{10}}{2}$ and $y_5 = y_4$, system (5.8) is equivalent to the following system:

$$\begin{cases} \dot{x}_5 = y_5, \\ \dot{y}_5 = n_{00} + m_{01}y_5 - x_5^2 + n_{11}x_5y_5 + o(|x_5, y_5|^2), \end{cases}$$

where

$$n_{00} = m_{00} + \frac{m_{10}^2}{4}, \quad n_{01} = m_{01} + \frac{m_{11}m_{10}}{2}, \quad n_{11} = m_{11}.$$

From the proof of Theorem 4.4, we obtain

$$n_{11}\Big|_{\lambda_1=\lambda_2=0} = \sqrt{-\frac{(2x_*^2 + sx_* - s)^2[(c+2)x_*^2 + (d+s+1)x_* + s]}{T_1(1+x_*)sx_*^3}} \neq 0.$$

Finally, letting

$$x_6 = -n_{11}^2 x_5, \quad y_6 = -n_{11}^3 y_5, \quad \tau = -\frac{1}{n_{11}}t,$$

we obtain the universal unfolding of system (5.4) as follows:

$$\begin{cases} \dot{x}_6 = y_6, \\ \dot{y}_6 = \mu_1 + \mu_2 y_6 + x_6^2 + x_6 y_6 + o(|x_6, y_6|^2), \end{cases}$$

where

$$\mu_1 = -n_{00}n_{11}^4, \quad \mu_2 = -n_{01}n_{11}.$$

Using Maple software, we have

$$\left|\frac{\partial(\mu_1, \mu_2)}{\partial(\lambda_1, \lambda_2)}\right|_{\lambda_1=\lambda_2=0} = -\frac{[(c+2)x_*^2 + (d+s+1)x_* + s]^4(2x_*^2 + sx_* - s)^5}{s^3x_*^8(1+x_*)^2T_1^4} \neq 0.$$

By the results in [28], system (1.4) undergoes a Bogdanov-Takens bifurcation of codimension two. The proof is completed.

The local expression of the bifurcation curves are given in [28] as follows:

(i) The saddle-node bifurcation curve

$$SN = \{(\lambda_1, \lambda_2) : \mu_1(\lambda_1, \lambda_2) = 0, \mu_2(\lambda_1, \lambda_2) \neq 0\};$$

(ii) The Hopf bifurcation curve

$$H = \left\{(\lambda_1, \lambda_2) : \mu_1(\lambda_1, \lambda_2) < 0, \mu_2(\lambda_1, \lambda_2) = \sqrt{-\mu_1(\lambda_1, \lambda_2)}\right\};$$

(iii) The homoclinic curve

$$HL = \left\{(\lambda_1, \lambda_2) : \mu_1(\lambda_1, \lambda_2) < 0, \mu_2(\lambda_1, \lambda_2) = \frac{5}{7}\sqrt{-\mu_1(\lambda_1, \lambda_2)}\right\}.$$

In what follows, we present the phase diagrams of system (5.4), as obtained by some numerical simulations. Choosing $c = 2, d = 1, s = 4, q = 20, k = \frac{1}{6}$ and $r = \frac{49}{3}$, and from Theorem 4.5(1), $E_*(1,1)$ is a cusp of codimension two. Figure 6 shows that system (1.4) undergoes a Bogdanov-Takens bifurcation of codimension two.

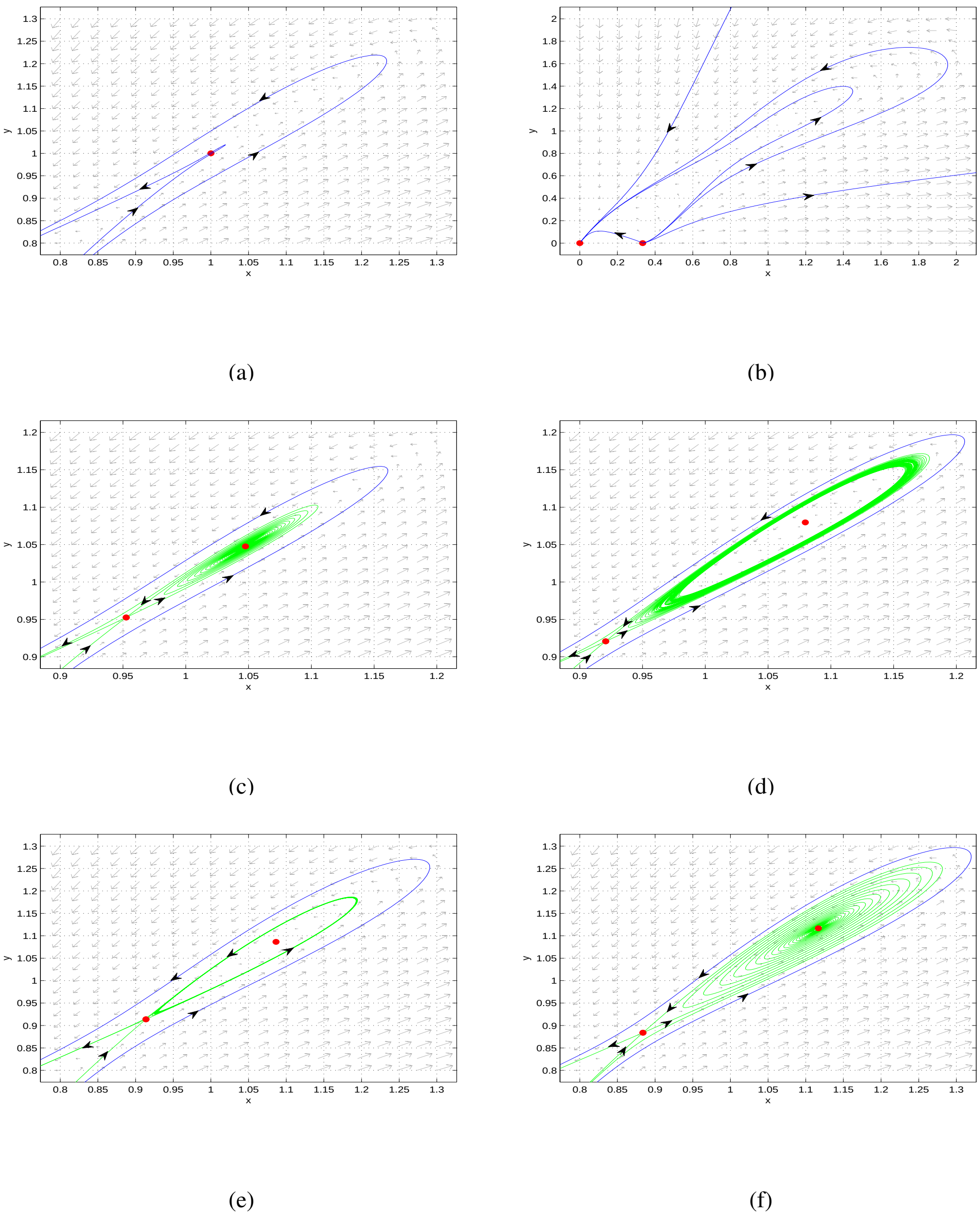


Figure 6. Phase portraits of system (5.4). (a) A cusp of codimension two when $(\lambda_1, \lambda_2) = (0, 0)$. (b) No positive equilibria when $(\lambda_1, \lambda_2) = (0.011, -0.08)$. (c) An unstable focus when $(\lambda_1, \lambda_2) = (-0.001, -0.08)$. (d) An unstable limit cycle when $(\lambda_1, \lambda_2) = (-0.028, -0.08)$. (e) An unstable homoclinic loop when $(\lambda_1, \lambda_2) = (-0.033, -0.08)$. (f) A stable focus when $(\lambda_1, \lambda_2) = (-0.06, -0.08)$.

6. Conclusions

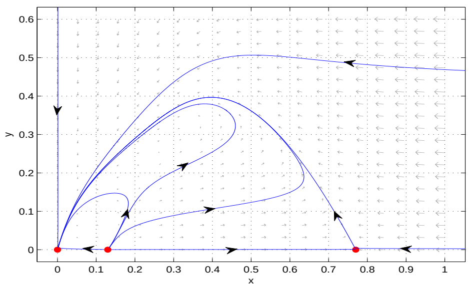
In this paper, we consider a Leslie-Gower predator-prey model with the fear effect and nonlinear harvesting. Fear of predator and nonlinear harvesting are the main factors affecting the dynamic behavior of system (1.4). Via numerical simulations, we show the influences of the fear effect and nonlinear harvesting on the dynamic behavior of system (1.4).

First, let $c = 0.1, d = 0.1, s = 0.04, q = 2$ and $r = 2$. When $k = 0.06$, system (1.4) has no positive equilibrium (see Figure 7(a)). In a biological sense, the prey and predator will become extinct when the fear effect is large. When $k = 0.053$, system (1.4) has two positive equilibria, where E_3 is a saddle and E_4 is an unstable node (see Figure 7(b)). Hence, in this case, the prey and predator are still extinct. When $k = 0.05$, E_4 becomes a stable node, and an unstable limit cycle appears around E_4 (see Figure 7(c)). Then, for system (1.4), a bistable phenomenon occurs, in which the prey and predator tend to steady states (or extinction), depending on the initial values lying inside (or outside) of the unstable limit cycle. When $k = 0.03$, E_4 is still a stable node and an unstable limit cycle disappears (see Figure 7(d)). Then, the prey and predator will survive or become extinct depending on the two stable manifolds of the saddle that act as a separatrix curve. When $k = 0$, that is, without the fear effect, the dynamic behavior of system (1.4) is similar to that shown in Figure 7(d) (see Figure 7(e)). Figure 7 shows that the prey and predator may survive or become extinct when the fear effect is small. With the increase of the fear effect, the survival area of species decreases, until finally, the prey and predator will become extinct if the fear effect is strong enough. Hence, a strong fear effect is not conducive to the survival of the species.

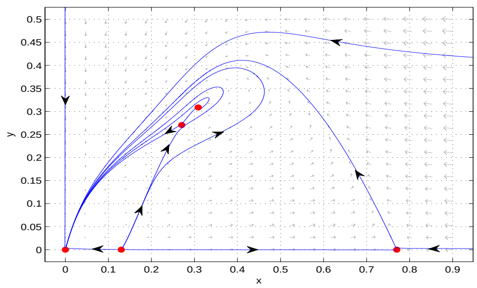
Second, we consider the impact of nonlinear harvesting on system (1.4). Let $c = 0.1, d = 0.1, s = 0.1, k = 0.1$ and $r = 3$. When $q = 3.5$, system (1.4) has no positive equilibrium and the origin is globally asymptotically stable (see Figure 8(a)). When $q = 3.3$, system (1.4) has two positive equilibria, where E_3 is a saddle and E_4 is an unstable node (see Figure 8(b)). In this case, the prey and predator are still extinct. When $q = 3.287$, system (1.4) has an unstable limit cycle and there is a bistable phenomenon (see Figure 8(c)). That is, an unstable limit cycle acts as a separatrix curve, where the prey and predator will become extinct or survive. When $q = 3.283$, there exists an unstable homoclinic loop in system (1.4) (see Figure 8(d)). When $q = 3.26$, the unstable limit cycle and homoclinic loop disappear. Hence, the prey and predator will tend to steady states (or extinction) if the initial values lies to the right (or left) of the two stable manifolds of the saddle (see Figure 8(e)). When $q = 0$, that is, without nonlinear harvesting, system (1.4) has only one positive equilibrium, which is globally asymptotically stable (see Figure 8(f)). This shows that overfishing can lead to the extinction of the predator and prey, so, maintaining proper harvesting can help the survival of the prey and predator.

By conducting numerical simulations, we were able to clearly observe that, when $k \leq 0.05$ and $q \leq 3.281$, the prey and predator tend to coexist around the stable positive equilibrium E_4. In other words, by effectively controlling the harvesting, we can ensure that the prey's fear of being caught remains within a smaller range, which benefits the survival of both populations. That is, weaker fear effects and less capture are beneficial to the survival of both predator and prey. We have conducted a theoretical analysis of system (1.4) and obtained some conclusions. However, when it comes to solving practical problems, there are many external factors. The actual application of the model may be difficult to achieve in the short term.

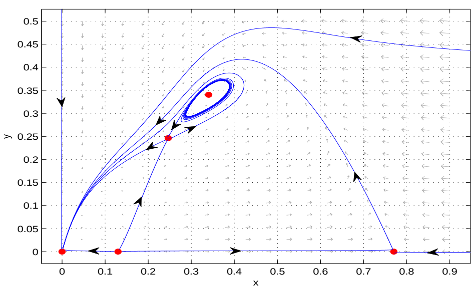
When $r - d > q$, the origin is a repeller, the only boundary equilibrium is a saddle point and the unique positive equilibrium may be stable or unstable. From Remark 4.1, the unique positive equilibrium is globally asymptotically stable if $q < 1$. Then, the prey and predator will tend to a positive coexistent steady state if the birth rate of the prey is high and the catchability coefficient is small. When $r - d = q \leq q_0$, from Remark 4.2, the origin is globally asymptotically stable, which



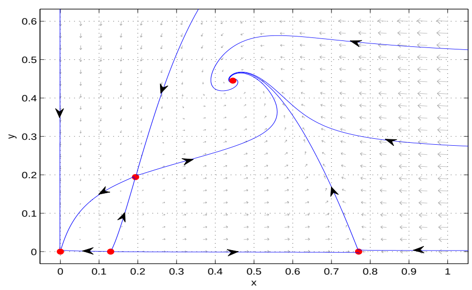
(a) $k = 0.06$



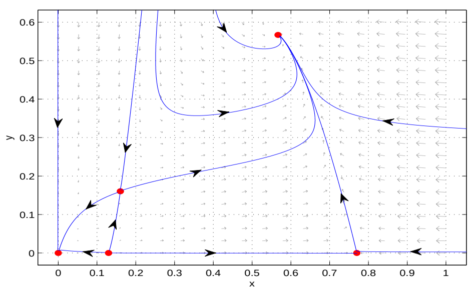
(b) $k = 0.053$



(c) $k = 0.05$



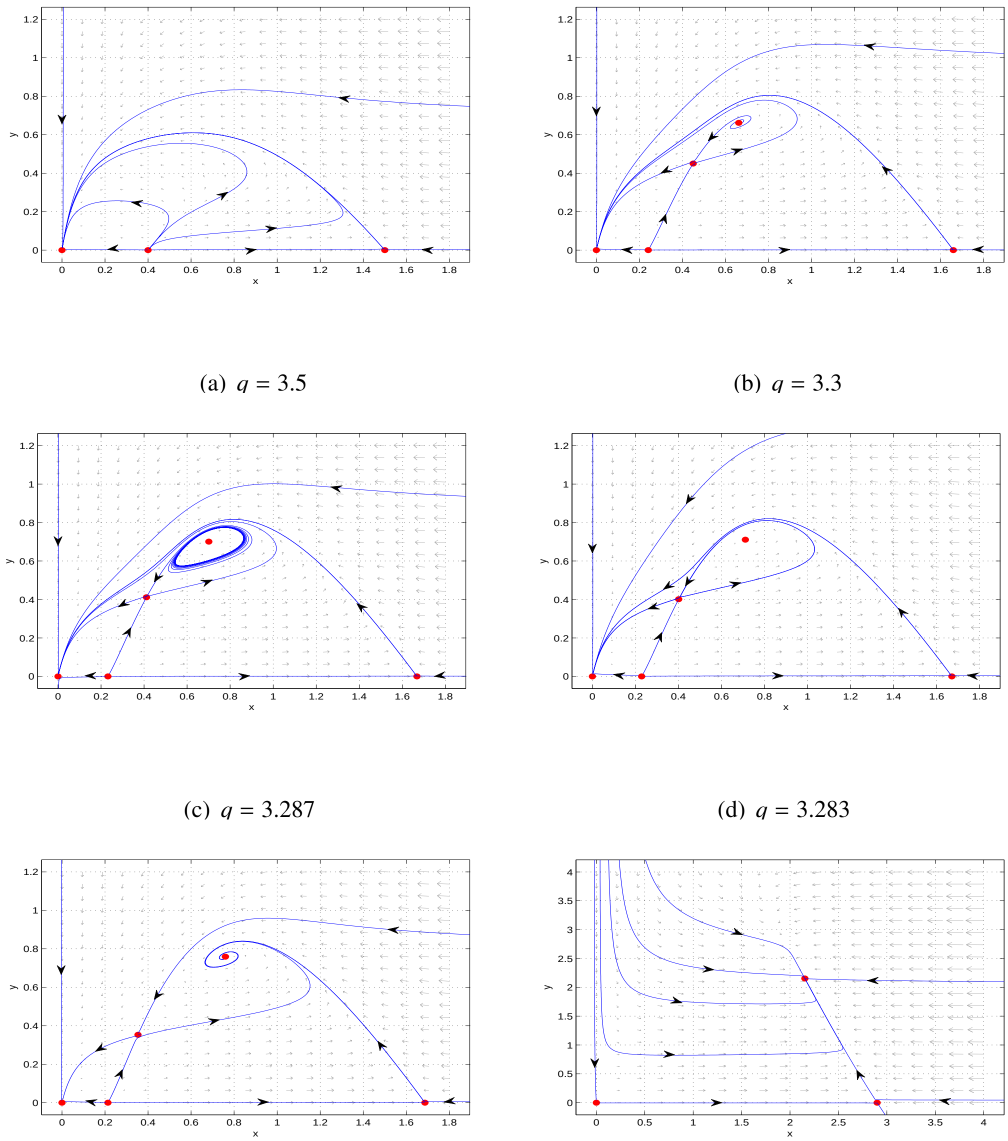
(d) $k = 0.03$



(e) $k = 0$

Figure 7. Phase portraits of system (1.4) with $c = 0.1, d = 0.1, s = 0.04, q = 2, r = 2$.

implies that the prey and predator will become extinct. When $r - d = q > q_0$ or $r - d < q$, system (1.4) may have zero, one or two positive equilibria, and these equilibria may be stable or unstable. We show that the unique equilibrium E_* is a saddle-node or a cusp of codimension two (or three). Moveover, system (1.4) undergoes saddle-node bifurcation and Bogdanov-Takens bifurcation around E_*. Also, system (1.4) undergoes a degenerate Hopf bifurcation and multiple limit cycles may appear around $\tilde{E}_4$. In Figure 5, we show that system (1.4) has two limit cycles (the inner one is stable and the outer is



(a) $q = 3.5$

(b) $q = 3.3$

(c) $q = 3.287$

(d) $q = 3.283$

(e) $q = 3.26$

(f) $q = 0$

Figure 8. Phase portraits of system (1.4) with $c = 0.1, d = 0.1, s = 0.1, k = 0.1, r = 3$.

unstable) around $\tilde{E}_4$, which implies the bistable phenomenon. That is a large amount of fear and prey harvesting are detrimental to the survival of the prey and predator. Additionally, the prey and predator will reach a steady state if the intrinsic growth rate of the prey is high and the catchability coefficient for the prey is low. However, the prey and predator will become extinct if the intrinsic growth rate for the prey and the catchability coefficient for the prey are small.

In [9], the authors studied the stability of the equilibria and demonstrated that there exists a limit cycle in system (1.1). Considering the Holling type II functional response, [10] showed that a unique equilibrium is a cusp of codimension two and a limit cycle appears. Unlike [9, 10], we show that a unique equilibrium is a cusp of codimension three and confirm the occurrence of Bogdanov-Takens bifurcation. We have found that system (1.4) has two limit cycles (the inner one is stable and the outer is unstable), which exhibit the bistable phenomenon. Also, we have proven that the origin and equilibrium are globally asymptotically stable under some conditions. The strong fear effect and nonlinear harvesting are not conducive to the survival of the species. These indicate that the dynamic behavior of system (1.4) is more complex than that of the systems in [9, 10].

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Acknowledgments

This work was supported by the Natural Science Foundation of Fujian Province (2021J01613, 2021J011032) and the Scientific Research Foundation of Minjiang University (MJY22027).

Conflict of interest

The authors declare that there is no conflict of interest.

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Appendix A. Coefficients in the proof of Theorem 4.5

$$
\begin{aligned}
P_0 = {} & 2(d+1)^2(13c^2-14d-14)(c^2-d-1),\\
P_1 = {} & (d+1)(76c^5-81c^4d+127c^4-390c^3d+147c^2d^2-390c^3-354c^2d+328cd^2-60d^3-501c^2\\
& +656cd+268d^2+328c+716d+388),\\
P_2 = {} & 58c^6-234c^5d+38c^4d^2+70c^5-1230c^4d+915c^3d^2-79c^2d^3-852c^4-1290c^3d+2439c^2d^2\\
& -612cd^3+18d^4-2205c^3+2523c^2d+2100cd^2-656d^3+5c^2+6036cd+612d^2+3324c\\
& +3264d+1978,\\
P_3 = {} & -177c^6+100c^5d+115c^4d^2-1158c^5+1979c^4d-63c^3d^2-74c^2d^3-2064c^4+7346c^3d\\
& -2310c^2d^2+16cd^3+26d^4+1169c^3+9954c^2d-5080cd^2+36d^3+8734c^2+5536cd-3024d^2
\end{aligned}
$$

$$+10632c+1116d+4150,$$

$$P_4=70c^6+350c^5d-130c^4d^2+1427c^5+883c^4d-1098c^3d^2+36c^2d^3+6845c^4-2440c^3d$$
$$-2088c^2d^2+344cd^3-6d^4+13906c^3-12212c^2d-1692cd^2+380d^3+13912c^2-16064cd$$
$$-696d^2+6964c-6780d+1470,$$

$$P_5=275c^6-380c^5d-11c^4d^2+1227c^5-3244c^4d+372c^3d^2+55c^2d^3+395c^4-9320c^3d$$
$$+2241c^2d^2+72cd^3-6d^4-7492c^3-12879c^2d+3884cd^2+44d^3-18425c^2-9696cd$$
$$+1992d^2-17220c-3276d-5730,$$

$$P_6=-290c^6-46c^5d+66c^4d^2-2724c^5+672c^4d+567c^3d^2-31c^2d^3-9888c^4+4870c^3d$$
$$+1203c^2d^2-148cd^3-18717c^3+11647c^2d+1224cd^2-124d^3-21411c^2+11340cd$$
$$+540d^2-14416c+3820d-4268,$$

$$P_7=-43c^6+204c^5d-23c^4d^2+312c^5+1603c^4d-303c^3d^2+3204c^4+4450c^3d-1116c^2d^2$$
$$+9837c^3+6564c^2d-1420cd^2+13344c^2+5192cd-576d^2+8084c+1664d+1728,$$

$$P_8=162c^6-70c^5d+1355c^5-761c^4d+4549c^4-2836c^3d+8456c^3-4580c^2d+9068c^2$$
$$-3328cd+5120c-896d+1152,$$

$$P_9=-(c+1)(55c^5+530c^4+1812c^3+2752c^2+1920c+512),$$

$$M_0=8(c+2)^9(3c+d+3),$$

$$M_1=4(c+2)^8(533c^2+184cd+15d^2+1106c+196d+573),$$

$$M_2=2(c+2)^7(9109c^3+4629c^2d+751cd^2+39d^3+28867c^2+10354cd+891d^2+30371c+5737d$$
$$+10613),$$

$$M_3=(c+2)^6(69015c^4+45510c^3d+10460c^2d^2+970cd^3+29d^4+289652c^3+154958c^2d+25896cd^2$$
$$+1334d^3+454154c^2+174818cd+15852d^2+315428c+65354d+81911),$$

$$M_4=2(c+2)^5(73321c^5+58214c^4d+16620c^3d^2+2014c^2d^3+87cd^4+378112c^4+263852c^3d$$
$$+62784c^2d^2+5788cd^3+156d^4+776396c^3+445684c^2d+78358cd^2+4098d^3+793634c^2$$
$$+332644cd+32310d^2+403923c+92598d+81894),$$

$$M_5=(c+2)^4(189851c^6+173592c^5d+57866c^4d^2+8320c^3d^3+435c^2d^4+1150784c^5+976160c^4d$$
$$+293568c^3d^2+36592c^2d^3+1560cd^4+2891094c^4+2177892c^3d+552334c^2d^2+52892cd^3$$
$$+1392d^4+3853380c^3+2410536c^2d+457164cd^2+25160d^3+2873687c^2+1324004cd$$
$$+140568d^2+1136796c+288792d+186328),$$

$$M_6=2(c+2)^3(3c^2+cd+5c+2d+2)(25505c^5+17632c^4d+3965c^3d^2+290c^2d^3+134140c^4$$
$$+83604c^3d+16324c^2d^2+980cd^3+280723c^3+147364c^2d+22110cd^2+824d^3+292258c^2$$
$$+114464cd+9852d^2+151382c+33072d+31212)$$

$$M_7=(c+2)^2(8325c^4+3848c^3d+435c^2d^2+36898c^3+14820c^2d+1380cd^2+61025c^2+18844cd$$
$$+1092d^2+44632c+7896d+12180)(3c^2+cd+5c+2d+2)^2,$$

$$M_8=2(c+2)(377c^3+87c^2d+1355c^2+258cd+1618c+192d+640)(3c^2+cd+5c+2d+2)^3,$$

$$M_9=(29c^2+80c+56)(3c^2+cd+5c+2d+2)^4.$$

Appendix B. Coefficients of system (5.3)

$$\tilde{c}_{20} = \frac{n_1k^2 + n_2k + c^2s + c^2}{(\alpha+1)(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^2},$$

$$\tilde{c}_{11} = \frac{2\sqrt{D}[n_3k^2 + (2\alpha cs + 2\alpha c + 4c^2 + 2cd + 2cs + 4c)k + c^2]}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^2},$$

$$\tilde{c}_{02} = -\frac{(\alpha s + \alpha + c + d + s + 2)k^2D}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^2},$$

$$\tilde{c}_{30} = -\frac{n_4k^3 + n_5k^2 + n_6k + c^3(s+1)(2\alpha+1)}{(\alpha+1)^2(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^3},$$

$$\tilde{c}_{21} = -\frac{\sqrt{D}[n_7k^3 + n_8k^2 + 3c^2(\alpha s + \alpha + 2c + d + s + 2)k + c^3]}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^3},$$

$$\tilde{c}_{12} = \frac{(s\alpha + \alpha + c + d + s + 2)k^2(2\alpha ks + 2\alpha k + 4ck + 2dk - ks + 2c + 4k)D}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^3},$$

$$\tilde{c}_{03} = -\frac{(s\alpha + \alpha + c + d + s + 2)k^3D^{\frac{3}{2}}}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^3},$$

$$\tilde{d}_{20} = -\frac{n_9k^2 + n_{10}k + \alpha c^2s - \alpha c s^2 + 2c^2s - c s^2 + c^2}{\sqrt{D}\,(\alpha+1)(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^2},$$

$$\tilde{d}_{11} = -\frac{[n_{11}k^2 + n_{12}k + 3c^2 - 2cs]s}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^2},$$

$$\tilde{d}_{02} = \frac{\sqrt{D}s[(2\alpha s + 2\alpha + 3c + 2d + 2s + 4)k^2 + (\alpha s + \alpha + 3c + d + s + 2)k + c]}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^2},$$

$$\tilde{d}_{30} = \frac{s[n_4k^3 + n_5k^2 + n_6k + c^3(s+1)(2\alpha+1)]}{\sqrt{D}(\alpha+1)^2(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^3},$$

$$\tilde{d}_{21} = \frac{s[n_{13}k^3 + n_{14}k^2 + 3c^2(\alpha s + \alpha + 2c + d + s + 2)k + c^3]}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^3},$$

$$\tilde{d}_{12} = -\frac{s(s\alpha + \alpha + c + d + s + 2)k^2(2\alpha ks + 2\alpha k + 4ck + 2dk - ks + 2c + 4k)\sqrt{D}}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^3},$$

$$\tilde{d}_{03} = \frac{s(s\alpha + \alpha + c + d + s + 2)k^3D}{(\alpha ks + \alpha k + 2ck + dk + ks + c + 2k)^3},$$

$$\begin{aligned} n_1 = {} & (2s^2 + 3s + 1)\alpha^2 + (3s^2c + d s^2 + 8sc + 4ds + 5s^2 + 4c + 2d + 10s + 4)\alpha + 4c^2s + 4cds + 3s^2c \\ & + d^2s + d s^2 + 4c^2 + 4dc + 12sc + d^2 + 6ds + 3s^2 + 8c + 4d + 8s + 4, \end{aligned}$$

$$n_2 = (2s^2c + 4sc + 2c)\alpha + 4c^2s + 2cds + 2s^2c + 4c^2 + 2dc + 6sc + 4c,$$

$$n_3 = (s^2+2s+1)\alpha^2 + (4cs+2ds+s^2+4c+2d+5s+4)\alpha + 4c^2+4dc+3cs+d^2+ds+8c+4d + 2s+4,$$
$$n_4 = (4s^3+10s^2+8s+2)\alpha^4 + w_1\alpha^3 + w_2\alpha^2 + w_3\alpha + w_4,$$
$$n_5 = 2c(s+1)(2s^2+6s+3)\alpha^3 + w_5\alpha^2 + w_6\alpha + c\,s^3 + (10c^2+4cd+11c)s^2 + (12c^3+12c^2d+3c\,d^2 + 36c^2+18cd+24c)s + 12c^3+12c^2d+3c\,d^2+24c^2+12cd+12c,$$
$$n_6 = 6c^2(s+1)^2\alpha^2 + 3c^2(s+1)(3s+4c+2d+5)\alpha + 3c^2(s+1)(s+2c+d+2),$$
$$n_7 = (s+1)^3\alpha^3 + (s+1)^2(6c+3d-s+6)\alpha^2 + (s+1)(12c^2+12cd+3d^2-2ds-2s^2+24c+12d - 4s+12)\alpha + 8c^3 + (12d+4s+24)c^2 + (6d^2-3s^2+24d+24)c + (d+2)(d+s+2)(d-2s+2),$$
$$n_8 = 3c(s+1)^2\alpha^2 + 2c(s+1)(s+6c+3d+6)\alpha + 12c^3 + (12d+8s+24)c^2 + (d+s+2)(3d-s+6)c,$$
$$n_9 = s(s+1)^2\alpha^3 + (s+1)(4cs+2ds+2s^2+7s+1)\alpha^2 + (s^3+(9c+4d+12)s^2 + (4c^2+4cd+d^2 + 20c+10d+18)s + 4c+2d+4)\alpha + (5c+2d+5)s^2 + 2(2c+d+3)(2c+d+2)s + (2c+d+2)^2,$$
$$n_{10} = s(s+1)(-s+2c)\alpha^2 + (-2s^3+(3c-d-3)s^2+(4c^2+2cd+10c)s+2c)\alpha + (8s+4)c^2 + (4ds + s^2+2d+10s+4)c - s^2(d+s+2),$$
$$n_{11} = 3(s+1)^2\alpha^2 + 2(s+1)(s+6c+3d+6)\alpha + 12c^2 + (12d+6s+24)c + (d+s+2)(3d-s+6),$$
$$n_{12} = 2(s+1)(-s+3c)\alpha + 12c^2 + (6d+12)c - 2s(d+s+2),$$
$$n_{13} = (s+1)^3\alpha^3 + (s+1)^2(6c+3d-s+6)\alpha^2 + w_7\alpha + w_8,$$
$$n_{14} = 3c(s+1)^2\alpha^2 + 2c(s+1)(s+6c+3d+6)\alpha + 12c^3 + (12d+8s+24)c^2 + (d+s+2)(3d-s+6)c,$$
$$w_1 = (6c+2d+15)s^3 + (30c+14d+48)s^2 + (36c+18d+46)s + 12c+6d+13,$$
$$w_2 = (13c+4d+19)s^3 + (20c^2+18cd+4d^2+90c+41d+79)s^2 + (48c^2+48cd+12d^2+138c + 69d+93)s + 3(4c+2d+5)(2c+d+2),$$
$$w_3 = (8c+2d+9)s^3 + (28c^2+24cd+5d^2+78c+34d+51)s^2 + (16c^3+24c^2d+12c\,d^2+2d^3 + 96c^2+96cd+24d^2+156c+78d+76)s + (4c+2d+7)(2c+d+2)^2,$$
$$w_4 = (c+1)s^3 + (8c^2+6cd+d^2+18c+7d+10)s^2 + (8c^3+12c^2d+6c\,d^2+d^3+36c^2+36cd + 9d^2+48c+24d+20)s + (2c+d+2)^3,$$
$$w_5 = 9c\,s^3 + (22c^2+10cd+49c)s^2 + (48c^2+24cd+69c)s + 3c(8c+4d+9),$$
$$w_6 = 6c\,s^3 + (32c^2+14cd+44c)s^2 + (24c^3+24c^2d+6c\,d^2+96c^2+48cd+78c)s + 6c(2c+d+3)(2c+d+2),$$
$$w_7 = (s+1)(12c^2+12cd+3d^2-2ds-2s^2+24c+12d-4s+12),$$
$$w_8 = (-3c-2d-4)s^2 + (4c^2-d^2-4d-4)s + (2c+d+2)^3.$$