Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(10): 18468–18490.
DOI: 10.3934/mbe.2023820
Received: 14 July 2023
Revised: 24 September 2023
Accepted: 25 September 2023
Published: 26 September 2023

Research article

Global Hopf bifurcation of a cholera model with media coverage

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Abstract: We propose a model for cholera under the impact of delayed mass media, including human-to-human and environment-to-human transmission routes. First, we establish the extinction and uniform persistence of the disease with respect to the basic reproduction number. Then, we conduct a local and global Hopf bifurcation analysis by treating the delay as a bifurcation parameter. Finally, we carry out numerical simulations to demonstrate theoretical results. The impact of the media with the time delay is found to not influence the threshold dynamics of the model, but is a factor that induces periodic oscillations of the disease.

Keywords: cholera model; media coverage; threshold dynamics; basic reproduction number; global Hopf bifurcation

1. Introduction

In modern society, media coverage has become an important strategy for controlling and preventing disease transmission. It can alter an individual's behavior, such as wearing protective masks, vaccination, self-isolating or avoiding gathering activities, and hence reduce the possibility of contracting the infection [1]. Additionally, the media coverage also influences the implementation of a public health policy intervention and control polices [2]. Therefore, how to quantify this impact through mathematical modeling is an important issue for epidemics control.

Many mathematical models have been developed to investigate the impact of media coverage on disease transmission and control. The most general approach to modeling the media impact is to change the transmission coefficient as a nonlinear decreasing function with respect to the number of infectious individuals (I). For example, Cui et al. [3] used a contact transmission rate μe^{-mI} to describe the impact of media coverage, where $\mu > 0$ is the probability of baseline transmission and $m > 0$ reflects the effect of the media. Li and Cui [4] employed $\frac{\beta_2 I}{m+I}$ to reflect the reduced amount of contact rate due to media coverage, producing the transmission rate of the form $\beta_1 - \frac{\beta_2 I}{m+I}$ ($\beta_1 > \beta_2 > 0, m > 0$). By considering the correlation between media impact and the number of infected individuals at different disease stages, Liu et al. [5] devised the media impact function $\beta e^{-a_1 E - a_2 I - a_3 H}$ with E denoting exposed individuals, I infectives, H hospitalized individuals and nonnegative constants a_1, a_2, a_3. However, it is noted that individuals can also change their behavior in response to the rate of change in case numbers. Xiao et al. [6] used an exponentially decreasing function $e^{-M(I, \frac{dI}{dt})}$ with $M(I, \frac{dI}{dt}) = \max\{0, p_1 I + p_2 \frac{dI}{dt}\}$ to model such media impacts. For more studies on media impacts, we refer readers to [1, 2, 7–10] and references therein.

However, as mentioned in [10], impact of media coverage on the population transmission dynamics has lag, describing both the lag time of individuals' response to the media and the lag time of media reports about an infectious disease outbreak. Some studies have been conducted to investigate the impact of delayed mass media, see [1, 8, 10, 11]. In this paper, we will consider the time delay of the media impact in the mathematical modeling of cholera, an acute intestinal infectious disease caused by the bacterium Vibrio cholerae (or, V. cholerae). Several mathematical models have been proposed to study cholera dynamics, including, but not limited to, [12–18]. However, to our knowledge, few of these have specifically taken into account the response delay in media impacts.

Our model is inspired by the disease transmission models in [10, 13]. We divide the population into three subclasses: Susceptible $S(t)$, infectives $I(t)$ and recovered $R(t)$. The concentration of bacteria V. cholerae in contaminated water is indicated by $B(t)$. To study the lag effect of the impact of the media on cholera transmission, we incorporate a decreasing factor $e^{-mI(t-\tau)}$ into the direct and indirect incidence rate. That is,

$$\beta_1 e^{-mI(t-\tau)} S(t) I(t) \text{ and } \beta_2 e^{-mI(t-\tau)} S(t) B(t).$$

Here, β_1 and β_2 are, respectively, the baseline direct and indirect transmission rate, τ represents the report delay and response time of individuals to the current infection and m is the weight of media impact sensitive to the case number. The basic SIRB model under consideration is then

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta_1 e^{-mI(t-\tau)} S(t) I(t) - \beta_2 e^{-mI(t-\tau)} S(t) B(t) - \mu S(t) + \sigma R(t), \\ \dfrac{dI(t)}{dt} = \beta_1 e^{-mI(t-\tau)} S(t) I(t) + \beta_2 e^{-mI(t-\tau)} S(t) B(t) - (\mu + \gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\mu + \sigma) R(t), \\ \dfrac{dB(t)}{dt} = \xi I(t) - \delta B(t), \end{cases} \tag{1.1}$$

where Λ stands for the influx rate of susceptible humans, μ is the natural death rate, σ is the immunity waning rate, γ is the recovery rate, ξ denotes the rate of contribution to V. cholerae in the aquatic environment and $\delta = \delta_1 - \delta_2 > 0$ is the net death rate of V. cholerae in the aquatic environment, where δ_1, δ_2 are the natural washout rate and the natural growth rate of V. cholerae in the aquatic environment, respectively. All the parameters of model (1.1) are assumed to be positive.

Note that the total population size $N(t) = S(t) + I(t) + R(t)$ satisfies

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t),$$

and has a unique equilibrium $N^* = \frac{\Lambda}{\mu}$ that is globally stable in $\mathbb{R}_+$. We then consider the following limiting system:

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} =\beta_1 e^{-mI(t-\tau)}(N^* - I(t) - R(t))I(t) + \beta_2 e^{-mI(t-\tau)}(N^* - I(t) - R(t))B(t) - (\mu + \gamma)I(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} =\gamma I(t) - (\mu + \sigma)R(t), \\ \dfrac{\mathrm{d}B(t)}{\mathrm{d}t} =\xi I(t) - \delta B(t). \end{cases} \tag{1.2}$$

The remainder of this work is organized as follows. In Section 2, we first give the well-posedness of system (1.2) and then establish its threshold dynamics concerning the basic reproduction number. In Section 3, we are devoted to the stability and Hopf bifurcation analysis of the positive equilibrium. In Section 4, we investigate the global continuation of a local branch. In Section 5, we perform numerical simulations to illustrate our analytical results. The last section is a brief discussion.

2. Well-posedness of the model and threshold dynamics

Let $C := C([-\tau, 0], \mathbb{R}^3)$ and $C^+ := C([-\tau, 0], \mathbb{R}^3_+)$. Then (C, C^+) is an ordered Banach space equipped with the supremum norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$ for $\phi \in C$. For any given continuous function $u = (u_1, u_2, u_3) : [-\tau, \rho) \to \mathbb{R}^3$ with $\rho > 0$, we define $u_t \in C$ by

$$u_t(\theta) = (u_1(t + \theta), u_2(t + \theta), u_3(t + \theta)), \quad \forall\theta \in [-\tau, 0],$$

for any $t \in [0, \rho)$. Set

$$\Gamma = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C^+ : \ \varphi_1(\theta) + \varphi_2(\theta) \leq N^*, \ \forall\theta \in [-\tau, 0]\}.$$

We then have the following result.

Lemma 2.1. *For any $\varphi \in \Gamma$, system* (1.2) *admits a unique nonnegative bounded solution $u(t, \varphi)$ on $[0, \infty)$ with $u_0 = \varphi$, and $u_t(\varphi) \in \Gamma$, $\forall t \geq 0$.*

Proof. For any $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \Gamma$, we define

$$f(\varphi) = \begin{pmatrix} e^{-m\varphi_1(-\tau)}(N^* - \varphi_1(0) - \varphi_2(0))(\beta_1\varphi_1(0) + \beta_2\varphi_3(0)) - (\mu + \gamma)\varphi_1(0) \\ \gamma\varphi_1(0) - (\mu + \sigma)\varphi_2(0) \\ \xi\varphi_1(0) - \delta\varphi_3(0) \end{pmatrix}.$$

Obviously, $f(\varphi)$ is continuous in $\varphi \in \Gamma$ and Lipschitz in φ on each compact subset of Γ. It follows from [19, Theorems 2.2.3] that system (1.2) has a unique solution $u(t, \varphi)$ on its maximal existence interval $[0, t_\varphi)$ with $u_0 = \varphi$.

For any given $\varphi \in \Gamma$, one sees that if $\varphi_i(0) = 0$ for some $i \in \{1, 2, 3\}$, then $f_i(\varphi) \geq 0$. By [20, Theorem 5.2.1], we obtain $u(t, \varphi) \geq 0$ for all $t \in [0, t_\varphi)$. Let $H(t) = I(t) + R(t)$. Then one has

$$\begin{aligned} \frac{\mathrm{d}H(t)}{\mathrm{d}t}\Big|_{H(t)=N^*} &= [\beta_1 e^{-mI(t-\tau)}(N^* - H(t))I(t) + \beta_2 e^{-mI(t-\tau)}(N^* - H(t))B(t) - \mu H(t) - \sigma R(t)]\Big|_{H(t)=N^*} \\ &= -\mu N^* - \sigma R(t) \leq 0, \quad \forall t \in (0, t_\varphi), \end{aligned}$$

which implies that $I_t + R_t \leq N^*$ for $t \in [0, t_\varphi)$, and hence, $u_t(\varphi) \in \Gamma$ for all $t \in [0, t_\varphi)$. Moreover, by the third equation of (1.2), one easily sees that $u_3(t, \varphi)$ is bounded on $[0, t_\varphi)$. As a result, [19, Theorem 2.3.1] means that $t_\varphi = \infty$.

Note that the disease-free equilibrium of (1.2) is $E_0 = (0,0,0)$. Since the R class do not participate in the transmission of cholera, we only consider the infected equations of the linearization of (1.2) at E_0:

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} =\beta_1 N^* I(t) + \beta_2 N^* B(t) - (\mu+\gamma)I(t), \\ \dfrac{\mathrm{d}B(t)}{\mathrm{d}t} =\xi I(t) - \delta B(t). \end{cases} \tag{2.1}$$

Based on the next generation matrix method [21], we define

$$F = \begin{pmatrix} \beta_1 N^* & \beta_2 N^* \\ 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} \mu+\gamma & 0 \\ -\xi & \delta \end{pmatrix}.$$

Hence, the basic reproduction number of system (1.2) is given by

$$\mathcal{R}_0 = r(FV^{-1}) = \mathcal{R}_1^d + \mathcal{R}_2^i = \frac{\beta_1 N^*}{\mu+\gamma} + \frac{\beta_2 \xi N^*}{\delta(\mu+\gamma)},$$

where $r(FV^{-1})$ is the spectral radius of FV^{-1}, and $\mathcal{R}_1^d$ (resp. $\mathcal{R}_2^i$ is the basic reproduction number for the direct (resp. indirect) transmission in the absence of indirect (resp. direct) transmission.

Now we begin to study the local stability of the disease-free equilibrium. The Jacobian matrix of system (1.2) at E_0 has the form

$$\begin{pmatrix} \beta_1 N^* - \mu - \gamma & 0 & \beta_2 N^* \\ \gamma & -\mu-\sigma & 0 \\ \xi & 0 & -\delta \end{pmatrix},$$

and the corresponding characteristic equation is

$$\lambda^3 + b\lambda^2 + c\lambda + d = 0, \tag{2.2}$$

where

$$\begin{aligned} b &= \delta + \mu + \sigma + (1-\mathcal{R}_0)(\mu+\gamma) + \frac{\beta_2 \xi N^*}{\delta}, \\ c &= (\mu+\sigma+\delta)(\mu+\gamma)(1-\mathcal{R}_0) + (\mu+\sigma)\left(\frac{\beta_2 \xi N^*}{\delta} + \delta\right), \\ d &= \delta(\mu+\sigma)(\mu+\gamma)(1-\mathcal{R}_0). \end{aligned}$$

When $\mathcal{R}_0 < 1$, one easily sees

$$b > 0, \ c > 0, \ d > 0, \ bc - d > 0.$$

Using the Routh-Hurwitz criterion, we know that all roots of (2.2) have negative real parts. Hence, E_0 is locally stable if $\mathcal{R}_0 < 1$. In the case where $\mathcal{R}_0 > 1$, the Eq (2.2) has at least one positive root since $d < 0$, and hence, E_0 is unstable. Now we proceed to the global stability of E_0.

Theorem 2.1. *If $\mathcal{R}_0 < 1$, then E_0 is globally asymptotically stable in Γ.*

Proof. From the second and third equations of (1.2), we see

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} \leq \beta_1 N^* I(t) + \beta_2 N^* B(t) - (\mu+\gamma) I(t), \\ \dfrac{\mathrm{d}B(t)}{\mathrm{d}t} = \xi I(t) - \delta B(t), \end{cases}$$

which is equivalent to

$$\begin{pmatrix} \frac{\mathrm{d}I(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}B(t)}{\mathrm{d}t} \end{pmatrix} \leq (F-V) \begin{pmatrix} I(t) \\ B(t) \end{pmatrix}. \tag{2.3}$$

Since the matrix $V^{-1}F$ is nonnegative, irreducible and $\mathcal{R}_0 = r(FV^{-1}) = r(V^{-1}F)$, there exists a positive left eigenvector v such that

$$vV^{-1}F = \mathcal{R}_0 v.$$

Define the functional $L: \Gamma \to \mathbb{R}$ as follows

$$L(\phi) = vV^{-1}(\phi_1(0), \phi_3(0))^{\mathrm{T}}, \quad \forall \phi = (\phi_1, \phi_2, \phi_3) \in \Gamma.$$

For any solution $u(t,\varphi)$ of (1.2) with $u_0 = \varphi \in \Gamma$, with the help of (2.3), direct calculation yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} L(u_t(\varphi)) &= vV^{-1}\Big(\frac{\mathrm{d}I(t)}{\mathrm{d}t}, \frac{\mathrm{d}B(t)}{\mathrm{d}t}\Big)^{\mathrm{T}} \leq vV^{-1}(F-V)(I(t), B(t))^{\mathrm{T}} \\ &= (\mathcal{R}_0 - 1) v (I(t), B(t))^{\mathrm{T}} \\ &\leq 0. \end{aligned} \tag{2.4}$$

Thus, L is a Lyapunov functional on Γ relative to system (1.2).

Let M be the largest compact invariant subset in the set $\{\phi \in \Gamma : \dot{L}_{(1.2)}(\phi) = 0\}$, where $\dot{L}_{(1.2)}(\phi)$ denotes the derivative of L along the solution of (1.2). It is easy to see from (2.4) that $\dot{L}_{(1.2)}(\phi) = 0$ implies that $I(t) = B(t) = 0$ for any $t \geq 0$, and hence, $M = \{\phi \in \Gamma : \phi_1 = \phi_3 = 0\}$. It then follows from the LaSalle invariance principle (see, e.g., [22, Theorem 1]) that $\lim_{t\to\infty}(I(t), B(t)) = (0,0)$ and further, $\lim_{t\to\infty} R(t) = 0$. Therefore, E_0 is globally attractive in Γ. This, together with the local stability of E_0, gives the desired result.

Next we are ready to show that the disease is uniformly persistent when $\mathcal{R}_0 > 1$. To this end, we need the following lemma.

Lemma 2.2. *For any given $\varphi \in \Gamma$, if there exists some $t_0 \geq 0$ such that $I(t_0,\varphi) > 0$ or $B(t_0,\varphi) > 0$, then $I(t,\varphi) > 0$ and $B(t,\varphi) > 0$ for all $t > t_0$.*

Proof. If $I(t_0,\varphi) > 0$ for some t_0, then

$$I(t,\varphi) > I(t_0,\varphi) e^{-(\mu+\gamma)(t-t_0)} > 0, \quad \forall t \geq t_0.$$

By the third equation of (1.2), we have

$$B(t,\varphi) = e^{-\delta(t-t_0)} B(t_0,\varphi) + \int_{t_0}^{t} e^{-\delta(t-s)} \xi I(s,\varphi) \mathrm{d}s > 0, \quad \forall t > t_0.$$

Similarly, if $B(t_0, \varphi) > 0$ for some t_0, we obtain that $B(t, \varphi) > 0$ for all $t \geq t_0$. We will show that $H(t, \varphi) := I(t, \varphi) + R(t, \varphi) < N^*$ for all $t > t_0$. If the assertion was false, then there exists $t_1 \in (t_0, \infty)$ such that

$$H(t, \varphi) < N^*, \ \forall t \in (t_0, t_1) \ \text{ and } \ H(t_1, \varphi) = N^*.$$

Clearly, $\frac{\mathrm{d}H(t,\varphi)}{\mathrm{d}t}\Big|_{t=t_1} \geq 0$ must hold. However, in view of the first two equations of (1.2), we find

$$\frac{\mathrm{d}H(t, \varphi)}{\mathrm{d}t}\Big|_{t=t_1} = -\mu N^* - \sigma R(t_1, \varphi) < 0.$$

This contradiction means that no such t_1 can exist. Accordingly, by the I-equation in (1.2), we have

$$\begin{aligned} I(t, \varphi) \geq & e^{-(\mu+\gamma)(t-t_0)} I(t_0, \varphi) \\ & + \int_{t_0}^{t} e^{-(\mu+\gamma)(t-s)} \beta_2 e^{-mI(s-\tau,\varphi)} (N^* - I(s, \varphi) - R(s, \varphi)) B(s, \varphi) \mathrm{d}s > 0 \end{aligned}$$

for all $t > t_0$. Summarizing these two cases, we establish the desired result.

Theorem 2.2. *If $\mathcal{R}_0 > 1$, then system* (1.2) *admits a unique componentwise positive equilibrium. Moreover, there exists $\eta > 0$ such that for any $\varphi \in \Gamma$ with $\varphi_1(0) \neq 0$ or $\varphi_3(0) \neq 0$, the solution $u(t, \varphi) = (I(t, \varphi), R(t, \varphi), B(t, \varphi))$ satisfies*

$$\liminf_{t \to \infty} \min\{I(t, \varphi), B(t, \varphi)\} \geq \eta. \tag{2.5}$$

Proof. Define the following two sets:

$$\begin{aligned} \Gamma_0 &= \{\varphi \in \Gamma : \ \varphi_1(0) > 0 \text{ and } \varphi_3(0) > 0\}, \\ \partial\Gamma_0 &= \Gamma \setminus \Gamma_0 = \{\varphi \in \Gamma : \ \varphi_1(0) = 0 \text{ or } \varphi_3(0) = 0\}. \end{aligned}$$

By the form of (1.2), it can be verified that both Γ and Γ_0 are positively invariant. Clearly, $\partial\Gamma_0$ is relatively closed in Γ. Let $\Phi(t)$ be the solution maps of system (1.2), namely

$$\Phi(t)\varphi = u_t(\varphi), \quad \forall t \geq 0, \ \varphi \in \Gamma.$$

Note that for each $t > \tau$, $\Phi(t)$ is continuous and compact (see [19, Theorem 3.6.1]). The ultimate boundedness of solutions implies that $\Phi(t)$ is point dissipative. It then follows from [23, Theorem 3.4.8] that $\Phi(t)$ has a global attractor K.

Let $\omega(\varphi)$ be the omega limit set of the forward orbit through φ for $\Phi(t)$ and define

$$M_\partial = \{\varphi \in \partial\Gamma_0 : \ \Phi(t)\varphi \in \partial\Gamma_0, \forall t \geq 0\}.$$

Then the following claim holds true.

Claim 1. E_0 is globally stable for $\Phi(t)$ in M_∂.

For any given $\varphi \in M_\partial$, we have $\Phi(t)\varphi \in \partial\Gamma_0, \forall t \geq 0$. Hence, for each $t \geq 0$, either $I(t, \varphi) = 0$ or $B(t, \varphi) = 0$. We further show

$$M_\partial \subset M_0 := \{\varphi \in \partial\Gamma_0 : \ \varphi_1(0) = 0 \text{ and } \varphi_3(0) = 0\}. \tag{2.6}$$

Suppose to the contrary that $\varphi \notin M_0$. Then either $\varphi_1(0) = I(0,\varphi) > 0$ or $\varphi_3(0) = B(0,\varphi) > 0$. With the help of Lemma 2.2, we have that $I(t,\varphi) > 0$ and $B(t,\varphi) > 0$ for all $t > 0$, which contradicts the fact that $\varphi \in M_\partial$. Thus, (2.6) holds. It then follows that $I(t,\varphi) = 0$ and $B(t,\varphi) = 0$ for all $\varphi \in M_\partial$ and $t \geq 0$, and further, $\lim_{t\to\infty} R(t,\varphi) = 0$. Therefore, $\omega(\varphi) = E_0$ for any $\varphi \in M_\partial$. In other words, E_0 is globally attractive for $\Phi(t)$ in M_∂. Note that system (1.2) restricted to M_∂ becomes $\frac{\mathrm{d}R(t)}{\mathrm{d}t} = -(\mu+\sigma)R(t)$. In view of [24, Lemma 2.2.1], E_0 is locally Lyapunov stable for $\Phi(t)$ in M_∂. So, the above claim is proved.

Since $\mathcal{R}_0 > 1$, we can choose $\epsilon > 0$ small enough such that

$$\mathcal{R}_0^\epsilon = \frac{e^{-m\epsilon}(\beta_1 + \beta_2\frac{\xi}{\delta})(N^* - 2\epsilon)}{\mu+\gamma} > 1.$$

Claim 2. $\limsup_{t\to\infty} \|\Phi(t)\varphi - E_0\| \geq \epsilon, \ \ \forall \varphi \in \Gamma_0$.

Suppose the claim is not true. Then there exists some $\psi \in \Gamma_0$ such that

$$\limsup_{t\to\infty} \|\Phi(t)\psi - E_0\| < \epsilon.$$

Thus, there is $\bar{t} > 0$ such that

$$0 < I(t,\psi), R(t,\psi), B(t,\psi) < \epsilon, \quad \forall t \geq \bar{t}.$$

It follows that for all $t \geq \bar{t} + \tau$, we have

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} \geq \beta_1 e^{-m\epsilon}(N^* - 2\epsilon)I(t) + \beta_2 e^{-m\epsilon}(N^* - 2\epsilon)B(t) - (\mu+\gamma)I(t), \\ \dfrac{\mathrm{d}B(t)}{\mathrm{d}t} = \xi I(t) - \delta B(t). \end{cases}$$

Let $(x(t), y(t))$ be the solution of the following system:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \beta_1 e^{-m\epsilon}(N^* - 2\epsilon)x(t) + \beta_2 e^{-m\epsilon}(N^* - 2\epsilon)y(t) - (\mu+\gamma)x(t), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \xi x(t) - \delta y(t), \end{cases} \tag{2.7}$$

and its Jacobian matrix is

$$M_\epsilon = \begin{pmatrix} \beta_1 e^{-m\epsilon}(N^* - 2\epsilon) - \mu - \gamma & \beta_2 e^{-m\epsilon}(N^* - 2\epsilon) \\ \xi & -\delta \end{pmatrix}.$$

Recall that the stability modulus of M_ϵ is defined by

$$s(M_\epsilon) := \max\{\mathrm{Re}\lambda : \lambda \text{ is an eigenvalue of } M_\epsilon\}.$$

Since M_ϵ is quasi-positive and irreducible, it follows from [20, Corollary 3.2] that $s(M_\epsilon)$ is a simple eigenvalue of M_ϵ with a strongly positive eigenvector v_ϵ. In particular, $e^{s(M_\epsilon)t}v_\epsilon$ is a solution of (2.7). By using the proof of [21, Theorem 2], we know that $s(M_\epsilon) > 0$ if and only if $\mathcal{R}_0^\epsilon > 1$.

Since $(I(t,\psi), B(t,\psi)) \gg (0,0)$ for all $t > 0$, there exists $\kappa > 0$ such that

$$(I(t,\psi), B(t,\psi)) \geq \kappa e^{s(M_\epsilon)t} v_\epsilon, \quad \forall t \in [\bar{t}, \bar{t} + \tau].$$

Hence by comparison,

$$(I(t,\psi), B(t,\psi)) \geq \kappa e^{s(M_\epsilon)t} v_\epsilon, \quad \forall t \geq \bar{t} + \tau.$$

In view of $s(M_\epsilon) > 0$, we see that $I(t,\psi)$ and $B(t,\psi)$ tend to infinity as $t \to \infty$. This contradicts the boundedness of solutions, and thus the claim holds.

Claim 1 shows that E_0 cannot form a cycle in $\partial\Gamma_0$, and claim 2 implies that E_0 is an isolated invariant set in Γ and $W^s(E_0) \cap \Gamma_0 = \emptyset$, where $W^s(E_0)$ is the stable set of E_0 for $\Phi(t)$. According to the acyclicity theorem of the uniform persistence of maps (see, e.g., [24, Theorem 1.3.1, Remark 1.3.1 and Remark 1.3.2]), we conclude that $\Phi(t) : \Gamma \to \Gamma$ is uniformly persistent with respect to $(\Gamma_0, \partial\Gamma_0)$. Moreover, by [25, Theorem 2.4], there exists a global attractor A_0 for $\Phi(t)$ in Γ_0 and system (1.2) has an stationary coexistence state $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3) \in \Gamma_0$, and $\Phi(t)\bar{\varphi} = \bar{\varphi}$ for all $t \geq 0$. Noting that $\bar{\varphi}$ is a constant function, we let $\bar{I} = \bar{\varphi}_1(0)$, $\bar{R} = \bar{\varphi}_2(0)$ and $\bar{B} = \bar{\varphi}_3(0)$. Then $\bar{R} \in \mathbb{R}_+$ and $(\bar{I}, \bar{B}) \in \mathrm{int}(\mathbb{R}^2_+)$. We further claim that $\bar{R} \in \mathbb{R}_+ \setminus \{0\}$. Suppose that $\bar{R} = 0$. By the R-equation in (1.2), we then get $0 = \gamma\bar{I}$, and hence, $\bar{I} = 0$, a contradiction. Therefore, $E_1 = (\bar{I}, \bar{R}, \bar{B})$ is a componentwise positive equilibrium of (1.2).

Next, we show the uniqueness of the positive equilibrium. Suppose that $(\hat{I}, \hat{R}, \hat{B})$ is the equilibrium of (1.2). Then $(\hat{I}, \hat{R}, \hat{B})$ satisfies

$$\begin{cases} \beta_1 e^{-m\hat{I}}(N^* - \hat{I} - \hat{R})\hat{I} + \beta_2 e^{-m\hat{I}}(N^* - \hat{I} - \hat{R})\hat{B} - (\mu + \gamma)\hat{I} = 0, \\ \gamma\hat{I} - (\mu + \sigma)\hat{R} = 0, \\ \xi\hat{I} - \delta\hat{B} = 0. \end{cases} \tag{2.8}$$

Set $\hat{S} = N^* - \hat{I} - \hat{R}$. A direct verification yields

$$\Lambda - \beta_1 e^{-m\hat{I}}\hat{S}\,\hat{I} - \beta_2 e^{-m\hat{I}}\hat{S}\,\hat{B} - \mu\hat{S} + \sigma\hat{R} = 0.$$

This implies that $(\hat{S}, \hat{I}, \hat{R}, \hat{B})$ is the equilibrium of (1.1). In view of (2.8), we see that

$$\hat{S} = \mathcal{F}(\hat{I}) := N^* - \hat{I} - \frac{\gamma}{\mu + \sigma}\hat{I} \quad \text{and} \quad \hat{S} = \mathcal{G}(\hat{I}) := \frac{\mu + \gamma}{\beta_1 e^{-m\hat{I}} + \beta_2 e^{-m\hat{I}}\xi/\delta}.$$

Therefore, $\mathcal{F}(\hat{I})$ is strictly decreasing in $\hat{I} \in \mathbb{R}_+$, and $\mathcal{G}(\hat{I})$ is strictly increasing in $\hat{I} \in \mathbb{R}_+$. If $\mathcal{R}_0 > 1$, then $\mathcal{F}(0) > \mathcal{G}(0)$, which means that there is a unique intersection in $\mathbb{R}^2_+$ between $\mathcal{F}(\hat{I})$ and $\mathcal{G}(\hat{I})$, and thus $(\hat{I}, \hat{R}, \hat{B})$ is the unique positive equilibrium of (1.2). Moreover, the uniqueness of positive equilibrium also implies that of E_1.

Finally, to derive the practical persistence, we define a continuous function $p : \Gamma \to \mathbb{R}_+$ by

$$p(\varphi) = \min\{\varphi_1(0), \varphi_3(0)\}, \quad \forall \varphi \in \Gamma.$$

Clearly, $\Gamma_0 = p^{-1}(0,\infty)$ and $\partial\Gamma_0 = p^{-1}(0)$. Since A_0 is a compact subset of Γ_0, it follows that $\inf_{\varphi\in A_0} = \min_{\varphi\in A_0} p(\varphi) > 0$. Consequently, there exists $\eta > 0$ such that

$$\liminf_{t\to\infty} \min\{I(t,\varphi), B(t,\varphi)\} = \liminf_{t\to\infty} p(\Phi(t)\varphi) \geq \eta, \quad \forall \varphi \in \Gamma_0.$$

Moreover, for any given $\varphi \in \Gamma$ with $\varphi_1(0) \neq 0$ or $\varphi_3(0) \neq 0$, Lemma 2.2 suggests that $I(t,\varphi) > 0$ and $B(t,\varphi) > 0$ for all $t > 0$. Fix a $t_0 > \tau$. By using the fact $\Phi(t)\varphi = \Phi(t - t_0)(\Phi(t_0)\varphi), \forall t > t_0$, we see that statement (2.5) holds true.

Remark 2.1. *The uniqueness of the positive equilibrium E_1 can be obtained directly. In fact, from* (1.2) *we get*

$$e^{\frac{m(\mu+\sigma)}{\mu+\sigma+\gamma}\frac{\Lambda}{\mu}-m\bar{I}}\left(\frac{m(\mu+\sigma)}{\mu+\sigma+\gamma}\frac{\Lambda}{\mu}-m\bar{I}\right)=\frac{m(\mu+\sigma)(\mu+\gamma)e^{\frac{m(\mu+\sigma)}{\mu+\sigma+\gamma}\frac{\Lambda}{\mu}}}{(\beta_1+\beta_2\frac{\xi}{\delta})(\mu+\sigma+\gamma)},\quad \bar{R}=\frac{\gamma}{\mu+\sigma}\bar{I},\quad \bar{B}=\frac{\xi}{\delta}\bar{I}.$$

Solving the above equation with respect to $\bar{I}$ gives rise to

$$\bar{I}=\frac{\mu+\sigma}{\mu+\sigma+\gamma}\frac{\Lambda}{\mu}-\frac{1}{m}\text{Lambert W}\left(\frac{m(\mu+\sigma)(\mu+\gamma)e^{\frac{m(\mu+\sigma)}{\mu+\sigma+\gamma}\frac{\Lambda}{\mu}}}{(\beta_1+\beta_2\frac{\xi}{\delta})(\mu+\sigma+\gamma)}\right),$$

where the definition of Lambert W function is seen in [26]. It is positive (i.e. $\bar{I}>0$) provided

$$\frac{m(\mu+\sigma)}{\mu+\sigma+\gamma}\frac{\Lambda}{\mu}>\text{Lambert W}\left(\frac{m(\mu+\sigma)(\mu+\gamma)e^{\frac{m(\mu+\sigma)}{\mu+\sigma+\gamma}\frac{\Lambda}{\mu}}}{(\beta_1+\beta_2\frac{\xi}{\delta})(\mu+\sigma+\gamma)}\right),$$

which is equivalent to $\mathcal{R}_0>1$. In addition, this explicit expression makes the numerical approximation of $(\bar{I},\bar{R},\bar{B})$ more convenient.

3. Stability analysis and local Hopf bifurcations

In this section, we will consider the global stability of the endemic equilibrium E_1 and the existence of local Hopf bifurcations of system (1.2).

3.1. Stability of E_1 for system (1.2) without delay

The linearized form of system (1.2) at $E_1=(\bar{I},\bar{R},\bar{B})$ is given by

$$\begin{cases}\dfrac{\mathrm{d}I(t)}{\mathrm{d}t}=\beta_1AI(t)-CI(t)-(\mu+\gamma)I(t)-m(\mu+\gamma)\bar{I}I(t-\tau)-CR(t)+\beta_2AB(t),\\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t}=\gamma I(t)-(\mu+\sigma)R(t),\\ \dfrac{\mathrm{d}B(t)}{\mathrm{d}t}=\xi I(t)-\delta B(t),\end{cases}\tag{3.1}$$

where $A=e^{-m\bar{I}}(N^*-\bar{I}-\bar{R})$ and $C=\beta_1e^{-m\bar{I}}\bar{I}+\beta_2e^{-m\bar{I}}\bar{B}$. The characteristic equation of system (3.1) is

$$p(\lambda,\tau):=\lambda^3+a_0\lambda^2+a_1\lambda+a_2+e^{-\lambda\tau}(b_0\lambda^2+b_1\lambda+b_2)=0,\tag{3.2}$$

where

$$\begin{aligned}&a_0=\mu+\sigma+C+\beta_2A\frac{\xi}{\delta}+\delta, && a_1=C(\gamma+\delta)+(\mu+\sigma)\left(\beta_2A\frac{\xi}{\delta}+\delta+C\right),\\ &a_2=C\delta(\mu+\sigma+\gamma), && b_0=m(\mu+\gamma)\bar{I},\\ &b_1=m(\mu+\gamma)(\mu+\sigma+\delta)\bar{I}, && b_2=m\delta(\mu+\gamma)(\mu+\sigma)\bar{I}.\end{aligned}$$

When $\tau=0$, the equation (3.2) becomes

$$\lambda^3+(a_0+b_0)\lambda^2+(a_1+b_1)\lambda+a_2+b_2=0.\tag{3.3}$$

In view of $a_0 + b_0 > 0$ and

$$(a_0 + b_0)(a_1 + b_1) - (a_2 + b_2)$$
$$= X^2(\mu + \sigma) + (\mu + \sigma)^2 X + C\gamma\Big(\beta_2 A\frac{\xi}{\delta} + C + m(\mu + \gamma)\bar{I} + \mu + \sigma\Big) + CX\delta + m\delta(\mu + \sigma)\bar{I}X > 0,$$

where $X = \beta_2 A\frac{\xi}{\delta} + \delta + C + m(\mu + \gamma)\bar{I}$, we deduce from the Routh-Hurwitz criterion that all roots of (3.3) have negative real parts. Hence, the following result holds true.

Theorem 3.1. *Suppose that $\mathcal{R}_0 > 1$. Then the endemic equilibrium E_1 of system* (1.2) *is locally asymptotically stable when $\tau = 0$.*

Now we present the geometric approach developed in [27] to study the global stability of E_1. Consider the following autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \quad x \in D,\ f \in C^1, \tag{3.4}$$

where $D \subset \mathbb{R}^n$ is a simply connected open set. Let $x(t, x_0)$ be the solution of (3.4) such that $x(0, x_0) = x_0$. The second compound equation with respect to $x(t, x_0) \in D$ is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f^{[2]}}{\partial x}(x(t, x_0))z,$$

where $\frac{\partial f^{[2]}}{\partial x}$ is the additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$ (see [28]).

Let $P(x)$ be a $k \times k$ matrix-valued C^1 function, where $k = \frac{1}{2}n(n-1)$, and suppose $P^{-1}(x)$ exists for $x \in D$. Set

$$Q = P_f P^{-1} + P\frac{\partial f^{[2]}}{\partial x}P^{-1},$$

where P_f is the matrix obtained by replacing each entry p_{ij} in P with its directional derivative in the direction of f. The Lozinskiĭ measure of Q with respect to the matrix norm $|\cdot|$ in $\mathbb{R}^{n\times n}$ is defined as

$$\mu(Q) = \lim_{h\to 0^+} \frac{|I + hQ| - 1}{h}.$$

Define a quantity $\bar{q}_2$ as

$$\bar{q}_2 = \limsup_{t\to\infty} \sup_{x_0\in K} \frac{1}{t}\int_0^t \mu(Q(x(s, x_0)))\mathrm{d}s,$$

where K denotes a compact absorbing set in D. The following result from [27, Theorem 3.5] is used to prove the global stability of E_1.

Lemma 3.1. *Assume that system* (3.4) *has a unique equilibrium x^* in D. Then x^* is globally stable in D if $\bar{q}_2 < 0$.*

Theorem 3.2. *If $\mathcal{R}_0 > 1$, $\tau = 0$ and $\mu + \gamma > \sigma$, then E_1 is globally asymptotically stable in $\Omega = \{(I, R, B) \in \mathbb{R}^3_+ :\ I + R \le N^*,\ I > 0,\ B > 0\}$.*

Proof. Note that system (1.2) is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} =\Lambda - \beta_1 e^{-mI(t)}S(t)I(t) - \beta_2 e^{-mI(t)}S(t)B(t) - \mu S(t) + \sigma(N^* - S(t) - I(t)), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} =\beta_1 e^{-mI(t)}S(t)I(t) + \beta_2 e^{-mI(t)}S(t)B(t) - (\mu+\gamma)I(t), \\ \dfrac{\mathrm{d}B(t)}{\mathrm{d}t} =\xi I(t) - \delta B(t). \end{cases} \tag{3.5}$$

Thus, under the condition $\mathcal{R}_0 > 1$, the global stability of E_1 is equivalent to that of the unique positive equilibrium $\hat{E}_1$ of system (3.5), where $\hat{E}_1 = (N^* - \bar{I} - \bar{R}, \bar{I}, \bar{B})$. Define

$$\tilde{\Omega} = \{(S, I, B) \in \mathbb{R}^3_+ : \ S + I \le N^*, \ I > 0, \ B > 0\}.$$

Clearly, $\tilde{\Omega}$ is a simply connected region in $\mathbb{R}^3$. Similar to the arguments in Theorem 2.2, one can prove that system (3.5) is uniformly persistent in $\tilde{\Omega}$. This, together with the boundedness of solutions to (3.5), implies the existence of a compact absorbing set $K \subset \tilde{\Omega}$. Therefore, by Lemma 3.1, it suffices to show that there exists a matrix-valued function $P(x)$ such that $\bar{q}_2 < 0$.

Define the diagonal matrix P as

$$P(S, I, B) = \mathrm{diag}\left(1, \frac{I}{B}, \frac{I}{B}\right).$$

Then P is C^1 and nonsigular in $\tilde{\Omega}$. Let f denote the vector field of (3.5). Then

$$P_f P^{-1} = \mathrm{diag}\left(0, \frac{I'}{I} - \frac{B'}{B}, \frac{I'}{I} - \frac{B'}{B}\right),$$

where $'$ is the derivative with respect to time t. The Jacobian matrix J associated with a general solution $(S(t), I(t), B(t))$ to (3.5) is

$$J = \begin{pmatrix} -g(I,B) - \mu - \sigma & -\beta_1 e^{-mI}S + mS\,g(I,B) - \sigma & -\beta_2 e^{-mI}S \\ g(I,B) & \beta_1 e^{-mI}S - mS\,g(I,B) - \mu - \gamma & \beta_2 e^{-mI}S \\ 0 & \xi & -\delta \end{pmatrix},$$

where $g(I, B) = \beta_1 e^{-mI} I + \beta_2 e^{-mI} B$. The second additive compound matrix of $J = (j_{ik})_{3\times3}$, denoted by $J^{[2]}$, is expressed by

$$\begin{aligned} J^{[2]} &= \begin{pmatrix} j_{11} + j_{22} & j_{23} & -j_{13} \\ j_{32} & j_{11} + j_{33} & j_{12} \\ -j_{31} & j_{21} & j_{22} + j_{33} \end{pmatrix} \\ &= \begin{pmatrix} -g(I,B) - \gamma - 2\mu - \sigma + \beta_1 e^{-mI}S - mS\,g(I,B) & \beta_2 e^{-mI}S & \beta_2 e^{-mI}S \\ \xi & -g(I,B) - \sigma - \mu - \delta & -\beta_1 e^{-mI}S + mS\,g(I,B) - \sigma \\ 0 & g(I,B) & \beta_1 e^{-mI}S - mS\,g(I,B) - \mu - \gamma - \delta \end{pmatrix}. \end{aligned}$$

As a result, the matrix $Q = P_f P^{-1} + P J^{[2]} P^{-1}$ can be written in block form as follows:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix},$$

where

$$
\begin{aligned}
Q_{11} &= -g(I,B) - \gamma - 2\mu - \sigma + \beta_1 e^{-mI} S - mS\, g(I,B),\\
Q_{12} &= \left(\beta_2 e^{-mI} S \frac{B}{I}, \beta_2 e^{-mI} S \frac{B}{I}\right),\\
Q_{21} &= \left(\frac{I}{B}\xi, 0\right)^{\mathrm{T}},\\
Q_{22} &= \begin{pmatrix} -g(I,B) - \sigma - \mu - \delta + \frac{I'}{I} - \frac{B'}{B} & -\beta_1 e^{-mI} S + mS\, g(I,B) - \sigma \\ g(I,B) & \beta_1 e^{-mI} S - mS\, g(I,B) - \mu - \gamma - \delta + \frac{I'}{I} - \frac{B'}{B} \end{pmatrix}.
\end{aligned}
$$

The vector norm $|\cdot|$ in $\mathbb{R}^3$ is taken as

$$
|(u,v,w)| = \max\{|u|, |v| + |w|\}.
$$

As described in [27], we have the following estimate

$$
\mu(Q) \le \sup\{g_1, g_2\},
$$

with

$$
\begin{aligned}
g_1 &= \mu_1(Q_{11}) + |Q_{12}| = -g(I,B) - \gamma - 2\mu - \sigma + \beta_1 e^{-mI} S - mS\, g(I,B) + \beta_2 e^{-mI} S \frac{B}{I},\\
g_2 &= |Q_{21}| + \mu_1(Q_{22}) = \frac{I}{B}\xi + \mu_1(Q_{22}).
\end{aligned}
$$

Here, $|Q_{12}|$ and $|Q_{21}|$ are matrix norms induced by the l_1 norm, and μ_1 denotes the Lozinskiĭ measure concerning the l_1 norm. Moreover, using the method in [29], we calculate $\mu_1(Q_{22})$ as

$$
\begin{aligned}
\mu_1(Q_{22}) &= \max\left\{-\mu - \sigma - \delta + \frac{I'}{I} - \frac{B'}{B}, H(S,I,B) - \mu - \gamma - \delta - \sigma + \frac{I'}{I} - \frac{B'}{B} + |H(S,I,B)|\right\},\\
&= -\mu - \sigma - \delta + \frac{I'}{I} - \frac{B'}{B} + \sup\{0, 2H(S,I,B) - \gamma\},
\end{aligned}
$$

where $H(S,I,B) = \beta_1 e^{-mI} S - mS\, g(I,B) + \sigma$. In view of

$$
\begin{aligned}
g_2 - g_1 &= \mu_1(Q_{22}) + \frac{I}{B}\xi + g(I,B) + \mu + \sigma + mS\, g(I,B) - \frac{I'}{I}\\
&\ge -\mu - \sigma - \delta + \frac{I'}{I} - \frac{B'}{B} + \frac{I}{B}\xi + g(I,B) + \mu + \sigma + mS\, g(I,B) - \frac{I'}{I}\\
&= -\mu - \sigma - \delta - \frac{B'}{B} + \delta + \frac{B'}{B} + g(I,B) + \mu + \sigma + mS\, g(I,B)\\
&= g(I,B) + mS\, g(I,B) > 0,
\end{aligned}
$$

we have

$$
\begin{aligned}
\mu(Q) &\le \sup\{g_1, g_2\} = g_2\\
&= -\mu - \sigma - \delta + \frac{I'}{I} - \frac{B'}{B} + \frac{I}{B}\xi + \sup\{0, 2H(S,I,B) - \gamma\}\\
&= -\mu - \sigma + \frac{I'}{I} + \sup\{0, 2\beta_1 e^{-mI} S - 2mS\, g(I,B) + 2\sigma - \gamma\}\\
&= -\mu - \sigma + \frac{I'}{I} + \sup\{0, 2S(\beta_1 e^{-mI} I)' - 2mS\beta_2 e^{-mI} B + 2\sigma - \gamma\}\\
&\le -\mu - \sigma + \frac{I'}{I} + \sup\{0, 2S(\beta_1 e^{-mI} I)' + 2\sigma - \gamma\}.
\end{aligned}
$$

Define the function

$$\tilde{f}(S,I)=\begin{cases}0, & \text{if } \ 2S(\beta_1 e^{-mI}I)'+2\sigma-\gamma\le 0,\\ 2S(\beta_1 e^{-mI}I)'+2\sigma-\gamma, & \text{if } \ 2S(\beta_1 e^{-mI}I)'+2\sigma-\gamma>0.\end{cases}$$

Along each solution $(S(t), I(t), B(t))$ to (3.5) such that $(S(0), I(0), B(0)) \in K$, we find that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \tilde{f}(S(s),I(s))\mathrm{d}s=0 \ \text{ or } \ 2\sigma-\gamma,$$
$$\limsup_{t\to\infty}\left(\frac{1}{t}\ln\frac{I(t)}{I(0)}-\mu-\sigma\right)=-\mu-\sigma.$$

Here we have used the boundedness of solutions to (3.5). Observe that

$$\frac{1}{t}\int_0^t \mu(Q)\mathrm{d}s\le\frac{1}{t}\ln\frac{I(t)}{I(0)}-\mu-\sigma+\frac{1}{t}\int_0^t \tilde{f}(S(s),I(s))\mathrm{d}s,$$

then $\bar{q}_2$ has two possibilities:

$$\text{(i) } \bar{q}_2\le-\mu-\sigma<0;\quad \text{(ii) } \bar{q}_2\le-\mu-\gamma+\sigma<0 \quad (\text{due to } \mu+\gamma>\sigma),$$

for all $(S(0), I(0), B(0)) \in K$. Therefore, the desired result holds.

3.2. Local Hopf bifurcations

From Theorem 3.1, we know that if $\mathcal{R}_0 > 1$ and $\tau = 0$, E_1 is locally asymptotically stable. Further, 0 cannot be an eigenvalue of (3.2) due to $p(0,\tau) = a_2 + b_2 > 0$ for any $\tau \ge 0$. Therefore, the stability of E_1 changes only when at least a pair of eigenvalues of (3.2) cross the imaginary axis to the right. We thus suppose that $\lambda = \mathrm{i}\omega$ ($\omega > 0$) is a purely imaginary solution of (3.2) for some $\tau > 0$, that is,

$$(-\omega^3+a_1\omega)\mathrm{i}-a_0\omega^2+a_2+e^{-\mathrm{i}\omega\tau}(-b_0\omega^2+\mathrm{i}b_1\omega+b_2)=0.$$

Separating the real and imaginary parts, we obtain

$$-\omega^3+a_1\omega+b_0\omega^2\sin(\omega\tau)+b_1\omega\cos(\omega\tau)-b_2\sin(\omega\tau)=0,$$
$$-a_0\omega^2+a_2-b_0\omega^2\cos(\omega\tau)+b_1\omega\sin(\omega\tau)+b_2\cos(\omega\tau)=0.$$

Equivalently,

$$\begin{cases}\cos(\omega\tau)=G(\omega)=-\dfrac{b_1\omega(a_1\omega-\omega^3)+(b_2-b_0\omega^2)(a_2-a_0\omega^2)}{(b_2-b_0\omega^2)^2+b_1^2\omega^2},\\[2ex] \sin(\omega\tau)=N(\omega)=\dfrac{(b_2-b_0\omega^2)(a_1\omega-\omega^3)-b_1\omega(a_2-a_0\omega^2)}{(b_2-b_0\omega^2)^2+b_1^2\omega^2}.\end{cases} \tag{3.6}$$

Squaring and adding both equations of (3.6) yields

$$\omega^6+p\omega^4+q\omega^2+r=0, \tag{3.7}$$

with

$$p = a_0^2 - 2a_1 - b_0^2, \quad q = a_1^2 - 2a_0a_2 - b_1^2 + 2b_0b_2, \quad r = a_2^2 - b_2^2.$$

Let $x = \omega^2$. Then equation (3.7) becomes

$$x^3 + px^2 + qx + r = 0.$$

Therefore, if $\mathrm{i}\omega$ is a purely imaginary root of (3.2), then the equation

$$h(x) := x^3 + px^2 + qx + r = 0$$

has a positive root $x = \omega^2$. Define the set W as

$$W = \{(p, q, r) \in \mathbb{R}^3 \mid h(x) = 0 \text{ has only one positive real root } x^* \text{ and } h'(x^*) > 0\}.$$

By using the properties for the general cubic equations given in [30, Lemma A.2], we have $(p, q, r) \in W$ if and only if one of the following conditions holds:

(C1) $\Delta = (pq - 9r)^2 - 4(p^2 - 3q)(q^2 - 3pr) > 0, r < 0$;

(C2) $r = 0, q = 0, p < 0$;

(C3) $\Delta < 0, q > 0, p > 0, r < 0$.

In the case where $(p, q, r) \in W$, let $\omega_0 = \sqrt{x^*}$. Solving (3.6) for τ and $\omega = \omega_0$, we obtain

$$\tau_n = \tau_0 + \frac{2n\pi}{\omega_0}, \quad \tau_0 = \begin{cases} \frac{\arccos G(\omega_0)}{\omega_0}, & N(\omega_0) \geq 0, \\ \frac{2\pi - \arccos G(\omega_0)}{\omega_0}, & N(\omega_0) < 0, \end{cases} \quad n = 0, 1, 2, \cdots. \tag{3.8}$$

Then we have the following result.

Theorem 3.3. *Let $\mathcal{R}_0 > 1$. For $(p, q, r) \in W$, E_1 is locally asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Besides, system* (1.2) *undergoes Hopf bifurcation at E_1 when $\tau = \tau_n$, $n = 0, 1, 2, \cdots$.*

Proof. Differentiating both sides of equation (3.2) with respect to τ yields

$$\left[\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\right]^{-1} = \frac{3\lambda^2 + 2a_0\lambda + a_1}{-\lambda^4 - a_0\lambda^3 - a_1\lambda^2 - a_2\lambda} - \frac{\tau}{\lambda} + \frac{2b_0\lambda + b_1}{b_0\lambda^3 + b_1\lambda^2 + b_2\lambda}.$$

Substituting $\tau = \tau_n$ into the above equality, we obtain

$$\begin{aligned} \left[\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}\right]^{-1}\Bigg|_{\tau=\tau_n} =& \mathrm{Re}\left(\frac{a_1 - 3\omega^2 + 2a_0\omega\mathrm{i}}{a_1\omega^2 - \omega^4 + a_0\omega^3\mathrm{i} - a_2\omega\mathrm{i}}\right) - \mathrm{Re}\left(\frac{2b_0\omega\mathrm{i} + b_1}{(b_2\omega - b_0\omega^3\mathrm{i}) - b_1\omega^2}\right) \\ =& \omega^2\frac{3\omega^2 + 2\omega^2(-2a_1 + a_0^2 - b_0^2) + a_1^2 - 2a_0a_2 + 2b_0b_2 - b_1^2}{\omega^2[(b_2 - b_0\omega^2)^2 + b_1^2\omega^2]} \\ =& \frac{\omega^2}{\omega^2[(b_2 - b_0\omega^2)^2 + b_1^2\omega^2]}\frac{\mathrm{d}h(x)}{\mathrm{d}x}\Bigg|_{x^*=\omega^2}. \end{aligned}$$

Since $\omega^2[(b_2 - b_0\omega^2)^2 + b_1^2\omega^2] > 0$, we have

$$\mathrm{sign}\left(\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}\Bigg|_{\tau=\tau_n}\right) = \mathrm{sign}\left(\left[\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}\right]^{-1}\Bigg|_{\tau=\tau_n}\right) = \mathrm{sign}\left(\frac{\mathrm{d}h(x)}{\mathrm{d}x}\Bigg|_{x^*=\omega^2}\right) = 1.$$

Therefore, the transversality condition holds, and so the desired result follows.

4. Global Hopf bifurcation

Theorem 3.3 states that periodic solutions can bifurcate from E_1 when τ is near the local Hopf bifurcation values τ_n, $n = 0, 1, 2, \cdots$. In this section, we use the global Hopf bifurcation theorem (see [31, Therorem 3.2]) to study the global continuation of these locally bifurcating periodic solutions. In the remainder of this section, we always assume that $\mathcal{R}_0 > 1$, $\mu + \gamma > \sigma$ and $(p, q, r) \in W$.

Our arguments are similar to those in [1, 10, 30, 32–35]. Let $z(t) = (I(\tau t), R(\tau t), B(\tau t))^{\mathrm{T}}$. Rewrite system (1.2) as the following functional differential equation

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = F(z_t, \tau, T), \quad (t, \tau, T) \in \mathbb{R}_+ \times (0, \infty) \times \mathbb{R}_+, \tag{4.1}$$

where $z_t \in Y := C([-1, 0], \mathbb{R}^3_+)$ with $z_t(\theta) = z(t + \theta)$ for $\theta \in [-1, 0]$, parameter T is the period of the non-constant periodic solution of (4.1), and

$$F(z_t, \tau, T) = \tau \begin{pmatrix} e^{-mz_{1t}(-1)}(N^* - z_{1t}(0) - z_{2t}(0))(\beta_1 z_{1t}(0) + \beta_2 z_{3t}(0)) - (\mu + \gamma)z_{1t}(0) \\ \gamma z_{1t}(0) - (\mu + \sigma)z_{2t}(0) \\ \xi z_{1t}(0) - \delta z_{3t}(0) \end{pmatrix} \tag{4.2}$$

with $z_t = (z_{1t}, z_{2t}, z_{3t}) \in Y$. Restricting F to the subspace of Y, we get a restricted function

$$\widetilde{F}(z, \tau, T) := F|_{\mathbb{R}^3 \times (0,\infty) \times \mathbb{R}_+} = \tau \begin{pmatrix} e^{-mz_1}(N^* - z_1 - z_2)(\beta_1 z_1 + \beta_2 z_3) - (\mu + \gamma)z_1 \\ \gamma z_1 - (\mu + \sigma)z_2 \\ \xi z_1 - \delta z_3 \end{pmatrix}.$$

Clearly, $\tilde{F}$ is twice continuously differentiable, that is, the assumption (A1) in [31] holds.

By Theorems 2.1 and 2.2, the set of stationary solutions of system (4.1) is given by

$$N(F) = \{(E_0, \tau, T) : \ (\tau, T) \in (0, \infty) \times \mathbb{R}_+\} \cup \{(E_1, \tau, T) : \ (\tau, T) \in (0, \infty) \times \mathbb{R}_+\}.$$

For any stationary solution $(\tilde{z}, \tau, T) \in N(F)$, the characteristic matrix is

$$\begin{aligned} \Delta_{(\tilde{z},\tau,T)}(\lambda) &= \lambda \mathrm{Id} - DF(\tilde{z}, \tau, T)(e^{\lambda \cdot}\mathrm{Id}) \\ &= \begin{pmatrix} -\tau\beta_1\tilde{A} + \tau\tilde{C} + \tau m(\mu + \gamma)\tilde{z}_1 e^{-\lambda} + \tau\gamma + \tau\mu + \lambda & \tau\tilde{C} & -\tau\beta_2\tilde{A} \\ -\tau\gamma & \tau\mu + \tau\sigma + \lambda & 0 \\ -\tau\xi & 0 & \tau\delta + \lambda \end{pmatrix}, \end{aligned}$$

where Id is the 3×3 identity matrix, $\tilde{A} = e^{-m\tilde{z}_1}(N^* - \tilde{z}_1 - \tilde{z}_2)$ and $\tilde{C} = \beta_1 e^{-m\tilde{z}_1}\tilde{z}_1 + \beta_2 e^{-m\tilde{z}_1}\tilde{z}_3$. Thus, for any stationary solution $(\tilde{z}, \tau, T)$, the characteristic equation reads

$$\det \Delta_{(\tilde{z},\tau,T)}(\lambda) = \lambda^3 + \tilde{a}_0\tau\lambda^2 + \tilde{a}_1\tau^2\lambda + \tilde{a}_2\tau^3 + e^{-\lambda}(\tilde{b}_0\tau\lambda^2 + \tilde{b}_1\tau^2\lambda + \tilde{b}_2\tau^3) = 0,$$

where

$$\begin{aligned} \tilde{a}_0 &= \mu + \sigma + \tilde{C} + \beta_2\tilde{A}\frac{\xi}{\delta} + \delta, & \tilde{a}_1 &= \tilde{C}(\gamma + \delta) + (\mu + \sigma)(\beta_2\tilde{A}\frac{\xi}{\delta} + \delta + \tilde{C}), \\ \tilde{a}_2 &= \tilde{C}\delta(\mu + \sigma + \gamma), & \tilde{b}_0 &= m(\mu + \gamma)\tilde{z}_1, \\ \tilde{b}_1 &= m(\mu + \gamma)(\mu + \sigma + \delta)\tilde{z}_1, & \tilde{b}_2 &= m\delta(\mu + \gamma)(\mu + \sigma)\tilde{z}_1. \end{aligned}$$

Under the condition $\mathcal{R}_0 > 1$, 0 cannot be an eigenvalue of any stationary solution of (4.1). Therefore, the condition (A2) in [31] holds. From Eq (4.2), it can be easily verified that the smoothness condition (A3) in [31] is also valid.

As defined in [36], the stationary solution $(\tilde{z}, \tilde{\tau}, \tilde{T})$ of (4.1) is called a center if $\det\Delta_{(\tilde{z},\tilde{\tau},\tilde{T})}(ik\frac{2\pi}{\tilde{T}}) = 0$ for some positive integer k. A center is isolated if no other center exists in some neighborhood of $(\tilde{z}, \tilde{\tau}, \tilde{T})$ and there are only finite pure imaginary eigenvalues of the form $ik\frac{2\pi}{\tilde{T}}$. Let $J(\tilde{z}, \tilde{\tau}, \tilde{T})$ represent the set of all such positive integers k. Theorem 3.3 implies that if $(p, q, r) \in W$, for any integer $n \geq 0$, $(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n})$ is an isolated center of (4.1), and there is only one pure imaginary root of the form $ik\frac{2\pi}{\tilde{T}}$ with $k = 1$ and $\tilde{T} = \frac{2\pi}{\omega_0\tau_n}$. Therefore,

$$J(\tilde{z}, \tilde{\tau}, \tilde{T}) = \{1\}. \tag{4.3}$$

Moreover, it follows from Theorem 3.3 that the crossing number at each of these center is

$$\gamma_1\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right) = -\text{sign}\,(\text{Re}\lambda'(\tau_n)) = -\text{sign}(h'(x^*)) = -1. \tag{4.4}$$

Thus the condition (A4) in [31] holds.

Next, we define a closed subset $\Sigma(F)$ of $Y \times (0, \infty) \times \mathbb{R}_+$ by

$$\Sigma(F) = Cl\{(z, \tau, T) \in Y \times (0, \infty) \times \mathbb{R}_+ : \ z \text{ is a } T\text{-periodic nontrivial solution of (4.1)}\}.$$

For each integer $n \geq 0$, let $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ denote the connected branch of $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ in $\Sigma(F)$. Theorem 3.3 guarantees that $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ is a nonempty subset of $\Sigma(F)$. The global bifurcation theorem [31, Theorem 3.4] means that one of the following holds:

(i) $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ is unbounded in $Y \times (0, \infty) \times \mathbb{R}_+$;

(ii) $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ is bounded, $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right) \cap N(F)$ is finite and

$$\sum_{(\tilde{z},\tau,T)\in C\left(E_1,\tau_n,\frac{2\pi}{\omega_0\tau_n}\right)\cap N(F)} \gamma_k(\tilde{z}, \tau, T) = 0,$$

for all $k = 1, 2, 3, \cdots$, where $\gamma_k(\tilde{z}, \tau, T)$ is the k-th crossing number of $(\tilde{z}, \tau, T)$ if $k \in J(\tilde{z}, \tau, T)$, otherwise, $\gamma_k(\tilde{z}, \tau, T)$ is zero.

For each $n = 1, 2, \cdots$, $(\tilde{z}, \tau, T) \in C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$, based on (4.3) and (4.4), we see

$$\sum_{(\tilde{z},\tau,T)\in C\left(E_1,\tau_n,\frac{2\pi}{\omega_0\tau_n}\right)\cap N(F)} \gamma_k(\tilde{z}, \tau, T) = \gamma_1(\tilde{z}, \tau, T) = -1.$$

Hence, (ii) fails and (i) holds. The following two lemmas help us confirm the boundedness of the projections of $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ on z-space and T-space.

Lemma 4.1. *For initial value $\phi = (\phi_1, \phi_2, \phi_3) \in Y$ with $\phi_1(\theta) + \phi_2(\theta) \leq \frac{\Lambda}{\mu}$, $\forall\theta \in [-1, 0]$, all periodic solutions of system* (4.1) *are uniformly bounded.*

The proof is a direct result of Lemma 2.1, and hence is omitted.

Lemma 4.2. *If $\mathcal{R}_0 > 1$ and $\mu + \gamma > \sigma$, then system* (4.1) *has no periodic solutions of period* 1.

Proof. Suppose that $z(t) = (z_1(t), z_2(t), z_3(t))$ is the periodic solution of system (4.1) with period 1, then $z(t)$ is a periodic solution of the following ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}z_1(t)}{\mathrm{d}t} =\tau e^{-mz_1(t)}(N^* - z_1(t) - z_2(t))(\beta_1 z_1(t) + \beta_2 z_3(t)) - \tau(\mu+\gamma)z_1(t), \\ \dfrac{\mathrm{d}z_2(t)}{\mathrm{d}t} =\tau\gamma z_1(t) - \tau(\mu+\sigma)z_2(t), \\ \dfrac{\mathrm{d}z_3(t)}{\mathrm{d}t} =\tau\xi z_1(t) - \tau\delta z_3(t). \end{cases} \tag{4.5}$$

By Theorem 3.2, system (4.5) has a unique positive equilibrium that is globally stable, and thus no periodic solutions appear. This leads to a contradiction.

Lemma 4.1 shows that for any integer $n \geq 0$, the projection of $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ onto z-space is bounded. Lemma 4.2 implies that system (4.1) also has no periodic solutions of period $\frac{1}{n+1}$ for any $n \geq 0$, Moreover, with the help of (3.8), we obtain

$$\frac{1}{n+1} < \frac{2\pi}{\omega_0\tau_n} < 1, \quad n = 1, 2, \cdots.$$

Therefore, the projection of $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ onto T-space is bounded. Accordingly, the projection of $C\left(E_1, \tau_n, \frac{2\pi}{\omega_0\tau_n}\right)$ onto τ-space is unbounded.

Summarizing the above discussion, we arrive at the following result.

Theorem 4.1. *Assume that $\mathcal{R}_0 > 1$, $\mu + \gamma > \sigma$ and $(p, q, r) \in W$, then for any $\tau > \tau_1$ system* (1.2) *has at least one nontrivial periodic solution.*

5. Numerical simulation

In this section, we carry out numerical simulations to demonstrate the theoretical results. Particularly, the global Hopf branches are computed by a Matlab package DDE-BIFTOOL developed by Engelborghs et al. [37, 38].

For illustrative purpose, we choose the parameters of system (1.1) as follows:

$$\begin{aligned} &\Lambda = 31,\ \beta_1 = 0.09,\ \beta_2 = 0.08,\ \mu = 0.81, \\ &m = 0.1,\ \sigma = 0.3,\ \gamma = 0.1,\ \xi = 0.5,\ \delta = 0.2. \end{aligned}$$

Direct calculation gives $\mathcal{R}_0 = 12.1964 > 1$ and $(p, q, r) \in W$. By using (3.8), we further obtain $\tau_0 = 5.5131$, $\tau_1 = 18.3517$, $\tau_2 = 31.1903, \cdots$. Figure 1 shows that the endemic equilibrium $E_1 = (17.8880, 1.6115, 44.7201)$ is asymptotically stable when $\tau = 4 < \tau_0$, while Figure 2 displays that E_1 loses stability and a Hopf bifurcation occurs when $\tau = 8 > \tau_0$. These numerical results agree with the result of Theorem 3.3.

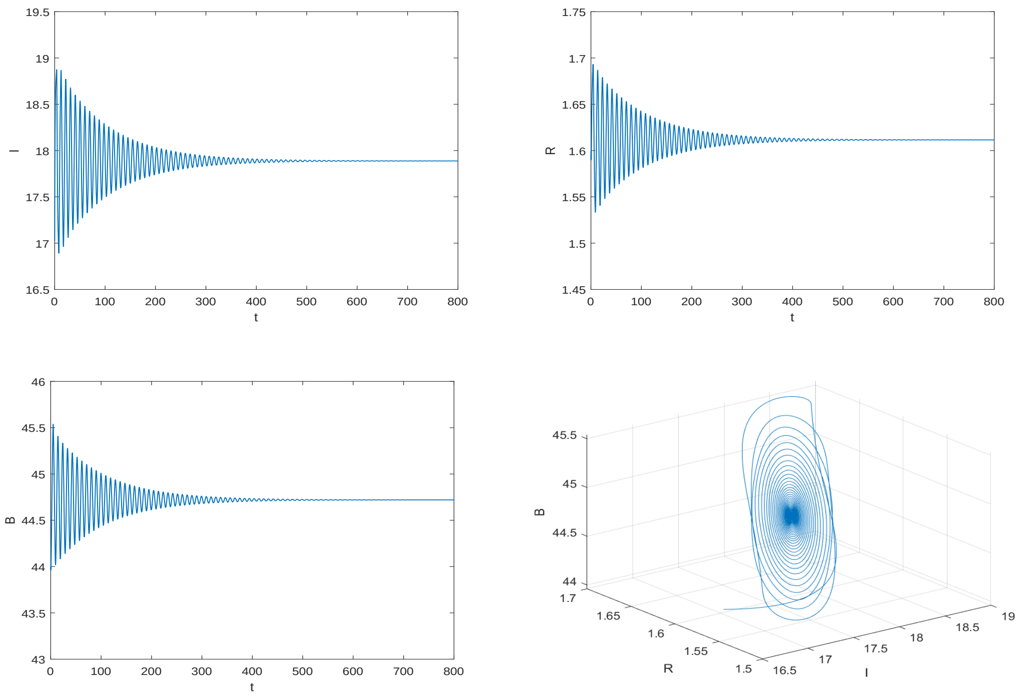

Figure 1. A solution converges to the stable equilibrium E_1 when $\tau = 4 < \tau_0$.

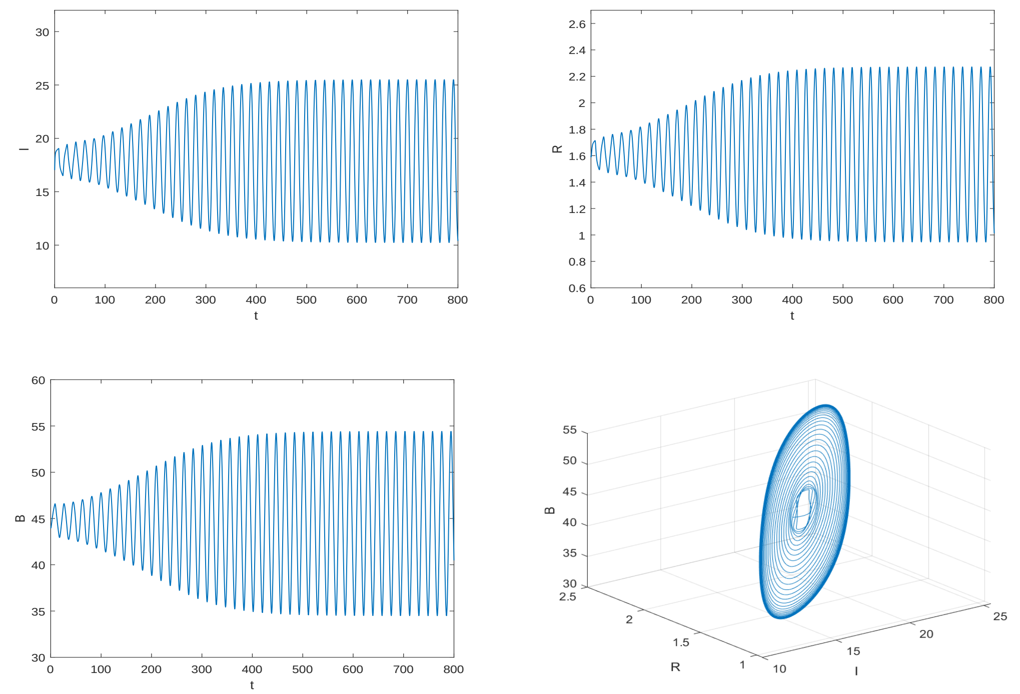

Figure 2. A solution converges to a stable periodic solution when $\tau = 8 > \tau_0$.

Moreover, we depict the global Hopf branches of periodic solutions emanating from the Hopf bifurcation points τ_0, τ_1 and τ_2. As seen in Figure 3, when $\tau_0 < \tau < \tau_1$, system (1.2) has only one periodic solution originating from τ_0. When τ lies between τ_1 and τ_2, the periodic solutions from τ_0 and τ_1 coexist. With the further increase of τ and $\tau > \tau_2$, three periodic solutions originating from τ_0, τ_1 and τ_2 coexist.

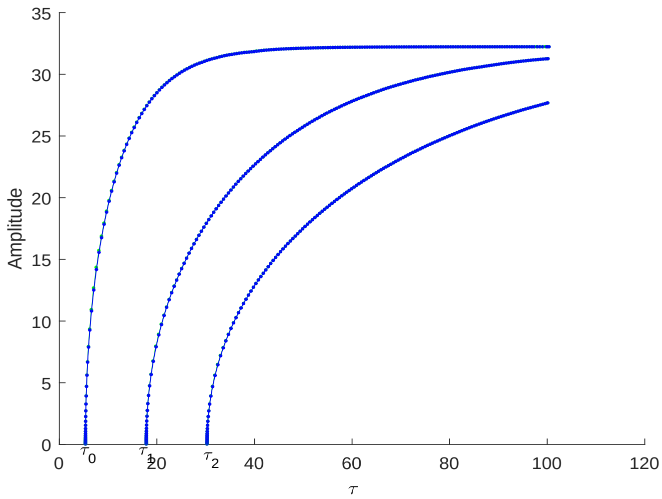

Figure 3. Global Hopf branches of system (1.2) at $\tau_0 = 5.5131$, $\tau_1 = 18.3517$, and $\tau_2 = 31.1903$, respectively.

Finally, we use the delay as the bifurcation parameter to plot the bifurcation diagram. Figure 4 demonstrates the onset and global continuation of Hopf bifurcations as τ varies.

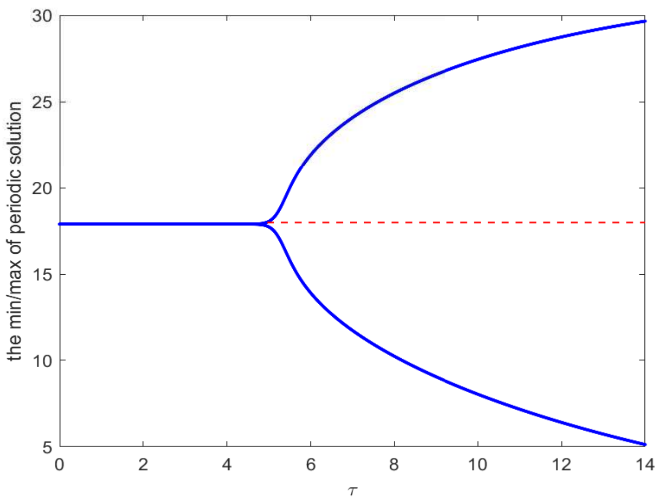

Figure 4. Bifurcation diagram of (1.2) with respect to τ, where red dashed line represents unstable equilibrium.

6. Discussion

Models related to the impact of media coverage on disease spread have shown great popularity in recent years. To study the effect of media coverage on cholera transmission, we considered a cholera model with delayed media impact. We showed that the basic reproduction number $\mathcal{R}_0$ is an epidemic threshold parameter that determines the extinction and uniform persistence of the disease. However, this threshold phenomenon is not influenced by the delayed media impact, since $\mathcal{R}_0$ is independent of m and τ. This observation motivates us to further explore the impact of media coverage. We proved that the positive equilibrium E_1 is locally asymptotically stable when $\mathcal{R}_0 > 1$ and $\tau \in [0, \tau_0)$, and is unstable when $\tau > \tau_0$ (see Theorem 3.3). Furthermore, system (1.2) undergoes a Hopf bifurcation at E_1 along the sequence τ_n, $n = 0, 1, 2, \cdots$. To examine the onset and termination of periodic solutions bifurcated from E_1, we use delay as the bifurcation parameter and establish the existence of global bifurcation (see Theorem 4.1).

The local stability and the Hopf bifurcation analysis of the positive equilibrium E_1 are critically dependent on the existence and distribution of the roots of the cubic equation $h(x) = 0$ [39]. In this paper, we considered only case where $h(x) = 0$ has only one simple positive root. Therefore, it is exciting to study properties of the global Hopf branches that accompany the stability switch when $h(x) = 0$ has exactly two or three simple positive roots. Another possible project is to consider a spatial version of model (1.1). For such a model with multiple compartments, the investigation of Hopf bifurcation is challenging. We leave these topics for future research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (No. 11971369) and the Fundamental Research Funds for the Central Universities (No. QTZX23004).

Conflict of interest

The authors declare that there is no conflict of interest.

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