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Research article

Emergent dynamics of various Cucker–Smale type models with a fractional derivative

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Abstract: In this paper, we demonstrate emergent dynamics of various Cucker–Smale type models, especially standard Cucker–Smale (CS), thermodynamic Cucker–Smale (TCS), and relativistic Cucker–Smale (RCS) with a fractional derivative in time variable. For this, we adopt the Caputo fractional derivative as a widely used standard fractional derivative. We first introduce basic concepts and previous properties based on fractional calculus to explain its unusual aspects compared to standard calculus. Thereafter, for each proposed fractional model, we provide several sufficient frameworks for the asymptotic flocking of the proposed systems. Unlike the flocking dynamics which occurs exponentially fast in the original models, we focus on the flocking dynamics that occur slowly at an algebraic rate in the fractional systems.

Keywords: asymptotic flocking; Caputo fractional derivative; Cucker–Smale; relativistic Cucker–Smale; thermodynamic Cucker–Smale

1. Introduction

Collective behaviors of interacting many-body systems are frequently observed in nature and human society, for example, synchronization of fireflies and pacemaker cells [1–3], aggregation of bacteria [4], flocking of birds [5], swarming of fish [6, 7], etc. To introduce these subjects, we refer to [8–14]. Among them, we are primarily interested in asymptotic "*flocking*" (see Definition 3.1, 4.1 and 5.1 for mathematical description), in which each agent uses limited environmental information and simple laws and its velocity converges to a common value. Since the joint work [15] on the flocking model for birds proposed by Vicsek et al., several mathematical models describing collective motion have been lively studied in community. After a seminal paper [5], mathematicians and physicists have focused on the Cucker–Smale (CS) model and its variants based on Newtonian second-order model for *position-velocity*. The time-evolutionary behavior of CS particles are governed by the following

Cauchy problem:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N] := \{1, \cdots, N\}, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N} \displaystyle\sum_{j=1}^{N} \psi(\|x_i - x_j\|) \left(v_j - v_i\right), \\ (x_i(0), v_i(0)) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

where (x_i, v_i) is position-velocity pair of i-th particle, N is the number of particles, κ is a nonnegative coupling strength, $\|\cdot\|$ denotes the standard l^2-norm, and $\psi : \mathbb{R}_+ := [0, \infty) \longrightarrow \mathbb{R}_+$ satisfies the following assumptions for the global well-posedness of (1.1):

$$0 \le \psi(r) \le \psi(0) = 1, \quad (\psi(r_1) - \psi(r_2))(r_1 - r_2) \le 0, \quad \psi(\cdot) \in C^{0,1}_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+).$$

To date, lots of studies on the CS model (1.1) and its variants have been conducted, to name a few, the mean-field limit [16–18], kinetic model [19, 20], hydrodynamic descriptions [21–23], particle analysis [19], time-delay effect [24], stochastic analysis [25], bi-cluster flocking [26], unit-speed constraint [27] and collision avoidance [28]. For the survey paper, we refer to [9]. Herein, we note that most frameworks for the CS model have proceeded via the Markovian approach. However, the behaviors of agents (human, animal, insects, etc) in ecosystem tend to be influenced by memories and past experiences. Fractional calculus can be a nice tool to mathematically describe this memory effect, and is being studied very actively to this day (refer to [29]). In addition, many models concerning fractional-order derivative can be applied in biology, chemistry, microbiology, electrochemistry, viscoelasticity, medical science and diffusion (see [30]). Therefore, it is natural to consider the CS-type flocking systems in non-Markovian sense under the interplay of memory effect. For this, we propose the following fractional CS system by changing usual time-derivative to the Caputo fractional derivative with $\kappa = 1$:

$$\begin{cases} D^c_\alpha x_i = v_i, \quad t > 0, \quad i \in [N], \\ D^c_\alpha v_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \psi(\|x_i - x_j\|) \left(v_j - v_i\right), \\ (x_i(0), v_i(0)) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.2}$$

where D^c_α is the Caputo fractional derivative of order $\alpha \in (0, 1)$ (See Definition 2.1). Unlike the standard CS model (1.1), which is the case of $\alpha = 1$ in (1.2), there are very few works on (1.2), to name a few, the discrete model analysis for (1.2) using the linearization method [31, 32], the Riemann-Liouville fractional derivative under constant communication weight based on optimal control problem [33], the Caputo derivative under general communication weight through optimal consensus control problem approach [34, 35] and flocking estimate under general communication weight applying the iterative method [36]. In this paper, we adopt the iterative method for the integral form of the fractional system (1.2), used in [36]. One of the difficulties in analysis is that the proposed fractional system is non-autonomous. However, using integral form makes it possible to analyze the model. We refer to [37, 38] for readers interested in the method employing an integral form. In addition, it is worth to mention that after deriving the flocking estimate of (1.2) through the proposed method, by taking $\alpha = 1$, one can obtain the flocking estimates of the standard CS model (1.1) (see [9]). Furthermore, we will present more improved sufficient frameworks than [36] for the flocking dynamics of (1.2) on general communication weights. For the detailed and rigorous descriptions, see Section 3.

To set up the second stage, we notice that the aforementioned model (1.2) was conducted without considering temperature field. In order to describe more realistic flocking dynamics, the authors of [39] extended the standard CS model (1.1) to a thermodynamic Cucker–Smale (TCS), which is derived from the system of gas mixtures using rational methods and reductions. In other words, they proposed the TCS model to deal with the collective behaviors of agents with a time varying internal variable by generalizing (1.1). Since then, the authors of a follow-up paper [40] reduced the TCS model to derive the approximated TCS model for *position–velocity–temperature* under the assumption that the diffusion velocities are sufficiently small. The approximated TCS model is given by the following Cauchy problem in terms of (x_i, v_i, T_i):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j=1}^{N} \phi(\|x_i - x_j\|) \left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i} \right), \\ \dfrac{dT_i}{dt} = \dfrac{\kappa_2}{N} \displaystyle\sum_{j=1}^{N} \zeta(\|x_i - x_j\|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j} \right), \\ (x_i(0), v_i(0), T_i(0)) \in \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}_+ - \{0\}), \end{cases} \tag{1.3}$$

where κ_1 and κ_2 are nonnegative coupling strengths. Here, $\phi, \zeta : [0, \infty) \to \mathbb{R}_+$ are communication weights which are nonnegative, uniformly bounded, locally Lipschitz continuous and monotonically decreasing, equivalently,

$$\begin{cases} 0 \le \phi(r) \le \phi(0) = 1, \quad (\phi(r_1) - \phi(r_2))(r_1 - r_2) \le 0, \quad \phi(\cdot) \in C^{0,1}_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+), \\ 0 \le \zeta(r) \le \zeta(0) = 1, \quad (\zeta(r_1) - \zeta(r_2))(r_1 - r_2) \le 0, \quad \zeta(\cdot) \in C^{0,1}_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+). \end{cases}$$

We introduce various literature concerning the flocking dynamics of (1.3) and its variants, to name a few, derivation of model [39], asymptotic behavior [40], uniform stability and uniform-in-time mean-field limit [41], hydrodynamic description [42], time-delay effect [43] and Riemannian manifold setting [44]. However, a fractional TCS system based on non-Markovian approach has not been studied. Hence, we suggest the following fractional TCS model with the Caputo derivative of order $\alpha \in (0, 1)$ and $\kappa_1 = \kappa_2 = 1$:

$$\begin{cases} D^c_\alpha x_i = v_i, \quad t > 0, \quad i \in [N], \\ D^c_\alpha v_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \phi(\|x_i - x_j\|) \left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i} \right), \\ D^c_\alpha T_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \zeta(\|x_i - x_j\|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j} \right), \\ (x_i(0), v_i(0), T_i(0)) \in \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}_+ - \{0\}). \end{cases} \tag{1.4}$$

To demonstrate suitable sufficient frameworks for the flocking dynamics of the above system (1.4), we provide several preparatory estimates for temperatures, and then we apply methodology used in Section 3 to (1.4). For detailed descriptions, see Section 4.

To set up the final stage, we point out that (1.1) and (1.3) are based on a classical Newtonian mechanics without suitable relativistic corrections. In other words, relativism is not considered in (1.1) and (1.3). For instance, we can imagine interaction system among space shuttles, hypersonic rockets, and satellites. Then, the velocity of each agent is comparable to the speed of light c so that one should consider appropriate relativistic corrections. With this motivation, the relativistic Cucker–Smale (RCS) model was rigorously proposed in [45] via reduction process from the Euler system of the homogeneous relativistic fluids mixture. This RCS model is given by the following Cauchy problem regarding (x_i, w_i):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\ \dfrac{dw_i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \rho(\|x_i - x_j\|)\left(v_j - v_i\right), \\ (x_i(0), w_i(0)) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.5}$$

where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a communication weight satisfying

$$0 \le \rho(r) \le \rho(0) = 1, \quad (\rho(r_1) - \rho(r_2))(r_1 - r_2) \le 0 \quad \text{and} \quad \rho(\cdot) \in C^{0,1}_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_+),$$

c denotes the speed of light, and w_i is a relativistic variable satisfying

$$w_i := F_i v_i = f(v_i), \quad F_i := \Gamma_i\left(1 + \frac{\Gamma_i}{c^2}\right) = g(\|v_i\|), \quad \Gamma_i := \frac{1}{\sqrt{1 - \frac{\|v_i\|^2}{c^2}}} \quad i \in [N].$$

Here, Γ_i is the Lorentz factor of i-th particle. Note that the bijective function f and strictly increasing function $g : [0, c) \to \mathbb{R}_+$ are expressed by

$$\begin{aligned} f(x) &= \frac{cx}{\sqrt{c^2 - \|x\|^2}} + \frac{x}{c^2 - \|x\|^2}, \quad x \in \{v \in \mathbb{R}^d \mid \|v\| < c\}, \\ g(x) &= \frac{c}{\sqrt{c^2 - x^2}} + \frac{1}{c^2 - x^2}, \quad x \in [0, c). \end{aligned} \tag{1.6}$$

Due to the bijectivity of f, the RCS model (1.5) can be formulated in terms of both (x_i, v_i) or (x_i, w_i).

To introduce the RCS model (1.5) and its variants, we refer to the uniform stability and uniform-in-time mean-field limit [46], Riemannian manifold setting [47], singular kernel [48], flocking dynamics [45], uniform-in-time nonrelativistic limit [49], time delay [50] and approximated RCS model and its kinetic and hydrodynamic descriptions [51]. However, as in the case of the TCS model (1.3), there is no research on the RCS system (1.5) in non-Markovian sense. Therefore, we propose the following fractional RCS system with the Caputo derivative of order $\alpha \in (0, 1)$:

$$\begin{cases} D^c_\alpha x_i = v_i, \quad t > 0, \quad i \in [N], \\ D^c_\alpha w_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \rho(\|x_i - x_j\|)\left(v_j - v_i\right), \\ (x_i(0), w_i(0)) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.7}$$

where we used the abuse of notation as follows: "c" in D^c_α is the abbreviation of Caputo derivative, and "c" in the definition of Γ denotes the speed of light. However, they can be distinguished without confusion in context. To obtain the flocking estimates of the above system (1.7), we follow the methodology used in Sections 3 and 4. For the detailed descriptions, refer to Section 5.

With these various CS-type model under memory effects, in this paper, we are primarily concerned with the following issue:

- (Question): *Under what sufficient conditions for initial data and system parameters do the proposed fractional systems* (1.2)*,* (1.4) *and* (1.7) *exhibit asymptotic flocking, respectively?*

To answer this question, as aforementioned, we adopt the iterative method for the integral form of the fractional systems (1.2), (1.4) and (1.7).

The rest of paper is organized as follows. In Section 2, we briefly review previous results on fractional calculus and (1.2), and present several conserved quantities for (1.4) and (1.7). In Section 3, we provide improved flocking estimates of (1.2). In Section 4, with the methodology employed in Section 3, we present a sufficient framework for the flocking dynamics of (1.4). In Section 5, we demonstrate the asymptotic flocking of (1.7) under admissible data. Finally, Section 6 is devoted to a brief summary of the main results and some discussion on the remaining issues to be investigated in a future work.

Notation We employ the following notation for simplicity.

$$\begin{aligned}
&[N] := \{1, \cdots, N\}, \quad I_d = d \times d \text{ identity matrix}, \quad x_i, v_i, w_i \in \mathbb{R}^{d\times 1},\\
&X =: (x_1, \cdots, x_N)^T, \quad V =: (v_1, \cdots, v_N)^T, \quad W =: (w_1, \cdots, w_N)^T, \quad T =: (T_1, \cdots, T_N)^T,\\
&\mathbb{C} =: \text{the set of complex numbers}, \quad \mathbb{Z} =: \text{the set of integers}, \quad [\cdot] =: \text{Gauss integer},\\
&\mathbb{N} =: \text{the set of natural numbers}, \quad \mathbb{Z}_{\le 0} =: \text{the set of nonpositive integers}, \quad \mathbb{R}_+ := [0, \infty),\\
&|z| =: \text{the modulus of } z \in \mathbb{C}, \quad \mathbb{R}^{m\times n} =: \text{the set of } m \times n \text{ matrices}, \quad A^T =: \text{the transpose of } A.
\end{aligned}$$

Note that $X, V, W \in \mathbb{R}^{N\times d}$ and $T \in \mathbb{R}^{N\times 1}$.

2. Preliminaries

In this section, we briefly introduce previous results covered in fractional calculus and analysis. Thereafter, we provide several conserved quantities in the fractional systems (1.2), (1.4) and (1.7), which will be used to represent the systems in matrix form.

2.1. Fractional calculus and analysis

In this subsection, we provide essential materials for fractional calculus and analysis to develop the main results in Sections 3–5. We begin with the definitions of the Caputo fractional derivative and Gamma and Beta functions.

Definition 2.1. [29] *For a given strictly positive number $\alpha \in (0, \infty)$, the Caputo fractional derivative*

$D^c_\alpha f$ of order α is defined by

$$D^c_\alpha f(t) =: \begin{cases} \dfrac{1}{\Gamma([\alpha]+1-\alpha)} \displaystyle\int_0^t \frac{f^{([\alpha]+1)}(s)}{(t-s)^{\alpha-[\alpha]}} ds, & \text{if } \alpha \neq [\alpha], \\ f^\alpha(t), & \text{if } \alpha = [\alpha], \end{cases}$$

if the right sides are well defined.

We note that when $\alpha = 1$, the fractional systems (1.2), (1.4), and (1.7) can be reduced to (1.1), (1.3) and (1.5), respectively. From Definition 2.1, we recall the definitions of the Gamma function and Beta function as follows:

Definition 2.2. *The Gamma function* $\Gamma = \Gamma(z) : \mathbb{C} - \mathbb{Z}_{\leq 0} \to \mathbb{C} - \{0\}$ *and Beta function* $B = B(z, w)$ *for* $\mathrm{Re}(z) > 0$ *and* $\mathrm{Re}(w) > 0$ *are given by*

1) *(Gamma function)* $\Gamma(z) =: \displaystyle\int_0^\infty e^{-t} t^{z-1} dt, \quad z \in \mathbb{C}.$

2) *(Beta function)* $B(z, w) =: \displaystyle\int_0^1 t^{z-1}(1-t)^{w-1} dt, \quad \text{for} \quad \mathrm{Re}(z), \mathrm{Re}(w) > 0.$

Then, we revisit the basic properties for the Gamma function and Beta function as follows because these appear in the proofs of main results studied in Sections 3–5:

Remark 2.1. *It is well known that for the Gamma function* $\Gamma(z)$ *and Beta function* $B(z, w)$*,*

1) $\Gamma(1) = 1, \quad \Gamma(n) = (n-1)!$ *for all positive integer* n*,* $\quad \Gamma(z+1) = z\Gamma(z).$

2) *B is symmetric, i.e.,* $B(z, w) = B(w, z), \quad B(z, w) = \dfrac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$

Now, we move on to the definition of the Mittag-Leffler function, which appears frequently as solutions of fractional ODE systems. In the fractional flocking systems (1.2), (1.4), and (1.7), we also have flocking estimates associated with the Mittag-Leffler function.

Definition 2.3. *For* $\alpha, \beta \in \mathbb{C}$*, the Mittag-Leffler function* $E_{\alpha,\beta}$ *is defined as*

$$E_{\alpha,\beta}(z) =: \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

We then present the following several relations and basic lemmas with respect to the Mittag-Leffler function:

Proposition 2.1. [29, 36] *Let* $\alpha, \beta, \gamma \in \mathbb{C}$*. Then, it follows that*

1) $E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \dfrac{1}{\Gamma(\beta)}.$

2) $\dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (t-s)^{\gamma-1} E_{\alpha,\beta}(\lambda s^\alpha) s^{\beta-1} ds = t^{\beta+\gamma-1} E_{\alpha,\beta+\gamma}(\lambda t^\alpha), \quad \lambda \in \mathbb{C}, \ \beta, \gamma > 0.$

3) For $y, z \in \mathbb{C}$, and $\beta, \gamma > 0$, we have

$$\int_0^t s^{\gamma-1} E_{\alpha,\gamma}(ys^\alpha)(t-s)^{\beta-1} E_{\alpha,\beta}(z(t-s)^\alpha)ds$$
$$= \frac{yE_{\alpha,\beta+\gamma}(yt^\alpha) - zE_{\alpha,\beta+\gamma}(zt^\alpha)}{y-z} t^{\beta+\gamma-1}, \quad \beta, \gamma > 0.$$

Proposition 2.1 will be crucially used to estimate the asymptotic flocking of the fractional systems (1.2), (1.4) and (1.7). Subsequently, the following proposition is related to the algebraic flocking rates of (1.2), (1.4) and (1.7) described in Sections 3–5.

Proposition 2.2. [29, 36] *Assume that the constants α, β, and γ satisfy*

$$\alpha \in (0,2), \quad \beta \in \mathbb{R}, \quad \frac{\pi\alpha}{2} < \gamma < \min(\pi, \pi\alpha).$$

Then, one has the following relations for all $n \in \mathbb{N}$:

1) For $|\arg(z)| \le \gamma$,

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\exp(z^{\frac{1}{\alpha}}) - \sum_{k=1}^{n} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-n}), \quad as \quad |z| \to \infty.$$

2) For $\gamma \le |\arg(z)| \le \pi$,

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{n} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-n}), \quad as \quad |z| \to \infty.$$

3) As a direct consequence of (2), the following assertion holds:

$$E_{\alpha,1}(-Ct^\alpha) \sim \frac{t^{-\alpha}}{C\Gamma(1-\alpha)}, \quad as \quad t \to \infty, \quad \text{for every positive constant } C.$$

Next, we present a concept of "completely monotone" and basic relationship with this and the Mittag-Leffler function to deal with the fractional Caputo derivative of order $\alpha \in (0,1)$ throughout the paper.

Definition 2.4. [52] *We say a C^∞-function f defined on $(0,\infty)$ is completely monotone if it satisfies*

$$(-1)^k f^{(k)}(t) \ge 0, \quad t \in (0,\infty), \quad \forall k \in \mathbb{N} \cup \{0\}.$$

Note that the completely monotone function is always nonnegative, monotonically decreasing, and convex. For the detailed proof, see [52]. The following proposition offers some sufficient condition to guarantee the complete monotonicity of $E_{\alpha,\beta}(-t)$.

Proposition 2.3. [53] *Let $E_{\alpha,\beta}(z)$ be the Mittag-Leffler function defined in Definition 2.4. Then, $E_{\alpha,\beta}(-t)$ is completely monotone on $t \in (0,\infty)$ if and only if $\alpha \in (0,1]$ and $\beta \ge \alpha$.*

In what follows, we display some properties regarding relationships between monotone functions and sign changes of their Caputo derivatives.

Proposition 2.4. [54] *Let a function $f \in C^1[a,b]$ be monotone on $[a,b]$ and $\alpha \in (0,1)$. Then, the following assertions hold:*

1) It follows that $\forall t \in [a,b]$,

$$D^c_\alpha f(t) \begin{cases} \le 0, & \textit{if f is monotonically decreasing}, \\ \ge 0, & \textit{if f is monotonically increasing}. \end{cases}$$

2) Assume that $f \in C^1[a,b]$ satisfies $D^c_\alpha f(t) \ge 0$ (≤ 0) for $\forall t \in [a,b]$ and all $\alpha \in (\alpha_0, 1)$ for some $\alpha_0 \in (0,1)$. Then, f is monotone increasing (decreasing).

3) Suppose that $f \in C^1[a,b]$ satisfies $D^c_\alpha f(t) \ge 0$ (≤ 0) for $\forall t \in [a,b]$ and all $\alpha \in (\alpha_0, 1)$ for some $\alpha_0 \in (0,1)$. Then, $D^c_\alpha f(t) \ge 0$ (≤ 0) for $\forall t \in [a,b]$ and all $\alpha \in (0,1)$.

From now on, we briefly introduce an alternative expression for fractional ODE system which generalizes the fractional flocking systems (1.2), (1.4) and (1.7). For $\alpha \in (0,1)$ and $T > 0$,

$$D^\alpha_c x(t) = f(t, x(t)), \quad t \in [0,T), \quad x(0) = x_0. \tag{2.1}$$

Then, it is widely known that the above equation (2.1) can be reformulated into the following integral equation when f is continuous:

$$x(t) = x_0 + D^{-\alpha}_c f(t, x(t)) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} ds. \tag{2.2}$$

In particular, when $f(t, x(t)) = Ax(t) + y(t)$ where $A \in \mathbb{R}^{d\times d}$ is a constant coefficient for some $d \in \mathbb{N}$, $x : [0,T) \to \mathbb{R}^d$, and $y : [0,T) \to \mathbb{R}^d$, we have that for $\alpha \in (0,1)$,

$$D^\alpha_c x(t) = Ax(t) + y(t), \quad t \in [0,T), \quad x(0) = x_0. \tag{2.3}$$

Then, there exists a unique solution $x(t)$ on $[0,T]$ to (2.3) under appropriate conditions.

Proposition 2.5. [36,55] *Let $x(t)$ be a solution to (2.3) of order $\alpha \in (0,1)$ and we assume that $y(t)$ is an element of a space $\mathscr{C}_{1-\alpha}([0,T])$, where $\mathscr{C}_{1-\alpha}([0,T])$ is given by*

$$\mathscr{C}_{1-\alpha}([0,T]) =: \left\{ f(t) \in C^0[0,T] \mid \|f\|_{\mathscr{C}_{1-\alpha}} =: \sup_{t\in[0,T]} \|t^{1-\alpha} f(t)\| < \infty \right\}.$$

Then, a unique solution of (2.3) is represented by

$$x(t) = E_{\alpha,1}(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) y(s) ds,$$

where $E_{\alpha,\beta}(A)$ is given by

$$E_{\alpha,\beta}(A) =: \sum_{k=0}^\infty \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad \textit{where } A \in \mathbb{R}^{d\times d}.$$

Before we end this subsection, we provide the standard Cauchy–Lipschitz theory for the fractional ODE system (2.1) of order $\alpha \in (0, 1]$.

Proposition 2.6. [56] *We consider the following fractional ODE system of order* $\alpha \in (0, 1]$*:*

$$D_c^\alpha x(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{2.4}$$

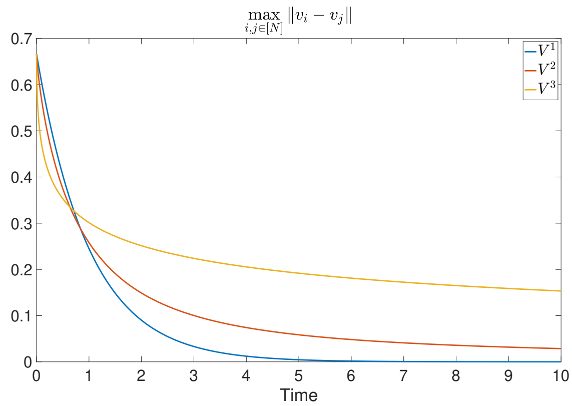
Further assume that f is locally Lipschitz continuous in x and continuous in t. Then, for some strictly positive number ϵ, there exists a unique solution $x(t)$ to (2.4) *on the interval* $[t_0 - \epsilon, t_0 + \epsilon]$.

2.2. *Comparison of* (1.1) *and* (1.2) *via numerical experiments*

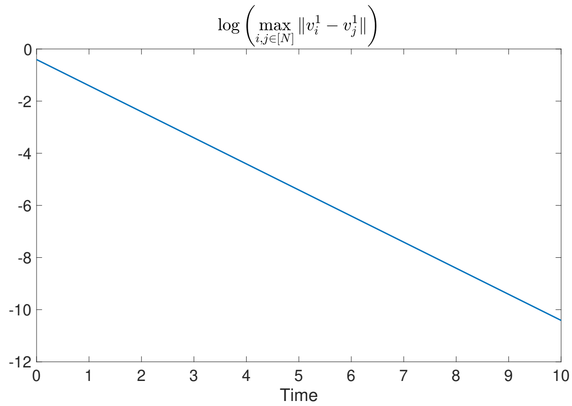
In this subsection, we provide some numeric examples comparing convergence rate of (1.1) and (1.2). For simplicity, we set $\kappa = 1$, $\psi \equiv 1$ and $d = 1$. For fixed $N = 10$, let (X^1, V^2) be a solution of (1.1) and (X^2, V^2) and (X^3, V^3) be solutions of (1.2) with $\alpha = 0.8$ and $\alpha = 0.4$, respectively. Furthermore, we choose initial data satisfying

$$x_i^1(0) = x_i^2(0) = x_i^3(0), \quad v_i^1(0) = v_i^2(0) = v_i^3(0), \quad i \in [10],$$

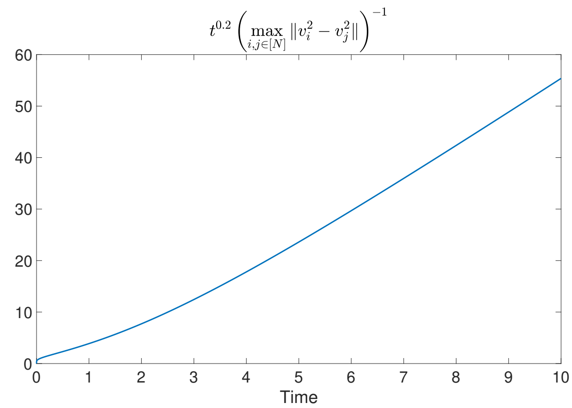
which are chosen randomly in $[0, 1]$. Under these setting, we observe the dynamics until $t = 10$.



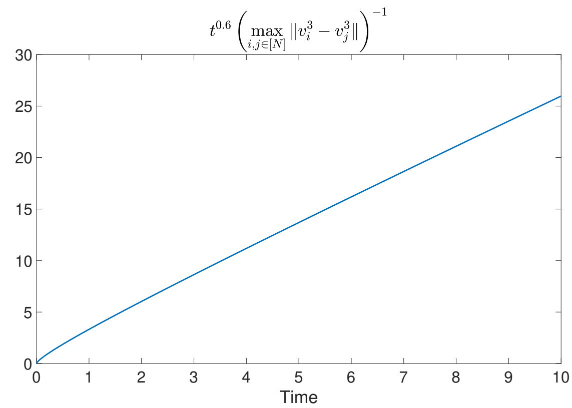
(a) Diameter of V^i's



(b) Exponential decaying of diameter of V^1



(c) Algebraic decaying of diameter of V^2



(d) Algebraic decaying of diameter of V^3

Figure 1. Various decaying rates of diameters of V^i's.

In Figure 1 (A), we plot time evolutionary behavior of

$$\max_{j,k\in[N]} \|v^i_j - v^i_k\|, \quad i = 1, 2, 3,$$

which are monotonically decreasing. Since (1.1) is a special case of (1.2) where $\alpha = 1$, it can be seen that the larger α implies the faster decay of diameter of V^i's. Moreover, from Figure 1(B)–(D), one can infer the exponential decay of $\max_{j,k\in[N]} \|v^1_j - v^1_k\|$ and the algebraic decay of $\max_{j,k\in[N]} \|v^i_j - v^i_k\|$, $i = 2, 3$, respectively.

2.3. Basic estimates of the fractional systems

In this subsection, we provide several conserved quantities of the proposed fractional systems (1.2), (1.4), and (1.7), which will be used to represent (1.2), (1.4), and (1.7) as matrices. First, we briefly introduce the following previous results of the fractional CS system (1.2):

Proposition 2.7. [36] *Let* (X, V) *be a solution to* (1.2)*. Then, for any* $t \geq 0$*, we have*

$$\sum_{i=1}^N v_i(t) = \sum_{i=1}^N v_i^0, \qquad \sum_{i=1}^N x_i(t) = \sum_{i=1}^N x_i^0 + \frac{t^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^N v_i^0.$$

Similarly, we also present the propagation of conserved quantities in the fractional TCS system (1.4) as follows:

Lemma 2.1. *Let* (X, V, T) *be a solution to* (1.4)*. Then, it follows that for any* $t \geq 0$*,*

$$\sum_{i=1}^N v_i(t) = \sum_{i=1}^N v_i^0, \qquad \sum_{i=1}^N x_i(t) = \sum_{i=1}^N x_i^0 + \frac{t^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^N v_i^0, \qquad \sum_{i=1}^N T_i(t) = \sum_{i=1}^N T_i^0.$$

Proof. We use the standard trick of interchanging i and j and dividing 2 in $(1.4)_2$ to get

$$D^c_\alpha\left(\sum_{i=1}^N v_i(t)\right) = \frac{1}{N}\sum_{i,j=1}^N \phi(\|x_i - x_j\|)\left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right) = \frac{1}{N}\sum_{i,j=1}^N \phi(\|x_i - x_j\|)\left(\frac{v_i}{T_i} - \frac{v_j}{T_j}\right) = 0,$$

$$D^c_\alpha\left(\sum_{i=1}^N T_i(t)\right) = \frac{1}{N}\sum_{i,j=1}^N \zeta(\|x_i - x_j\|)\left(\frac{1}{T_j} - \frac{1}{T_i}\right) = \frac{1}{N}\sum_{i,j=1}^N \zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right) = 0.$$

Thus, we combine the above results with (2.2) to have the desired first and third assertions.

$$\sum_{i=1}^N v_i(t) = \sum_{i=1}^N v_i^0, \qquad \sum_{i=1}^N T_i(t) = \sum_{i=1}^N T_i^0.$$

Now, for the second assertion, we note from $(1.4)_1$ that

$$D^c_\alpha\left(\sum_{i=1}^N x_i(t)\right) = \sum_{i=1}^N v_i(t).$$

Here, we apply this and the first assertion to (2.2) to yield that

$$
\begin{aligned}
\sum_{i=1}^N x_i(t) &= \sum_{i=1}^N x_i^0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\sum_{i=1}^N v_i(s) \right) ds \\
&= \sum_{i=1}^N x_i^0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\sum_{i=1}^N v_i^0 \right) ds \\
&= \sum_{i=1}^N x_i^0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=1}^N v_i^0 = \sum_{i=1}^N x_i^0 + \frac{t^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^N v_i^0,
\end{aligned}
$$

where we used the relation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Hence, we obtain the desired second assertion. □

Finally, we describe the following lemma on the fractional RCS system (1.7), which is a counterpart of Proposition 2.7 and Lemma 2.1:

Lemma 2.2. *Let* (X, W) *be a solution to* (1.7)*. Then, we attain that for* $t \geq 0$*,*

$$
\sum_{i=1}^N w_i(t) = \sum_{i=1}^N w_i^0.
$$

Proof. As in the proof of Lemma 2.1, we again use the standard trick of interchanging i and j and dividing 2 to $(1.7)_2$ to find

$$
D_\alpha^c \left(\sum_{i=1}^N w_i(t) \right) = \frac{1}{N} \sum_{i,j=1}^N \rho(\|x_i - x_j\|) \left(v_j - v_i \right) = \frac{1}{N} \sum_{i,j=1}^N \rho(\|x_i - x_j\|) \left(v_i - v_j \right) = 0.
$$

Therefore, by using (2.2), we can show that

$$
\sum_{i=1}^N w_i(t) = \sum_{i=1}^N w_i^0.
$$

□

Throughout the paper, we assume that

$$
\sum_{i=1}^N v_i^0 = 0 \text{ in (1.2) and (1.4)}, \quad \sum_{i=1}^N w_i^0 = 0 \text{ in (1.7)}, \quad \text{and} \quad \sum_{i=1}^N T_i^0 =: NT^\infty > 0 \text{ in (1.4)}.
$$

3. Improved flocking dynamics of fractional CS model

In this section, we improve the asymptotic flocking dynamics of (1.2) addressed in the previous paper [36]. To do this, we first present the following basic notion for the asymptotic flocking:

Definition 3.1. *Let* $Z =: (X, V)$ *be a solution to the fractional CS system* (1.2)*. Then, the configuration* Z *exhibits asymptotic flocking if the following assertions hold:*

$$
(i) \text{ (Group formation)} \iff \sup_{t \in \mathbb{R}_+} \max_{i,j \in [N]} \|x_i(t) - x_j(t)\| < \infty.
$$

$$
(ii) \text{ (Velocity alignment)} \iff \lim_{t \to \infty} \max_{i,j \in [N]} \|v_j(t) - v_i(t)\| = 0.
$$

Subsequently, we give the following simple matrix representation in terms of X, V in the system (1.2) to obtain its asymptotic flocking estimate.

$$D_\alpha^c X(t) = V(t), \quad D_\alpha^c V(t) = \Psi(X(t))V(t). \tag{3.1}$$

To explicitly write a matrix $\Psi(X(t)) \in \mathbb{R}^{N\times N}$, we set each (i, j)-th element, $(\Psi(X(t)))_{ij}$, as

$$(\Psi(X(t)))_{ij} = \frac{1}{N}\begin{cases} \psi(\|x_i - x_j\|), & \text{if} \quad i \neq j, \\ -\sum_{k\neq i} \psi(\|x_i - x_k\|), & \text{if} \quad i = j, \end{cases} \qquad \text{for} \quad i, j \in [N].$$

Then, we introduce the previous result on asymptotic flocking of (1.2) as follows:

Proposition 3.1. [36] *Suppose that*

$$d = 1, \quad \psi_M =: \sup_{x\in\mathbb{R}_+} \psi(x), \quad \psi_m^0 =: \min_{i\neq j, i, j\in[N]} \psi(\|x_i^0 - x_j^0\|),$$

and that initial data and communication weight satisfy the following relations: there exists $\epsilon \in (0, 1)$ such that

$$\psi_M < 2\psi_*, \quad \psi(X_\infty) > \psi_* > 0,$$

where ψ_* and X_∞ are given by

$$\psi_* =: \frac{(1+\epsilon)}{2}\psi_m^0, \quad X_\infty =: \sqrt{2}\left(\|X^0\| + \frac{\|V^0\|}{2\psi_* - \psi_M}\right).$$

Let (X, V) be a solution to (1.2) with $d = 1$. Then, for any $t \geq 0$, we have

$$\|V(t)\| \leq \|V(0)\|E_{\alpha,1}(-\lambda t^\alpha), \quad \|X(t)\| \leq \|X(0)\| + \frac{\|V(0)\|}{\lambda},$$

where $\lambda =: 2\psi_* - \psi_M$ is a positive constant.

Note that the sufficient conditions for the asymptotic flocking of Proposition 3.1 is somewhat restrictive due to the positivity of λ. Indeed, when we assume $\alpha = 1$ in Proposition 3.1, it may not match the flocking dynamics of the standard CS model (1.1) (see [9]). Hence, we need to describe more improved flocking estimate of (1.2) in $\mathbb{R}^d$. To achieve this, we provide the following new flocking dynamics of (1.2) as the main result of this section.

Theorem 3.1. *Assume that there exists a nonnegative constant X^∞ satisfying*

$$\|X(0)\| + \frac{\|V(0)\|}{\psi(\sqrt{2}X^\infty)} \leq X^\infty \tag{3.2}$$

and let (X, V) be a solution to (1.2) on $[0, \tau)$ with the initial data $(X(0), V(0))$. Then, we have the global well-posedness of (1.2) and the following asymptotic flocking estimate holds:

$$\tau = \infty, \quad \textit{and} \quad \|X(t)\| \leq X^\infty, \quad \|V(t)\| \leq \|V(0)\|E_{\alpha,1}(-\psi(\sqrt{2}X^\infty)t^\alpha), \quad \forall t \in \mathbb{R}_+.$$

For the detailed proof of Theorem 3.1, using the first assertion of Proposition 2.7, we reformulate the fractional CS system (3.1) as

$$D^c_\alpha X(t) = V(t), \quad D^c_\alpha V(t) = -V(t) + \tilde{\Psi}(X(t))V(t),$$

where

$$(\tilde{\Psi}(X(t)))_{ij} = \frac{1}{N}\begin{cases} \Big(\psi(\|x_i - x_j\|) - 1\Big), & \text{if} \quad i \neq j, \\ -\sum_{k\neq i} (\psi(\|x_i - x_k\|) - 1), & \text{if} \quad i = j, \end{cases} \qquad \text{for} \quad i, j \in [N].$$

Then, we use (2.2) and Proposition 2.5 to yield

$$\begin{aligned} X(t) &= X(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}V(s)ds, \\ V(t) &= E_{\alpha,1}(-t^\alpha)V(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha)\tilde{\Psi}(X(s))V(s)ds. \end{aligned} \tag{3.3}$$

Now, we set

$$\mathcal{S} =: \{t \in (0,\tau] \mid \|X(s)\| \le X^\infty, \quad \forall s \in [0,t)\}, \tag{3.4}$$

where the set $\mathcal{S}$ is nonempty due to the condition for X^∞ in (3.2) and the continuity of X. Then, we claim that

$$\sup \mathcal{S} =: \bar{\tau} = \tau. \tag{3.5}$$

If we prove (3.5), by employing Proposition 2.6, the Cauchy–Lipschitz theory for the fractional system, we can immediately obtain $\tau = \infty$. For the proof by contradiction, suppose that

$$\sup \mathcal{S} = \bar{\tau} < \tau.$$

Based on the iterative method employed in [36], we construct the following sequence $\{V^{(i)}\}_{i=0}^\infty$ from the second equation of (3.3):

$$\begin{aligned} V^{(0)} &= 0, \quad i \in \mathbb{N} \cup \{0\}, \\ V^{(i+1)} &= E_{\alpha,1}(-t^\alpha)V(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha)\tilde{\Psi}(X(s))V^{(i)}(s)ds. \end{aligned} \tag{3.6}$$

As a preparatory lemma, we estimate the operator norm $\|\cdot\|_{\text{op}}$ of the aforementioned matrix $\tilde{\Psi}(X(t))$ on $t \in [0, \bar{\tau}]$ to investigate the recurrence relation (3.6).

Lemma 3.1. $\tilde{\Psi}(X) \in \mathbb{R}^{N\times N}$ *is a positive semi-definite matrix and moreover,*

$$\|\tilde{\Psi}(X)\|_{\text{op}} \le 1 - \psi(\sqrt{2}X^\infty),$$

where $\|\cdot\|_{\text{op}}$ *denotes the operator norm of a matrix.*

Proof. We recall $\tilde{\Psi}(X(t)) \in \mathbb{R}^{N\times N}$ as

$$(\tilde{\Psi}(X(t)))_{ij} = \frac{1}{N}\begin{cases} \Big(\psi(\|x_i - x_j\|) - 1\Big), & \text{if} \quad i \neq j, \\ -\sum_{k\neq i}\left(\psi(\|x_i - x_k\|) - 1\right), & \text{if} \quad i = j \end{cases} \qquad \text{for} \quad i, j \in [N].$$

We note that $\tilde{\Psi}(X)$ is symmetric and $\mathbf{1} := (1, \cdots, 1)^T \in \mathbb{R}^{N\times 1}$ is the eigenvector of $\tilde{\Psi}(X)$ corresponding to an eigenvalue 0. For any vector $v = (v_1, \cdots, v_N)^T \in \mathbf{1}^\perp$, where $v_i \in \mathbb{R},\ i \in [N]$, one has

$$\begin{aligned} \langle \tilde{\Psi}(X)v, v\rangle &= \sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{\Psi}(X))_{ij}\langle v_i, v_j\rangle \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{\Psi}(X))_{ij}\langle v_i - v_j, v_j\rangle + \sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{\Psi}(X))_{ij}\|v_j\|^2 \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{\Psi}(X))_{ij}\langle v_i - v_j, v_j\rangle = -\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{\Psi}(X))_{ij}\frac{\|v_i - v_j\|^2}{2} \geq 0. \end{aligned}$$

Therefore, $\tilde{\Psi}(X)$ becomes a positive semi-definite matrix. Since $\tilde{\Psi}(X)$ is symmetric and positive semi-definite, the operator norm of $\tilde{\Psi}(X)$ is bounded by its maximum eigenvalue. Hence, it follows from $\sum_{i=1}^{N} v_i = 0$, $\|x_i - x_j\| \leq \sqrt{2}\|X\| \leq \sqrt{2}X^\infty$, and monotonicity of ψ that

$$\langle \tilde{\Psi}(X)v, v\rangle \leq \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}(1 - \psi(\sqrt{2}X^\infty))\|v_i - v_j\|^2 = (1 - \psi(\sqrt{2}X^\infty))\|v\|^2,$$

which implies the desired inequality. □

Motivated from the proof of Lemma 3.4 in the previous literature [36], we present the following useful lemma estimating $\|V^{(i+1)} - V^{(i)}\|$:

Lemma 3.2. *For $t \in [0, \bar{\tau}]$, the following assertion holds for $i \in \mathbb{N} \cup \{0\}$.*

$$\begin{aligned} \|V^{(i+1)} - V^{(i)}\| &\leq \Big(1 - \psi(\sqrt{2}X^\infty)\Big)^i \|V(0)\| \sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j \cdot (-1)^j}{\Gamma(\alpha(j+i)+1)}\right)\cdot t^{\alpha(j+i)} \\ &= \left(\frac{1 - \psi(\sqrt{2}X^\infty)}{-1}\right)^i \|V(0)\| \sum_{j=i}^{\infty}\left(\frac{{}_{j}C_i \cdot (-1)^j}{\Gamma(\alpha j+1)}\right)\cdot t^{\alpha j}, \end{aligned}$$

where ${}_nC_r$ is the number of r-combinations from a given set of n elements.

Proof. We employ an inductive method to prove this lemma.

• (The case of $i = 0$): Definition 2.3 and (3.6) imply

$$\|V^{(1)} - V^{(0)}\| \leq \|V(0)\|E_{\alpha,1}(-t^\alpha) = \|V(0)\| \sum_{j=0}^{\infty}\left(\frac{(-1)^j}{\Gamma(\alpha j+1)}\right)\cdot t^{\alpha j}.$$

• (The case of $i > 0$): Assume that the desired assertion holds for all $k \le i$, $k \in \mathbb{N} \cup \{0\}$. Then, it follows from the inductive assumption, Definition 2.3, and Lemma 3.1 that

$$\begin{aligned}
&\|V^{(i+2)} - V^{(i+1)}\| \\
&\le (1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \\
&\quad \times \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha}) \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot (-1)^j}{\Gamma(\alpha(j+i)+1)} \right) \cdot s^{\alpha(j+i)} ds \\
&= (1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \\
&\quad \times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot (-1)^{k+j}}{\Gamma(\alpha(k+1))\Gamma(\alpha(j+i)+1)} \right) \cdot \int_0^t (t-s)^{\alpha(k+1)-1} s^{\alpha(j+i)} ds.
\end{aligned}$$

Using the Beta function of Definition 2.2 with the second assertion of Remark 2.1, we can obtain that for $z, w \in \mathbb{C}$ satisfying $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$,

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(z,w) =: \int_0^1 t^{z-1}(1-t)^{w-1} dt = \int_0^t \frac{(t-s)^{w-1} s^{z-1}}{t^{z+w-1}} ds. \tag{3.7}$$

This yields the following equations:

$$\begin{aligned}
&(1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \\
&\quad \times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot (-1)^{k+j}}{\Gamma(\alpha(k+1))\Gamma(\alpha(j+i)+1)} \right) \cdot \int_0^t (t-s)^{\alpha(k+1)-1} s^{\alpha(j+i)} ds \\
&= (1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \\
&\quad \times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot (-1)^{k+j} \cdot \Gamma(\alpha(k+1))\Gamma(\alpha(j+i)+1)}{\Gamma(\alpha(k+1))\Gamma(\alpha(j+i)+1)\Gamma(\alpha(k+j+i+1)+1)} \right) \cdot t^{\alpha(k+j+i+1)} \\
&= (1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot (-1)^{k+j}}{\Gamma(\alpha(k+j+i+1)+1)} \right) \cdot t^{\alpha(k+j+i+1)} \\
&= (1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \sum_{j=0}^{\infty} \sum_{l=j}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot (-1)^{l}}{\Gamma(\alpha(l+i+1)+1)} \right) \cdot t^{\alpha(l+i+1)} \\
&= (1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \sum_{l=0}^{\infty} \sum_{j=0}^{l} {}_{(j+i)}C_j \left(\frac{(-1)^{l}}{\Gamma(\alpha(l+i+1)+1)} \right) \cdot t^{\alpha(l+i+1)} \\
&= (1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \sum_{l=0}^{\infty} \left(\frac{{}_{(l+i+1)}C_l \cdot (-1)^{l}}{\Gamma(\alpha(l+i+1)+1)} \right) \cdot t^{\alpha(l+i+1)} \\
&= (1 - \psi(\sqrt{2}X^{\infty}))^{i+1}\|V(0)\| \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i+1)}C_j \cdot (-1)^{j}}{\Gamma(\alpha(j+i+1)+1)} \right) \cdot t^{\alpha(j+i+1)},
\end{aligned}$$

where we used the following relation:

$$\sum_{j=0}^{l} {}_{(j+i)}C_j = {}_{(l+i+1)}C_l.$$

Indeed, for a moment, we assume that the following summation,

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j\cdot(-1)^{k+j}}{\Gamma(\alpha(k+j+i+1)+1)}\right)\cdot t^{\alpha(k+j+i+1)},$$

is absolutely convergent in the second equation to obtain

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{l=j}^{\infty}\left(\frac{{}_{(j+i)}C_j\cdot(-1)^{l}}{\Gamma(\alpha(l+i+1)+1)}\right)\cdot t^{\alpha(l+i+1)}\\&=\sum_{l=0}^{\infty}\sum_{j=0}^{l}{}_{(j+i)}C_j\left(\frac{(-1)^{l}}{\Gamma(\alpha(l+i+1)+1)}\right)\cdot t^{\alpha(l+i+1)}.\end{aligned}$$

This will be rigorously verified in the proof of Theorem 3.1. Therefore, we have the desired result by the above inductive arguments. □

Finally, we are ready to demonstrate the asymptotic flocking result in Theorem 3.1 with technical estimates and continuous argument.

Proof of Theorem 3.1. Employing Lemma 3.2 and Definition 2.3 with the following relation,

$${}_jC_i \le {}_jC_{[\frac{j}{2}]} \le 2^{j-1}, \quad \text{for} \quad j\ge i, \quad i,j\in\mathbb{N}, \tag{3.8}$$

leads to the following estimates:

$$\begin{aligned}&\|V^{(i+1)}-V^{(i)}\|\\&\le\left(1-\psi(\sqrt{2}X^\infty)\right)^i\|V(0)\|\sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j\cdot(-1)^j}{\Gamma(\alpha(j+i)+1)}\right)\cdot t^{\alpha(j+i)},\\&\le\left(1-\psi(\sqrt{2}X^\infty)\right)^i\|V(0)\|\sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j}{\Gamma(\alpha(j+i)+1)}\right)\cdot t^{\alpha(j+i)}\\&\le\frac{1}{2}\left(1-\psi(\sqrt{2}X^\infty)\right)^i\|V(0)\|\sum_{j=0}^{\infty}\left(\frac{(2t^\alpha)^{j+i}}{\Gamma(\alpha(j+i)+1)}\right)\\&\le\frac{1}{2}\left(1-\psi(\sqrt{2}X^\infty)\right)^i\|V(0)\|\sum_{j=0}^{\infty}\left(\frac{(2t^\alpha)^{j}}{\Gamma(\alpha j+1)}\right)\\&=\frac{1}{2}\left(1-\psi(\sqrt{2}X^\infty)\right)^i\|V(0)\|E_{\alpha,1}(2t^\alpha),\end{aligned}$$

where we used the definition of the Mittag-Leffler function offered in Definition 2.3. Herein, we observe that $E_{\alpha,1}(2t^\alpha)$ is continuous on $[0,\bar{\tau}]$ to get the following estimate for some positive constant $C =: C(\bar{\tau},\psi,X^\infty,V(0))$:

$$\sup_{0\le t\le\bar{\tau}}\|V^{(i+1)}-V^{(i)}\|\le C(1-\psi(\sqrt{2}X^\infty))^i. \tag{3.9}$$

This implies that $\{V^{(i)}\}_{i=0}^{\infty}$ is a Cauchy sequence on $[0, \bar{\tau}]$ because

$$|1 - \psi(\sqrt{2}X^{\infty})| = 1 - \psi(\sqrt{2}X^{\infty}) < 1.$$

Hence, there exists a unique continuous solution V^{∞} satisfying the second equation of (3.3). Moreover, due to Proposition 2.6, V^{∞} is also unique continuous solution to (1.2) on $[0, \bar{\tau}]$. In addition, we note that (3.9) makes the proof of Lemma 3.2 rigorous.

Next, because $\|V\|$ can be estimated as follows:

$$\|V\| = \lim_{i\to\infty} \|V^{(i)}\| \le \sum_{i=0}^{\infty} \|V^{(i+1)} - V^{(i)}\|,$$

if we combine this with Lemma 3.2, then we attain that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \|V^{(k+1)} - V^{(k)}\| \\
&\le \|V(0)\| \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} \left(\frac{1-\psi(\sqrt{2}X^{\infty})}{-1}\right)^{i} \left(\frac{{}_jC_i \cdot (-1)^j}{\Gamma(\alpha j+1)}\right) \cdot t^{\alpha j} \\
&= \|V(0)\| \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j}}{\Gamma(\alpha j+1)} \sum_{i=0}^{j} \left(\frac{1-\psi(\sqrt{2}X^{\infty})}{-1}\right)^{i} {}_jC_i \\
&= \|V(0)\| \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j}}{\Gamma(\alpha j+1)} \left(\frac{1-\psi(\sqrt{2}X^{\infty})}{-1} + 1\right)^{j} \\
&= \|V(0)\| \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j}}{\Gamma(\alpha j+1)} \cdot \frac{(-\psi(\sqrt{2}X^{\infty}))^j}{(-1)^j} \\
&= \|V(0)\| \sum_{j=0}^{\infty} \frac{\left(-\psi(\sqrt{2}X^{\infty})t^{\alpha}\right)^j}{\Gamma(\alpha j+1)} = \|V(0)\| E_{\alpha,1}\left(-\psi(\sqrt{2}X^{\infty})t^{\alpha}\right),
\end{aligned}$$

where we used Definition 2.3. Then, we employ the first and second results of Proposition 2.1 with (3.2) and (3.4) to obtain that for $t \in [0, \bar{\tau}]$,

$$\begin{aligned}
\|X(t)\| &\le \|X(0)\| + \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (t-s)^{\alpha-1} \|V(s)\| ds \\
&\le \|X(0)\| + \frac{\|V(0)\|}{\Gamma(\alpha)} \int_0^{\infty} (t-s)^{\alpha-1} E_{\alpha,1}(-\psi(\sqrt{2}X^{\infty})s^{\alpha}) ds \\
&= \|X(0)\| + \|V(0)\| t^{\alpha} E_{\alpha,\alpha+1}(-\psi(\sqrt{2}X^{\infty})t^{\alpha}) \\
&= \|X(0)\| + \frac{\|V(0)\|}{\psi(\sqrt{2}X^{\infty})} \left(1 - E_{\alpha,1}\left(-\psi(\sqrt{2}X^{\infty})t^{\alpha}\right)\right) < \|X(0)\| + \frac{\|V(0)\|}{\psi(\sqrt{2}X^{\infty})} \le X^{\infty}.
\end{aligned}$$

Hence,

$$\lim_{t\to\bar{\tau}-} \|X(t)\| = \|X(\bar{\tau})\| < X^{\infty}.$$

However, the above estimate gives a contradiction to the definition of $\bar{\tau}$ because the definition of $\bar{\tau}$ yields

$$\lim_{t\to\bar{\tau}-} \|X(t)\| = \|X(\bar{\tau})\| = X^{\infty}.$$

Consequently, one has $\bar{\tau} = \tau$. Finally, we reach

$$\|X(t)\| \le X^{\infty}, \quad \|V(t)\| \le \|V(0)\| E_{\alpha,1}(-\psi(\sqrt{2}X^{\infty})t^{\alpha}), \quad \forall t \in [0,\tau).$$

Besides, from Proposition 2.6, the Cauchy–Lipschitz theory for fractional ODE, we can demonstrate the desired assertion, that is, $\tau = \infty$. □

4. Flocking dynamics of fractional TCS model

In this section, we provide several sufficient frameworks to guarantee the asymptotic flocking of the fractional TCS system (1.4). For this, we first present the following definition of the asymptotic flocking of (1.4):

Definition 4.1. *Let $Z =: (X, V, T)$ be a solution to the fractional TCS system* (1.4). *Then, the configuration Z exhibits asymptotic flocking if the following assertions hold:*

$$(i)\ \text{(Group formation)} \iff \sup_{t\in\mathbb{R}_+} \max_{i,j\in[N]} \|x_i(t) - x_j(t)\| < \infty.$$

$$(ii)\ \text{(Velocity alignment)} \iff \lim_{t\to\infty} \max_{i,j\in[N]} \|v_j(t) - v_i(t)\| = 0.$$

$$(iii)\ \text{(Temperature equilibrium)} \iff \lim_{t\to\infty} \max_{i,j\in[N]} |T_j(t) - T_i(t)| = 0.$$

Subsequently, we reformulate (1.4) using matrix representation in terms of X, V and $\bar{T}$ to derive its asymptotic flocking estimate:

$$\begin{cases} D^c_{\alpha} X(t) = V(t), \quad t > 0, \\ D^c_{\alpha} V(t) = \Phi(X(t), T(t))V(t), \\ D^c_{\alpha} \bar{T}(t) = Z(X(t), T(t))\bar{T}(t), \quad T_i(0) \in \mathbb{R}_+ - \{0\},\ i \in [N], \quad \mathbf{1}^T \bar{T}(0) = 0 \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $\mathbf{1} := (1, \cdots, 1)^T \in \mathbb{R}^{N\times 1}$, $\bar{T} =: (T_1 - T^{\infty}, \cdots, T_N - T^{\infty})^T$, and Φ and Z are defined as follows. To represent the following matrices concretely,

$$\Phi(X(t), T(t)) \in \mathbb{R}^{N\times N}, \quad Z(X(t), T(t)) \in \mathbb{R}^{N\times N},$$

we set each (i, j)-th element of $\Phi(X(t), T(t))$, $(\Phi(X(t), T(t)))_{ij} \in \mathbb{R}$, as

$$(\Phi(X(t), T(t)))_{ij} = \frac{1}{N} \begin{cases} \dfrac{\phi(\|x_i - x_j\|)}{T_j}, & \text{if} \quad i \ne j, \\ -\displaystyle\sum_{k\ne i} \dfrac{\phi(\|x_i - x_k\|)}{T_i}, & \text{if} \quad i = j, \end{cases} \quad \text{for} \quad i, j \in [N],$$

and each (i,j)-th element of $Z(X(t),T(t))$, $(Z(X(t),T(t)))_{ij}\in\mathbb{R}$, as

$$(Z(X(t),T(t)))_{ij}=\frac{1}{N}\begin{cases}\dfrac{\zeta(\|x_i-x_j\|)}{T_jT_i}, & \text{if}\quad i\neq j,\\ -\displaystyle\sum_{k\neq i}\dfrac{\zeta(\|x_i-x_k\|)}{T_kT_i}, & \text{if}\quad i=j,\end{cases}\qquad \text{for}\quad i,j\in[N].$$

Then, we present the desired main result on the asymptotic flocking dynamics of (1.4).

Theorem 4.1. *Suppose that there is a a nonnegative constant X^∞ such that $\zeta(\sqrt{2}X^\infty)>0$,*

$$\phi^\infty:=\frac{1}{T_m^\infty}-\frac{(2N-2)}{\sqrt{N}}\left(\frac{1}{T_m^\infty}-\frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}\right)>0,\quad \text{and}\quad \|X(0)\|+\frac{\|V(0)\|}{\phi^\infty}\leq X^\infty, \tag{4.2}$$

where we set two positive constants T_M^∞ and T_m^∞ as

$$T_M^\infty:=T^\infty+\|\bar{T}(0)\|>0,\quad T_m^\infty:=T^\infty-\|\bar{T}(0)\|>0.$$

Let (X,V,T) be a solution to (1.4) *on* $[0,\tau)$ *with the initial data* $(X(0),V(0),T(0))$. *Then, the global well-posedness of* (1.4) *holds, that is, $\tau=\infty$. Furthermore, we have the following asymptotic flocking estimate for $t\in\mathbb{R}_+$:*

$$\|X(t)\|\leq X^\infty,\quad \|V(t)\|\leq\|V(0)\|E_{\alpha,1}(-\phi^\infty t^\alpha),\quad \|\bar{T}(t)\|\leq\|\bar{T}(0)\|E_{\alpha,1}\left(-\frac{\zeta(\sqrt{2}X^\infty)}{T_M^\infty}\right).$$

Note that the sufficient conditions of Theorem 4.1 are admissible if $T_M^\infty-T_m^\infty$ is somewhat close to 0 and $\phi(\sqrt{2}X^\infty)$ is close to 1. To rigorously verify Theorem 4.1, we reorganize the fractional TCS system (4.1) as a matrix representation to get the flocking dynamics of (1.4) as follows:

$$\begin{cases}D_\alpha^cX(t)=V(t),\\ D_\alpha^cV(t)=-\dfrac{V(t)}{T_m^\infty}+\tilde{\Phi}(X(t),T(t))V(t),\\ D_\alpha^c\bar{T}(t)=-\dfrac{\bar{T}(t)}{(T_m^\infty)^2}+\tilde{Z}(X(t),T(t))\bar{T}(t),\end{cases}$$

where

$$(\tilde{\Phi}(X(t),T(t)))_{ij}=\frac{1}{N}\begin{cases}\left(\dfrac{\phi(\|x_i-x_j\|)}{T_j}-\dfrac{1}{T_m^\infty}\right), & \text{if}\quad i\neq j,\\ -\displaystyle\sum_{k\neq i}\left(\dfrac{\phi(\|x_i-x_k\|)}{T_i}-\dfrac{1}{T_m^\infty}\right), & \text{if}\quad i=j,\end{cases}\qquad \text{for}\quad i,j\in[N],$$

$$(\tilde{Z}(X(t),T(t)))_{ij}=\frac{1}{N}\begin{cases}\left(\dfrac{\zeta(\|x_i-x_j\|)}{T_jT_i}-\dfrac{1}{(T_m^\infty)^2}\right), & \text{if}\quad i\neq j,\\ -\displaystyle\sum_{k\neq i}\left(\dfrac{\zeta(\|x_i-x_k\|)}{T_kT_i}-\dfrac{1}{(T_m^\infty)^2}\right), & \text{if}\quad i=j,\end{cases}\qquad \text{for}\quad i,j\in[N].$$

Then, we apply Proposition 2.5 to express $X(t)$, $V(t)$ and $T(t)$ as the following integral equations:

$$\begin{aligned} X(t) &= X(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}V(s)ds, \\ V(t) &= E_{\alpha,1}\left(-\frac{t^\alpha}{T_m^\infty}\right)V(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t-s)^\alpha}{T_m^\infty}\right)\tilde{\Phi}(X(s),T(s))V(s)ds, \\ \bar{T}(t) &= E_{\alpha,1}\left(-\frac{t^\alpha}{(T_m^\infty)^2}\right)\bar{T}(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t-s)^\alpha}{(T_m^\infty)^2}\right)\tilde{Z}(X(s),T(s))\bar{T}(s)ds. \end{aligned} \tag{4.3}$$

As in the same method studied in Section 3, we construct the following set $\mathcal{S}$:

$$\mathcal{S} =: \{t \in (0,\tau] \mid \|X(s)\| \le X^\infty \text{ and } T_m^\infty \le T_i(s) \le T_M^\infty,\ i \in [N], \quad \forall s \in [0,t)\},$$

where it is nonempty due to the continuity of $X(s)$ and (4.2). Then, we first claim that

$$\sup \mathcal{S} =: \bar{\tau} = \tau. \tag{4.4}$$

For the proof by contradiction, suppose that

$$\sup \mathcal{S} = \bar{\tau} < \tau.$$

If we prove (4.4), then the Cauchy–Lipschitz theory for fractional ODE (see Proposition 2.6) with asymptotic flocking estimates leads to $\tau = \infty$.

Based on the iterative method used in Section 3, we can consider the following sequence $\{\bar{T}^{(i)}\}_{i=0}^\infty$ from the third equation of (4.3).

$$\begin{aligned} &\bar{T}^{(0)} = 0, \quad i \in \mathbb{N} \cup \{0\}, \\ &\bar{T}^{(i+1)} = E_{\alpha,1}\left(-\frac{t^\alpha}{(T_m^\infty)^2}\right)\bar{T}(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t-s)^\alpha}{(T_m^\infty)^2}\right)\tilde{Z}(X(s),T(s))\bar{T}^{(i)}(s)ds. \end{aligned} \tag{4.5}$$

To obtain a temperature equilibrium estimate from (4.5), we need to estimate the operator norm of $\tilde{Z}(X(t),T(t))$, $\|\tilde{Z}(X(t),T(t))\|_{\text{op}}$, on $t \in [0,\bar{\tau}]$ as in Lemma 3.1.

Lemma 4.1. *$\tilde{Z}(X,T) \in \mathbb{R}^{N\times N}$ is a positive semi-definite matrix satisfying*

$$\|\tilde{Z}(X,T)\|_{\text{op}} \le \frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}.$$

Proof. We omit this rigorous proof because it can be proved in the same way as Lemma 3.1. We observe that $\tilde{Z}(X,T)$ is symmetric and $\mathbf{1} := (1,\cdots,1)^T \in \mathbb{R}^{N\times 1}$ is the eigenvector of $\tilde{Z}(X,T)$ having eigenvalue 0. Then, we see that for any vector $v = (v_1,\cdots,v_N)^T \in \mathbf{1}^\perp$, where $v_i \in \mathbb{R},\ i \in [N]$,

$$\langle \tilde{Z}v, v\rangle \ge 0.$$

Hence, $\tilde{Z}(X,T)$ is a positive semi-definite matrix. Moreover,

$$\langle \tilde{Z}(X)v, v\rangle \le \frac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N\left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)\|v_i - v_j\|^2 = \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)\|v\|^2.$$

We get the desired assertion. □

In what follows, we concern the upper bound of $\|\bar{T}^{(i+1)} - \bar{T}^{(i)}\|$.

Lemma 4.2. *For $t \in [0, \bar{\tau}]$, the following assertion holds for $i \in \mathbb{N} \cup \{0\}$.*

$$\|\bar{T}^{(i+1)} - \bar{T}^{(i)}\| \leq \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)^i \|\bar{T}(0)\| \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^j}{\Gamma(\alpha(j+i)+1)}\right) \cdot t^{\alpha(j+i)}$$
$$= \left(\frac{(T_m^\infty)^2 \zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2} - 1\right)^i \|\bar{T}(0)\| \sum_{j=i}^{\infty} \left(\frac{{}_jC_i \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^j}{\Gamma(\alpha j+1)}\right) \cdot t^{\alpha j}.$$

Proof. We again employ inductive argument used in the proof of Lemma 3.2 to prove the desired result.

• (The case of $i = 0$): It follows from (4.5) and Definition 2.3 that

$$\|\bar{T}^{(1)} - \bar{T}^{(0)}\| \leq \|\bar{T}(0)\| E_{\alpha,1}\left(-\frac{t^\alpha}{(T_m^\infty)^2}\right) = \|\bar{T}(0)\| \sum_{j=0}^{\infty} \left(\frac{\left(-\frac{1}{(T_m^\infty)^2}\right)^j}{\Gamma(\alpha j+1)}\right) \cdot t^{\alpha j}.$$

• (The case of $i > 0$): Suppose that the desired result holds for all $k \leq i$, $k \in \mathbb{N} \cup \{0\}$. Then, using the inductive assumption, Definition 2.3, Lemma 4.1, and (3.7) yields

$$\begin{aligned}
&\|\bar{T}^{(i+2)} - \bar{T}^{(i+1)}\| \\
&\leq \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)^{i+1} \|\bar{T}(0)\| \\
&\quad \times \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{(t-s)^\alpha}{(T_m^\infty)^2}\right) \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^j}{\Gamma(\alpha(j+i)+1)}\right) \cdot s^{\alpha(j+i)} ds \\
&= \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)^{i+1} \|\bar{T}(0)\| \\
&\quad \times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^{k+j}}{\Gamma(\alpha(k+1))\Gamma(\alpha(j+i)+1)}\right) \cdot \int_0^t (t-s)^{\alpha(k+1)-1} s^{\alpha(j+i)} ds \\
&= \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)^{i+1} \|\bar{T}(0)\| \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^{k+j}}{\Gamma(\alpha(k+j+i+1)+1)}\right) \cdot t^{\alpha(k+j+i+1)}.
\end{aligned}$$

For the same reason as the proof of Theorem 3.1, one can show that the following term is absolutely convergent:

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^{k+j}}{\Gamma(\alpha(k+j+i+1)+1)}\right) \cdot t^{\alpha(k+j+i+1)}.$$

Then, we can obtain

$$\left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)^{i+1} \|\bar{T}(0)\| \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^{k+j}}{\Gamma(\alpha(k+j+i+1)+1)}\right) \cdot t^{\alpha(k+j+i+1)}$$

$$= \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2} \right)^{i+1} \|\bar{T}(0)\| \sum_{j=0}^{\infty} \sum_{l=j}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^l}{\Gamma(\alpha(l+i+1)+1)} \right) \cdot t^{\alpha(l+i+1)}$$

$$= \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2} \right)^{i+1} \|\bar{T}(0)\| \sum_{l=0}^{\infty} \sum_{j=0}^{l} {}_{(j+i)}C_j \left(\frac{\left(-\frac{1}{(T_m^\infty)^2}\right)^l}{\Gamma(\alpha(l+i+1)+1)} \right) \cdot t^{\alpha(l+i+1)}$$

$$= \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2} \right)^{i+1} \|\bar{T}(0)\| \sum_{l=0}^{\infty} \left(\frac{{}_{(l+i+1)}C_l \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^l}{\Gamma(\alpha(l+i+1)+1)} \right) \cdot t^{\alpha(l+i+1)},$$

where we used

$$\sum_{j=0}^{l} {}_{(j+i)}C_j = {}_{(l+i+1)}C_l.$$

In conclusion, we reach the desired assertion. □

Thus, the above Lemma 4.2, (4.5) and (3.8) induce that

$$\begin{aligned}
&\|\bar{T}^{(i+1)} - \bar{T}^{(i)}\| \\
&\le \left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2} \right)^i \|\bar{T}(0)\| \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{(T_m^\infty)^2}\right)^j}{\Gamma(\alpha(j+i)+1)} \right) \cdot t^{\alpha(j+i)} \\
&\le \frac{\left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2} \right)^i}{\left(\frac{1}{(T_m^\infty)^2}\right)^i} \|\bar{T}(0)\| \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(\frac{1}{(T_m^\infty)^2}\right)^{j+i}}{\Gamma(\alpha(j+i)+1)} \right) \cdot t^{\alpha(j+i)} \\
&\le \frac{1}{2} \frac{\left(\frac{1}{(T_m^\infty)^2} - \frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2} \right)^i}{\left(\frac{1}{(T_m^\infty)^2}\right)^i} \|\bar{T}(0)\| \sum_{j=0}^{\infty} \left(\frac{\left(\frac{2t^\alpha}{(T_m^\infty)^2}\right)^{j+i}}{\Gamma(\alpha(j+i)+1)} \right) \\
&= \frac{1}{2} \left(1 - \frac{(T_m^\infty)^2 \zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2} \right)^i \|\bar{T}(0)\| E\left(\frac{2t^\alpha}{(T_m^\infty)^2} \right),
\end{aligned}$$

where we employed the definition of the Mittag-Leffler stated in Definition of 2.3. Hence, one can obtain a unique continuous solution $\bar{T}^\infty$ satisfying the third equation of (4.3) in the same way as the proof of Theorem 3.1. Moreover, $\bar{T}^\infty$ is a also unique continuous solution to (1.4) on $[0, \bar{\tau}]$ because of Proposition 2.6 and the uniform boundedness of T_i.

Then, we have from Definition 2.3 that for $t \in [0, \bar{\tau}]$,

$$
\begin{aligned}
\|\bar{T}\| &= \lim_{i\to\infty}\|\bar{T}^{(i)}\| \le \sum_{i=0}^{\infty}\|\bar{T}^{(i+1)}-\bar{T}^{(i)}\| \\
&\le \sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\left(\frac{(T_m^\infty)^2\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}-1\right)^i \|\bar{T}(0)\|\left(\frac{{}_jC_i\cdot\left(-\frac{1}{(T_m^\infty)^2}\right)^j}{\Gamma(\alpha j+1)}\right)\cdot t^{\alpha j} \\
&= \|\bar{T}(0)\|\sum_{j=0}^{\infty}\frac{\left(-\frac{1}{(T_m^\infty)^2}\right)^j t^{\alpha j}}{\Gamma(\alpha j+1)}\sum_{i=0}^{j}\left(\frac{(T_m^\infty)^2\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}-1\right)^i {}_jC_i \\
&= \|\bar{T}(0)\|\sum_{j=0}^{\infty}\frac{\left(-\frac{1}{(T_m^\infty)^2}\right)^j t^{\alpha j}}{\Gamma(\alpha j+1)}\left(\frac{(T_m^\infty)^2\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)^j \\
&= \|\bar{T}(0)\|\sum_{j=0}^{\infty}\frac{\left(\frac{-\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\right)^j t^{\alpha j}}{\Gamma(\alpha j+1)} = \|\bar{T}(0)\|E_{\alpha,1}\left(-\frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}t^\alpha\right).
\end{aligned}
\tag{4.6}
$$

Finally, we move on to study estimates regarding $\|V^{(i+1)}-V^{(i)}\|$ to conclude the proof of Theorem 4.1. To do this, we consider the following sequence $\{V^{(i)}\}_{i=0}^\infty$ in the third equation of (4.3) to estimate $\|V\|$:

$$
\begin{aligned}
&V^{(0)} = 0, \quad i\in\mathbb{N}\cup\{0\}, \\
&V^{(i+1)} = E_{\alpha,1}\left(-\frac{t^\alpha}{T_m^\infty}\right)V(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t-s)^\alpha}{T_m^\infty}\right)\tilde{\Phi}(X(s),T(s))V^{(i)}(s)ds,
\end{aligned}
\tag{4.7}
$$

where we crucially notice that $\tilde{\Phi}(X(s),T(s))$ on $t\in[0,\bar{\tau}]$ can not be estimated in the same way as the proof of Lemma 3.1 and Lemma 4.1. Indeed, $\tilde{\Phi}(X(s),T(s))$ is not symmetric (thus, not positive semi-definite matrix). Therefore, we have to use a different method.

Lemma 4.3. $\tilde{\Phi}(X,T)\in\mathbb{R}^{N\times N}$ *satisfies the following inequality:*

$$
\|\tilde{\Phi}(X,T)\|_{\rm op} \le \frac{(2N-2)}{\sqrt{N}}\left(\frac{1}{T_m^\infty}-\frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}\right).
$$

Proof. We use the following well-known fact that for a matrix A contained in $\mathbb{R}^{N\times N}$,

$$
\|A\|_{\rm op} \le \sqrt{N}\|A\|_\infty := \sqrt{N}\max_{i\in[N]}\sum_{j=1}^{N}|A_{ij}|,
$$

to find that

$$
\begin{aligned}
&\|\tilde{\Phi}(X,T)\|_{\rm op} \\
&\le \frac{1}{\sqrt{N}}\max_{i\in[N]}\left(\sum_{j\in[N],j\neq i}\left|\frac{\phi(\|x_i-x_j\|)}{T_j}-\frac{1}{T_m^\infty}\right| + \sum_{k\neq i}\left|\frac{\phi(\|x_i-x_k\|)}{T_i}-\frac{1}{T_m^\infty}\right|\right).
\end{aligned}
$$

Here, we use the definition of $\mathcal{S}$ to lead to

$$
\|\tilde{\Phi}(X,T)\|_{\rm op} \le \frac{(2N-2)}{\sqrt{N}}\left(\frac{1}{T_m^\infty}-\frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}\right),
$$

where we applied the following estimate:

$$\left|\frac{\phi(\|x_i - x_j\|)}{T_j} - \frac{1}{T_m^\infty}\right| \leq \frac{1}{T_m^\infty} - \frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}.$$

We conclude the desired lemma. □

Next, we provide the following lemma to deal with $\|V^{(i+1)} - V^{(i)}\|$:

Lemma 4.4. *For $t \in [0, \bar{\tau}]$, the following assertion holds for $i \in \mathbb{N} \cup \{0\}$.*

$$\begin{aligned}
&\|V^{(i+1)} - V^{(i)}\| \\
&\leq \left(\frac{(2N-2)}{\sqrt{N}}\left(\frac{1}{T_m^\infty} - \frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}\right)\right)^i \|V(0)\| \sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{T_m^\infty}\right)^j}{\Gamma(\alpha(j+i)+1)}\right) \cdot t^{\alpha(j+i)} \\
&= \left(-\frac{(2N-2)}{\sqrt{N}}\left(1 - \frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty}\right)\right)^i \|V(0)\| \sum_{j=i}^{\infty}\left(\frac{{}_{j}C_i \cdot \left(-\frac{1}{T_m^\infty}\right)^j}{\Gamma(\alpha j+1)}\right) \cdot t^{\alpha j}.
\end{aligned}$$

Proof. We use induction to obtain the desired estimate.

• (The case of $i = 0$): It follows from (4.7) and Definition 2.3 that

$$\|V^{(1)} - V^{(0)}\| \leq \|V(0)\| E_{\alpha,1}\left(-\frac{t^\alpha}{T_m^\infty}\right) = \|V(0)\| \sum_{j=0}^{\infty}\left(\frac{\left(-\frac{1}{T_m^\infty}\right)^j}{\Gamma(\alpha j+1)}\right) \cdot t^{\alpha j}.$$

• (The case of $i > 0$): We assume that the desired estimate holds for all $k \leq i$, $k \in \mathbb{N} \cup \{0\}$. Then, we apply the same way as in Lemma 3.2 and Lemma 4.2 to have that

$$\begin{aligned}
&\|V^{(i+2)} - V^{(i+1)}\| \\
&\leq \left(\frac{(2N-2)}{\sqrt{N}}\left(\frac{1}{T_m^\infty} - \frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}\right)\right)^{i+1} \|V(0)\| \\
&\quad \times \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{(t-s)^\alpha}{T_m^\infty}\right) \sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{T_m^\infty}\right)^j}{\Gamma(\alpha(j+i)+1)}\right) \cdot s^{\alpha(j+i)} ds \\
&= \left(\frac{(2N-2)}{\sqrt{N}}\left(\frac{1}{T_m^\infty} - \frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}\right)\right)^{i+1} \|V(0)\| \\
&\quad \times \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{T_m^\infty}\right)^{k+j}}{\Gamma(\alpha(k+j+i+1)+1)}\right) \cdot t^{\alpha(k+j+i+1)}.
\end{aligned}$$

Hence, using the same arguments as in the proofs of Lemma 3.2 and Lemma 4.2, we get that

$$\left(\frac{(2N-2)}{\sqrt{N}}\left(\frac{1}{T_m^\infty} - \frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}\right)\right)^{i+1} \|V(0)\|$$

$$\times \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{1}{T_m^\infty}\right)^{k+j}}{\Gamma(\alpha(k+j+i+1)+1)} \right) \cdot t^{\alpha(k+j+i+1)}$$

$$= \left(\frac{(2N-2)}{\sqrt{N}} \left(\frac{1}{T_m^\infty} - \frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty} \right) \right)^{i+1} \|V(0)\| \sum_{l=0}^{\infty} \left(\frac{{}_{(l+i+1)}C_l \cdot \left(-\frac{1}{T_m^\infty}\right)^{l}}{\Gamma(\alpha(l+i+1)+1)} \right) \cdot t^{\alpha(l+i+1)}.$$

Thus, we acquire the desired result. □

Then, we verify Theorem 4.1 for the asymptotic flocking of (1.4) employing the previous lemmas and propositions.

Proof of Theorem 4.1. We use (3.8), (4.7), Lemma 4.3, and Lemma 4.4 to estimate that

$$\begin{aligned}
&\|V^{(i+1)} - V^{(i)}\| \\
&\le \left(-\frac{(2N-2)}{\sqrt{N}} \left(1 - \frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty} \right) \right)^{i} \|V(0)\| \sum_{j=i}^{\infty} \left(\frac{{}_jC_i \cdot \left(-\frac{1}{T_m^\infty}\right)^{j}}{\Gamma(\alpha j+1)} \right) \cdot t^{\alpha j}, \\
&\le \left(-\frac{(2N-2)}{\sqrt{N}} \left(1 - \frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty} \right) \right)^{i} \|V(0)\| \sum_{j=0}^{\infty} \left(\frac{{}_{(j+i)}C_j \cdot \left(\frac{1}{T_m^\infty}\right)^{j+i}}{\Gamma(\alpha(j+i)+1)} \right) \cdot t^{\alpha(j+i)} \\
&\le \frac{1}{2} \left(\frac{(2N-2)}{\sqrt{N}} \left(1 - \frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty} \right) \right)^{i} \|V(0)\| E_{\alpha,1}\left(\frac{2t^\alpha}{T_m^\infty} \right).
\end{aligned}$$

Due to the continuity of $E_{\alpha,1}\left(\frac{2t^\alpha}{T_m^\infty}\right)$ on $[0,\bar{\tau}]$, there exists a positive real number $C =: C(\bar{\tau}, T(0), \phi, X^\infty)$ such that

$$\sup_{0\le t\le \bar{\tau}} \|V^{(i+1)} - V^{(i)}\| \le C \left(\frac{(2N-2)}{\sqrt{N}} \left(1 - \frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty} \right) \right)^{i},$$

which yields that $\{V^{(i)}\}_{i=0}^\infty$ is a Cauchy sequence on $[0,\bar{\tau}]$ from (4.2), that is,

$$0 \le \frac{(2N-2)}{\sqrt{N}} \left(1 - \frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty} \right) < 1.$$

Then, there exists a unique continuous solution V^∞ satisfying the third equation of (4.3). Therefore, V^∞ is a also the unique continuous solution to (1.4) on $[0,\bar{\tau}]$ by Proposition 2.6. Moreover, we can estimate $\|V\|$ as follows:

$$\begin{aligned}
\|V\| &= \lim_{i\to\infty} \|V^{(i)}\| \le \sum_{i=0}^{\infty} \|V^{(i+1)} - V^{(i)}\|, \\
&\le \|V(0)\| \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \left(-\frac{(2N-2)}{\sqrt{N}} \left(1 - \frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty} \right) \right)^{i} \left(\frac{{}_jC_i \cdot \left(-\frac{1}{T_m^\infty}\right)^{j}}{\Gamma(\alpha j+1)} \right) \cdot t^{\alpha j}
\end{aligned}$$

$$
\begin{aligned}
&= \|V(0)\| \sum_{j=0}^{\infty} \frac{\left(-\frac{1}{T_m^\infty}\right)^j t^{\alpha j}}{\Gamma(\alpha j+1)} \sum_{i=0}^{j} \left(-\frac{(2N-2)}{\sqrt{N}}\left(1-\frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty}\right)\right)^i {}_jC_i \\
&= \|V(0)\| \sum_{j=0}^{\infty} \frac{\left(-\frac{1}{T_m^\infty}\right)^j t^{\alpha j}}{\Gamma(\alpha j+1)} \left(1-\frac{(2N-2)}{\sqrt{N}}\left(1-\frac{\phi(\sqrt{2}X^\infty)T_m^\infty}{T_M^\infty}\right)\right)^j \\
&= \|V(0)\| E_{\alpha,1}\left(-\left(\frac{1}{T_m^\infty}-\frac{(2N-2)}{\sqrt{N}}\left(\frac{1}{T_m^\infty}-\frac{\phi(\sqrt{2}X^\infty)}{T_M^\infty}\right)\right)t^\alpha\right) \\
&= \|V(0)\| E_{\alpha,1}\left(-\phi^\infty t^\alpha\right).
\end{aligned}
$$

Accordingly, the first and second assertions of Proposition 2.1, (4.2), and the definition of $\mathcal{S}$ imply that

$$
\begin{aligned}
\|X(t)\| &\le \|X(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^\infty (t-s)^{\alpha-1}\|V(s)\|ds \\
&\le \|X(0)\| + \frac{\|V(0)\|}{\Gamma(\alpha)}\int_0^\infty (t-s)^{\alpha-1}E_{\alpha,1}(-\phi^\infty s^\alpha)ds \\
&= \|X(0)\| + \|V(0)\| t^\alpha E_{\alpha,\alpha+1}(-\phi^\infty t^\alpha) \\
&= \|X(0)\| + \frac{\|V(0)\|}{\phi^\infty}\left(1-E_{\alpha,1}\left(-\phi^\infty t^\alpha\right)\right) \\
&< \|X(0)\| + \frac{\|V(0)\|}{\phi^\infty} \le X^\infty, \quad t\in[0,\bar{\tau}].
\end{aligned}
$$

Hence, we can reach

$$
\lim_{t\to\bar{\tau}-}\|X(t)\| = \|X(\bar{\tau})\| < X^\infty. \tag{4.8}
$$

Subsequently, we observe from (4.6) that

$$
\lim_{t\to\bar{\tau}-}\|\bar{T}(t)\| = \|\bar{T}(\bar{\tau})\| \le \|\bar{T}(0)\|E_{\alpha,1}\left(-\frac{\zeta(\sqrt{2}X^\infty)}{(T_M^\infty)^2}\bar{\tau}^\alpha\right) < \|\bar{T}(0)\|.
$$

Then, because

$$
T_i(\bar{\tau}) - T^\infty \le \|\bar{T}(\bar{\tau})\| \quad \text{and} \quad T^\infty - T_i(\bar{\tau}) \le \|\bar{T}(\bar{\tau})\|, \quad i\in[N],
$$

we attain that for $i \in [N]$,

$$
T_i(\bar{\tau}) < T^\infty + \|\bar{T}(0)\| = T_M^\infty, \quad T_i(\bar{\tau}) > T^\infty - \|\bar{T}(0)\| = T_m^\infty. \tag{4.9}
$$

Thus, if we combine (4.8) and (4.9), then $\bar{\tau} < \tau$ is contradictory, and we obtain

$$
\bar{\tau} = \tau.
$$

Finally, we have the following estimates from the above arguments for $t \in [0,\tau)$:

$$
\|X(t)\| \le X^\infty, \quad \|V(t)\| \le \|V(0)\|E_{\alpha,1}(-\phi^\infty t^\alpha), \quad \|\bar{T}(t)\| \le \|\bar{T}(0)\|E_{\alpha,1}\left(-\frac{\zeta(\sqrt{2}X^\infty)}{T_M^\infty}\right).
$$

Using the Cauchy–Lipschitz theory for fractional ODE (see Proposition 2.6), one has $\tau = \infty$ and therefore, we demonstrate the desired theorem. □

5. Flocking dynamics of fractional RCS model

In this section, we describe appropriate sufficient framework for the flocking dynamics of the fractional RCS system (1.7). To do this, we first provide the following basic concept for the asymptotic flocking of (1.7):

Definition 5.1. *Let* $Z =: (X, W)$ *be a solution to* (1.7).

$$(i) \text{ (Group formation) } \iff \sup_{t\in\mathbb{R}_+} \max_{i,j\in[N]} \|x_i(t) - x_j(t)\| < \infty.$$

$$(ii) \text{ (Relativistic velocity alignment) } \iff \lim_{t\to\infty} \max_{i,j\in[N]} \|w_j(t) - w_i(t)\| = 0.$$

$$(iii) \text{ (Velocity alignment) } \iff \lim_{t\to\infty} \max_{i,j\in[N]} \|v_j(t) - v_i(t)\| = 0.$$

If (i) *and* (ii) *hold, then the configuration Z exhibits asymptotic flocking, and if* (i) *and* (ii) *hold, then the configuration Z exhibits relativistic asymptotic flocking.*

In fact, (ii) and (iii) in Definition 5.1 are equivalent if each speed of all particles along (1.7) is strictly less than the speed of light, c. Indeed,

Lemma 5.1. [49] *For two vectors* $w, w_* \in \mathbb{R}^d$, *let* $w := Fv$ *and* $w_* := F_* v_*$, *where*

$$F := \Gamma\left(1 + \frac{\Gamma}{c^2}\right), \quad F_* := \Gamma_*\left(1 + \frac{\Gamma_*}{c^2}\right), \quad \Gamma := \frac{c}{\sqrt{c^2 - \|v\|^2}}, \quad \Gamma_* := \frac{c}{\sqrt{c^2 - \|v_*\|^2}}.$$

If there exists a nonnegative constant $v^\infty \geq 0$ *such that*

$$\|v\|, \|v_*\| \leq v^\infty < c,$$

then $\|v - v_*\|$ *and* $\|w - w_*\|$ *are equivalent. Moreover,*

$$\frac{c^2 + 1}{c^2}\|v - v_*\| \leq \|w - w_*\| \leq (g'(v^\infty)v^\infty + g(v^\infty))\|v - v_*\|,$$

where g is defined in (1.6).

Due to the above lemma, if we can guarantee that velocity of each particle is strictly less than c uniformly in time, then we can easily check that the relativistic velocity alignment and velocity alignment are equivalent and furthermore, the relativistic asymptotic flocking and asymptotic flocking are equivalent in (1.7).

Subsequently, we give the following matrix representation of (1.7) to construct suitable sufficient framework for the asymptotic flocking of (1.7):

$$D^c_\alpha X(t) = V(t), \quad D^c_\alpha W(t) = P(X(t))V(t), \tag{5.1}$$

where $P(X(t)) \in \mathbb{R}^{N\times N}$ is a matrix whose each (i, j)-th element, $(P(X(t)))_{ij} \in \mathbb{R}$ is as follows:

$$(P(X(t)))_{ij} = \frac{1}{N}\begin{cases} \rho(\|x_i - x_j\|), & \text{if} \quad i \neq j, \\ -\sum_{k\neq i}\rho(\|x_i - x_k\|), & \text{if} \quad i = j, \end{cases} \qquad \text{for} \quad i, j \in [N].$$

Then, the goal of this section is to prove the following asymptotic flocking of (1.7):

Theorem 5.1. *Assume that there exist two nonnegative numbers X^∞ and v^∞ such that*

$$\|X(0)\| + \frac{c^2\|W(0)\|}{(c^2+1)\rho^\infty} \le X^\infty \quad \text{and} \quad \max_{i\in[N]}\|w_i(0)\| + \frac{\sqrt{2}(N-1)c^2\|W(0)\|}{(c^2+1)N\rho^\infty} \le g(v^\infty) < \infty, \tag{5.2}$$

where

$$\rho^\infty =: \frac{c^2}{c^2+1} - \frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1} - \frac{\rho(\sqrt{2}X^\infty)}{g(v^\infty)}\right) > 0,$$

and let (X, W) be a solution to (1.7) on $[0,\tau)$ with the initial data $(X(0), W(0))$. Then, we obtain the global well-posedness of (1.7)

$$\tau = \infty$$

and moreover, the following asymptotic flocking with an algebraic decay holds:

$$\|X(t)\| \le X^\infty, \quad \|W(t)\| \le \|W(0)\|E_{\alpha,1}(-\rho^\infty t^\alpha), \quad \forall t \in \mathbb{R}_+.$$

We note that the sufficient conditions described in Theorem 5.1 are admissible by taking $v^\infty \ll c$. For the rigorous verification of Theorem 5.1, we transform (5.1) to the following system to obtain the flocking dynamics with an algebraic decay:

$$D^c_\alpha X(t) = V(t), \quad D^c_\alpha W(t) = -\frac{c^2}{c^2+1}W(t) + \tilde{P}(X(t), W(t))W(t),$$

where

$$(\tilde{P}(X(t), W(t)))_{ij} = \frac{1}{N}\begin{cases} \left(\dfrac{\rho(\|x_i - x_j\|)}{F_j} - \dfrac{c^2}{c^2+1}\right), & \text{if } \ i \neq j, \\ -\displaystyle\sum_{k\neq i}\left(\dfrac{\rho(\|x_i - x_k\|)}{F_i} - \dfrac{c^2}{c^2+1}\right), & \text{if } \ i = j, \end{cases} \quad \text{for } \ i, j \in [N],$$

Then, by Proposition 2.5, we can represent two matrix solutions $X(t)$ and $W(t)$ as the following integral forms:

$$\begin{aligned} X(t) &= X(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}V(s)ds, \\ W(t) &= E_{\alpha,1}\left(-\frac{c^2}{c^2+1}t^\alpha\right)W(0) \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{c^2}{c^2+1}(t-s)^\alpha\right)\tilde{P}(X(s), W(s))W(s)ds. \end{aligned} \tag{5.3}$$

Next, we define a set $\mathcal{S}$ by

$$\mathcal{S} =: \left\{ t \in (0,\tau] \,\middle|\, \|X(s)\| \le X^\infty \quad \text{and} \quad \max_{i\in[N]}\|v_i(s)\| \le v^\infty, \quad \forall s \in [0,t) \right\},$$

where we observe that the set $\mathcal{S}$ is nonempty from the continuity of $X(s)$ and (5.2). From now on, we claim that

$$\sup\mathcal{S} =: \bar{\tau} = \tau.$$

For the proof by contradiction, we suppose $\sup \mathcal{S} = \bar{\tau} < \tau$. Then, we can attain the desired estimates by showing that this statement is contradictory. Based on the iterative method as in Sections 3 and 4, we consider the following recurrence relation in terms of sequence $\{W^{(i)}\}_{i=0}^{\infty}$ from the second assertion of (5.3):

$$\begin{aligned} &W^{(0)} = 0, \quad i \in \mathbb{N} \cup \{0\}, \\ &W^{(i+1)} = E_{\alpha,1}\left(-\frac{c^2}{c^2+1}t^{\alpha}\right)W(0) \\ &\qquad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{c^2}{c^2+1}(t-s)^{\alpha}\right)\tilde{P}(X(s),W(s))W^{(i)}(s)ds. \end{aligned} \tag{5.4}$$

To study the recurrence relation (5.4), we estimate the operator norm of $\tilde{P}(X, W)$ on $t \in [0, \bar{\tau}]$ as follows:

Lemma 5.2. *The operator norm of $\tilde{P}(X, W) \in \mathbb{R}^{N\times N}$ can be estimated as*

$$\|\tilde{P}(X,W)\|_{\mathrm{op}} \le \frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1} - \frac{\rho(\sqrt{2}X^{\infty})}{g(v^{\infty})}\right), \tag{5.5}$$

where v^{∞} is defined in Theorem 5.1.

Proof. Employing the following relation that for a matrix $A \in \mathbb{R}^{N\times N}$,

$$\|A\|_{\mathrm{op}} \le \sqrt{N}\|A\|_{\infty} := \sqrt{N}\max_{i\in[N]}\sum_{j=1}^{N}|A_{ij}|,$$

yields that

$$\begin{aligned} \|\tilde{P}(X,W)\|_{\mathrm{op}} &\le \frac{1}{\sqrt{N}}\max_{i\in[N]}\left(\sum_{j=1}^{N}\left|\frac{\rho(\|x_i-x_j\|)}{F_j} - \frac{c^2}{c^2+1}\right| + \sum_{k\neq i}\left|\frac{\rho(\|x_i-x_k\|)}{F_i} - \frac{c^2}{c^2+1}\right|\right) \\ &\le \frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1} - \frac{\rho(\sqrt{2}X^{\infty})}{g(v^{\infty})}\right), \end{aligned}$$

where we used

$$\|x_i - x_j\| \le \sqrt{2}\|X\| \le \sqrt{2}X^{\infty}, \quad 1 + \frac{1}{c^2} \le F_j = g(\|v_j\|) \le g(v^{\infty}).$$

□

Subsequently, we prove the following lemma to estimate $\|W^{(i+1)} - W^{(i)}\|$, which will be crucially used to derive the relativistic velocity alignment of (1.7).

Lemma 5.3. *For $t \in [0, \bar{\tau}]$, the following assertion holds for $i \in \mathbb{N} \cup \{0\}$:*

$$\|W^{(i+1)} - W^{(i)}\|$$

$$\leq \left(\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(v^\infty)}\right)\right)^i \|W(0)\| \sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{c^2}{c^2+1}\right)^j}{\Gamma(\alpha(j+i)+1)}\right)\cdot t^{\alpha(j+i)}$$

$$= \left(-\frac{(2N-1)}{\sqrt{N}}\left(1-\frac{(c^2+1)\rho(\sqrt{2}X^\infty)}{c^2 g(v^\infty)}\right)\right)^i \|W(0)\| \sum_{j=i}^{\infty}\left(\frac{{}_jC_i \cdot \left(-\frac{c^2}{c^2+1}\right)^j}{\Gamma(\alpha j+1)}\right)\cdot t^{\alpha j}.$$

Proof. First, we use induction to have the desired result.

• (The case of $i = 0$): From (5.4) and Definition 2.3, we immediately obtain

$$\|W^{(1)} - W^{(0)}\| \leq \|W(0)\| E_{\alpha,1}\left(-\frac{c^2}{c^2+1}t^\alpha\right) = \|V(0)\| \sum_{j=0}^{\infty}\left(\frac{\left(-\frac{c^2}{c^2+1}\right)^j}{\Gamma(\alpha j+1)}\right)\cdot t^{\alpha j}.$$

• (The case of $i > 0$): We suppose that the desired result holds for all $k \leq i$, $k \in \mathbb{N}\cup\{0\}$. Then, it follows from (5.4), (5.5), and (3.7) that

$$\begin{aligned}
&\|W^{(i+2)} - W^{(i+1)}\| \\
&\leq \left(\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(v^\infty)}\right)\right)^{i+1} \|W(0)\| \\
&\quad \times \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{c^2}{c^2+1}(t-s)^\alpha\right)\sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{c^2}{c^2+1}\right)^j}{\Gamma(\alpha(j+i)+1)}\right)\cdot s^{\alpha(j+i)}ds \\
&= \left(\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(v^\infty)}\right)\right)^{i+1} \|W(0)\| \\
&\quad \times \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{{}_{(j+i)}C_j \cdot \left(-\frac{c^2}{c^2+1}\right)^{k+j}}{\Gamma(\alpha(k+j+i+1)+1)}\right)\cdot t^{\alpha(k+j+i+1)} =: I,
\end{aligned}$$

Here, we can show the following inequality applying the same methods as in the proofs of Lemma 3.2, Lemma 4.2, and Lemma 4.4:

$$\begin{aligned}
I &\leq \left(\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(v^\infty)}\right)\right)^{i+1} \|W(0)\| \\
&\quad \times \sum_{l=0}^{\infty}\left(\frac{{}_{(l+i+1)}C_l \cdot \left(-\frac{c^2}{c^2+1}\right)^l}{\Gamma(\alpha(l+i+1)+1)}\right)\cdot t^{\alpha(l+i+1)}.
\end{aligned}$$

Accordingly, we acquire the desired result. □

As a final step, we verify the main result of this section, Theorem 5.1, with preparatory frameworks and continuous arguments.

Proof of Theorem 5.1. We employ (3.8), (5.2), (5.4), and Lemma 5.3 to deduce that

$$\|W^{(i+1)} - W^{(i)}\|$$

$$\le \left(\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(\nu^\infty)}\right)\right)^i \|W(0)\| \sum_{j=i}^{\infty}\left(\frac{{}_jC_i\cdot\left(-\frac{c^2t^\alpha}{c^2+1}\right)^j}{\Gamma(\alpha j+1)}\right)$$

$$\le \left(\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(\nu^\infty)}\right)\right)^i \|W(0)\| \sum_{j=i}^{\infty}\left(\frac{{}_jC_i\cdot\left(\frac{c^2t^\alpha}{c^2+1}\right)^j}{\Gamma(\alpha j+1)}\right)$$

$$\le \left(\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(\nu^\infty)}\right)\right)^i \|W(0)\| \sum_{j=i}^{\infty}\left(\frac{2^{j-1}\cdot\left(\frac{c^2t^\alpha}{c^2+1}\right)^j}{\Gamma(\alpha j+1)}\right)$$

$$\le \frac{1}{2}\left(\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(\nu^\infty)}\right)\right)^i \|W(0)\| E_{\alpha,1}\left(\frac{2c^2t^\alpha}{c^2+1}\right).$$

Using the continuity of $E_{\alpha,1}\left(\frac{2c^2t^\alpha}{c^2+1}\right)$ on $[0,\bar{\tau}]$ and definition of ρ^∞, i.e.,

$$0 \le \frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(\nu^\infty)}\right) < 1,$$

it follows that $\{W^{(i)}\}_{i=0}^\infty$ is a Cauchy sequence on $[0,\bar{\tau}]$, which gives a unique existence of continuous solution W^∞ of the second equation of (5.3) and $(1.7)_2$ due to Proposition 2.6. In addition, $\|W\|$ can be estimated as below using the methodologies employed in the proof of Theorem 3.1 and Theorem 4.1:

$$\begin{aligned}
\|W\| &= \lim_{i\to\infty}\|W^{(i)}\| \le \sum_{i=0}^{\infty}\|W^{(i+1)}-W^{(i)}\|,\\
&\le \|W(0)\|\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\left(-\frac{(2N-1)}{\sqrt{N}}\left(1-\frac{(c^2+1)\rho(\sqrt{2}X^\infty)}{c^2g(\nu^\infty)}\right)\right)^i\left(\frac{{}_jC_i\cdot\left(-\frac{c^2t^\alpha}{c^2+1}\right)^j}{\Gamma(\alpha j+1)}\right)\\
&= \|W(0)\|\sum_{j=0}^{\infty}\frac{\left(-\frac{c^2}{c^2+1}\right)^j t^{\alpha j}}{\Gamma(\alpha j+1)}\sum_{i=0}^{j}\left(-\frac{(2N-1)}{\sqrt{N}}\left(1-\frac{(c^2+1)\rho(\sqrt{2}X^\infty)}{c^2g(\nu^\infty)}\right)\right)^i {}_jC_i \qquad (5.6)\\
&= \|W(0)\|\sum_{j=0}^{\infty}\frac{\left(-\frac{c^2}{c^2+1}\right)^j t^{\alpha j}}{\Gamma(\alpha j+1)}\left(1-\frac{(2N-1)}{\sqrt{N}}\left(1-\frac{(c^2+1)\rho(\sqrt{2}X^\infty)}{c^2g(\nu^\infty)}\right)\right)^j\\
&= \|W(0)\|E_{\alpha,1}\left(-\frac{c^2}{c^2+1}+\frac{(2N-1)}{\sqrt{N}}\left(\frac{c^2}{c^2+1}-\frac{\rho(\sqrt{2}X^\infty)}{g(\nu^\infty)}\right)\right) = \|W(0)\|E_{\alpha,1}\left(-\rho^\infty t^\alpha\right).
\end{aligned}$$

From the above assertion, we apply the first and second results of Proposition 2.1 and (5.2) to demonstrate that for $t\in[0,\bar{\tau}]$,

$$\begin{aligned}
\|X(t)\| &\le \|X(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^\infty (t-s)^{\alpha-1}\|V(s)\|ds\\
&\le \|X(0)\| + \frac{c^2}{(c^2+1)\Gamma(\alpha)}\int_0^\infty (t-s)^{\alpha-1}\|W(s)\|ds\\
&\le \|X(0)\| + \frac{c^2\|W(0)\|}{(c^2+1)\Gamma(\alpha)}\int_0^\infty (t-s)^{\alpha-1}E_{\alpha,1}(-\rho^\infty s^\alpha)ds
\end{aligned}$$

$$
\begin{aligned}
&= \|X(0)\| + \frac{c^2\|W(0)\|t^\alpha}{(c^2+1)} E_{\alpha,\alpha+1}(-\rho^\infty t^\alpha) \\
&= \|X(0)\| + \frac{c^2\|W(0)\|}{(c^2+1)\rho^\infty}\left(1 - E_{\alpha,1}\left(-\rho^\infty t^\alpha\right)\right) < \|X(0)\| + \frac{c^2\|W(0)\|}{(c^2+1)\rho^\infty} \le X^\infty,
\end{aligned}
$$

where we used $F_i \ge 1 + \frac{1}{c^2}$ to estimate $\|V\| \le \frac{c^2}{c^2+1}\|W\|$. Therefore, one has

$$
\lim_{t\to\bar{\tau}-} \|X(t)\| = \|X(\bar{\tau})\| < X^\infty. \tag{5.7}
$$

Subsequently, because

$$
\frac{c^2+1}{c^2}\|V(t)\| \le \|W(t)\| \le \|W(0)\|E_{\alpha,1}\left(-\rho^\infty t^\alpha\right), \quad t \in [0,\bar{\tau}],
$$

it follows from the first and second assertions of Proposition 2.1, $(1.7)_2$, (2.2), $\rho \le 1$ and (5.6) that for $t \in [0,\bar{\tau}]$,

$$
\begin{aligned}
\|w_i(t)\| &\le \|w_i(0)\| + \frac{1}{\Gamma(\alpha)N}\sum_{j=1}^N \int_0^\infty \rho(\|x_i - x_j\|)(t-s)^{\alpha-1}\|v_j(s) - v_i(s)\|ds \\
&\le \|w_i(0)\| + \frac{\sqrt{2}(N-1)}{\Gamma(\alpha)N}\int_0^\infty (t-s)^{\alpha-1}\|V(s)\|ds \\
&\le \|w_i(0)\| + \frac{\sqrt{2}(N-1)c^2}{(c^2+1)\Gamma(\alpha)N}\int_0^\infty (t-s)^{\alpha-1}\|W(s)\|ds \\
&\le \|w_i(0)\| + \frac{\sqrt{2}(N-1)c^2\|W(0)\|}{(c^2+1)\Gamma(\alpha)N}\int_0^\infty (t-s)^{\alpha-1}E_{\alpha,1}\left(-\rho^\infty s^\alpha\right)ds \\
&< \|w_i(0)\| + \frac{\sqrt{2}(N-1)c^2\|W(0)\|}{(c^2+1)N\rho^\infty} \le g(v^\infty) < \infty.
\end{aligned}
$$

Then, we get

$$
\lim_{t\to\bar{\tau}-} \max_{i\in[N]} \|v_i(t)\| < v^\infty. \tag{5.8}
$$

Hence, combining (5.7) and (5.8) implies that the assumption $\bar{\tau} < \tau$ is contradictory. Thus, $\bar{\tau} = \tau$ and moreover, we see the following relativistic asymptotic flocking for $t \in [0,\tau)$:

$$
\|X(t)\| \le X^\infty, \quad \|W(t)\| \le \|W(0)\|E_{\alpha,1}(-\rho^\infty t^\alpha).
$$

In conclusion, one has $\tau = \infty$ by Proposition 2.6, and we obtain the desired result. □

6. Conclusions

In this paper, we have provided suitable sufficient frameworks for an algebraic asymptotic flocking of the Cucker–Smale, thermodynamic Cucker–Smale and relativistic Cucker–Smale systems with a Caputo derivative using fractional calculus and continuous arguments with the iterative method.

Herein, we first presented more improved sufficient framework for asymptotic flocking on the fractional Cucker–Smale system than previous paper [36]. Using this method, we also demonstrated appropriate sufficient frameworks for the asymptotic flocking of the fractional relativistic Cucker–Smale and fractional thermodynamic Cucker–Smale systems. However, we still have several topics to study as a future research. Examples include

- (Question 1): Can we improve the sufficient frameworks for the asymptotic flocking of fractional thermodynamic Cucker–Smale and fractional relativistic Cucker–Smale systems addressed in this paper?
- (Question 2): Can we extend the proposed fractional systems presented in this paper to Vlasov-type kinetic systems via the mean-field limit regime?
- (Question 3): Can we establish fractional calculus theory on Riemannian manifold setting using geometric quantities from fractional calculus on Euclidean space?
- (Question 4): If an answer to (Question 3) is affirmative, can we prove the asymptotic flocking of fractional Cucker–Smale type systems on Riemannian manifolds?

We leave the above questions as future work.

Use of AI tools declaration

We have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no conflict of interest.

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