Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 19(9): 9636–9657.
DOI: 10.3934/mbe.2022448
Received: 06 May 2022
Revised: 23 June 2022
Accepted: 27 June 2022
Published: 04 July 2022

Research article

Existence and uniqueness results for fractional Langevin equations on a star graph

Wei Zhang*, **Jifeng Zhang and Jinbo Ni**

School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan 232001, China

* **Correspondence:** Email: zhangwei_azyw@163.com.

Abstract: This paper discusses a class of fractional Langevin equations on a star graph with mixed boundary conditions. Using Schaefer's fixed point theorem and Banach contraction mapping principle, the existence and uniqueness of solutions are established. Finally, two examples are constructed to illustrate the application of the obtained results. This study provides new results that enrich the existing literature on the fractional boundary value problem for graphs.

Keywords: fractional Langevin equation; boundary value problem; star graph; fixed point theorem; existence and uniqueness

---

1. Introduction

In the last two decades, the topic in the study of fractional calculus theory has attracted significant attention from researchers. The strong interest stems not only from the important application of the theory, but also from the consideration of its mathematical nature. Indeed, many phenomena arising from scientific fields, including biology, physics, chemistry, financial economics, control theory, materials, medicine, and anomalous diffusion, are precisely described by fractional differential equations [1–3]. As an important topic for the theory of fractional differential equations, the existence results of fractional boundary value problems (BVPs) have been investigated comprehensively by scholars [4–7].

On the other hand, the theory of differential equations on graphs originated from Lumer's research work in the framework of ramification spaces in the 1980s [8]. Differential equations on graphs appear in various fields, including chemical engineering, biology, physics, and ecology [9–12]. For this reason, many scholars study mathematical models described by fractional BVPs on graphs.

In 2014 [10], Graef et al. investigated the existence of solutions for fractional BVPs on a star graph, which is composed of three nodes and two edges, that is $G{=}V \cup E$ with $V{=}\{\gamma_0, \gamma_1, \gamma_2\}$ and

$E = \left\{\overrightarrow{\gamma_1\gamma_0}, \overrightarrow{\gamma_2\gamma_0}\right\}$, where γ_0 represents the junction node, $\overrightarrow{\gamma_i\gamma_0}$ is the edge connecting γ_i and γ_0 with length $l_i = \left|\overrightarrow{\gamma_i\gamma_0}\right|, i = 1, 2$. On each edge $\overrightarrow{\gamma_i\gamma_0}, i = 1, 2$, the authors considered the fractional BVPs in a local coordinate system with γ_i as origin on $x \in (0, l_i)$, given by

$$\begin{cases} -D^{\alpha}_{0+}\mathfrak{u}_i = \mathfrak{m}_i(x)\mathfrak{f}_i(x, u_i),\ 0 < x < l_i,\ i = 1, 2. \\ \mathfrak{u}_1(0) = \mathfrak{u}_2(0) = 0,\ \mathfrak{u}_1(l_1) = \mathfrak{u}_2(l_2),\ D^{\beta}_{0+}\mathfrak{u}_1(l_1) + D^{\beta}_{0+}\mathfrak{u}_2(l_2) = 0, \end{cases} \tag{1.1}$$

where $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are Riemann-Liouville fractional derivative operators, $1 < \alpha \leq 2,\ 0 < \beta < \alpha$, $\mathfrak{m}_i \in C[0, l_i], i = 1, 2$ with $\mathfrak{m}_i(x) \not\equiv 0$ on $[0, l_i]$ and $\mathfrak{f}_i \in C([0, l_i] \times \mathbb{R}, \mathbb{R}), i = 1, 2$. By using Schauder fixed point theorem and Banach contraction mapping theorem, the existence and uniqueness of solutions of BVP (1.1) are obtained.

Later in 2019 [11], Mehandiratta et al. extended the results of Graef et al. on a general star graph (see Figure 1), which is a graph consisting of $k + 1$ nodes and k edges, that is, the authors considered a graph $G = V \cup E,\ V = \{v_0, v_1, \cdots, v_k\},\ E = \{e_i = \overrightarrow{v_iv_0}, i = 1, 2, \cdots, k\}$, where v_0 is the junction node, $\overrightarrow{v_iv_0}$ represents the edge connecting v_i and v_0 with length $l_i = \left|\overrightarrow{v_iv_0}\right|, i = 1, 2, \cdots, k$. The author investigated the following fractional BVPs on the star graph G given by

$$\begin{cases} {}^{C}D^{\alpha}_{0,x}\mathfrak{u}_i(x) = \mathfrak{f}_i(x, \mathfrak{u}_i(x), {}^{C}D^{\beta}_{0,x}\mathfrak{u}_i(x)),\ 0 < x < l_i,\ i = 1, 2, \cdots, k, \\ \mathfrak{u}_i(0) = 0,\ i = 1, 2, \cdots, k, \\ \mathfrak{u}_i(l_i) = \mathfrak{u}_j(l_j),\ i, j = 1, 2, \cdots, k,\ i \neq j, \\ \sum_{i=1}^{k} \mathfrak{u}'_i(l_i) = 0,\ i = 1, 2, \cdots, k, \end{cases} \tag{1.2}$$

where ${}^{C}D^{\alpha}_{0,x}, {}^{C}D^{\beta}_{0,x}$ are Caputo fractional derivative, $1 < \alpha \leq 2,\ 0 < \beta \leq \alpha - 1$, $\mathfrak{f}_i, i = 1, 2, \cdots, k$ are continuous functions on $[0, l_i] \times \mathbb{R} \times \mathbb{R}$. The existence and uniqueness results for BVP (1.2) are established using Schaefer's fixed point theorem and Banach contraction mapping theorem.

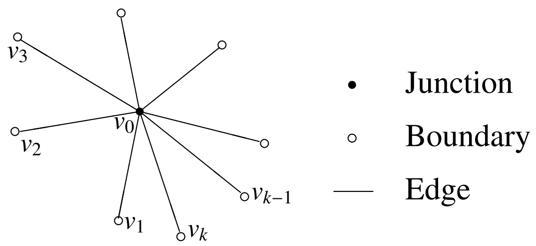

Figure 1. A general star graph with k edges.

Based on the two studies mentioned above, the subject of fractional BVPs on graphs has received significant research attention, and various interesting results have been recently established [12–19]. For example, in [12], Zhang and Liu discussed BVPs of fractional differential equations on a star graph with $n + 1$ nodes and n edges. The existence and uniqueness of solutions are established using Schaefer's fixed point theorem and Banach contraction mapping principle. Etemad and Rezapour in [13] studied the BVPs of fractional differential equations on ethane graph. The existence results of solutions were obtained using Schaefer's fixed point theorem and Krasnoselskii's fixed point theorem. In [14], Baleanu et al. investigated the existence of solutions for BVPs of fractional differential equations on the glucose graphs. In [15], Ali et al. studied the existence of solutions of BVPs for fractional differential equations on the cyclohexane graphs using the fixed point theory. In [16], Mehandiratta et al. considered a nonlinear fractional BVPs on a particular metric graph. They proved the existence and uniqueness of solutions using Krasnoselskii's fixed point theorem and Banach contraction principle.

It is well known that Langevin first formulated the Langevin equation in 1908. Langevin equation is an important tool for describing the evolution of physical phenomena in fluctuating environments [20]. However, people have realized that the traditional integer Langevin equation cannot accurately describe dynamic systems for complex phenomena. Therefore, one way to overcome this disadvantage is to use fractional derivative instead of integer derivative [21]. This gives rise to the fractional Langevin equation. Studies of BVPs on fractional Langevin equations have increased in recent years, and new research is constantly emerging [22–25]. For example, in [22], Fazli et al. studied the anti-periodic BVPs of fractional Langevin equation and obtained the existence and uniqueness solutions using the coupled fixed point theorem for mixed monotone mappings. In [23], Matar et al. established the existence, uniqueness and stability of solutions for the coupled Caputo-Hadamard fractional Langevin equation with the help of the fixed point theorem. In [24], Salem et al. considered the fractional Langevin equation with three-point boundary value conditions and obtained the existence of solutions by using Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative theorem.

From the literature review, no result is concerned with fractional Langevin equations on graphs. To fill this knowledge gap, this study aims to establish the existence and uniqueness results for fractional Langevin equations on a star graph subject to mixed boundary conditions. Precisely, we investigate the following problems:

$$\begin{cases} {}^C D^{\alpha}_{0,x}(D+\lambda_i)\mathfrak{y}_i(x) = \mathfrak{g}_i(x, \mathfrak{y}_i(x), {}^C D^{\gamma}_{0,x}\mathfrak{y}_i(x)),\ 0 < x < \rho_i,\ i = 1, 2, \cdots, k, \\ \mathfrak{y}_i(0) = 0, \quad i = 1, 2, \cdots, k, \\ \mathfrak{y}_i(\rho_i) = \mathfrak{y}_j(\rho_j), \quad i, j = 1, 2, \cdots, k,\ i \neq j, \\ \sum\limits_{i=1}^{k} \mathfrak{y}'_i(\rho_i) = 0, \quad i = 1, 2, \cdots, k, \end{cases} \tag{1.3}$$

where $0 < \alpha < 1,\ 0 < \gamma < \alpha,\ \lambda_i \in \mathbb{R}^+,\ i = 1, 2, \cdots, k,\ {}^C D^{\alpha}_{0,x}, {}^C D^{\gamma}_{0,x}$ are Caputo fractional derivative, D is the ordinary derivative, $\mathfrak{g}_i \in C([0, \rho_i] \times \mathbb{R}^2, \mathbb{R}),\ i = 1, 2, \cdots, k$. The star graph has $k + 1$ nodes and k edges, that is $G = V \cup E,\ V = \{v_0, v_1, \cdots, v_k\},\ E = \{e_i = \overrightarrow{v_i v_0},\ i = 1, 2, \cdots, k\}$, where v_0 is the junction node, $e_i = \overrightarrow{v_i v_0}$ represents the edge connecting v_i and v_0 with length $\rho_i = \left|\overrightarrow{v_i v_0}\right|,\ i = 1, 2, \cdots, k$. We consider a local coordinate system with v_i as origin and $x \in (0, \rho_i)$ as the coordinate. The existence and uniqueness of the solution of BVP (1.3) are discussed using Schaefer's fixed point theorem and Banach contraction mapping principle.

The rest of paper is organized as follows: In Section 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma (Lemma 2.6), which transforms the problem (1.3) to BVP (2.1). In Section 3, we study the existence and uniqueness results of BVP (2.1) by using Schaefer's fixed point theorem and Banach contraction principle, respectively. Finally, two illustrative examples are discussed at the end of this paper.

2. Preliminaries

In this section, we recall some definitions of fractional calculus and provide preliminary results which we will use in the rest of the paper.

Definition 2.1 [1]. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f \in C(a, b)$ is defined by

$$I^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} f(s)ds, \quad a < t < b.$$

Definition 2.2 [1]. The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C^{n}(a, b)$ is presented by

$${}^{C}D^{\alpha}_{a,t}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad a < t < b,$$

where $n = [\alpha] + 1$.

Lemma 2.1 [1]. Let $\alpha > 0$. Suppose that $u \in AC^{n}[0, 1]$. Then

$$I^{\alpha}_{0+}\,{}^{C}D^{\alpha}_{0,t}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 1, 2, \cdots, n,\ n = [\alpha] + 1$.

Lemma 2.2 [26]. Let $\alpha>0,\ n\in N$, and $D = d/dx$. Suppose that $(D^{n}x)(t)$ and $({}^{C}D^{\alpha+n}_{a,t}x)(t)$ are exist. Then

$$({}^{C}D^{\alpha}_{a,t}D^{n}x)(t) = ({}^{C}D^{\alpha+n}_{a,t}\ x)(t).$$

Lemma 2.3 [1]. If $\beta > 0,\ \gamma > \beta - 1,\ t > 0$, then

$${}^{C}D^{\beta}_{0,t}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta)} t^{\gamma-\beta}.$$

Theorem 2.4 [27]. (Scheafer's fixed point theorem) Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $\Omega = \{x \in X,\ x = \mu T x,\ \mu \in (0, 1)\}$ is bounded. Then T has a fixed point in X.

Lemma 2.5 [11]. Suppose that $\mathfrak{y}$ is a function defined on $[0, \rho]$ such that ${}^{C}D^{\alpha}_{0,x}\mathfrak{y}$ exists on $[0, \rho]$ with $\alpha > 0$ and let $x \in [0, \rho],\ t = x/\rho \in [0, 1],\ y(t) = \mathfrak{y}(\rho t)$. Then

$${}^{C}D^{\alpha}_{0,x}\mathfrak{y}(x) = \rho^{-\alpha}({}^{C}D^{\alpha}_{0,t}y(t)).$$

Lemma 2.6 Suppose that $\mathfrak{y}$ be a function defined on $[0, \rho]$ such that ${}^{C}D^{\alpha}_{0,x}\mathfrak{y}$ exists on $[0, \rho]$ with $\alpha \in (n-1, n)$ and let $x \in [0, \rho],\ t = x/\rho \in [0, 1],\ y(t) = \mathfrak{y}(\rho t)$. Then

$${}^{C}D^{\alpha}_{0,x}(D + \lambda)\mathfrak{y}(x) = \rho^{-\alpha-1}\,{}^{C}D^{\alpha}_{0,t}(D + \lambda\rho)y(t).$$

Proof. By using the Definition 2.2 and Lemma 2.2, we can obtain

$$
\begin{aligned}
&{}^{C}D_{0,x}^{\alpha}(D+\lambda)\mathfrak{y}(x)={}^{C}D_{0,x}^{\alpha+1}\mathfrak{y}(x)+\lambda{}^{C}D_{0,x}^{\alpha}\mathfrak{y}(x)\\
=&\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-s)^{n-\alpha-1}\mathfrak{y}^{(n+1)}(s)ds+\frac{\lambda}{\Gamma(n-\alpha)}\int_0^x(x-s)^{n-\alpha-1}\mathfrak{y}^{(n)}(s)ds\\
=&\frac{1}{\Gamma(n-\alpha)}\int_0^{\rho t}(\rho t-s)^{n-\alpha-1}\mathfrak{y}^{(n+1)}(s)ds+\frac{\lambda}{\Gamma(n-\alpha)}\int_0^{\rho t}(\rho t-s)^{n-\alpha-1}\mathfrak{y}^{(n)}(s)ds\quad(x=\rho t)\\
=&\frac{\rho^{n-\alpha}}{\Gamma(n-\alpha)}\int_0^t(t-\hat{s})^{n-\alpha-1}\mathfrak{y}^{(n+1)}(\rho\hat{s})d\hat{s}+\frac{\lambda\rho^{n-\alpha}}{\Gamma(n-\alpha)}\int_0^t(t-\hat{s})^{n-\alpha-1}\mathfrak{y}^{(n)}(\rho\hat{s})d\hat{s}\quad(\hat{s}=s/\rho)\\
=&\frac{\rho^{-\alpha-1}}{\Gamma(n-\alpha)}\int_0^t(t-\hat{s})^{n-\alpha-1}y^{(n+1)}(\hat{s})d\hat{s}+\frac{\lambda\rho^{-\alpha}}{\Gamma(n-\alpha)}\int_0^t(t-\hat{s})^{n-\alpha-1}y^{(n)}(\hat{s})d\hat{s}\quad(y^{(n)}(t)=\rho^n\mathfrak{y}^{(n)}(\rho t))\\
=&\rho^{-\alpha-1}{}^{C}D_{0,t}^{\alpha+1}y(t)+\lambda\rho^{-\alpha}{}^{C}D_{0,t}^{\alpha}y(t)\\
=&\rho^{-\alpha-1}{}^{C}D_{0,t}^{\alpha}(D+\lambda\rho)y(t),
\end{aligned}
$$

This completes the proof of Lemma 2.6.

By a direct calculation with help of Lemmas 2.5 and 2.6, BVP (1.3) can be transformed into a BVP defined on [0, 1] given by

$$
\begin{cases}
{}^{C}D_{0,t}^{\alpha}(D+\lambda_i\rho_i)y_i(t)=\rho_i^{\alpha+1}g_i(t,y_i(t),\rho_i^{-\gamma}{}^{C}D_{0,t}^{\gamma}y_i(t)),\ t\in(0,1),\\
y_i(0)=0,\quad i=1,2,\cdots,k,\\
y_i(1)=y_j(1),\quad i,j=1,2,\cdots,k,\ i\neq j,\\
\sum\limits_{i=1}^{k}\rho_i^{-1}y_i'(1)=0,\quad i=1,2,\cdots,k,
\end{cases}
\tag{2.1}
$$

where $y_i(t)=\mathfrak{y}_i(\rho_i t),\ g_i(t,u,v)=\mathfrak{g}_i(\rho_i t,u,v),\ i=1,2,\cdots,k.$

3. Main results

In this section, we investigate the existence and uniqueness results of problem (2.1). To this end, we consider the space $Y=\{y: y\in C[0,1],\ {}^{C}D_{0,t}^{\gamma}y\in C[0,1]\}$, endowed with the norm

$$\|y\|_Y=\|y\|+\|{}^{C}D_{0,t}^{\gamma}y\|,$$

where $\|y\|=\max\limits_{t\in[0,1]}|y(t)|,\ \|{}^{C}D_{0,t}^{\gamma}y\|=\max\limits_{t\in[0,1]}|{}^{C}D_{0,t}^{\gamma}y(t)|$. Then $(Y,\|\cdot\|_Y)$ is a Banach space, and the product space $(Y^k,\|\cdot\|_{Y^k})$ equipped with the norm

$$\|(y_1,y_2,\cdots,y_k)\|_{Y^k}=\sum\nolimits_{i=1}^{k}\|y_i\|_Y,\ (y_1,y_2,\cdots,y_k)\in Y^k$$

is also a Banach space, where $Y^k=\overbrace{Y\times Y\times\cdots\times Y}^{k}$.

Lemma 3.1 Let $h_i \in C[0,1],\ i = 1, 2, \cdots, k$. Then the BVP of fractional Langevin equations

$$\begin{cases} {}^{C}D^{\alpha}_{0,t}(D+\lambda_i\rho_i)y_i(t) = h_i(t),\ t \in (0,1),\ \alpha \in (0,1),\ i = 1,2,\cdots,k, \\ y_i(0) = 0,\quad i = 1,2,\cdots,k, \\ y_i(1) = y_j(1),\quad i,j = 1,2,\cdots,k,\ i \neq j, \\ \sum\limits_{i=1}^{k} \rho_i^{-1} y_i'(1) = 0,\quad i = 1,2,\cdots,k, \end{cases} \tag{3.1}$$

is equivalent to the integral equations

$$\begin{aligned} y_i(t) =& -\lambda_i\rho_i \int_0^t y_i(s)ds + I^{\alpha+1}_{0+}h_i(t) + t\sum_{j=1}^{k} \ell_j(\lambda_j\rho_j y_j(1) - I^{\alpha}_{0+}h_j(t)|_{t=1}) \\ &+ t\sum_{j=1,j\neq i}^{k} \ell_j\Big(-\lambda_j\rho_j\int_0^1 y_j(s)ds + \lambda_i\rho_i\int_0^1 y_i(s)ds + I^{\alpha+1}_{0+}h_j(t)|_{t=1} - I^{\alpha+1}_{0+}h_i(t)|_{t=1}\Big), \end{aligned}$$

where $\ell_j := \frac{\rho_j^{-1}}{\sum_{j=1}^{k}\rho_j^{-1}},\ i,j = 1,2,\cdots,k.$

Proof. Applying the operator I^{α}_{0+} on both sides of Eq (3.1) and combining with the Lemma 2.1, we obtain

$$(D+\lambda_i\rho_i)y_i(t) = I^{\alpha}_{0+}h_i(t) + c_1^i,$$

where $c_1^i \in \mathbb{R},\ i = 1,2,\cdots,k$. The above equation can be rewritten as

$$y_i'(t) = -\lambda_i\rho_i y_i(t) + I^{\alpha}_{0+}h_i(t) + c_1^i. \tag{3.2}$$

Integrating both sides of Eq (3.2) from 0 to t, we get

$$y_i(t) = -\lambda_i\rho_i\int_0^t y_i(s)ds + I^{\alpha+1}_{0+}h_i(t) + c_1^i t + y_i(0).$$

By conditions $y_i(0) = 0,\ i = 1,2,\cdots,k$, we conclude

$$y_i(t) = -\lambda_i\rho_i\int_0^t y_i(s)ds + I^{\alpha+1}_{0+}h_i(t) + c_1^i t. \tag{3.3}$$

Applying the conditions $\sum\limits_{i=1}^{k}\rho_i^{-1}y_i'(1) = 0$ and $y_i(1) = y_j(1),\ i,j = 1,2,\cdots,k,\ i \neq j$ in Eqs (3.2) and (3.3), respectively, we find

$$\sum_{i=1}^{k}\rho_i^{-1}(-\lambda_i\rho_i y_i(1) + I^{\alpha}_{0+}h_i(t)|_{t=1} + c_1^i) = 0,$$

and

$$\begin{aligned} &-\lambda_i\rho_i\int_0^1 y_i(s)ds + I^{\alpha+1}_{0+}h_i(t)|_{t=1} + c_1^i \\ &= -\lambda_j\rho_j\int_0^1 y_j(s)ds + I^{\alpha+1}_{0+}h_j(t)|_{t=1} + c_1^j, i,j = 1,2,\cdots,k,\ i \neq j. \end{aligned}$$

Combining the above two equations, we get

$$\sum_{j=1}^{k}\rho_j^{-1}(-\lambda_j\rho_jy_j(1)+I_{0+}^{\alpha}h_j(t)|_{t=1})+\rho_i^{-1}c_1^i$$
$$=-\sum_{j=1,j\neq i}^{k}\rho_j^{-1}c_1^i+\sum_{j=1,j\neq i}^{k}\rho_j^{-1}\Big(-\lambda_j\rho_j\int_0^1 y_j(s)ds+I_{0+}^{\alpha+1}h_j(t)|_{t=1}+\lambda_i\rho_i\int_0^1 y_i(s)ds-I_{0+}^{\alpha+1}h_i(t)|_{t=1}\Big).$$

This yields

$$\sum_{j=1}^{k}\rho_j^{-1}c_1^i=-\sum_{j=1}^{k}\rho_j^{-1}(-\lambda_j\rho_jy_j(1)+I_{0+}^{\alpha}h_j(t)|_{t=1})$$
$$+\sum_{j=1,j\neq i}^{k}\rho_j^{-1}\Big(-\lambda_j\rho_j\int_0^1 y_j(s)ds+\lambda_i\rho_i\int_0^1 y_i(s)ds+I_{0+}^{\alpha+1}h_j(t)|_{t=1}-I_{0+}^{\alpha+1}h_i(t)|_{t=1}\Big),$$

from which we deduce that

$$c_1^i=\sum_{j=1,j\neq i}^{k}\ell_j\Big(-\lambda_j\rho_j\int_0^1 y_j(s)ds+\lambda_i\rho_i\int_0^1 y_i(s)ds+I_{0+}^{\alpha+1}h_j(t)|_{t=1}-I_{0+}^{\alpha+1}h_i(t)|_{t=1}\Big)$$
$$-\sum_{j=1}^{k}\ell_j(-\lambda_j\rho_jy_j(1)+I_{0+}^{\alpha}h_j(t)|_{t=1}),\ i=1,2,\cdots,k.$$

Substituting c_1^i $(i=1,2,\cdots,k)$ into the Eq (3.3), we get the desired result. The converse of the lemma is calculated directly. The proof is completed.

In view of Lemma 3.1, we define the operator $T:Y^k\to Y^k$ by

$$T(y_1,y_2,\cdots,y_k)(t):=(T_1(y_1,y_2,\cdots,y_k)(t),T_2(y_1,y_2,\cdots,y_k)(t),\cdots,T_k(y_1,y_2,\cdots,y_k)(t)),$$

for $t\in[0,1]$ and $y_i\in Y,\ i=1,2,\cdots,k$, where

$$\begin{aligned}
&T_i(y_1,y_2,\cdots,y_k)(t)\\
&=-\lambda_i\rho_i\int_0^t y_i(s)ds+\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha}g_i(s,y_i(s),\rho_i^{-\gamma C}D_{0,s}^{\gamma}y_i(s))ds\\
&\quad+t\sum_{j=1}^{k}\ell_j\Big(\lambda_j\rho_jy_j(1)-\frac{\rho_j^{\alpha+1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}g_j(s,y_j(s),\rho_j^{-\gamma C}D_{0,s}^{\gamma}y_j(s))ds\Big)\\
&\quad+t\sum_{j=1,j\neq i}^{k}\ell_j\Big(-\lambda_j\rho_j\int_0^1 y_j(s)ds+\frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}g_j(s,y_j(s),\rho_j^{-\gamma C}D_{0,s}^{\gamma}y_j(s))ds\Big)\\
&\quad+t\sum_{j=1,j\neq i}^{k}\ell_j\Big(-\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}g_i(s,y_i(s),\rho_i^{-\gamma C}D_{0,s}^{\gamma}y_i(s))ds+\lambda_i\rho_i\int_0^1 y_i(s)ds\Big).
\end{aligned}\tag{3.4}$$

In the following part, for convenience of presentation, we denote the notations:

$$M_1=\frac{1}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)},$$
$$M_2=\frac{2}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}.$$

Theorem 3.1 Assume that

(H_1) The functions $g_i:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, $(i=1,2,\cdots,k)$ are continuous and there exist functions $a_i(t)\in C([0,1],[0,+\infty))$, $i=1,2,\cdots,k$, such that

$$|g_i(t,u,v)-g_i(t,u_1,v_1)|\le a_i(t)(|u-u_1|+|v-v_1|),$$

for all $t\in[0,1]$ and $(u,v),(u_1,v_1)\in\mathbb{R}^2$. Then the BVP (2.1) has a unique solution on [0, 1], provided that

$$\sum_{i=1}^{k}P_i\Big(\sum_{i=1}^{k}A_i\Big)+\sum_{i=1}^{k}Q_i<1,$$

where

$$P_i=M_1\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+1}+\rho_j^{\alpha-\gamma+1})+M_2(\rho_i^{\alpha+1}+\rho_i^{\alpha-\gamma+1}),$$
$$Q_i=3\lambda_i\rho_i+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j+\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}\Big),\ A_i=\max_{t\in[0,1]}|a_i(t)|.$$

Proof. Applying the Banach contraction mapping principle, we have to prove that T is a contractive mapping. To prove this, we let $y=(y_1,y_2,\cdots,y_k)$, $\bar{y}=(\bar{y}_1,\bar{y}_2,\cdots,\bar{y}_k)\in Y^k$, $t\in[0,1]$. By Eq (3.4), we have

$$\begin{aligned}
&|T_iy(t)-T_i\bar{y}(t)|\\
\le&\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha}|g_i(s,y_i(s),\rho_i^{-\gamma C}D_{0,s}^{\gamma}y_i(s))-g_i(s,\bar{y}_i(s),\rho_i^{-\gamma C}D_{0,s}^{\gamma}\bar{y}_i(s))|ds\\
&+\lambda_i\rho_i\int_0^t|y_i(s)-\bar{y}_i(s)|ds+t\sum_{j=1}^{k}\ell_j(\lambda_j\rho_j|y_j(1)-\bar{y}_j(1)|)\\
&+t\sum_{j=1}^{k}\frac{\ell_j\rho_j^{\alpha+1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|g_j(s,y_j(s),\rho_j^{-\gamma C}D_{0,s}^{\gamma}y_j(s))ds-g_j(s,\bar{y}_j(s),\rho_j^{-\gamma C}D_{0,s}^{\gamma}\bar{y}_j(s))|ds\\
&+t\sum_{j=1,j\neq i}^{k}\ell_j\Big(\lambda_j\rho_j\int_0^1|y_j(s)-\bar{y}_j(s)|ds\Big)+t\sum_{j=1,j\neq i}^{k}\lambda_i\rho_i\ell_j\int_0^1|y_i(s)-\bar{y}_i(s)|ds\\
&+t\sum_{j=1,j\neq i}^{k}\frac{\ell_j\rho_j^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}|g_j(s,y_j(s),\rho_j^{-\gamma C}D_{0,s}^{\gamma}y_j(s))-g_j(s,\bar{y}_j(s),\rho_j^{-\gamma C}D_{0,s}^{\gamma}\bar{y}_j(s))|ds\\
&+t\sum_{j=1,j\neq i}^{k}\frac{\ell_j\rho_i^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}|g_i(s,y_i(s),\rho_i^{-\gamma C}D_{0,s}^{\gamma}y_i(s))-g_i(s,\bar{y}_i(s),\rho_i^{-\gamma C}D_{0,s}^{\gamma}\bar{y}_i(s))|ds.
\end{aligned}$$

By using the assumption (H_1) and $t \in [0,1]$, $\ell_j \in (0,1)$, $j = 1, 2, \cdots, k$, we deduce

$$
\begin{aligned}
&|T_i y(t) - T_i \bar{y}(t)| \\
&\le \frac{2\rho_i^{\alpha+1}}{\Gamma(\alpha+2)} A_i \|y_i - \bar{y}_i\| + \frac{2\rho_i^{\alpha-\gamma+1}}{\Gamma(\alpha+2)} A_i \|{}^C D_{0,t}^{\gamma} y_i - {}^C D_{0,t}^{\gamma} \bar{y}_i\| + 2\lambda_i\rho_i \|y_i - \bar{y}_i\| \\
&\quad + \sum_{j=1}^{k} \lambda_j\rho_j \|y_j - \bar{y}_j\| + \sum_{j=1, j\neq i}^{k} \lambda_j\rho_j \|y_j - \bar{y}_j\| + \sum_{j=1}^{k} \frac{\rho_j^{\alpha+1} A_j}{\Gamma(\alpha+1)} \|y_j - \bar{y}_j\| \\
&\quad + \sum_{j=1}^{k} \frac{\rho_j^{\alpha-\gamma+1} A_j}{\Gamma(\alpha+1)} \|{}^C D_{0,t}^{\gamma} y_j - {}^C D_{0,t}^{\gamma} \bar{y}_j\| \\
&\quad + \sum_{j=1, j\neq i}^{k} \frac{\rho_j^{\alpha+1} A_j}{\Gamma(\alpha+2)} \|y_j - \bar{y}_j\| + \sum_{j=1, j\neq i}^{k} \frac{\rho_j^{\alpha-\gamma+1} A_j}{\Gamma(\alpha+2)} \|{}^C D_{0,t}^{\gamma} y_j - {}^C D_{0,t}^{\gamma} \bar{y}_j\| \\
&\le \frac{2A_i}{\Gamma(\alpha+2)} (\rho_i^{\alpha+1} + \rho_i^{\alpha-\gamma+1}) (\|y_i - \bar{y}_i\| + \|{}^C D_{0,t}^{\gamma} y_i - {}^C D_{0,t}^{\gamma} \bar{y}_i\|) \\
&\quad + 3\lambda_i\rho_i \|y_i - \bar{y}_i\| + \sum_{j=1, j\neq i}^{k} 2\lambda_j\rho_j \|y_j - \bar{y}_j\| \\
&\quad + \sum_{j=1}^{k} \frac{A_j}{\Gamma(\alpha+1)} (\rho_j^{\alpha+1} + \rho_j^{\alpha-\gamma+1}) (\|y_j - \bar{y}_j\| + \|{}^C D_{0,t}^{\gamma} y_j - {}^C D_{0,t}^{\gamma} \bar{y}_j\|) \\
&\quad + \sum_{j=1, j\neq i}^{k} \frac{A_j}{\Gamma(\alpha+2)} (\rho_j^{\alpha+1} + \rho_j^{\alpha-\gamma+1}) (\|y_j - \bar{y}_j\| + \|{}^C D_{0,t}^{\gamma} y_j - {}^C D_{0,t}^{\gamma} \bar{y}_j\|).
\end{aligned}
$$

Then for any $y, \bar{y} \in Y^k$, we obtain

$$
\begin{aligned}
&\|T_i y - T_i \bar{y}\| \\
&\le \Big(\frac{2}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)(\rho_i^{\alpha+1} + \rho_i^{\alpha-\gamma+1}) A_i \|y_i - \bar{y}_i\|_Y + 3\lambda_i\rho_i \|y_i - \bar{y}_i\|_Y \\
&\quad + \sum_{j=1, j\neq i}^{k} \Big(\frac{1}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)(\rho_j^{\alpha+1} + \rho_j^{\alpha-\gamma+1}) A_j \|y_j - \bar{y}_j\|_Y + \sum_{j=1, j\neq i}^{k} 2\lambda_j\rho_j \|y_j - \bar{y}_j\|_Y.
\end{aligned}
\tag{3.5}
$$

On the other hand, by using Lemma 2.3, we have

$$
\begin{aligned}
&|{}^C D_{0,t}^{\gamma} T_i y(t) - {}^C D_{0,t}^{\gamma} T_i \bar{y}(t)| \\
&\le \frac{\rho_i^{\alpha+1}}{\Gamma(\alpha-\gamma+1)} \int_0^t (t-s)^{\alpha-\gamma} |g_i(s, y_i(s), \rho_i^{-\gamma C} D_{0,s}^{\gamma} y_i(s)) - g_i(s, \bar{y}_i(s), \rho_i^{-\gamma C} D_{0,s}^{\gamma} \bar{y}_i(s))| ds \\
&\quad + \frac{\lambda_i\rho_i}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |y_i(s) - \bar{y}_i(s)| ds + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \sum_{j=1}^{k} \ell_j\lambda_j\rho_j |y_j(1) - \bar{y}_j(1)| \\
&\quad + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(\alpha)} \sum_{j=1}^{k} \ell_j\rho_j^{\alpha+1} \int_0^1 (1-s)^{\alpha-1} |g_j(s, y_j(s), \rho_j^{-\gamma C} D_{0,s}^{\gamma} y_j(s)) ds - g_j(s, \bar{y}_j(s), \rho_j^{-\gamma C} D_{0,s}^{\gamma} \bar{y}_j(s))| ds
\end{aligned}
$$

$$+\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\ell_j\lambda_j\rho_j\int_0^1|y_j(s)-\bar{y}_j(s)|ds+\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\lambda_i\rho_i\ell_j\int_0^1|y_i(s)-\bar{y}_i(s)|ds$$
$$+\frac{t^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^{k}\ell_j\rho_j^{\alpha+1}\int_0^1(1-s)^{\alpha}|g_j(s,y_j(s),\rho_j^{-\gamma C}D_{0,s}^{\gamma}y_j(s))-g_j(s,\bar{y}_j(s),\rho_j^{-\gamma C}D_{0,s}^{\gamma}\bar{y}_j(s))|ds$$
$$+\frac{t^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^{k}\rho_i^{\alpha+1}\ell_j\int_0^1(1-s)^{\alpha}|g_i(s,y_i(s),\rho_i^{-\gamma C}D_{0,s}^{\gamma}y_i(s))-g_i(s,\bar{y}_i(s),\rho_i^{-\gamma C}D_{0,s}^{\gamma}\bar{y}_i(s))|ds.$$

In a similar manner, we deduce

$$\begin{aligned}
&|{}^CD_{0,t}^{\gamma}T_iy(t)-{}^CD_{0,t}^{\gamma}T_i\bar{y}(t)|\\
&\leq\frac{\rho_i^{\alpha+1}A_i}{\Gamma(\alpha-\gamma+2)}\|y_i-\bar{y}_i\|+\frac{\rho_i^{\alpha-\gamma+1}A_i}{\Gamma(\alpha-\gamma+2)}\|{}^CD_{0,t}^{\gamma}y_i-{}^CD_{0,t}^{\gamma}\bar{y}_i\|+\frac{2\lambda_i\rho_i}{\Gamma(2-\gamma)}\|y_i-\bar{y}_i\|\\
&\quad+\frac{1}{\Gamma(2-\gamma)}\sum_{j=1}^{k}\lambda_j\rho_j\|y_j-\bar{y}_j\|+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\sum_{j=1}^{k}\rho_j^{\alpha+1}A_j\|y_j-\bar{y}_j\|\\
&\quad+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\sum_{j=1}^{k}\rho_j^{\alpha-\gamma+1}A_j\|{}^CD_{0,t}^{\gamma}y_j-{}^CD_{0,t}^{\gamma}\bar{y}_j\|\\
&\quad+\frac{1}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j\|y_j-\bar{y}_j\|+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+1}A_j\|y_j-\bar{y}_j\|\\
&\quad+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha-\gamma+1}A_j\|{}^CD_{0,t}^{\gamma}y_j-{}^CD_{0,t}^{\gamma}\bar{y}_j\|\\
&\quad+\frac{\rho_i^{\alpha+1}A_i}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\|y_i-\bar{y}_i\|+\frac{\rho_i^{\alpha-\gamma+1}A_i}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\|{}^CD_{0,t}^{\gamma}y_i-{}^CD_{0,t}^{\gamma}\bar{y}_i\|\\
&\leq\frac{1}{\Gamma(\alpha-\gamma+2)}(\rho_i^{\alpha+1}+\rho_i^{\alpha-\gamma+1})A_i(\|y_i-\bar{y}_i\|+\|{}^CD_{0,t}^{\gamma}y_i-{}^CD_{0,t}^{\gamma}\bar{y}_i\|)\\
&\quad+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}\|y_i-\bar{y}_i\|+\frac{2}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j\|y_j-\bar{y}_j\|\\
&\quad+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\sum_{j=1}^{k}(\rho_j^{\alpha+1}+\rho_j^{\alpha-\gamma+1})A_j(\|y_j-\bar{y}_j\|+\|{}^CD_{0,t}^{\gamma}y_j-{}^CD_{0,t}^{\gamma}\bar{y}_j\|)\\
&\quad+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+1}+\rho_j^{\alpha-\gamma+1})A_j(\|y_j-\bar{y}_j\|+\|{}^CD_{0,t}^{\gamma}y_j-{}^CD_{0,t}^{\gamma}\bar{y}_j\|)\\
&\quad+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}(\rho_i^{\alpha+1}+\rho_i^{\alpha-\gamma+1})A_i(\|y_i-\bar{y}_i\|+\|{}^CD_{0,t}^{\gamma}y_i-{}^CD_{0,t}^{\gamma}\bar{y}_i\|).
\end{aligned}$$

This implies that, for any $y,\bar{y}\in Y^k$,

$$\begin{aligned}
&\|{}^CD_{0,t}^{\gamma}T_iy(t)-{}^CD_{0,t}^{\gamma}T_i\bar{y}(t)\|\\
&\leq\Big(\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)(\rho_i^{\alpha+1}+\rho_i^{\alpha-\gamma+1})A_i\|y_i-\bar{y}_i\|_Y
\end{aligned}$$

$$+\Big(\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+1}+\rho_j^{\alpha-\gamma+1})A_j\|y_j-\bar{y}_j\|_Y$$
$$+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}\|y_i-\bar{y}_i\|_Y+\sum_{j=1,j\neq i}^{k}\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}\|y_j-\bar{y}_j\|_Y. \tag{3.6}$$

By a direct calculation with help of (3.5) and (3.6), we get

$$\begin{aligned}&\|T_iy-T_i\bar{y}\|+\|{}^CD_{0,t}^{\gamma}T_iy-{}^CD_{0,t}^{\gamma}T_i\bar{y}\|\\ &\le M_2(\rho_i^{\alpha+1}+\rho_i^{\alpha-\gamma+1})A_i\|y_i-\bar{y}_i\|_Y+\Big(3\lambda_i\rho_i+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}\Big)\|y_i-\bar{y}_i\|_Y\\ &+M_1\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+1}+\rho_j^{\alpha-\gamma+1})A_j\|y_j-\bar{y}_j\|_Y+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j+\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}\Big)\|y_j-\bar{y}_j\|_Y.\end{aligned}$$

From this it follows that

$$\begin{aligned}&\|T_iy-T_i\bar{y}\|_Y\\ &\le\Big(M_2(\rho_i^{\alpha+1}+\rho_i^{\alpha-\gamma+1})+M_1\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+1}+\rho_j^{\alpha-\gamma+1})\Big)\Big(\sum_{i=1}^{k}A_i\Big)\sum_{j=1}^{k}\|y_j-\bar{y}_j\|_Y\\ &\quad+\Big(3\lambda_i\rho_i+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j+\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}\Big)\Big)\sum_{j=1}^{k}\|y_j-\bar{y}_j\|_Y\\ &=\Big(P_i\Big(\sum_{i=1}^{k}A_i\Big)+Q_i\Big)\sum_{j=1}^{k}\|y_j-\bar{y}_j\|_Y.\end{aligned}$$

As a consequence, we obtain

$$\|Ty-T\bar{y}\|_{Y^k}=\sum_{i=1}^{k}\|T_iy-T_i\bar{y}\|_Y\le\Big(\sum_{i=1}^{k}P_i\Big(\sum_{i=1}^{k}A_i\Big)+\sum_{i=1}^{k}Q_i\Big)\|y-\bar{y}\|_{Y^k}.$$

It follows from the condition $\sum_{i=1}^{k}P_i\Big(\sum_{i=1}^{k}A_i\Big)+\sum_{i=1}^{k}Q_i<1$ that T is a contractive mapping. Hence, T has a unique fixed point on Y^k, that is, BVP (2.1) has a unique solution. Therefore, we obtain the conclusion of the theorem.

Theorem 3.2 Assume that

(H_2) The functions $g_i:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, $(i=1,2,\cdots,k)$ are continuous and there exist functions $p_i(t),q_i(t),r_i(t)\in C([0,1],[0,+\infty))$, $(i=1,2,\cdots,k)$ such that

$$|g_i(t,u,v)|\le p_i(t)+q_i(t)|u(t)|+r_i(t)|v(t)|,$$

for all $t\in[0,1]$, $u,v\in\mathbb{R}$. Then the BVP (2.1) admits at least one solution in Y provided that

$$\sum_{i=1}^{k}\theta_i<1,$$

where

$$\theta_i = \Delta_i(q_i^* + \rho_i^{-\gamma} r_i^*) + \varpi_i + \sum_{j=1, j\neq i}^{k} (\tilde{\Delta}_j(q_j^* + \rho_j^{-\gamma} r_j^*) + \tilde{\varpi}_j),$$

and

$$p_i^* = \max_{t\in[0,1]} |p_i(t)|,\ q_i^* = \max_{t\in[0,1]} |q_i(t)|,\ r_i^* = \max_{t\in[0,1]} |r_i(t)|,\ \Delta_i = M_2\rho_i^{\alpha+1},$$
$$\varpi_i = 3\lambda_i\rho_i + \frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)},\ \tilde{\varpi}_j = 2\lambda_j\rho_j + \frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)},\ \tilde{\Delta}_j = M_1\rho_j^{\alpha+1}.$$

Proof. We divide the proof into two steps.
Step 1. We need to verify that the operator T is a completely continuous. In fact, since the functions $g_i(i = 1, 2, \cdots, k)$ are continuous, we can easily prove that the operators $T_i(i = 1, 2, \cdots k)$ are continuous, and thus T is continuous. Next, we have to show that T is compact. To see this, we define the bounded subset $\Lambda = \{y_i \in Y,\ \|y_i\|_Y \le \varepsilon_i\}$ on Y, then for any $y = (y_1, y_2, \cdots, y_k) \in \Lambda$, by (H_2), we find that

$$\begin{aligned}
|T_i y(t)| \le& \lambda_i\rho_i \int_0^t |y_i(s)|ds + \frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha} |g_i(s, y_i(s), \rho_i^{-\gamma C} D_{0,s}^{\gamma} y_i(s))|ds \\
&+ \sum_{j=1}^{k} \ell_j\Big(\lambda_j\rho_j|y_j(1)| + \frac{\rho_j^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |g_j(s, y_j(s), \rho_j^{-\gamma C} D_{0,s}^{\gamma} y_j(s))|ds\Big) \\
&+ \sum_{j=1, j\neq i}^{k} \ell_j\Big(\lambda_j\rho_j \int_0^1 |y_j(s)|ds + \frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} |g_j(s, y_j(s), \rho_j^{-\gamma C} D_{0,s}^{\gamma} y_j(s))|ds\Big) \\
&+ \sum_{j=1, j\neq i}^{k} \ell_j\Big(\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} |g_i(s, y_i(s), \rho_i^{-\gamma C} D_{0,s}^{\gamma} y_i(s))|ds + \lambda_i\rho_i \int_0^1 |y_i(s)|ds\Big) \\
\le& \lambda_i\rho_i\|y_i\| + \frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+2)} (p_i^* + q_i^*\|y_i\| + \rho_i^{-\gamma} r_i^* \|{}^C D_{0,t}^{\gamma} y_i\|) \\
&+ \sum_{j=1}^{k} \lambda_j\rho_j\|y_j\| + \sum_{j=1}^{k} \frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+1)} (p_j^* + q_j^*\|y_j\| + \rho_j^{-\gamma} r_j^* \|{}^C D_{0,t}^{\gamma} y_j\|) \\
&+ \sum_{j=1, j\neq i}^{k} \lambda_j\rho_j\|y_j\| + \sum_{j=1, j\neq i}^{k} \frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+2)} (p_j^* + q_j^*\|y_j\| + \rho_j^{-\gamma} r_j^* \|{}^C D_{0,t}^{\gamma} y_j\|) \\
&+ \frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+2)} (p_i^* + q_i^*\|y_i\| + \rho_i^{-\gamma} r_i^* \|{}^C D_{0,t}^{\gamma} y_i\|) + \lambda_i\rho_i\|y_i\| \\
\le& \lambda_i\rho_i\|y_i\|_Y + \frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+2)} (p_i^* + (q_i^* + \rho_i^{-\gamma} r_i^*)\|y_i\|_Y) \\
&+ \sum_{j=1}^{k} \lambda_j\rho_j\|y_j\|_Y + \sum_{j=1}^{k} \frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+1)} (p_j^* + (q_j^* + \rho_j^{-\gamma} r_j^*)\|y_j\|_Y) \\
&+ \sum_{j=1, j\neq i}^{k} \lambda_j\rho_j\|y_j\|_Y + \sum_{j=1, j\neq i}^{k} \frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+2)} (p_j^* + (q_j^* + \rho_j^{-\gamma} r_j^*)\|y_j\|_Y)
\end{aligned}$$

$$+\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+2)}(p_i^* + (q_i^* + \rho_i^{-\gamma} r_i^*)\|y_i\|_Y) + \lambda_i\rho_i\|y_i\|_Y.$$

From which we can deduce that

$$\begin{aligned}
\|T_i y\| \le & 3\lambda_i\rho_i\|y_i\|_Y + \Big(\frac{2}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)\rho_i^{\alpha+1}(p_i^* + (q_i^* + \rho_i^{-\gamma} r_i^*)\|y_i\|_Y) \\
& + \sum_{j=1,j\neq i}^{k} 2\lambda_j\rho_j\|y_j\|_Y + \sum_{j=1,j\neq i}^{k}\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}(p_j^* + (q_j^* + \rho_j^{-\gamma} r_j^*)\|y_j\|_Y) \\
\le & \Big(3\lambda_i\rho_i + \Big(\frac{2}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)\rho_i^{\alpha+1}(q_i^* + \rho_i^{-\gamma} r_i^*)\Big)\|y_i\|_Y \\
& + \sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j + \Big(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}(q_j^* + \rho_j^{-\gamma} r_j^*)\Big)\|y_j\|_Y \\
& + \Big(\frac{2}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)\rho_i^{\alpha+1} p_i^* + \sum_{j=1,j\neq i}^{k}\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1} p_j^*.
\end{aligned} \tag{3.7}$$

On the other hand, by Lemma 2.3 and (H_2), we also can get the estimate

$$\begin{aligned}
& |{}^C D_{0,t}^{\gamma} T_i y(t)| \\
\le & \frac{\lambda_i\rho_i}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma}|y_i(s)|ds + \frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1-\gamma)}\int_0^t (t-s)^{\alpha-\gamma}|g_i(s, y_i(s), \rho_i^{-\gamma C} D_{0,s}^{\gamma} y_i(s))|ds \\
& + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{j=1}^{k}\ell_j\Big(\lambda_j\rho_j|y_j(1)| + \frac{\rho_j^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|g_j(s, y_j(s), \rho_j^{-\gamma C} D_{0,s}^{\gamma} y_j(s))|ds\Big) \\
& + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\ell_j\Big(\lambda_j\rho_j\int_0^1 |y_j(s)|ds + \frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}|g_j(s, y_j(s), \rho_j^{-\gamma C} D_{0,s}^{\gamma} y_j(s))|ds\Big) \\
& + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\ell_j\Big(\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}|g_i(s, y_i(s), \rho_i^{-\gamma C} D_{0,s}^{\gamma} y_i(s))|ds + \lambda_i\rho_i\int_0^1 |y_i(s)|ds\Big) \\
\le & \frac{\lambda_i\rho_i}{\Gamma(2-\gamma)}\|y_i\| + \frac{\rho_i^{\alpha+1}}{\Gamma(\alpha-\gamma+2)}(p_i^* + q_i^*\|y_i\| + \rho_i^{-\gamma} r_i^*\|{}^C D_{0,t}^{\gamma} y_i\|) \\
& + \frac{1}{\Gamma(2-\gamma)}\sum_{j=1}^{k}\lambda_j\rho_j\|y_j\| + \frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\sum_{j=1}^{k}\rho_j^{\alpha+1}(p_j^* + q_j^*\|y_j\| + \rho_j^{-\gamma} r_j^*\|{}^C D_{0,t}^{\gamma} y_j\|) \\
& + \frac{1}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j\|y_j\| + \frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+1}(p_j^* + q_j^*\|y_j\| + \rho_j^{-\gamma} r_j^*\|{}^C D_{0,t}^{\gamma} y_j\|) \\
& + \frac{\rho_i^{\alpha+1}}{\Gamma(2-\gamma)\Gamma(\alpha+2)}(p_i^* + q_i^*\|y_i\| + \rho_i^{-\gamma} r_i^*\|{}^C D_{0,t}^{\gamma} y_i\|) + \frac{\lambda_i\rho_i}{\Gamma(2-\gamma)}\|y_i\|.
\end{aligned}$$

In a similar manner, we deduce

$$\begin{aligned}
& \|{}^C D_{0,t}^{\gamma} T_i y\| \\
& \le \Big(\frac{1}{\Gamma(\alpha-\gamma+2)} + \frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)} + \frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_i^{\alpha+1}(q_i^* + \rho_i^{-\gamma} r_i^*)\|y_i\|_Y
\end{aligned}$$

$$+\sum_{j=1,j\neq i}^{k}\Big(\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}+\Big(\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}(q_j^*+\rho_j^{-\gamma}r_j^*)\Big)\|y_j\|_Y$$
$$+\sum_{j=1,j\neq i}^{k}\Big(\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}p_j^*+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}\|y_i\|_Y \tag{3.8}$$
$$+\Big(\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_i^{\alpha+1}p_i^*.$$

From (3.7) and (3.8), we get that

$$\begin{aligned}
&\|T_iy\|+\|^CD_{0,t}^{\gamma}T_iy\|\\
\leq\ & M_2\rho_i^{\alpha+1}(q_i^*+\rho_i^{-\gamma}r_i^*)\|y_i\|_Y+\Big(3\lambda_i\rho_i+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}\Big)\|y_i\|_Y+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j+\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}\Big)\|y_j\|_Y\\
&+\sum_{j=1,j\neq i}^{k}M_1\rho_j^{\alpha+1}(q_j^*+\rho_j^{-\gamma}r_j^*)\|y_j\|_Y+M_2\rho_i^{\alpha+1}p_i^*+\sum_{j=1,j\neq i}^{k}M_1\rho_j^{\alpha+1}p_j^*\\
=\ &\Delta_i(q_i^*+\rho_i^{-\gamma}r_i^*)\|y_i\|_Y+\varpi_i\|y_i\|_Y+\sum_{j=1,j\neq i}^{k}\tilde{\Delta}_j(q_j^*+\rho_j^{-\gamma}r_j^*)\|y_j\|_Y+\sum_{j=1,j\neq i}^{k}\tilde{\varpi}_j\|y_j\|_Y+\Delta_ip_i^*+\sum_{j=1,j\neq i}^{k}\tilde{\Delta}_jp_j^*\\
\leq\ &(\Delta_i(q_i^*+\rho_i^{-\gamma}r_i^*)+\varpi_i)\|y_i\|_Y+\sum_{j=1,j\neq i}^{k}(\tilde{\Delta}_j(q_j^*+\rho_j^{-\gamma}r_j^*)+\tilde{\varpi}_j)\|y_j\|_Y+\Delta_ip_i^*+\sum_{j=1,j\neq i}^{k}\tilde{\Delta}_jp_j^*\\
\leq\ &\Big[(\Delta_i(q_i^*+\rho_i^{-\gamma}r_i^*)+\varpi_i)+\sum_{j=1,j\neq i}^{k}(\tilde{\Delta}_j(q_j^*+\rho_j^{-\gamma}r_j^*)+\tilde{\varpi}_j)\Big]\sum_{j=1}^{k}\|y_j\|_Y+N_i\\
=\ &\theta_i\sum_{j=1}^{k}\varepsilon_j+N_i,
\end{aligned}$$

where

$$N_i=\Delta_ip_i^*+\sum_{j=1,j\neq i}^{k}\tilde{\Delta}_jp_j^*,\ i=1,2,\cdots k. \tag{3.9}$$

Form this it follows that

$$\|Ty\|_{Y^k}=\sum_{i=1}^{k}\|T_iy\|_Y\leq\sum_{i=1}^{k}\theta_i\Big(\sum_{j=1}^{k}\varepsilon_j\Big)+\sum_{i=1}^{k}N_i.$$

Hence, the operator T is uniformly bounded on Λ.

Now, We will show that the operator T is equicontinuous on Λ. Indeed, for $y=(y_1,y_2,\cdots,y_k)\in\Lambda,\ t_1,t_2\in[0,1],\ t_1<t_2$, we have

$$\begin{aligned}
&|T_iy(t_2)-T_iy(t_1)|\\
&\leq\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1)}\Big(\int_0^{t_1}((t_2-s)^{\alpha}-(t_1-s)^{\alpha})ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha}ds\Big)(p_i^*+q_i^*\|y_i\|+\rho_i^{-\gamma}r_i^*\|^CD_{0,t}^{\gamma}y_i\|)
\end{aligned}$$

$$
\begin{aligned}
&+(t_2-t_1)\sum_{j=1}^{k}\Big(\lambda_j\rho_j\|y_j\|+\frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+1)}(p_j^*+q_j^*\|y_j\|+\rho_j^{-\gamma}r_j^*\|^C D_{0,t}^{\gamma}y_j\|)\Big)\\
&+(t_2-t_1)\sum_{j=1,j\neq i}^{k}\Big(\lambda_j\rho_j\|y_j\|+\frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+2)}(p_j^*+q_j^*\|y_j\|+\rho_j^{-\gamma}r_j^*\|^C D_{0,t}^{\gamma}y_j\|)\Big)\\
&+(t_2-t_1)\Big(\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+2)}(p_i^*+q_i^*\|y_i\|+\rho_i^{-\gamma}r_i^*\|^C D_{0,t}^{\gamma}y_i\|)+\lambda_i\rho_i\|y_i\|\Big)+\lambda_i\rho_i\|y_i\|(t_2-t_1)\\
\le{}&\lambda_i\rho_i\varepsilon_i(t_2-t_1)+\frac{\rho_i^{\alpha+1}(p_i^*+(q_i^*+r_i^*\rho_i^{-\gamma})\varepsilon_i)}{\Gamma(\alpha+2)}(t_2^{\alpha+1}-t_1^{\alpha+1})\\
&+(t_2-t_1)\sum_{j=1}^{k}\Big(\lambda_j\rho_j\varepsilon_j+\frac{\rho_j^{\alpha+1}(p_j^*+(q_j^*+r_j^*\rho_j^{-\gamma})\varepsilon_j)}{\Gamma(\alpha+1)}\Big)\\
&+(t_2-t_1)\sum_{j=1,j\neq i}^{k}\Big(\lambda_j\rho_j\varepsilon_j+\frac{\rho_j^{\alpha+1}(p_j^*+(q_j^*+r_j^*\rho_j^{-\gamma})\varepsilon_j)}{\Gamma(\alpha+2)}\Big)+(t_2-t_1)\Big(\frac{\rho_i^{\alpha+1}(p_i^*+(q_i^*+r_i^*\rho_i^{-\gamma})\varepsilon_i)}{\Gamma(\alpha+2)}+\lambda_i\rho_i\varepsilon_i\Big),
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
&|^C D_{0,t}^{\gamma}T_iy(t_2)-{}^C D_{0,t}^{\gamma}T_iy(t_1)|\\
\le{}&\frac{\lambda_i\rho_i}{\Gamma(1-\gamma)}\|y_i\|\Big(\int_0^{t_1}((t_1-s)^{-\gamma}-(t_2-s)^{-\gamma})ds+\int_{t_1}^{t_2}(t_2-s)^{-\gamma}ds\Big)\\
&+\frac{\rho_i^{\alpha+1}(p_i^*+q_i^*\|y_i\|+\rho_i^{-\gamma}r_i^*\|^C D_{0,t}^{\gamma}y_i\|)}{\Gamma(\alpha-\gamma+1)}\Big(\int_0^{t_1}((t_2-s)^{\alpha-\gamma}-(t_1-s)^{\alpha-\gamma})ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-\gamma}ds\Big)\\
&+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})}{\Gamma(2-\gamma)}\sum_{j=1}^{k}\lambda_j\rho_j\|y_j\|+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\sum_{j=1}^{k}\rho_j^{\alpha+1}(p_j^*+q_j^*\|y_j\|+\rho_j^{-\gamma}r_j^*\|^C D_{0,t}^{\gamma}y_j\|)\\
&+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j\|y_j\|+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+1}(p_j^*+q_j^*\|y_j\|+\rho_j^{-\gamma}r_j^*\|^C D_{0,t}^{\gamma}y_j\|)\\
&+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})\rho_i^{\alpha+1}}{\Gamma(2-\gamma)\Gamma(\alpha+2)}(p_i^*+q_i^*\|y_i\|+\rho_i^{-\gamma}r_i^*\|^C D_{0,t}^{\gamma}y_i\|)+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})\lambda_i\rho_i\|y_i\|}{\Gamma(2-\gamma)}\\
\le{}&\frac{\lambda_i\rho_i\varepsilon_i}{\Gamma(2-\gamma)}(t_1^{1-\gamma}-t_2^{1-\gamma}+2(t_2-t_1)^{1-\gamma})+\frac{\rho_i^{\alpha+1}(p_i^*+(q_i^*+\rho_i^{-\gamma}r_i^*)\varepsilon_i)}{\Gamma(\alpha-\gamma+2)}(t_2^{\alpha-\gamma+1}-t_1^{\alpha-\gamma+1})\\
&+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})}{\Gamma(2-\gamma)}\sum_{j=1}^{k}\lambda_j\rho_j\varepsilon_j+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\sum_{j=1}^{k}\rho_j^{\alpha+1}(p_j^*+(q_j^*+\rho_j^{-\gamma}r_j^*)\varepsilon_j)\\
&+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})}{\Gamma(2-\gamma)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j\varepsilon_j+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+1}(p_j^*+(q_j^*+\rho_j^{-\gamma}r_j^*)\varepsilon_j)\\
&+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})\rho_i^{\alpha+1}}{\Gamma(2-\gamma)\Gamma(\alpha+2)}(p_i^*+(q_i^*+\rho_i^{-\gamma}r_i^*)\varepsilon_i)+\frac{(t_2^{1-\gamma}-t_1^{1-\gamma})\lambda_i\rho_i\varepsilon_i}{\Gamma(2-\gamma)}.
\end{aligned}
\tag{3.11}
$$

Form (3.10) and (3.11), we get

$$
\begin{aligned}
&\|T_iy(t_2)-T_iy(t_1)\|_Y\\
\le&\Big(3\lambda_i\rho_i\varepsilon_i+\Big(\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+1)}+\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+2)}\Big)(p_i^*+(q_i^*+\rho_i^{-\beta}r_i^*)\varepsilon_i)\Big)(t_2-t_1)\\
&+\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha+2)}(p_i^*+(q_i^*+\rho_i^{-\gamma}r_i^*)\varepsilon_i)(t_2^{\alpha+1}-t_1^{\alpha+1})+\frac{2\lambda_i\rho_i\varepsilon_i}{\Gamma(2-\gamma)}(t_2^{1-\gamma}-t_1^{1-\gamma})\\
&+\Big(\Big(\frac{\rho_i^{\alpha+1}}{\Gamma(2-\gamma)\Gamma(\alpha+2)}+\frac{\rho_i^{\alpha+1}}{\Gamma(2-\gamma)\Gamma(\alpha+1)}\Big)(p_i^*+(q_i^*+\rho_i^{-\gamma}r_i^*)\varepsilon_i)\Big)(t_2^{1-\gamma}-t_1^{1-\gamma})\\
&+\frac{\lambda_i\rho_i\varepsilon_i}{\Gamma(2-\gamma)}(t_1^{1-\gamma}-t_2^{1-\gamma}+2(t_2-t_1)^{1-\gamma})+\frac{\rho_i^{\alpha+1}}{\Gamma(\alpha-\gamma+2)}(p_i^*+(q_i^*+\rho_i^{-\gamma}r_i^*)\varepsilon_i)(t_2^{\alpha-\gamma+1}-t_1^{\alpha-\gamma+1})\\
&+\sum_{j=1,j\neq i}^{k}\Big(\frac{2\lambda_j\rho_j\varepsilon_j}{\Gamma(2-\gamma)}+\Big(\frac{\rho_j^{\alpha+1}}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{\rho_j^{\alpha+1}}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)(p_j^*+(q_j^*+\rho_j^{-\gamma}r_j^*)\varepsilon_j)\Big)(t_2^{1-\gamma}-t_1^{1-\gamma})\\
&+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j\varepsilon_j+\Big(\frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{\rho_j^{\alpha+1}}{\Gamma(\alpha+1)}\Big)(p_j^*+(q_j^*+\rho_j^{-\gamma}r_j^*)\varepsilon_j)\Big)(t_2-t_1),
\end{aligned}
$$

which implies $\|T_iy(t_2)-T_iy(t_1)\|_Y\to 0$ as $t_2\to t_1$, and so $\|Ty(t_2)-Ty(t_1)\|_{Y^k}\to 0$ as $t_2\to t_1$. Therefore, the operator T is equicontinuous on Λ. According to Arzelá-Ascoli theorem that T is completely continuous.

Step 2. By applying Scheafer's fixed point theorem, we now prove that T has fixed point in Y. To this aim, we define $\Omega=\{(y_1,y_2,\cdots,y_k)\in Y^k:(y_1,y_2,\cdots,y_k)=\mu T(y_1,y_2,\cdots,y_k),\mu\in(0,1)\}$ and show that Ω is bounded. In fact, for $(y_1,y_2,\cdots,y_k)\in\Omega$, then $(y_1,y_2,\cdots,y_k)=\mu T(y_1,y_2,\cdots,y_k)$, that is, for $t\in[0,1]$, we have $y_i(t)=\mu T_i(y_1,y_2,\cdots,y_k),\ i=1,2,\cdots,k$. Similarly in the proof of (3.7), by assumption (H_2), we deduce

$$
\begin{aligned}
|y_i(t)|\le&\mu\Big[\Big(3\lambda_i\rho_i+\Big(\frac{2}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\rho_i^{\alpha+1}(q_i^*+\rho_i^{-\gamma}r_i^*)\Big)\|y_i\|_Y\\
&+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j+\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}(q_j^*+\rho_j^{-\gamma}r_j^*)\Big)\|y_j\|_Y\\
&+\Big(\frac{2}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\rho_i^{\alpha+1}p_i^*+\sum_{j=1,j\neq i}^{k}\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}p_j^*\Big],
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
\|y_i\|\le&\Big(3\lambda_i\rho_i+\Big(\frac{2}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\rho_i^{\alpha+1}(q_i^*+\rho_i^{-\gamma}r_i^*)\Big)\|y_i\|_Y\\
&+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j+\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}(q_j^*+\rho_j^{-\gamma}r_j^*)\Big)\|y_j\|_Y\\
&+\Big(\frac{2}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\rho_i^{\alpha+1}p_i^*+\sum_{j=1,j\neq i}^{k}\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}p_j^*.
\end{aligned}
\tag{3.12}
$$

In a similar manner of deduce (3.8), by assumption (H_2), we also can obtain the estimate

$$
\begin{aligned}
&|{}^C D_{0,t}^{\gamma} y_i(t)| \\
&\leq \mu\Big[\Big(\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_i^{\alpha+1}(q_i^*+\rho_i^{-\gamma}r_i^*)\|y_i\|_Y \\
&\quad+\sum_{j=1,j\neq i}^{k}\Big(\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}+\Big(\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}(q_j^*+\rho_j^{-\gamma}r_j^*)\Big)\|y_j\|_Y \\
&\quad+\sum_{j=1,j\neq i}^{k}\Big(\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}p_j^*+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}\|y_i\|_Y \\
&\quad+\Big(\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_i^{\alpha+1}p_i^*\Big].
\end{aligned}
$$

Then for $t \in [0, 1]$, we get

$$
\begin{aligned}
&\|{}^C D_{0,t}^{\gamma} y_i\| \\
&\leq \Big(\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_i^{\alpha+1}(q_i^*+\rho_i^{-\gamma}r_i^*)\|y_i\|_Y \\
&\quad+\sum_{j=1,j\neq i}^{k}\Big(\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}+\Big(\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}(q_j^*+\rho_j^{-\gamma}r_j^*)\Big)\|y_j\|_Y \\
&\quad+\sum_{j=1,j\neq i}^{k}\Big(\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_j^{\alpha+1}p_j^*+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}\|y_i\|_Y \\
&\quad+\Big(\frac{1}{\Gamma(\alpha-\gamma+2)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+1)}+\frac{1}{\Gamma(2-\gamma)\Gamma(\alpha+2)}\Big)\rho_i^{\alpha+1}p_i^*.
\end{aligned}
$$

Combining this with (3.12) gives

$$
\begin{aligned}
&\|y_i\|+\|{}^C D_{0,t}^{\gamma} y_i\| \\
&\leq M_2\rho_i^{\alpha+1}(q_i^*+\rho_i^{-\gamma}r_i^*)\|y_i\|_Y+\Big(3\lambda_i\rho_i+\frac{3\lambda_i\rho_i}{\Gamma(2-\gamma)}\Big)\|y_i\|_Y+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j+\frac{2\lambda_j\rho_j}{\Gamma(2-\gamma)}\Big)\|y_j\|_Y \\
&\quad+\sum_{j=1,j\neq i}^{k}M_1\rho_j^{\alpha+1}(q_j^*+\rho_j^{-\gamma}r_j^*)\|y_j\|_Y+M_2\rho_i^{\alpha+1}p_i^*+\sum_{j=1,j\neq i}^{k}M_1\rho_j^{\alpha+1}p_j^* \\
&\leq \theta_i\sum_{j=1}^{k}\|y_j\|_Y+N_i,
\end{aligned}
$$

where N_i is defined as in (3.9), from which we deduce that

$$\|y\|_{Y^k}=\sum_{i=1}^{k}\|y_i\|_Y\leq\sum_{i=1}^{k}\theta_i\|y\|_{Y^k}+\sum_{i=1}^{k}N_i.$$

It follows from $\sum_{i=1}^{k}\theta_i<1$ that Ω is bounded. By Theorem 2.4, the operator T has at least one fixed point, that is, the BVP (2.1) has at least one solution.

4. Examples

Example 4.1 Consider the BVP (1.3) with $k=3$, $\alpha=\frac{1}{2}, \gamma=\frac{1}{3}$, $\lambda_1=\lambda_2=\lambda_3=\rho_1=\frac{1}{10}$, $\rho_2=\frac{1}{20}$, $\rho_3=\frac{1}{30}$, and

$$\begin{cases} \mathfrak{g}_1(x,\mathfrak{u},\mathfrak{v}) = \cos x + \dfrac{1}{(x+2)^2}(\sin\mathfrak{u}+\mathfrak{v}), \ (x,\mathfrak{u},\mathfrak{v}) \in [0,\rho_1]\times\mathbb{R}\times\mathbb{R}, \\ \mathfrak{g}_2(x,\mathfrak{u},\mathfrak{v}) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{2(x^2+4)^2}(|\mathfrak{u}|+|\mathfrak{v}|), \ (x,\mathfrak{u},\mathfrak{v}) \in [0,\rho_2]\times\mathbb{R}\times\mathbb{R}, \\ \mathfrak{g}_3(x,\mathfrak{u},\mathfrak{v}) = 1 + x^2 + \dfrac{1}{3(x+3)^3}\Big(\dfrac{\mathfrak{u}}{1+\mathfrak{u}}+\mathfrak{v}\Big), \ (x,\mathfrak{u},\mathfrak{v}) \in [0,\rho_3]\times\mathbb{R}\times\mathbb{R}. \end{cases}$$

In view of Lemma 2.6, we get the equivalent system

$$\begin{cases} {}^{C}D_{0,t}^{1/2}\Big(D+\dfrac{1}{100}\Big)y_1(t)=\Big(\dfrac{1}{10}\Big)^{3/2}\Big[\cos t+\dfrac{1}{(t+2)^2}\Big(\sin y_1(t)+\Big(\dfrac{1}{10}\Big)^{-1/3}{}^{C}D_{0,t}^{1/3}y_1(t)\Big)\Big], \\ {}^{C}D_{0,t}^{1/2}\Big(D+\dfrac{1}{200}\Big)y_2(t)=\Big(\dfrac{1}{20}\Big)^{3/2}\Big[\dfrac{1}{\sqrt{t}}+\dfrac{1}{2(t^2+4)^2}\Big(|y_2(t)|+\Big(\dfrac{1}{20}\Big)^{-1/3}|{}^{C}D_{0,t}^{1/3}y_2(t)|\Big)\Big], \\ {}^{C}D_{0,t}^{1/2}\Big(D+\dfrac{1}{300}\Big)y_3(t)=\Big(\dfrac{1}{30}\Big)^{3/2}\Big[1+t^2+\dfrac{1}{3(t+3)^3}\Big(\dfrac{y_3(t)}{1+y_3(t)}+\Big(\dfrac{1}{30}\Big)^{-1/3}{}^{C}D_{0,t}^{1/3}y_3(t)\Big)\Big], \\ y_1(0)=y_2(0)=y_3(0)=0, \quad y_1(1)=y_2(1)=y_3(1), \\ (1/10)^{-1}y_1'(1)+(1/20)^{-1}y_2'(1)+(1/30)^{-1}y_3'(1)=0. \end{cases} \tag{4.1}$$

From (4.1), for $t \in [0,1], u, v, u_1, v_1 \in \mathbb{R}$, we can conclude that

$$|g_1(t,u,v)-g_1(t,u_1,v_1)| \le \frac{1}{(t+2)^2}(|u-u_1|+|v-v_1|),$$
$$|g_2(t,u,v)-g_2(t,u_1,v_1)| \le \frac{1}{2(t^2+4)^2}(|u-u_1|+|v-v_1|),$$
$$|g_3(t,u,v)-g_3(t,u_1,v_1)| \le \frac{1}{3(t+3)^3}(|u-u_1|+|v-v_1|).$$

So, we get

$$a_1(t)=\frac{1}{(t+2)^2}, \ a_2(t)=\frac{1}{2(t^2+4)^2}, \ a_3(t)=\frac{1}{3(t+3)^3}.$$

By simple calculation, we obtain

$$A_1=\max_{t\in[0,1]}|a_1(t)|=\frac{1}{4}, \ A_2=\max_{t\in[0,1]}|a_2(t)|=\frac{1}{32}, \ A_3=\max_{t\in[0,1]}|a_3(t)|=\frac{1}{81}.$$

$$P_1 \doteq 0.8256, \ P_2 \doteq 0.7281, \ P_3 \doteq 0.7183, \ Q_1 \doteq 0.0983, \ Q_2 \doteq 0.0878, \ Q_3 \doteq 0.0843.$$

Then

$$\Big(\sum_{i=1}^{3}P_i\Big)\Big(\sum_{i=1}^{3}A_i\Big)+\sum_{i=1}^{3}Q_i \doteq 0.9374 < 1.$$

From Theorem 3.1 that the BVP (4.1) has a unique solution.

Example 4.2 Consider the BVP (1.3) with k=3, $\alpha=\frac{1}{2}$, $\gamma=\frac{1}{3}$, $\lambda_1=\lambda_2=\lambda_3=\frac{1}{20}$, $\rho_2=\frac{1}{5}$, $\rho_1=\rho_3=\frac{1}{10}$, and

$$\begin{cases} \mathfrak{g}_1(x,\mathfrak{u},\mathfrak{v}) = \dfrac{x}{10} + \dfrac{1}{2(x+3)^3}\mathfrak{u} + \dfrac{1}{5\sqrt[3]{10}(x+2)^2}\mathfrak{v}, \ (x,\mathfrak{u},\mathfrak{v}) \in [0,\rho_1]\times\mathbb{R}\times\mathbb{R}, \\ \mathfrak{g}_2(x,\mathfrak{u},\mathfrak{v}) = \sin x + \dfrac{1}{3(x+2)^2}\mathfrak{u} + \dfrac{x}{20\sqrt[3]{5}}\mathfrak{v}, \ (x,\mathfrak{u},\mathfrak{v}) \in [0,\rho_2]\times\mathbb{R}\times\mathbb{R}, \\ \mathfrak{g}_3(x,\mathfrak{u},\mathfrak{v}) = 2x + \dfrac{1}{(x+5)^2}\mathfrak{u} + \dfrac{1}{12\sqrt[3]{10}(x+2)}\mathfrak{v}, \ (x,\mathfrak{u},\mathfrak{v}) \in [0,\rho_3]\times\mathbb{R}\times\mathbb{R}. \end{cases}$$

Then by Lemma 2.6, we obtain the equivalent system

$$\begin{cases} {}^CD^{1/2}_{0,t}\Big(D + \dfrac{1}{200}\Big)y_1(t) = \Big(\dfrac{1}{10}\Big)^{3/2}\Big[\dfrac{t}{10} + \dfrac{\sin y_1(t)}{2(t+3)^3} + \dfrac{{}^CD^{1/3}_{0,t}y_1(t)}{5\sqrt[3]{10}(t+2)^2}\Big], \\ {}^CD^{1/2}_{0,t}\Big(D + \dfrac{1}{100}\Big)y_2(t) = \Big(\dfrac{1}{5}\Big)^{3/2}\Big[\sin t + \dfrac{y_2(t)}{3(t+2)^2} + \dfrac{t}{20\sqrt[3]{5}}{}^CD^{1/3}_{0,t}y_2(t)\Big], \\ {}^CD^{1/2}_{0,t}\Big(D + \dfrac{1}{200}\Big)y_3(t) = \Big(\dfrac{1}{10}\Big)^{3/2}\Big[2t + \dfrac{y_3(t)}{(t+5)^2} + \dfrac{D^{1/3}_{0,t}y_3(t)}{12\sqrt[3]{10}(t+2)}\Big], \\ y_1(0) = y_2(0) = y_3(0), \ y_1(1) = y_2(1) = y_3(1), \\ (1/10)^{-1}y_1'(1) + (1/5)^{-1}y_2'(1) + (1/10)^{-1}y_3'(1) = 0. \end{cases} \tag{4.2}$$

Then

$$q_1(t) = \frac{1}{2(t+3)^3}, \ r_1(t) = \frac{1}{5(t+2)^2}, \ q_2(t) = \frac{1}{3(t+2)^2},$$
$$r_2(t) = \frac{t}{20}, \ q_3(t) = \frac{1}{(t+5)^2}, \ r_3(t) = \frac{1}{12(t+2)}.$$

For $t \in [0,1]$, we have $q_1^*=\frac{1}{54}$, $q_2^*=\frac{1}{12}$, $q_3^*=\frac{1}{25}$, $r_1^*=r_2^*=\frac{1}{20}$, $r_3^*=\frac{1}{24}$. By calculation, we get

$$\Delta_1 \doteq 0.1783, \ \tilde{\Delta}_1 \doteq 0.1253, \ \varpi_1 \doteq 0.0316, \ \tilde{\varpi}_1 \doteq 0.0211,$$
$$\Delta_2 \doteq 0.4043, \ \tilde{\Delta}_2 \doteq 0.3544, \ \varpi_2 \doteq 0.0632, \ \tilde{\varpi}_2 \doteq 0.0421,$$
$$\Delta_3 \doteq 0.1783, \ \tilde{\Delta}_3 \doteq 0.1253, \ \varpi_3 \doteq 0.0316, \ \tilde{\varpi}_3 \doteq 0.0211,$$

So,

$$\theta_1 = \Delta_1(q_1^* + \rho_1^{-\gamma}r_1^*) + \varpi_1 + (\tilde{\Delta}_2(q_2^* + \rho_2^{-\gamma}r_2^*) + \tilde{\varpi}_2) + (\tilde{\Delta}_3(q_3^* + \rho_3^{-\gamma}r_3^*) + \tilde{\varpi}_3) \doteq 0.1934,$$
$$\theta_2 = \Delta_2(q_2^* + \rho_2^{-\gamma}r_2^*) + \varpi_2 + (\tilde{\Delta}_1(q_1^* + \rho_1^{-\gamma}r_1^*) + \tilde{\varpi}_1) + (\tilde{\Delta}_3(q_3^* + \rho_3^{-\gamma}r_3^*) + \tilde{\varpi}_3) \doteq 0.2057,$$
$$\theta_3 = \Delta_3(q_3^* + \rho_3^{-\gamma}r_3^*) + \varpi_3 + (\tilde{\Delta}_1(q_1^* + \rho_1^{-\gamma}r_1^*) + \tilde{\varpi}_1) + (\tilde{\Delta}_2(q_2^* + \rho_2^{-\gamma}r_2^*) + \tilde{\varpi}_2) \doteq 0.1935.$$

Thus,

$$\theta_1+\theta_2+\theta_3 \doteq 0.5926 < 1.$$

According to Theorem 3.2, the BVP (4.2) has at least one solution.

5. Conclusions

This paper considers the fractional Langevin equations on a star graph of the form (1.3). By using Lemma 2.6, the problem (1.3) is transformed into an equivalent system of fractional Langevin equations supplemented with mixed boundary conditions defined on $[0,1]$, that is, problem (2.1). Making use of the fixed point theorems (Schauder's fixed point theorem, Banach's contraction mapping principle), sufficient criteria for the existence and uniqueness results are derived. Finally, we present two examples to illustrate the validity of the obtained results. As a possible extension of this paper, we will study the higher-order fractional Langevin-type equations on star graphs in the future, such as

$$ {}^{C}D_{0,x}^{\alpha}(D^{2}+\lambda_i)\mathfrak{y}_i(x)=\mathfrak{g}_i(x,\mathfrak{y}_i(x),{}^{C}D_{0,x}^{\beta}\mathfrak{y}_i(x)),\ 0<x<l_i,\ i=1,2,\cdots,k, $$

supplemented with the boundary conditions

$$ \begin{cases} \mathfrak{y}'_i(0)=\mathfrak{y}_i(0)=0, \quad i=1,2,\cdots,k,\\ \mathfrak{y}'_i(l_i)=\mathfrak{y}'_j(l_j), \quad i,j=1,2,\cdots,k, i\neq j,\\ \sum_{i=1}^{k}\mathfrak{y}''_i(l_i)=0, \quad i=1,2,\cdots,k, \end{cases} $$

and

$$ \begin{cases} \mathfrak{y}'_i(0)=\mathfrak{y}_i(1)=0, \quad i=1,2,\cdots,k,\\ \mathfrak{y}''_i(l_i)=\mathfrak{y}''_j(l_j), \quad i,j=1,2,\cdots,k, i\neq j,\\ \sum_{i=1}^{k}\mathfrak{y}''_i(l_i)=0,\ i=1,2,\cdots,k, \end{cases} $$

where $0<\alpha<1,\ 0<\beta<\alpha,\ \lambda_i\in\mathbb{R}^{+},\ i=1,2,\cdots,k,\ {}^{C}D_{0,x}^{\alpha},{}^{C}D_{0,x}^{\beta}$ are Caputo fractional derivative, D^2 is the ordinary second-order derivative, $\mathfrak{g}_i\in C([0,l_i]\times\mathbb{R}^2,\mathbb{R}),\ i=1,2,\cdots,k$. The star graph has $k+1$ nodes and k edges, that is $G=V\cup E$, $V=\{v_0,v_1,\cdots,v_k\}$, $E=\{e_i=\overrightarrow{v_iv_0}, i=1,2,\cdots,k\}$, where v_0 is the junction node, $e_i=\overrightarrow{v_iv_0}$ represents the edge connecting v_i and v_0 with length $l_i=\left|\overrightarrow{v_iv_0}\right|, i=1,2,\cdots,k$.

Acknowledgments

The authors wish to express their sincere appreciation to the editor and the anonymous referees for their valuable comments and suggestions. This research is supported by the National Natural Science Foundation of China (11601007) and the Key Program of University Natural Science Research Fund of Anhui Province (KJ2020A0291).

Conflict of interest

The authors declare that they have no competing interests.

References

1. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science Ltd., **204** (2006), 1–523. https://doi.org/10.1016/S0304-0208(06)80001-0

2. X. Zheng, H. Wang, An optimal-order numerical approximation to variable-order space-fractional diffusion equations on uniform or graded meshes, *SIAM J. Numer. Anal.*, **58** (2020), 330–352. https://doi.org/10.1137/19M1245621
3. V. J. Ervin, J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Methods Partial Differ. Equations*, **22** (2006), 558–576. https://doi.org/10.1002/num.20112
4. B. Ahmad, J. Henderson, R. Luca, *Boundary Value Problems for Fractional Differential Equations and Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, **9** (2021). https://doi.org/10.1142/11942
5. B. Ahmad, M. Alghanmi, A. Alsaedi, J. J. Nieto, Existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary conditions, *Appl. Math. Lett.*, **116** (2021), 1–10. https://doi.org/10.1016/j.aml.2021.107018
6. R. Luca, On a class of nonlinear singular Riemann-Liouville fractional differential equations, *Results Math.*, **73** (2018), 1–15. https://doi.org/10.1007/s00025-018-0887-5
7. R. Luca, Positive solutions for a system of fractional differential equations with p-Laplacian operator and multi-point boundary conditions, *Nonlinear Anal. Model. Control*, **23** (2018), 771–801. https://doi.org/10.15388/NA.2018.5.8
8. G. Lumer, Connecting of local operators and evolution equations on networks, in *Potential Theory Copenhagen*, *Lect. Notes Math.*, Springer, Berlin, Heidelberg, **787** (1979), 219–234. https://doi.org/10.1007/BFb0086338
9. A. I. Vol'pert, Differential equations on graphs, *Math. Model. Nat. Phenom.*, **10** (2015), 6–15. https://doi.org/10.1051/mmnp/201510502
10. J. R. Graef, L. Kong, M. Wang, Existence and uniqueness of solutions for a fractional boundary value problem on a graph, *Fract. Calc. Appl. Anal.*, **17** (2014), 499–510. https://doi.org/10.2478/s13540-014-0182-4
11. V. Mehandiratta, M. Mehra, G. Leugering, Existence and uniqueness results for a nonlinear Caputo fractional boundary value problem on a star graph, *J. Math. Anal. Appl.*, **477** (2019), 1243–1264. https://doi.org/10.1016/j.jmaa.2019.05.011
12. W. Zhang, W. Liu, Existence and Ulam's type stability results for a class of fractional boundary value problems on a star graph, *Math. Methods Appl. Sci.*, **43** (2020), 8568–8594. https://doi.org/10.1002/mma.6516
13. S. Etemad, S. Rezapour, On the existence of solutions for fractional boundary value problems on the ethane graph, *Adv. Differ. Equations*, **2020** (2020), 1–20. https://doi.org/10.1186/s13662-020-02736-4
14. D. Baleanu, S. Etemad, H. Mohammadi, S. Rezapour, A novel modeling of boundary value problems on the glucose graph, *Commun. Nonlinear Sci. Numer. Simul.*, **100** (2021), 1–13. https://doi.org/10.1016/j.cnsns.2021.105844
15. W. Ali, A. Turab, J. J. Nieto, On the novel existence results of solutions for a class of fractional boundary value problems on the cyclohexane graph, *J. Inequal. Appl.*, **2022** (2022), 1–19. https://doi.org/10.1186/s13660-021-02742-4

16. V. Mehandiratta, M. Mehra, G. Leugering, Existence results and stability analysis for a nonlinear fractional boundary value problem on a circular ring with an attached edge: A study of fractional calculus on metric graph, *Networks Heterogen. Media*, **16** (2021), 155–185. https://doi.org/10.3934/nhm.2021003

17. A. Turab, W. Sintunavarat, The novel existence results of solutions for a nonlinear fractional boundary value problem on the ethane graph, *Alexandria Eng. J.*, **60**, 2021, 5365–5374. https://doi.org/10.1016/j.aej.2021.04.020

18. G. Mophou, G. Leugering, P. S. Fotsing, Optimal control of a fractional Sturm-Liouville problem on a star graph, *Optimization*, **70** (2021), 659–687. https://doi.org/10.1080/02331934.2020.1730371

19. A. Turab, Z. D. Mitrović, A. Savić, Existence of solutions for a class of nonlinear boundary value problems on the hexasilinane graph, *Adv. Differ. Equations*, **2021** (2021), 1–20. https://doi.org/10.1186/s13662-021-03653-w

20. W. Coffey, Y. P. Kalmykov, *The Langevin Equation: With Applications to Stochastic Problems in Physics, Chemistry and Electrical Engineering*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, **27** (2012). https://doi.org/10.1142/8195

21. R. Zwanzig, *Nonequilibrium Statistical Mechanics*, Oxford University Press, New York, 2001.

22. H. Fazli, J. J. Nieto, Fractional Langevin equation with anti-periodic boundary conditions, *Chaos Solitons Fractals*, **114** (2018), 332–337. https://doi.org/10.1016/j.chaos.2018.07.009

23. A. Salem, F. Alzahrani, B. Alghamdi, Langevin equation involving two fractional orders with three-point boundary conditions, *Differ. Integr. Equations*, **33** (2020), 163–180.

24. M. M. Matar, J. Alzabut, J. M. Jonnalagadda, A coupled system of nonlinear Caputo-Hadamard Langevin equations associated with nonperiodic boundary conditions, *Math. Methods Appl. Sci.*, **44** (2021), 2650–2670. https://doi.org/10.1002/mma.6711

25. Y. Liu, R. Agarwal, Existence of solutions of BVPs for impulsive fractional Langevin equations involving Caputo fractional derivatives, *Turk. J. Math.*, **43** (2019), 2451–2472. https://doi.org/10.3906/mat-1905-23

26. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Academic Press, Inc., San Diego, CA, **198** (1999), 1–340. https://doi.org/10.1016/s0076-5392(99)x8001-5

27. A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003. https://doi.org/10.1007/978-0-387-21593-8

AIMS Press