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Research article

Hyers-Ulam-Rassias-Kummer stability of the fractional integro-differential equations

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Abstract: In this paper, using the fractional integral with respect to the Ψ function and the Ψ-Hilfer fractional derivative, we consider the Volterra fractional equations. Considering the Gauss Hypergeometric function as a control function, we introduce the concept of the Hyers-Ulam-Rassias-Kummer stability of this fractional equations and study existence, uniqueness, and an approximation for two classes of fractional Volterra integro-differential and fractional Volterra integral. We apply the Cădariu-Radu method derived from the Diaz-Margolis alternative fixed point theorem. After proving each of the main theorems, we provide an applied example of each of the results obtained.

Keywords: Hyers-Ulam-Rassias-Kummer stability; fractional Volterra integro-differential equation; alternative fixed-point theorem.

1. Introduction

Fractional calculus is of particular importance due to its many applications in various fields, including issues related to the effects of memory, engineering, physics, and medicine. Hence, one of the most important topics studied by scientists is fractional calculus and its application [1–4]. There are so many definitions of fractional operators today that we believe more general fractional operators like Ψ-Hilfer are more useful to study, see [5, 6]. Given that solutions of fractional differential equations better evaluate the results in different fields, they are therefore useful in modeling various phenomena. The first studies to investigate the stability of equations were conducted by Ulam in 1940 and after him, Hyers and Rassias researched in this field. In general, many authors have proposed and proved the existence, uniqueness, and Ulam-Hyers stability of the solution of fractional differential equations using several methods.

For example, in 2017, Benchohra and Lazreg investigated the existence of a unique solution and the stability of the following fractional differential equations

$$\begin{cases} {}^{H}\mathcal{D}^{\tau}\phi(\ell) = \omega(\ell, \phi(\ell), {}^{H}\mathcal{D}^{\tau}\phi(\ell)), \\ \phi(1) = \phi_1, \end{cases}$$

where ${}^{H}\mathcal{D}^{\tau}\phi(\ell)$ is the Hadamard fractional derivative of order $0 < \tau \leq 1$, $\omega : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $\phi_1 \in \mathbb{R}$ and $t \in J = [1, T]$, $T > 0$. Also, Sousa and Oliveira considered the following fractional Volterra integro-differential equation,

$$\begin{cases} {}^{H}\mathcal{D}^{\tau,\kappa,\Psi}\mathrm{g}(\ell) = \omega(\ell, \mathrm{g}(\ell)) + \int_0^{\ell} \mathrm{M}(\ell, \jmath, \mathrm{g}(\ell))d\jmath, \\ I_{0_+}^{1-\gamma}\mathrm{g}(0) = \zeta, \end{cases}$$

which is defined using the Hilfer fractional derivative, and invastigate the stability of Hyers-Ulam and Hyers-Ulam Rassias for this equation. In this equation, $\ell \in J = [0, T]$, where $\omega(\ell, u)$ is a continuous function with respect to the variables ℓ and g on $J \times \mathbb{R}$ and $\mathrm{M}(\ell, \jmath, \mathrm{g}(\ell))$ is continuous with respect to ℓ, $\jmath$, g on $J \times \mathbb{R} \times \mathbb{R}$ and $I_{0_+}^{1-\gamma}\mathrm{g}(0)$ with $0 < \gamma \leq 1$ is Ψ-Riemann-Liouville fractional integral.

Utilizing the continuous functions $\phi : [0, L] \to Y$, $\mathrm{M} : [0, L] \times [0, L] \times Y \to Y$, $\omega : [0, L] \times Y \to Y$, that Y is a Banach's space and also Ψ-Hilfer fractional derivative ${}^{H}\mathcal{D}_{a^+}^{\tau,\kappa;\Psi}\phi(\ell)$ that $0 < \tau < 1$, $0 \leq \kappa \leq 1$ and $\Delta_{\Psi}^{\tau}(\ell, \jmath) := \Psi'(\jmath)(\Psi(\ell) - \Psi(\jmath))^{\tau-1}$ that the function $\Psi(\ell)$ is an increasing and positive function with a continuous derivative, we define two equations of the fractional Volterra Integro-differential and the fractional Volterra integral as follows

$${}^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell) = \omega(\ell, \phi(\ell)) + \int_0^{\ell} \mathrm{M}(\ell, \jmath, \phi(\jmath))d\jmath, \tag{1.1}$$

and

$$\phi(\ell) = \omega(\ell, \phi(\ell)) + \frac{1}{\Gamma(\tau)} \int_0^{\ell} \Delta_{\Psi}^{\tau}(\ell, \jmath)\mathrm{M}(\ell, \jmath, \phi(\jmath))d\jmath. \tag{1.2}$$

In this paper, we consider the Gauss Hypergeometric function as a control function and stabilize the fractional equations with this control function, which we call the Kummer control function and apply the Cădariu-Radu method derived from the Diaz-Margolis alternative fixed point theorem to investigate the existence, uniqueness of solutions for fractional Eqs (1.1) and (1.2). We have two methods for investigating the solution of fractional differential equations: the Picard method and the Cădariu-Radu method. Picard's method which uses the Banach fixed point it shows only the existence of a unique solution. But Cădariu-Radu can be shown both existence of a unique solution and stability.

In the second section, we present the basic definitions and theorems. In the third section, we investigate the uniqueness and Hyers-Ulam-Rassias-Kummer stability of Eq (1.1), and at the end of this section, we provide a numerical example to illustrate main results. In the fourth section, we show the existence of a unique solution and the stability of Eq (1.2) with a numerical example.

2. Preliminaries

We provide some definitions of the Gauss hypergeometric function, fractional integrals, and derivatives.

Definition 2.1 (see [7])**.** Let $|\ell| < 1$ and consider the generic parameters ρ, σ, ς. We define the Kummer function by the infinite sum (that is convergent)

$$ {}_2F_1(\rho,\sigma;\varsigma;\ell) = \sum_{k=0}^{\infty} \frac{(\rho)_k(\sigma)_k}{(\varsigma)_k}\frac{\ell^k}{k!} = \frac{\Gamma(\varsigma)}{\Gamma(\rho)\Gamma(\sigma)} \sum_{k=0}^{\infty} \frac{\Gamma(\rho+k)\Gamma(\sigma+k)}{\Gamma(\varsigma+k)}\frac{\ell^k}{k!}. $$

Now, we present the concept of the Hyers-Ulam-Rassias-Kummer stability of Eqs (1.1) and (1.2).

Definition 2.2 (see [8–10])**.** Suppose that ${}_2F_1(\rho,\sigma;\varsigma;\ell)$ is the Kummer function in which $\ell \in [0, L]$. We say that Eq (1.1) has the Hyers-Ulam-Rassias-Kummer stability property if the following inequality holds for the differentiable function $\phi(\ell)$

$$ |{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell) - \omega(\ell,\phi(\ell)) - \int_0^{\ell} \mathrm{M}(\ell, \jmath, \phi(\jmath))d\jmath| \le {}_2F_1(\rho,\sigma;\varsigma;\ell), \tag{2.1} $$

then, there exists a solution $\vartheta(\ell)$ of Eq (1.1) such that for some $P > 0$,

$$ |{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell) - {}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\vartheta(\ell)| + |\phi(\ell) - \vartheta(\ell)| \le P\, {}_2F_1(\rho,\sigma;\varsigma;\ell). $$

Definition 2.3 (see [8–10])**.** Suppose that ${}_2F_1(\rho,\sigma;\varsigma;\ell)$ is the Kummer function in which $\ell \in [0, L]$. If the following inequality holds for the differentiable function $\phi(\ell)$

$$ |\phi(\ell) - \omega(\ell,\phi(\ell)) - \frac{1}{\Gamma(\tau)}\int_0^{\ell} \Delta_{\Psi}^{\tau}(\ell,\jmath)\mathrm{M}(\ell, \jmath, \phi(\jmath))d\jmath| \le {}_2F_1(\rho,\sigma;\varsigma;\ell), \tag{2.2} $$

then, there exists a solution $\vartheta(\ell)$ of Eq (1.2) such that for some $P > 0$

$$ |\phi(\ell) - \vartheta(\ell)| \le P\, {}_2F_1(\rho,\sigma;\varsigma;\ell), $$

then we say that Eq (1.2) is Hyers-Ulam-Rassias-Kummer stable, [11–13].

In the sequel, we present the fractional integral and the fractional derivative for more details and application we refer to [1–4].

Definition 2.4. The left-sided fractional integral of a function θ with respect to a function $\Psi(\ell)$ on interval $[a, b]$ for $\tau > 0$ is defined by

$$ \mathcal{I}_{a^+}^{\tau;\Psi}\theta(\ell) = \frac{1}{\Gamma(\tau)}\int_a^{\ell} \Delta_{\Psi}^{\tau}(\ell,\jmath)\theta(\jmath)d\jmath, \tag{2.3} $$

that the function $\Psi(\ell)$ is an increasing and positive function with a continuous derivative on (a, b). Also, the $\Delta_{\Psi}^{\tau}(\ell,\jmath)$ function used in the upper integral is $\Delta_{\Psi}^{\tau}(\ell,\jmath) := \Psi'(\jmath)(\Psi(\ell) - \Psi(\jmath))^{\tau-1}$. We can define the right-sided fractional integral in a similar way.

Definition 2.5. Consider the functions $\theta, \Psi \in C^n[a, b]$ where $n - 1 < \tau < n$ and $n \in \mathbb{N}$, the left-sided Ψ-Hilfer fractional derivative of a function θ of order τ on $[a, b]$ and for $0 \le \kappa \le 1$ is defined by

$$ {}^H\mathcal{D}_{a^+}^{\tau,\kappa;\Psi}\theta(\ell) = \mathcal{I}_{a^+}^{\kappa(n-\tau);\Psi}\left(\frac{1}{\Psi'(\ell)}\frac{d}{d\ell}\right)^n \mathcal{I}_{a^+}^{(1-\kappa)(n-\tau);\Psi}\theta(\ell). \tag{2.4} $$

wherein Ψ is an increasing function and $\Psi'(\ell) \neq 0$. We can define the right-sided Ψ-Hilfer fractional derivative in a similar way.

Theorem 2.6 (see [4]). *Let $\theta \in C^1[a,b]$, $\tau > 0$ and $0 \leq \kappa \leq 1$, we have*

$$ {}^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi} \mathcal{I}_{0^+}^{\tau;\Psi} \theta(\ell) = \theta(\ell). $$

Definition 2.7. Let $\mathcal{D}$ be a nonempty set, and let $\delta : \mathcal{D} \times \mathcal{D} \to [0,\infty]$ be a mapping such that for all $\mathfrak{d}, \mathfrak{e}, \mathfrak{c} \in \mathcal{D}$,

(i) $\delta(\mathfrak{d},\mathfrak{e}) = 0 \longleftrightarrow \mathfrak{d} = \mathfrak{e}$;
(ii) $\delta(\mathfrak{d},\mathfrak{e}) = \delta(\mathfrak{e},\mathfrak{d})$;
(iii) $\delta(\mathfrak{d},\mathfrak{e}) \leq \delta(\mathfrak{d},\mathfrak{c}) + \delta(\mathfrak{c},\mathfrak{e})$.

Then $(\mathcal{D},\delta)$ is called a generalized metric space (g.m.s., for short).

Now, we present an alternative fixed point theorem ([32,33]). In the alternative fixed point theorem, an unbounded state occurs for the meter, but we consider the bounded state.

Theorem 2.8 (Diaz-Margolis Theorem). *Consider $\mathcal{Y} \neq \emptyset$ with the complete $[0,\infty]$-valued metric δ and also consider the self map $\mathcal{L}$ on $\mathcal{Y}$ satisfy*

$$ \delta(\mathcal{L}y, \mathcal{L}t) \leq \kappa\delta(t,y), \quad \kappa < 1 $$

where $\kappa < 1$ is a Lipschitz constant. Assume that $y \in \mathcal{Y}$, so in this situation either $\delta(\mathcal{L}^m y, \mathcal{L}^{m+1} y) = \infty$, for all $m \in \mathbb{N}$, or $\delta(\mathcal{L}^m y, \mathcal{L}^{m+1} y) < \infty$, for all $m \geq m_0$. If $\delta(\mathcal{L}^m y, \mathcal{L}^{m+1} y) < \infty$, the following conditions apply to us simultaneously

(1) the fixed point t^* of $\mathcal{L}$ is the convergence point of the sequence $\{\mathcal{L}^m y\}$;
(2) in the set $V = \{t \in \mathcal{Y} \mid \delta(\mathcal{L}^{m_0} y, t) < \infty\}$, t^* is the unique fixed point of $\mathcal{L}$;
(3) $(1-\kappa)\delta(t,t^*) \leq \delta(t,\mathcal{L}t)$ for every $t \in \mathcal{Y}$.

3. Fractional Volterra integro-differential equations

This section is devoted to a class of fractional Volterra integro-differential equations [14–16]. We stabilize the mentioned equation by a Kummer function and apply the Cădariu-Radu Method to guarantee the existence of a unique solution of Eq (1.1). Our results improve and generalize the results of [17], see also [3,18–26].

Theorem 3.1. *Consider the positive constants E, E_1 and E_2 such that $0 < E_1+(E_1+E_2+E_2E)E < 1$. Also consider the Banach space Y, the continuous functions $\omega : [0,L] \times Y \to Y$, $\mathrm{M} : [0,L] \times [0,L] \times Y \to Y$ and ${}_2F_1 : \mathbb{R}^3 \times [0,L] \to (0,\infty)$ for all $\jmath, \ell \in [0,L]$ and $\phi, \vartheta \in Y$ and assume that the following conditions hold*

(1) $|\omega(\ell,\phi) - \omega(\ell,\vartheta)| \leq E_1|\phi - \vartheta|$,
(2) $|\mathrm{M}(\ell,\jmath,\phi(\jmath)) - \mathrm{M}(\ell,\jmath,\vartheta(\jmath))| \leq E_1|\phi - \vartheta|$,
(3) $\frac{1}{\Gamma(\tau)} \int_0^{\ell} \Delta_{\Psi}^{\tau}(\ell,\jmath)\, {}_2F_1(\rho,\sigma;\varsigma;\jmath) d\jmath \leq E\, {}_2F_1(\rho,\sigma;\varsigma;\ell)$.

If $\theta : [0,L] \to Y$ be a differentiable function satisfying (2.1), then Eq (1.1) has a unique solution $\theta_0 : [0,L] \to Y$. Therefore, we have

$$ |{}^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\theta(\ell) - {}^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\theta_0(\ell)| + |\theta(\ell) - \theta_0(\ell)| $$

$$\leq \frac{1+E}{1-E_1+(E_1+E_2+E_2E)E}\ {}_2F_1(\rho,\sigma;\varsigma;\ell),$$

and this means that Eq (1.1)*, has Hyers-Ulam-Rassias-Kummer stability property.*

Proof. At the first, we define the set

$$\mathrm{W}=\{\phi:[0,L]\to Y,\phi \text{ is differentiable}\}$$

and on this set, consider the mapping $\delta:\mathrm{W}\times\mathrm{W}\to[0,\infty]$ as follows

$$\begin{aligned}\delta(\phi,\vartheta)=\inf\{P\in[0,\infty]:|{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell)-{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\vartheta(\ell)|+|\phi(\ell)-\vartheta(\ell)|\\ \leq P\ {}_2F_1(\rho,\sigma;\varsigma;\ell),\ell\in[0,L]\}.\end{aligned} \tag{3.1}$$

We show that (W,δ) is $[0,\infty]$-valued complete metric space. Therefore, for functions $\phi,\vartheta\in\mathrm{W}$ if we have $\delta(\phi,\vartheta)>\delta(\phi,\nu)+\delta(\nu,\vartheta)$, then there exists $\ell_0\in[0,L]$ such that

$$\begin{aligned}&|{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell_0)-{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\vartheta(\ell_0)|+|\phi(\ell_0)-\vartheta(\ell_0)|\\ &\qquad>(\delta(\phi,\nu)+\delta(\nu,\vartheta))\ {}_2F_1(\rho,\sigma;\varsigma;\ell_0).\end{aligned}$$

according to the definition 3.1, we have

$$\begin{aligned}&|{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell_0)-{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\vartheta(\ell_0)|+|\phi(\ell_0)-\vartheta(\ell_0)|\\ &>|{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell_0)-{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\nu(\ell_0)|+|\phi(\ell_0)-\nu(\ell_0)|\\ &\quad+|{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\nu(\ell_0)-{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\vartheta(\ell_0)|+|\nu(\ell_0)-\vartheta(\ell_0)|,\end{aligned}$$

which is a contradiction.

In (W,δ), we consider a Cauchy sequence $\{\phi_n\}$. Then for every $\varepsilon>0$, we can find $N_\varepsilon\in\mathbb{N}$, such that for all $m,n\geq N_\varepsilon$ and $\ell\in[0,L]$, we have

$$|{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi_n(\ell)-{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi_m(\ell)|+|\phi_n(\ell)-\phi_m(\ell)|<\varepsilon\ {}_2F_1(\rho,\sigma;\varsigma;\ell). \tag{3.2}$$

Due to the continuity of the function ${}_2F_1$, on the compact interval $[0,L]$, the sequence $\{\phi_n\}$ is uniformly convergent to differentiable function $\phi\in\mathrm{W}$ and the sequence $\{{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi_n\}$ also is uniformly convergent to $\{{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi\}$. Therefore when $m\to\infty$ for $\ell\in[0,L]$ and for $\varepsilon>0$, we can find $N_\varepsilon\in\mathbb{N}$, such that if $n\geq N_\varepsilon$ we have

$$|{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi_n(\ell)-{}^H\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell)|+|\phi_n(\ell)-\phi(\ell)|<\varepsilon\ {}_2F_1(\rho,\sigma;\varsigma;\ell).$$

Thus $\delta(\phi_n,\phi)\leq\varepsilon$. Therefore, we have proved that the space (W,δ) is a $[0,\infty]$-valued complete metric space.

Now, we define the operator $\Omega:\mathrm{W}\to\mathrm{W}$ as follows

$$\Omega(\phi(\ell))=\mathcal{I}_{0^+}^{\tau;\Psi}\omega(\ell,\phi(\ell))+\mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^\ell \mathrm{M}(\ell,\jmath,\phi(\jmath))d\jmath\Big], \tag{3.3}$$

and show that Ω is a contraction operator. For this purpose, suppose that $\phi, \wp \in \mathrm{W}$, $P_{\phi\wp} \in [0,\infty]$ and $\delta(\phi,\vartheta) \leq P_{\phi\vartheta}$. Then

$$|^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\phi(\ell) -^{H} \mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\vartheta(\ell)| + |\phi(\ell) - \vartheta(\ell)| < P_{\phi\vartheta}\ {}_2F_1(\rho,\sigma;\varsigma;\ell),$$

for all $\ell \in [0, L]$. Using conditions (1) and (2) in the hypothesis, we have

$$\begin{aligned}
&|^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}(\Omega\phi(\ell) - \Omega\vartheta(\ell))| + |\Omega\phi(\ell) - \Omega\vartheta(\ell)| \\
&\leq |\omega(\ell,\phi(\ell)) - \omega(\ell,\vartheta(\ell))| + \int_0^{\ell} |\mathrm{M}(\ell,\jmath,\phi(\jmath)) - \mathrm{M}(\ell,\jmath,\vartheta(\jmath))|d\jmath \\
&\quad + |\mathcal{I}_{0^+}^{\tau;\Psi}(\omega(\ell,\phi(\ell)) - \omega(\ell,\vartheta(\ell)))| + |\mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^{\ell}(\mathrm{M}(\ell,\jmath,\phi(\jmath)) - \mathrm{M}(\ell,\jmath,\vartheta(\jmath)))\Big]d\jmath| \\
&\leq E_1|\phi(\ell) - \vartheta(\ell)| + E_2\int_0^{\ell} |\phi(\jmath) - \vartheta(\jmath)|d\jmath \\
&\quad + \mathcal{I}_{0^+}^{\tau;\Psi}(E_1|\phi(\ell) - \vartheta(\ell)|) + \mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^{\ell} E_2|\phi(\ell) - \vartheta(\ell)|d\jmath\Big] \\
&\leq E_1P_{\phi\vartheta}\ {}_2F_1(\rho,\sigma;\varsigma;\ell) + E_2P_{\phi\vartheta}\int_0^{\ell} {}_2F_1(\rho,\sigma;\varsigma;\jmath)d\jmath + E_1P_{\phi\vartheta}\mathcal{I}_{0^+}^{\tau;\Psi}\ {}_2F_1(\rho,\sigma;\varsigma;\ell) \\
&+ E_2\mathcal{I}_{0^+}^{\tau;\Psi}\Big[P_{\phi\vartheta}\int_0^{\ell} {}_2F_1(\rho,\sigma;\varsigma;\jmath)d\jmath\Big] \\
&\leq [E_1 + (E_2 + E_1 + E_2E)E]P_{\phi\vartheta}\ {}_2F_1(\rho,\sigma;\varsigma;\ell).
\end{aligned}$$

Or equivalent

$$\delta(\Omega\phi,\Omega\vartheta) \leq [E_1 + (E_2 + E_1 + E_2E)E]\delta(\phi,\vartheta). \tag{3.4}$$

From $0 < E_1 + (E_2 + E_1 + E_2E)E < 1$, we can conclude that Ω is a contractions mapping.

In the sequel, consider the function $\theta \in \mathrm{W}$ and use the inequality (2.1), then we have

$$\begin{aligned}
&|^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}(\Omega\theta(\ell) - \theta(\ell)| + |\Omega\theta(\ell) - \theta(\ell)| \\
&= |\omega(\ell,\theta(\ell)) + \int_0^{\ell} \mathrm{M}(\ell,\jmath,\theta(\jmath))d\jmath -^{H} \mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\theta(\ell)| \\
&\quad + |\mathcal{I}_{0^+}^{\tau;\Psi}\omega(\ell,\theta(\ell)) + \mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^{\ell} \mathrm{M}(\ell,\jmath,\theta(\jmath))d\jmath\Big] - \theta(\ell)| \\
&\leq {}_2F_1(\rho,\sigma;\varsigma;\ell) + |\mathcal{I}_{0^+}^{\tau;\Psi}\omega(\ell,\theta(\ell)) + \mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^{\ell} \mathrm{M}(\ell,\jmath,\theta(\jmath))d\jmath\Big] - \theta(\ell)| \\
&\leq (1 + E)\ {}_2F_1(\rho,\sigma;\varsigma;\ell).
\end{aligned}$$

Thus,

$$\delta(\Omega\theta,\theta) \leq 1 + E < \infty,\ E < 1. \tag{3.5}$$

Now all the conditions for the alternative Theorem 2.8 hold. Then

- The mapping Ω has a fixed point like θ_0. It means $\Omega\theta_0 = \theta_0$ or equivalently

$$\theta_0(\ell) = \mathcal{I}_{0^+}^{\tau;\Psi}(\omega(\ell, \theta_0(\ell))) + \mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^\ell \mathrm{M}(\ell, \jmath, \theta_0(\jmath))d\jmath\Big]. \tag{3.6}$$

- The fixed point θ_0 is unique in the set $\mathrm{W}^* = \{\vartheta \in \mathrm{W} : \delta(\Omega\theta, \vartheta) < \infty\}$.
- With respect to the continuity of the functions ω, M and the differentiability of the function θ_0 by taking the Ψ-Hilfer fractional derivative of Eq (3.6) and also utilizing Theorem 2.8, we have

$${}^{H}\mathcal{D}_{0^+}^{\theta,\kappa;\Psi}\theta_0(\ell) = \omega(\ell, \theta_0(\ell)) + \int_0^\ell \mathrm{M}(\ell, \jmath, \theta_0(\jmath))d\jmath, \tag{3.7}$$

then, by Eq (3.5), we get

$$\begin{aligned}\delta(\theta, \theta_0) &\leq \frac{1}{1 - [E_1 + (E_2 + E_1 + E_2E)E]}\delta(\Omega\theta, \theta)\\ &\leq \frac{1 + E}{1 - [E_1 + (E_2 + E_1 + E_2E)E]}.\end{aligned}$$

Thus, Eq (1.1) has Hyers-Ulam-Rassias-Kummer stability property.

Now, we prove $\mathrm{W}^* = \mathrm{W}$ to show the fixed point of mapping Ω is unique in W. Put

$$F = \frac{1 + E}{1 - [E_1 + (E_2 + E_1 + E_2E)E]},$$

and consider the differentiable function d, which holds in Eq (3.7) and $\theta \in \mathrm{W}$, then we have $\delta(\theta, \mathrm{d}) < F$ and

$${}^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\mathrm{d}(\ell) = \omega(\ell, \mathrm{d}(\ell)) + \int_0^\ell \mathrm{M}(\ell, \jmath, \mathrm{d}(\jmath))d\jmath. \tag{3.8}$$

We show that d is a fixed point of Ω and $\mathrm{d} \in \mathrm{W}^*$. From Eq (3.8), we have $\Omega\mathrm{d} = \mathrm{d}$.

Now, we show $\delta(\Omega\theta, \mathrm{d}) < \infty$. By $\delta(\theta, \mathrm{d}) < F$ and Eq (3.8), we get

$$\begin{aligned}&|{}^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}(\Omega\theta(\ell) - \mathrm{d}(\ell))| + |\Omega\theta(\ell) - \mathrm{d}(\ell)|\\ &= |\omega(\ell, \theta(\ell)) + \int_0^\ell \mathrm{M}(\ell, \jmath, \theta(\jmath))d\jmath - \omega(\ell, \mathrm{d}(\ell)) - \int_0^\ell \mathrm{M}(\ell, \jmath, \mathrm{d}(\jmath))d\jmath|\\ &\quad + \Big|\mathcal{I}_{0^+}^{\tau;\Psi}\omega(\ell, \theta(\ell)) + \mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^\ell \mathrm{M}(\ell, \jmath, \theta(\jmath))d\jmath\Big]\\ &\qquad - \mathcal{I}_{0^+}^{\tau;\Psi}\omega(\ell, \mathrm{d}(\ell)) - \mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^\ell \mathrm{M}(\ell, \jmath, \mathrm{d}(\jmath))d\jmath\Big]\Big|\\ &\leq |\omega(\ell, \theta(\ell)) - \omega(\ell, \mathrm{d}(\ell))| + \int_0^\ell |\mathrm{M}(\ell, \jmath, \theta(\jmath)) - \mathrm{M}(\ell, \jmath, \mathrm{d}(\jmath))|d\jmath\\ &\quad + |\mathcal{I}_{0^+}^{\tau;\Psi}[\omega(\ell, \theta(\ell)) - \omega(\ell, \mathrm{d}(\ell))]| + |\mathcal{I}_{0^+}^{\tau;\Psi}\Big[\int_0^\ell [\mathrm{M}(\ell, \jmath, \theta(\jmath)) - \mathrm{M}(\ell, \jmath, \mathrm{d}(\jmath))]d\jmath\Big]\\ &\leq E_1|\theta(\ell) - \mathrm{d}(\ell)| + E_2\int_0^\ell |\theta(\jmath) - \mathrm{d}(\jmath)|d\jmath\end{aligned}$$

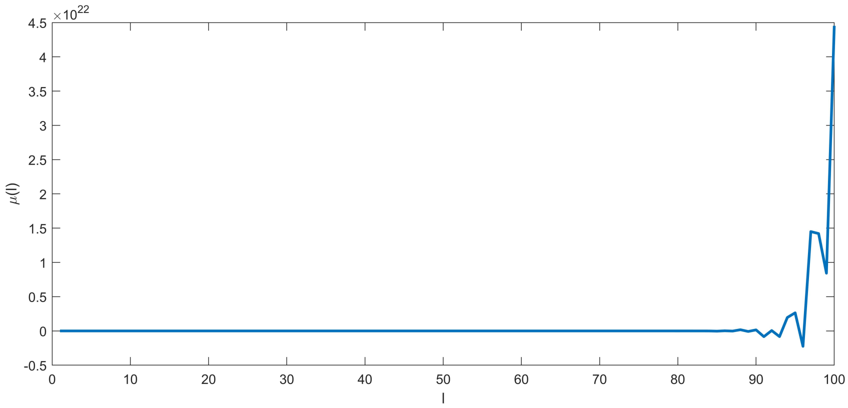

Figure 1. Diagram of the solution of the fractional Volterra integro-differential Eq (3.9).

$$+ E_1 \mathcal{I}_{0^+}^{\tau;\Psi}(|\theta(\ell) - \mathrm{d}(\ell)| + E_2 \mathcal{I}_{0^+}^{\tau;\Psi}\Big(\int_0^\ell |\theta(\jmath) - \mathrm{d}(\jmath)| d\jmath\Big)$$
$$\leq [E_1 + (E_2 + E_1 + E_2 E) E] F,$$

which implies that

$$\delta(\Omega\theta, \mathrm{d}) \leq [E_1 + (E_2 + E_1 + E_2 E)E]F < \infty.$$

□

Now, we provide an example to illustrate Theorem 3.1.

Example 3.2. Consider the following fractional Volterra integro-differential equation

$$^{H}\mathcal{D}_{0^+}^{\frac{1}{2},\frac{1}{4};\Psi}\mu(\ell) = \frac{1}{100}\mu(\ell)\sin(\mu(\ell)) - 2\ell + \int_0^\ell \frac{1}{200}\cos(\ell + \jmath)\mu(\jmath)\sin(\mu(\jmath))d\jmath, \tag{3.9}$$

where μ is a differentiable function. Define $\omega : [0, L] \times Y \longrightarrow Y$ by $\omega(\ell, \mu) = \frac{1}{100}\mu\sin(\mu) - 2\ell$ and $\mathrm{M} : [0, L] \times Y \times Y \longrightarrow Y$ by $\mathrm{M}(\ell, \jmath, \mu) = \frac{1}{200}\mu\sin(\mu)\cos(\ell + \jmath)$ that $a, b, L > 0$ and $\mu(\jmath) = \jmath^2$. Consider the positive coefficients $E = \frac{1}{10}, E_1 = \frac{1}{100}, E_2 = \frac{1}{200}$ such that $0 < E_1 + (E_1 + E_2 + E_2E)E < 1$ and for continuous functions $\omega, \mathrm{M}, {}_2F_1$ we have

(1) $|\omega(\ell, \mu) - \omega(\ell, \mu_0)| = |\frac{1}{100}\mu\sin(\mu) - 2\ell - \frac{1}{100}\mu_0\sin(\mu_0) + 2\ell| \leq |\frac{1}{100}||\mu - \mu_0|$,
(2) $|\mathrm{M}(\ell, \jmath, \mu) - \mathrm{M}(\ell, \jmath, \mu_0)| = |\frac{1}{200}\mu\sin(\mu)\cos(\ell + \jmath) - \frac{1}{200}\mu_0\sin(\mu_0)\cos(\ell + \jmath)| \leq |\frac{1}{200}||\mu - \mu_0|$,
(3) $\frac{1}{\Gamma(\frac{1}{2})}\int_0^\ell \Delta_\Psi^{\frac{1}{2}}(\ell, \jmath)\, {}_2F_1(\rho, \sigma; \varsigma; \jmath)d\jmath \leq \frac{1}{10}\, {}_2F_1(\rho, \sigma; \varsigma; \ell)$.

If the following inequality holds for the differentiable function θ

$$|^{H}\mathcal{D}_{0^+}^{\frac{1}{2},\frac{1}{4};\Psi}\theta(\ell) - \frac{1}{100}\theta(\ell)\sin(\theta(\ell)) + 2\ell - \int_0^\ell \frac{1}{200}\cos(\ell + \jmath)\theta(\jmath)\sin(\theta(\jmath))d\jmath| \leq {}_2F_1(\rho, \sigma; \varsigma; \ell),$$

and $E_1 = |\frac{1}{100}|$ and $E_2 = |\frac{1}{100}|$, Theorem 3.1 implies that, there is a solution θ_0, such that

- θ_0 is the fixed point of the operator Ω, which is defined as follows

$$\Omega(\theta_0(\ell)) = \mathcal{I}_{0^+}^{\frac{1}{2};\Psi}(\frac{1}{100}\theta_0\sin(\theta_0) - 2\ell) + \mathcal{I}_{0^+}^{\frac{1}{2};\Psi}\Big[\int_0^\ell \frac{1}{200}\theta_0\sin(\theta_0)\cos(\ell + \jmath)d\jmath\Big].$$

- The fixed point θ_0 is unique.
- θ_0 satisfies in Eq (3.9).
- $$\delta(\theta,\theta_0) \leq \frac{1+E}{1-E_1+(E_1+E_2+E_2E)E},$$

  or

$$|^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\theta(\ell) -^{H}\mathcal{D}_{0^+}^{\tau,\kappa;\Psi}\theta_0(\ell)| + |\theta(\ell)-\theta_0(\ell)| \leq \frac{1+E}{1-E_1+(E_1+E_2+E_2E)E}\ {}_2F_1(\rho,\sigma;\varsigma;\ell),$$

where $E_1+(E_1+E_2+E_2E)E = \frac{231}{20000}$ and $\frac{1+E}{1-E_1+(E_1+E_2+E_2E)E} \approx 1.112$. Then Eq (3.9) is Hyers-Ulam-Rassias-Kummer stable.

4. Fractional Volterra integral equation

This section is devoted to a class of fractional Volterra integral equations [14–16]. We stabilize the mentioned equation by a Kummer function and apply the Cădariu-Radu Method to guarantee the existence of a unique solution of Eq (1.2). Our results improve and generalize the results of [17].

Theorem 4.1. *Consider the positive constants E, E_1 and E_2 such that* $0 < (E_1+E_2E) < 1$. *On Banach space Y, consider the continuous functions $\omega : [0,L]\times Y \to Y$,* $\mathrm{M} : [0,L]\times[0,L]\times Y \to Y$ *and* ${}_2F_1 : \mathbb{R}^3\times[0,L] \to (0,\infty)$, *for all $\jmath,\ell \in [0,L]$ and $\phi,\vartheta \in Y$ and assume that the following conditions are true for them*

(4) $|\omega(\ell,\phi)-\omega(\ell,\vartheta)| \leq E_1|\phi-\vartheta|$,
(5) $|\mathrm{M}(\ell,\jmath,\phi)-\mathrm{M}(\ell,\jmath,\vartheta)| \leq E_2|\phi-\vartheta|$,
(6) $\frac{1}{\Gamma(\tau)}\int_0^\ell \Delta_\Psi^\tau(\ell,\jmath)\ {}_2F_1(\rho,\sigma;\varsigma;\jmath)d\jmath \leq E\ {}_2F_1(\rho,\sigma;\varsigma;\ell)$.

If $\theta : [0,L] \to Y$ be a differentiable function satisfying (2.2), *then Eq* (1.2) *has a unique solution $\theta_0 : [0,L] \to Y$. Thus, we have*

$$|\theta(\ell)-\theta_0(\ell)| \leq \frac{1}{1-(E_1+E_2E)}\ {}_2F_1(\rho,\sigma;\varsigma;\ell). \tag{4.1}$$

and this means that Eq (1.2) *has Hyers-Ulam-Rassias-Kummer stability property.*

Proof. We define the set W as follows

$$\mathrm{W} = \{\phi : [0,L] \to Y : \phi \text{ is continuous}\},$$

Also, we define $\delta : \mathrm{W}\times\mathrm{W} \to [0,\infty]$ as follows

$$\delta(\phi,\vartheta) = \inf\{P \in [0,\infty] : |\phi(\ell)-\vartheta(\ell)| \leq P\ {}_2F_1(\rho,\sigma;\varsigma;\ell),\ \ell \in [0,L]\}.$$

Similar to the argument of the previous theorem, we can easily show that the space (W,δ) is a $[0,\infty]$-valued complete metric space. Since the equation is integral, we consider the functions continuously and define the meter on the set of these functions. In Theorem 3.1, since we have a fractional differential integral equation, we consider the differentiable functions and define the meter on this set of functions.

Now, we define the mapping $\Omega : \mathrm{W} \longrightarrow \mathrm{W}$ by

$$\Omega(\phi(\ell)) = \omega(\ell, \phi(\ell)) + \frac{1}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath)\mathrm{M}(\ell, \jmath, \phi(\jmath))d\jmath,$$

and show that Ω is a contraction operator. Suppose that for $\phi, \vartheta \in \mathrm{W}$, $\delta(\phi, \vartheta) < P_{\phi\vartheta}$ that $P_{\phi\vartheta} > 0$, then we have

$$|\phi(\ell) - \vartheta(\ell)| \leq P_{\phi\vartheta}\ {}_2F_1(\rho, \sigma; \varsigma; \ell),$$

for all $\ell \in [0, L]$. By conditions (4) and (5), we have

$$\begin{aligned}
|\Omega(\phi(\ell)) - \Omega(\vartheta(\ell))| &\leq |\omega(\ell, \phi(\ell)) - \omega(\ell, \vartheta(\ell))| \\
&\quad + \frac{1}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath)|\mathrm{M}(\ell, \jmath, \phi(\jmath)) - \mathrm{M}(\ell, \jmath, \vartheta(\jmath))|d\jmath \\
&\leq E_1|\phi(\ell) - \vartheta(\ell)| + E_2\frac{1}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath)|\phi(\jmath) - \vartheta(\jmath)|d\jmath \\
&\leq E_1 P_{\phi\vartheta}\ {}_2F_1(\rho, \sigma; \varsigma; \ell) + \frac{P_{\phi\vartheta}E_2}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath)\ {}_2F_1(\rho, \sigma; \varsigma; \jmath)d\jmath \\
&\leq E_1 P_{\phi\vartheta}\ {}_2F_1(\rho, \sigma; \varsigma; \ell) + E_2 P_{\phi\vartheta 2}F_1(\rho, \sigma; \varsigma; \ell)E \\
&\leq (E_1 + E_2E)P_{\phi\vartheta}\ {}_2F_1(\rho, \sigma; \varsigma; \ell),
\end{aligned}$$

and so

$$\delta(\Omega\phi, \Omega\vartheta) \leq (E_1 + E_2E)\delta(\phi, \vartheta).$$

Since $0 < (E_1 + E_2E) < 1$, then Ω is a contraction mapping. In the sequel, considering the function $\theta \in \mathrm{W}$ and utilizing the inequality (2.2), we get

$$\delta(\Omega\theta, \theta) \leq 1 < \infty. \tag{4.2}$$

Now, all the conditions of Theorem 2.8 hold. Then

- The mapping Ω has a fixed point named by θ_0. It means $\Omega\theta_0 = \theta_0$ or equivalently

$$\theta_0(\ell) = \omega(\ell, \theta_0(\ell)) + \frac{1}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath)\mathrm{M}(\ell, \jmath, \theta_0(\jmath))d\jmath. \tag{4.3}$$

- The fixed point θ_0 is unique in the set $\mathrm{W}^* = \{\vartheta \in \mathrm{W}(\Omega\theta, \vartheta) < \infty\}$.
- By Eq (4.2) and Theorem (2.8), we get

$$\delta(\theta, \theta_0) = \leq \frac{1}{1 - (E_1 + E_2E)}\delta(\Omega\theta, \theta) \leq \frac{1}{1 - (E_1 + E_2E)}.$$

Thus, Eq (1.2) is Hyers-Ulam-Rassias-Kummer stable.

Now, we prove $W^* = W$ to show the fixed point of mapping Ω is unique in W. Consider another continuous function d that satisfying

$$d(\ell) = \omega(\ell, d(\ell)) + \frac{1}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath) M(\ell, \jmath, d(\jmath)) d\jmath.$$

In the following, we show that d is a fixed point of Ω and $d \in W^*$. From Eq (4.3) we have, $\Omega d = d$. Now, we show $\delta(\Omega\theta, d) < \infty$. From

$$\delta(\theta, d) \leq \frac{1}{1 - (E_1 + E_2 E)},$$

and conditions (4), (5) and (6), we get

$$\begin{aligned}
&|\Omega\theta(\ell) - d(\ell)| \\
&= |\omega(\ell, \theta(\ell)) - \frac{1}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath) M(\ell, \jmath, d(\jmath)) d\jmath - \omega(\ell, d(\ell)) \\
&+ \frac{1}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath) M(\ell, \jmath, \theta(\jmath)) d\jmath| \\
&\leq |\omega(\ell, \theta(\ell)) - \omega(\ell, d(\ell))| + \frac{1}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath) |M(\ell, \jmath, \theta(\jmath)) - M(\ell, \jmath, d(\jmath))| d\jmath \\
&\leq E_1 |\theta(\ell) - d(\ell)| + \frac{E_2}{\Gamma(\tau)} \int_0^\ell \Delta_\Psi^\tau(\ell, \jmath) |\theta(\jmath) - d(\jmath)| d\jmath \\
&\leq E_1 P_{\theta d}\, {}_2F_1(\rho, \sigma; \varsigma; \ell) + E_2 P_{\theta d} E\, {}_2F_1(\rho, \sigma; \varsigma; \ell) \\
&\leq (E_1 + E_2 E) \delta(\theta, d)\, {}_2F_1(\rho, \sigma; \varsigma; \ell) \\
&\leq \frac{(E_1 + E_2 E)}{1 - (E_1 + E_2 E)}\, {}_2F_1(\rho, \sigma; \varsigma; \ell),
\end{aligned}$$

which implies that $\delta(\Omega\theta, d) < \infty$. □

Example 4.2. Consider the following fractional Volterra integral equation

$$\mu(\ell) = \sqrt{\ell} + \frac{1}{100}\mu(\ell) - \frac{1}{\Gamma(\frac{1}{2})} \int_0^\ell \Delta_\Psi^{\frac{1}{2}}(\ell, \jmath) \frac{\mu(\jmath)}{\sqrt{\ell - \jmath}} d\jmath, \tag{4.4}$$

where in μ is a continuous function. Define $\omega : [0, L] \times Y \to Y$ by $\omega(\ell, \mu(\ell)) = \sqrt{\ell} + \frac{1}{100}\mu(\ell)$, and $M : [0, L] \times Y \times Y \to Y$ by $M(\ell, \jmath, \mu(\jmath)) = \frac{\mu(\jmath)}{\sqrt{\ell - \jmath}}$ that $L > 0$ and $\mu(j) = j^2$. Consider the positive coefficients E, E_1, E_2, such that $0 < E_1 + E_2 E < 1$ and for continuous functions $\omega, M, {}_2F_1$ we have

(4) $|\omega(\ell, \mu(\ell)) - \omega(\ell, \mu_0(\ell))| = |\sqrt{\ell} + \frac{1}{100}\mu - \sqrt{\ell} - \frac{1}{100}\mu_0| \leq \frac{1}{100}|\mu - \mu_0|$,
(5) $|M(\ell, \jmath, \mu(\jmath)) - M(\ell, \jmath, \mu_0(\jmath))| = |\frac{\mu}{\sqrt{\ell - \jmath}} - \frac{\mu_0}{\sqrt{\ell - \jmath}}| \leq \frac{1}{\sqrt{\ell - \jmath}}|\mu - \mu_0|$,
(6) $\frac{1}{\Gamma(\frac{1}{2})} \int_0^\ell \Delta_\Psi^{\frac{1}{2}}(\ell, \jmath)\, {}_2F_1(\rho, \sigma; \varsigma; \ell) d\jmath \leq \frac{1}{10}\, {}_2F_1(\rho, \sigma, \varsigma; \ell)$.

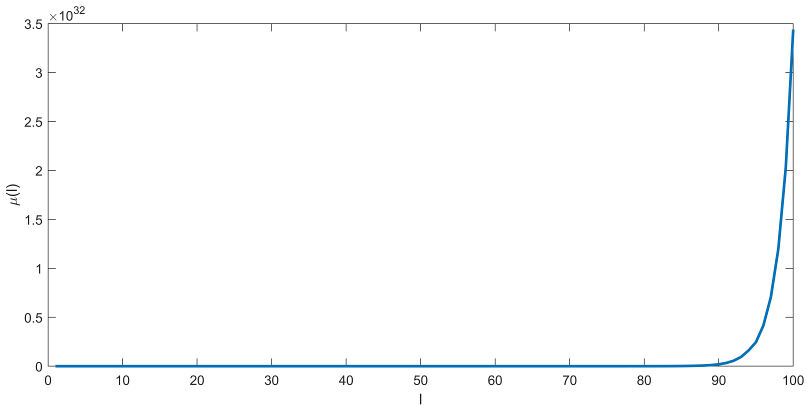

Figure 2. Diagram of the solution of the fractional Volterra integral Eq (4.4).

If the following inequality holds for the differentiable function θ

$$|\theta(\ell) - \sqrt{\ell} - \frac{1}{100}\theta(\ell) + \int_0^\ell \frac{\theta(\jmath)}{\sqrt{\ell-\jmath}}d\jmath| \leq {}_2F_1(\rho,\sigma,\varsigma;\ell)$$

and $E_1 = \frac{1}{100}$ and $E_2 = \frac{1}{\sqrt{\ell-\jmath}} = \frac{1}{200}$. Theorem (4.1) implies that there is a unique solution θ_0, such that

- θ_0 is the fixed point of the operator Ω, which is defined as follows

$$\Omega(\theta_0) = \sqrt{\ell} + \frac{1}{100}\theta_0(\ell) - \frac{1}{\Gamma(\frac{1}{2})}\int_0^\ell \Delta_\Psi^{\frac{1}{2}}(\ell,\jmath)\frac{\theta_0(\jmath)}{\sqrt{\ell-\jmath}}d\jmath.$$

- The fixed point θ_0 is unique.
- θ_0 satisfies in Eq (4.4).
- $$\delta(\theta,\theta_0) \leq \frac{1}{1-(E_1+E_2E)},$$

  or

  $$|\theta(\ell) - \theta_0(\ell)| \leq \frac{1}{1-(E_1+E_2E)}\ {}_2F_1(\rho,\sigma;\varsigma;\ell),$$

  where $E_1 + E_2E = \frac{21}{2000}$ and $\frac{1}{1-(E_1+E_2E)} \approx 1.011$, which implies that Equation (4.4) has Hyers-Ulam-Rassias-Kummer stability property.

5. Concluding remarks

Given that fractional differential equations are used in a variety of fields, including physics, chemistry, economics, medicine, and engineering, many authors have inspected these equations in recent years. Researchers have done a lot of research on the stability of fractional equations, including fractional differential equations [8, 27–30]. We applied the concept of the Hyers-Ulam-Rassias-Kummer stability of fractional equations to investigate existence, uniqueness, and an approximation with the optimum errors for two classes of fractional Volterra integro-differential and fractional Volterra integral via the Cădariu-Radu method derived from the Diaz-Margolis alternative fixed point theorem.

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Conflict of interest

The authors declare there is no conflict of interest.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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