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Research article

Differential equations of arbitrary order under Caputo-Fabrizio derivative: some existence results and study of stability

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Abstract: In this work, we consider the problem of the existence and uniqueness of solution, and also the simple existence of solution, for implicit differential equations of arbitrary order involving Caputo-Fabrizio derivative. The main tools for this study are contraction mapping principle and Schaefer's fixed point result. We also study the stability of the equations in the sense of Ulam-Hyers and also from the perspective of Ulam-Hyers-Rassias.

Keywords: fractional differential equations; Caputo-Fabrizio fractional derivative; fixed point theorems; contractive maps; functional equations; Gronwall inequality; Ulam-Hyers stability; Ulam-Hyers-Rassias stability

1. Introduction

In the recent years, some concepts have been proposed in relation with fractional calculus based on kernels that are non-singular functions. In particular, a new concept of fractional derivative was presented by Caputo and Fabrizio, with the main feature that it is defined without the use of a singular kernel, see [1, 2]. This new definition has immediately attracted the interest of scientific researchers and numerous papers has been published following this approach in the last years (see, for instance, [3–7]). Losada and Nieto in [8] considered differential equations of arbitrary order under the derivative of Caputo-Fabrizio and, in particular, they studied the problem of the existence and uniqueness of solutions. Mozyrska et al. studied in [9] the solutions to differential systems on time scales considering the Caputo-Fabrizio fractional delta derivative. In [10], Atangana and Baleanu developed some properties of the new concept.

Nowadays, it is possible to enumerate a large list of references based on this notion whose objective is to study its applications to the modeling of several phenomena in several scientific and engineering fields such as heat-transfer, steady heat flow, science of materials, biological problems, physical phenomena, dynamics occurring in fluids, control systems, etc. For more detailed information about these applications based on the operators of Caputo-Fabrizio type, we refer to the following works [4, 6, 7, 11–14], and the references given therein.

On the other hand, this paper is also connected with the concept of Hyers-Ulam stability. This notion is related with the existence of exact solutions close enough to an approximate solution. According to [15, 16], the pioneers in the investigation of Hyers-Ulam stability for differential equations were Alsina and Ger, see [17]. This type of stability has been used and studied intensively in the recent decade. For more detailed definitions, we mention the works [3, 16, 18–20], as well as the monographs [21, 22].

The interest of implicit differential problems arises in different examples of real-world applications such as the case of nonlinear electrical circuits, for instance (see [23]). In this reference, the authors study the existence of impasse solutions for electrical circuits modeled by this type of equations.

On the other hand, the use of fractional derivatives in the model allows to consider hereditary properties in the system or anomalous phenomena in the process. We find in the literature several different works whose main objectives are the study of implicit equations of fractional type. For instance, in [24], the authors consider the study of a boundary value problem for an implicit Caputo-type differential equation. Afterwards, in [25], some nonlinear implicit problems of Caputo-type were studied through coincidence theory.

The importance of the study of existence and stability results for implicit fractional differential equations with different types of fractional operators can be inferred from the existence of a large number of research works devoted to the topic and even monographs specially focused on the subject, see, for instance, [26]. For example, in [27], Riemann-Liouville operators are considered and Ulam stability for the implicit problem is studied. Some coupled implicit equations under Riemann-Liouville operators are also considered in [28]. On the other hand, Ulam-Hyers stability is investigated for generalized Hilfer derivatives in [29]. This approach has been also used for impulsive problems under Hadamard fractional derivative in [30]. Caputo-Fabrizio fractional derivative is another possibility and it will be the concept used in the paper in order to consider implicit differential equations with a derivative depending on a non-necessarily integer parameter.

Some of the research works mentioned conform the motivation for this contribution, which is devoted to the problems of existence, uniqueness, and stability, from the point of view of Hyers-Ulam and Hyers-Ulam-Rassias, of the solutions to the following differential equation of arbitrary order subject to initial conditions

$$ {}^{CF}\mathbb{D}^{\beta}x(t) = \mathcal{H}(t, x(t), {}^{CF}\mathbb{D}^{\beta}x(t)), \quad t \in \mathcal{J} := [0, b], \tag{1.1} $$

$$ x(0) = x_0 \in \mathbb{R}, \tag{1.2} $$

where ${}^{CF}\mathbb{D}^{\beta}$ denotes the Caputo-Fabrizio differential operator of fractional order $0 < \beta < 1$, and $\mathcal{H} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.

Concerning the motivation of following this approach, related to the proximity between approximate solutions and exact solutions, the study of approximate solutions and error estimates have demonstrated to be an interesting framework in the development of the theory of differential and integro-differential equations. In fact, in the study of the solutions to the Cauchy problem for a differential equation, one useful approach consists on the justification that the limit of a certain sequence of approximate solutions is precisely a solution to the problem (for instance, by following the Euler polygonal method).

In the context of implicit differential equations, iteration schemes and the process of approximating solutions constitute interesting methods. When the explicit resolution of the problem is too complicated or computationally inefficient, the localization of approximate solutions and the possibility of finding solutions close enough to them is a useful procedure to deduce some relevant properties concerning the localization of the solutions.

In practical applications, this method allows to guarantee the existence of a solution to the implicit problem of fractional type in a neighborhood of an approximate solution previously fixed. Since the obtention of approximate solutions is far too simple than solving the problem explicitly, then the concepts and results in the paper can easily give hints concerning the existence of solutions of the problem of interest and allow the development of numerical or even optimization processes.

In relation with particular real-world applications, in [31], approximate solutions and this type of stability are shown useful for the study of a thermostat control model, and in [32], a heat equation is considered.

2. Preliminaries

In this first section, we present some definitions and fundamental results that are essential to the developments in the paper, serving also to fix the notation.

To start with, we represent by $C([0,b],\mathbb{R})$, or also $C(\mathcal{J},\mathbb{R})$, the set of all real functions that are continuous on the interval $\mathcal{J}$. By considering the supremum norm in this space, that is,

$$\|x\|_\infty := \sup_{t\in[0,b]} |x(t)|,$$

$(C([0,b],\mathbb{R}),\|\cdot\|_\infty)$ conforms a Banach space.

We also denote by $C^1(\mathcal{J},\mathbb{R})$ the set of all real functions x that are differentiable on the interval $\mathcal{J}$ (assuming one-sided differentiability at the endpoints of $\mathcal{J}$) with derivative $x' \in C(\mathcal{J},\mathbb{R})$.

Definition 2.1 ([8]). Let $0 < \beta < 1$, and $\phi \in C^1(\mathcal{J},\mathbb{R})$. The Caputo-Fabrizio fractional derivative of order β for ϕ is given by

$${}^{CF}\mathbb{D}^\beta\phi(t) = \frac{(2-\beta)N(\beta)}{2(1-\beta)} \int_0^t \phi'(s)\exp\left(-\frac{\beta}{1-\beta}(t-s)\right) ds.$$

In the previous expression, $N(\beta)$ represents a constant that depends on the order β and provides a normalization factor.

Lemma 2.2. *The following assertions are equivalent:*

(i) ${}^{CF}\mathbb{D}^\beta\phi(t) \equiv 0$.
(ii) ϕ *is a function taking a constant value.*

Definition 2.3 ([8]). The fractional integral of Caputo-Fabrizio type and order β for the function ϕ is given by

$${}^{CF}\mathbb{I}^\beta\phi(t) = \frac{2}{(2-\beta)N(\beta)}\left[(1-\beta)\phi(t) + \beta\int_0^t \phi(s)\,ds\right].$$

Lemma 2.4 ([8]). *Let* $0 < \beta < 1$ *and* $x_0 \in \mathbb{R}$ *fixed. Consider the Cauchy problem*

$$\begin{cases} {}^{CF}\mathbb{D}^{\beta} x(t) = \varphi(t), & \text{for } t \in [0, b], \\ x(0) = x_0. \end{cases}$$

The above mentioned problem is uniquely solvable, with solution defined by the following expression

$$x(t) = x_0 + \gamma_\beta(\varphi(t) - \varphi(0)) + \sigma_\beta I^1 \varphi(t), \quad t \in [0, b],$$

where $I^1\varphi$ *denotes the primitive of* φ *passing through* $(0, 0)$*, and*

$$\gamma_\beta = \frac{2(1-\beta)}{(2-\beta)N(\beta)}, \qquad \sigma_\beta = \frac{2\beta}{(2-\beta)N(\beta)}.$$

One of the fixed point results used in this paper is the contraction mapping principle. We also use the result due to Schaefer.

Theorem 2.5 (Schaefer's fixed point result, [33]). *Consider* $\mathcal{X}$ *a Banach space and suppose that* $\Phi : \mathcal{X} \longrightarrow \mathcal{X}$ *is a completely continuous mapping. Let the set*

$$\mathcal{F} := \{x \in \mathcal{X} : x = \lambda \Phi x, \text{ for a certain } \lambda \in (0, 1)\}.$$

Then $\mathcal{F}$ *is unbounded, or the mapping* Φ *has at least a fixed point.*

In the following results, we denote $\mathbb{R}_+ = [0, \infty)$.

Lemma 2.6 (Gronwall's Lemma, [34, 35]). *Let* $b > 0$*, and* $c \geqslant 0$*. Suppose that the functions* $\varphi, v : [0, b] \longrightarrow \mathbb{R}_+$ *are continuous. If*

$$\varphi(t) \leqslant c + \int_0^t v(s)\varphi(s)\, ds, \text{ for all } t \in [0, b],$$

then

$$\varphi(t) \leqslant c \exp\Big(\int_0^t v(s)\, ds\Big), \text{ for every } t \in [0, b].$$

Lemma 2.7 (Generalized Gronwall's Lemma [36]). *Let* $b > 0$*. Suppose that the functions* φ*,* v*,* $c : [0, b] \longrightarrow \mathbb{R}_+$ *are continuous. If*

$$\varphi(t) \leqslant c(t) + \int_0^t v(s)\varphi(s)\, ds, \text{ for every } t \in [0, b],$$

then

$$\varphi(t) \leqslant c(t) + \int_0^t c(s)v(s) exp\Big(\int_s^t v(r)\, dr\Big) ds, \text{ for all } t \in [0, b].$$

Moreover, if $c(t)$ *is nondecreasing, then*

$$\varphi(t) \leqslant c(t) \exp\Big(\int_0^t v(s)\, ds\Big), \text{ for all } t \in [0, b].$$

Now, we recall the concept of Hyers-Ulam stability and some of its generalizations.

Definition 2.8 ([37]). It is said that the Eq (1.1) is Hyers-Ulam stable if it is guaranteed the existence of a real positive constant $c > 0$ such that the next property holds: for every $\varepsilon > 0$ and every function $z \in C^1([0,b],\mathbb{R})$ satisfying the following inequality

$$|^{CF}\mathbb{D}^{\beta}z(t) - \mathcal{H}(t, z(t), ^{CF}\mathbb{D}^{\beta}z(t))| \leqslant \varepsilon, \text{ for } t \in [0,b], \tag{2.1}$$

we can affirm the existence of a solution $x \in C^1([0,b],\mathbb{R})$ to the Eq (1.1) such that

$$|x(t) - z(t)| \leqslant c\,\varepsilon, \ t \in [0,b].$$

Definition 2.9. On the other hand, it is said that the Eq (1.1) is generalized Hyers-Ulam stable if we can guarantee the existence of a family of expressions $\xi_{\mathcal{H},\varepsilon} \in C(\mathbb{R}_+,\mathbb{R}_+)$, with $\xi_{\mathcal{H},0} \equiv 0$, and such that the following property holds: for every $\varepsilon > 0$ and each function $z \in C^1([0,b],\mathbb{R})$ satisfying the inequality (Eq 2.1), there exists $x \in C^1([0,b],\mathbb{R})$ a solution to the Eq (1.1) such that

$$|x(t) - z(t)| \leqslant \xi_{\mathcal{H},\varepsilon}(t), \ t \in [0,b].$$

Definition 2.10. It is said that the Eq (1.1) is stable, in terms of Hyers-Ulam-Rassias, with respect to a function $\xi \in C([0,b],\mathbb{R}_+)$ if it is possible to guarantee the existence of a real constant $c_{\mathcal{H}} > 0$ with the property that: for every $\varepsilon > 0$ and every function $z \in C^1([0,b],\mathbb{R})$ satisfying the inequality

$$|^{CF}\mathbb{D}^{\beta}z(t) - \mathcal{H}(t, z(t), ^{CF}\mathbb{D}^{\beta}z(t))| \leqslant \varepsilon\ \xi(t), \text{ for } t \in [0,b], \tag{2.2}$$

we can affirm the existence of a solution $x \in C^1([0,b],\mathbb{R})$ to the Eq (1.1) such that

$$|x(t) - z(t)| \leqslant c_{\mathcal{H}}\,\varepsilon\,\xi(t), \text{ for } t \in [0,b].$$

Definition 2.11. It is said that the Eq (1.1) is generalized stable, in terms of Hyers-Ulam-Rassias, with respect to a function $\xi \in C([0,b],\mathbb{R}_+)$ if it is possible to guarantee the existence of a family of expressions $\xi_{\mathcal{H},\varepsilon} \in C(\mathbb{R}_+,\mathbb{R}_+)$, with $\xi_{\mathcal{H},0} \equiv 0$, and such that the following property holds: for every $\varepsilon > 0$ and every function $z \in C^1([0,b],\mathbb{R})$ satisfying the inequality

$$|^{CF}\mathbb{D}^{\beta}z(t) - \mathcal{H}(t, z(t), ^{CF}\mathbb{D}^{\beta}z(t))| \leqslant \varepsilon\,\xi(t), \text{ for } t \in [0,b], \tag{2.3}$$

we can affirm the existence of a solution $x \in C^1([0,b],\mathbb{R})$ to the Eq (1.1) such that

$$|x(t) - z(t)| \leqslant \xi_{\mathcal{H},\varepsilon}(t), \text{ for } t \in [0,b].$$

The concept of Hyers-Ulam stability is connected with the possibility of approximating any approximate solution to a functional equation by an exact solution of the equation. In Definition 2.8, the maximum error between the exact solution and the approximate solution is bounded from above by a linear function of the error ε inherent to the approximate solution, that is, the error corresponding to the approximate solution suffers a linear increment when this function is compared with a certain solution. On the other hand, in Definition 2.9, the notion of stability is extended by admitting the variation of the error in terms of the (nonnecessarily linear) function $\xi_{\mathcal{H},\varepsilon} \in C(\mathbb{R}_+,\mathbb{R}_+)$ (depending on ε).

The notion of stability in terms of Hyers-Ulam-Rassias (Definition 2.10) is an extension of Definition 2.8 that computes the error corresponding to the approximate solution as a nonconstant function of the form $\varepsilon\xi(t)$, that is, it is computed with respect to a continuous function ξ, and there exists an exact solution whose maximum error with respect to the approximate solution is a linear function of the variable error $\varepsilon\xi(t)$. Hyers-Ulam stability is Hyers-Ulam-Rassias stability with respect to the constant function $\xi \equiv 1$. The extension of Hyers-Ulam-Rassias stability by admitting the variation of the error in terms of a (nonnecessarily linear) function $\xi_{\mathcal{H},\varepsilon} \in C(\mathbb{R}_+, \mathbb{R}_+)$ is referred to as generalized Hyers-Ulam-Rassias stability (see Definition 2.11).

3. Fundamental existence results

We take, throughout this paper, the following notation:

$$N(\beta) := \frac{2\beta}{\beta + 1} + \frac{1 - \beta}{2\beta + 1}.$$

We shall now give and prove an existence and uniqueness result for problems (1.1) and (1.2) that uses, as a fundamental tool, the contraction mapping principle. The main idea is to express the problem as a fix point formulation, by defining a mapping that takes each function x into an integral term involving the expression of a solution to an implicit functional equation depending on x and related to the implicit fractional problem of interest. Then, it is proved the contractive character of the mapping.

Theorem 3.1. *Suppose that the following two assumptions are satisfied:*

(C_1) *$\mathcal{H}$ is a continuous function.*
(C_2) *There exist two constants $m_1 > 0$ and $0 < m_2 < 1$ such that the next inequality holds*

$$|\mathcal{H}(t, x, z) - \mathcal{H}(t, \hat{x}, \hat{z})| \leqslant m_1|x - \hat{x}| + m_2|z - \hat{z}|,$$

for every $t \in \mathcal{J}$ and x, $\hat{x}$, z, $\hat{z} \in \mathbb{R}$.

If

$$\frac{m_1(2\gamma_\beta + b\sigma_\beta)}{1 - m_2} < 1, \tag{3.1}$$

then the Cauchy problems (1.1) *and* (1.2) *has a unique solution.*

Proof. We define the mapping $\Phi : C([0, b], \mathbb{R}) \longrightarrow C([0, b], \mathbb{R})$, given by

$$(\Phi x)(t) = \delta_0 + \gamma_\beta\varphi(t) + \sigma_\beta \int_0^t \varphi(s)\, ds, \tag{3.2}$$

where

$$\delta_0 := x_0 - \gamma_\beta\varphi(0),$$

and $\varphi \in C([0, b], \mathbb{R})$ (dependent on x) satisfies that

$$\varphi(s) = \mathcal{H}(s, x(s), \varphi(s)).$$

For each $t \in \mathcal{J}$, it follows that

$$(\Phi x_1)(t) - (\Phi x_2)(t) = -\gamma_\beta\Big(\varphi(0) - \psi(0)\Big) + \gamma_\beta\Big(\varphi(t) - \psi(t)\Big) + \sigma_\beta \int_0^t \Big(\varphi(s) - \psi(s)\Big)\, ds,$$

where $\varphi, \psi \in C([0, b], \mathbb{R})$ are such that

$$\varphi(s) = \mathcal{H}(s, x_1(s), \varphi(s)), \quad \psi(s) = \mathcal{H}(s, x_2(s), \psi(s)).$$

From the condition (C_2), we get, for $s \in [0, b]$,

$$|\varphi(s) - \psi(s)| \leqslant m_1|x_1(s) - x_2(s)| + m_2|\varphi(s) - \psi(s)|,$$

and, hence,

$$|\varphi(s) - \psi(s)| \leqslant \frac{m_1}{1 - m_2}|x_1(s) - x_2(s)|.$$

Then, we obtain

$$\begin{aligned}
\Big|(\Phi x_1)(t) - (\Phi x_2)(t)\Big| &\leqslant \gamma_\beta\Big|\varphi(0) - \psi(0)\Big| + \gamma_\beta\Big|\varphi(t) - \psi(t)\Big| + \sigma_\beta \int_0^t \Big|\varphi(s) - \psi(s)\Big|\, ds \\
&\leqslant \frac{m_1}{1 - m_2}\Big(\gamma_\beta\Big|x_1(0) - x_2(0)\Big| + \gamma_\beta\Big|x_1(t) - x_2(t)\Big| + \sigma_\beta \int_0^t \Big|x_1(s) - x_2(s)\Big|\, ds\Big) \\
&\leqslant \frac{m_1}{1 - m_2}\|x_1 - x_2\|_\infty\Big(2\gamma_\beta + \sigma_\beta \int_0^t ds\Big) \\
&\leqslant \frac{m_1(2\gamma_\beta + b\sigma_\beta)}{1 - m_2}\|x_1 - x_2\|_\infty.
\end{aligned}$$

Finally, we have

$$\|\Phi x_1 - \Phi x_2\|_\infty \leqslant \frac{m_1(2\gamma_\beta + b\sigma_\beta)}{1 - m_2}\|x_1 - x_2\|_\infty.$$

Since $\frac{m_1(2\gamma_\beta + b\sigma_\beta)}{1 - m_2} < 1$, then Φ is contractive, in consequence, there exists a unique fixed point for Φ. Therefore, there exists a unique solution to problems (1.1) and (1.2).

By weakening one of the restrictions in the previous theorem, namely, the inequality (Eq 3.1), which is replaced by a more general one, we can still prove the existence of solutions to problems (1.1) and (1.2) by using Schaefer's fixed point result (see Theorem 2.5), as illustrated below. The procedure is based on showing the continuity and compactness of the mapping Φ defined in the proof of Theorem 3.1, and the bounded character of the set of fixed points of the mappings $\lambda\Phi$, for $\lambda \in (0, 1)$.

Theorem 3.2. *Suppose that the conditions* (C_1) *and* (C_2) *are valid and assume that*

$$1 - m_2 - m_1\gamma_\beta > 0.$$

Then we can affirm the existence of at least one solution to problems (1.1) *and* (1.2).

Proof. We take the mapping Φ as given in (3.2) and divide the procedure into several parts in order to show different properties of the mapping.
Part 1. We prove that Φ is a continuous mapping.

Consider $\{x_n\}$ any convergent sequence, with limit, say, a function x, that is, $x_n \longrightarrow x$, with convergence in the space $C([0,b],\mathbb{R})$, and let $\varphi_n(s) = \mathcal{H}(s, x_n(s), \varphi_n(s))$, and $\varphi(s) = \mathcal{H}(s, x(s), \varphi(s))$. For every $t \in \mathcal{J}$, we get

$$\begin{aligned}
\Big|(\Phi x_n)(t) - (\Phi x)(t)\Big| &= \Big|-\gamma_\beta(\varphi_n(0) - \varphi(0)) + \gamma_\beta(\varphi_n(t) - \varphi(t)) + \sigma_\beta \int_0^t (\varphi_n(s) - \varphi(s))\,ds\Big| \\
&\leqslant \gamma_\beta\Big|\varphi_n(0) - \varphi(0)\Big| + \gamma_\beta\Big|\varphi_n(t) - \varphi(t)\Big| + \sigma_\beta \int_0^t \Big|\varphi_n(s) - \varphi(s)\Big|\,ds \\
&\leqslant \frac{m_1}{1-m_2}\Big(\gamma_\beta\Big|x_n(0) - x(0)\Big| + \gamma_\beta\Big|x_n(t) - x(t)\Big| + \sigma_\beta \int_0^t \Big|x_n(s) - x(s)\Big|\,ds\Big) \\
&\leqslant \frac{m_1(2\gamma_\beta + b\sigma_\beta)}{1-m_2}\|x_n - x\|_\infty.
\end{aligned}$$

This shows that Φ is continuous since $\|x_n - x\|_\infty \to 0$ as $n \to \infty$ implies that $\|\Phi x_n - \Phi x\| \to 0$ as $n \to \infty$.
Part 2. The image of a bounded set in $C([0,b],\mathbb{R})$ through the mapping Φ is a bounded set in $C([0,b],\mathbb{R})$.

Take $r > 0$ and consider the closed ball in $C([0,b],\mathbb{R})$ with center the null function and radius r, that is, $\mathrm{B}_r = \{z \in C([0,b],\mathbb{R}) : r \geqslant \|z\|_\infty\}$. For any $t \in \mathcal{J}$ and $x \in \mathrm{B}_r$, we have

$$\begin{aligned}
|\varphi(t)| &\leqslant |\mathcal{H}(t, x(t), \varphi(t)) - \mathcal{H}(t,0,0)| + |\mathcal{H}(t,0,0)| \\
&\leqslant m_1|x(t)| + m_2|\varphi(t)| + |\mathcal{H}(t,0,0)| \\
&\leqslant m_1 r + m_2|\varphi(t)| + |\mathcal{H}(t,0,0)|,
\end{aligned} \tag{3.3}$$

thus,

$$|\varphi(t)| \leqslant \frac{m_1 r}{1-m_2} + \frac{1}{1-m_2}|\mathcal{H}(t,0,0)|,$$

and

$$\sigma_\beta \int_0^t |\varphi(s)|\,ds \leqslant \frac{m_1 r\sigma_\beta}{1-m_2}\int_0^t ds + \frac{\sigma_\beta}{1-m_2}\int_0^t |\mathcal{H}(s,0,0)|\,ds,$$

hence

$$\begin{aligned}
|(\Phi x)(t)| &\leqslant |\delta_0| + \gamma_\beta|\varphi(t)| + \sigma_\beta\int_0^t |\varphi(s)|\,ds \\
&\leqslant |x_0| + \gamma_\beta|\varphi(0)| + \gamma_\beta|\varphi(t)| + \sigma_\beta\int_0^t |\varphi(s)|\,ds \\
&\leqslant |x_0| + \frac{m_1 r\gamma_\beta}{1-m_2} + \frac{\gamma_\beta}{1-m_2}|\mathcal{H}(0,0,0)| + \frac{m_1 r\gamma_\beta}{1-m_2} + \frac{\gamma_\beta}{1-m_2}|\mathcal{H}(t,0,0)| \\
&\quad + \frac{m_1 r\sigma_\beta}{1-m_2}\int_0^t ds + \frac{\sigma_\beta}{1-m_2}\sup_{s\in\mathcal{J}}|\mathcal{H}(s,0,0)|\int_0^t ds \\
&\leqslant |x_0| + \frac{m_1 r}{1-m_2}(2\gamma_\beta + b\sigma_\beta) + \frac{2\gamma_\beta + b\sigma_\beta}{1-m_2}\sup_{s\in\mathcal{J}}|\mathcal{H}(s,0,0)| =: m,
\end{aligned}$$

which implies that

$$\|\Phi x\| \leqslant m.$$

Part 3. The image of a bounded set in $C([0,b],\mathbb{R})$ through the mapping Φ is an equicontinuous set in $C([0,b],\mathbb{R})$.

Consider $t_1, t_2 \in \mathcal{J}$, with $t_2 > t_1$, and take $x \in \mathrm{B}_r$. So, we get

$$\begin{aligned}
|(\Phi x)(t_2) - (\Phi x)(t_1)| &= \left|\gamma_\beta(\varphi(t_2) - \varphi(t_1)) + \sigma_\beta\Big(\int_0^{t_2}\varphi(s)ds - \int_0^{t_1}\varphi(s)ds\Big)\right| \\
&\leqslant \gamma_\beta\Big|\varphi(t_2) - \varphi(t_1)\Big| + \sigma_\beta\left|\int_0^{t_2}\varphi(s)ds - \int_0^{t_1}\varphi(s)ds\right| \\
&\leqslant \gamma_\beta\Big|\varphi(t_2) - \varphi(t_1)\Big| + \sigma_\beta\int_{t_1}^{t_2}|\varphi(s)|ds \\
&\leqslant \gamma_\beta\Big|\varphi(t_2) - \varphi(t_1)\Big| + \frac{m_1 r\sigma_\beta}{1-m_2}\int_{t_1}^{t_2}ds + \frac{\sigma_\beta}{1-m_2}\int_{t_1}^{t_2}|\mathcal{H}(s,0,0)|\,ds \\
&= \gamma_\beta\Big|\varphi(t_2) - \varphi(t_1)\Big| + \frac{m_1 r\sigma_\beta}{1-m_2}(t_2 - t_1) + \frac{\sigma_\beta}{1-m_2}\int_{t_1}^{t_2}|\mathcal{H}(s,0,0)|\,ds.
\end{aligned}$$

Since φ is uniformly continuous on $\mathcal{J}$ and $s \in \mathcal{J} \to |\mathcal{H}(s,0,0)|$ is bounded, then, as $|t_1 - t_2| \to 0$, the expression on the right tends to zero. By Parts 1–3 and the application of Arzelà- Ascoli theorem, it is proved that the mapping Φ is completely continuous, as desired.

Part 4. Now, we calculate a priori bounds, which are cruzial for the application of the fixed point result.

Let $\mathcal{X} = C([0,b],\mathbb{R})$, and take $x \in \mathcal{F}$, that is, $x \in \mathcal{X}$ such that $x = \lambda\Phi x$ for a certain $\lambda \in (0,1)$. Thus, for each $t \in \mathcal{J}$, we have, from the second inequality in (3.3),

$$\begin{aligned}
|x(t)| &= \lambda\left|\delta_0 + \gamma_\beta\varphi(t) + \sigma_\beta\int_0^t\varphi(s)ds\right| \\
&\leqslant |x_0| + \gamma_\beta|\varphi(0)| + \gamma_\beta|\varphi(t)| + \sigma_\beta\int_0^t|\varphi(s)|ds \\
&\leqslant |x_0| + \frac{m_1\gamma_\beta}{1-m_2}|x(0)| + \frac{m_1\gamma_\beta}{1-m_2}|x(t)| + \frac{m_1\sigma_\beta}{1-m_2}\int_0^t|x(s)|ds \\
&\quad + \frac{\gamma_\beta}{1-m_2}|\mathcal{H}(0,0,0)| + \frac{\gamma_\beta}{1-m_2}|\mathcal{H}(t,0,0)| + \frac{\sigma_\beta}{1-m_2}\int_0^t|\mathcal{H}(s,0,0)|\,ds \\
&\leqslant |x_0| + \frac{2\gamma_\beta + b\sigma_\beta}{1-m_2}\sup_{s\in\mathcal{J}}|\mathcal{H}(s,0,0)| + \frac{m_1\gamma_\beta}{1-m_2}|x(0)| + \frac{m_1\gamma_\beta}{1-m_2}|x(t)| + \frac{m_1\sigma_\beta}{1-m_2}\int_0^t|x(s)|\,ds,
\end{aligned}$$

and, thus,

$$\begin{aligned}
|x(t)| &\leqslant \frac{(1-m_2)}{1-m_2-m_1\gamma_\beta}\Big(|x_0| + \frac{2\gamma_\beta + b\sigma_\beta}{1-m_2}\sup_{s\in\mathcal{J}}|\mathcal{H}(s,0,0)|\Big) \\
&\quad + \frac{m_1\gamma_\beta}{1-m_2-m_1\gamma_\beta}|x(0)| + \frac{m_1\sigma_\beta}{1-m_2-m_1\gamma_\beta}\int_0^t|x(s)|ds.
\end{aligned}$$

Since $|x(0)| = \lambda|(\Phi x)(0)| \leq |\delta_0 + \gamma_\beta\varphi(0)| = |x_0|$, then

$$|x(t)| \leqslant \frac{1-m_2+m_1\gamma_\beta}{1-m_2-m_1\gamma_\beta}|x_0| + \frac{2\gamma_\beta + b\sigma_\beta}{1-m_2-m_1\gamma_\beta}\sup_{s\in\mathcal{J}}|\mathcal{H}(s,0,0)| + \frac{m_1\sigma_\beta}{1-m_2-m_1\gamma_\beta}\int_0^t|x(s)|ds.$$

Then, by applying the Gronwall's inequality, we have, for every $t \in [0, b]$,

$$|x(t)| \leqslant \left(\frac{1 - m_2 + m_1\gamma_\beta}{1 - m_2 - m_1\gamma_\beta} |x_0| + \frac{2\gamma_\beta + b\sigma_\beta}{1 - m_2 - m_1\gamma_\beta} \sup_{s\in\mathcal{J}} |\mathcal{H}(s, 0, 0)| \right) \exp \left(\frac{bm_1\sigma_\beta}{1 - m_2 - m_1\gamma_\beta} \right) < \infty.$$

Therefore, by Schaefer's fixed point result, the mapping Φ has at least one fixed point, and, hence, there exists at least one solution to the problems (1.1) and (1.2).

4. Stability in the sense of Hyers-Ulam

In this section, we start the study of the stability of Eq (1.1), in particular, of Hyers-Ulam type. In the proof of the following result, we use the integral formulation of the implicit fractional problem to give an estimate for the distance between an approximate and an exact solution. Gronwall's Lemma (Lemma 2.6) is also a useful tool to conclude.

Theorem 4.1. *Suppose that the hypotheses* (C_1) *and* (C_2) *are satisfied, and that condition* (3.1) *holds. Under these restrictions, the Eq* (1.1) *is stable in the sense of Hyers-Ulam.*

Proof. Take an arbitrary $\varepsilon > 0$ and consider $z \in C^1(\mathcal{J}, \mathbb{R})$ a function satisfying the inequality

$$|{}^{CF}\mathbb{D}^\beta z(t) - \mathcal{H}(t, z(t), {}^{CF}\mathbb{D}^\beta z(t))| \leqslant \varepsilon, \text{ for every } t \in \mathcal{J}. \tag{4.1}$$

Let us denote by $x \in C(\mathcal{J}, \mathbb{R})$ the unique function that is a solution to the following Cauchy problem

$${}^{CF}\mathbb{D}^\beta x(t) = \mathcal{H}(t, x(t), {}^{CF}\mathbb{D}^\beta x(t)), \text{ for } t \in \mathcal{J},$$

$$x(0) = z(0).$$

By applying Lemma 2.4, we get

$$x(t) = \delta_{0,x} + \gamma_\beta\varphi_x(t) + \sigma_\beta \int_0^t \varphi_x(s)ds,$$

where $\varphi_x \in C(\mathcal{J}, \mathbb{R})$ satisfies the functional identity

$$\varphi_x(t) = \mathcal{H}(t, x(t), \varphi_x(t)).$$

By formula (4.1), we have

$$\left| z(t) - \delta_{0,z} - \gamma_\beta\varphi_z(t) - \sigma_\beta \int_0^t \varphi_z(s)ds \right| \leqslant c_\varepsilon,$$

where

$$\varphi_z(t) = \mathcal{H}(t, z(t), \varphi_z(t)),$$

and

$$c_\varepsilon = \frac{2(1 - \beta + b\beta)\varepsilon}{(2 - \beta)N(\beta)}.$$

Furthermore, we derive, for each $t \in \mathcal{J}$,

$$
\begin{aligned}
|z(t)-x(t)| &= \left|z(t)-\delta_{0,x}-\gamma_\beta\varphi_x(t)-\sigma_\beta\int_0^t \varphi_x(s)ds\right| \\
&= \left|z(t)+(\delta_{0,z}-\delta_{0,x})-\delta_{0,z}+\gamma_\beta(\varphi_z(t)-\varphi_x(t))-\gamma_\beta\varphi_z(t)\right. \\
&\quad \left. -\sigma_\beta\int_0^t \varphi_z(s)ds+\sigma_\beta\int_0^t(\varphi_z(s)-\varphi_x(s))ds\right| \\
&\leqslant \left|z(t)-\delta_{0,z}-\gamma_\beta\varphi_z(t)-\sigma_\beta\int_0^t \varphi_z(s)ds\right| \\
&\quad +\left|(\delta_{0,z}-\delta_{0,x})+\gamma_\beta(\varphi_z(t)-\varphi_x(t))+\sigma_\beta\int_0^t(\varphi_z(s)-\varphi_x(s))ds\right| \\
&\leqslant \frac{2(1-\beta+b\beta)\varepsilon}{(2-\beta)N(\beta)}+\gamma_\beta\left|\varphi_z(0)-\varphi_x(0)\right|+\gamma_\beta\left|\varphi_z(t)-\varphi_x(t)\right| \\
&\quad +\sigma_\beta\int_0^t\left|\varphi_z(s)-\varphi_x(s)\right|ds.
\end{aligned}
$$

However, according to (C_2), we have

$$
\begin{aligned}
|\varphi_z(s)-\varphi_x(s)| &= |\mathcal{H}(s,z(s),\varphi_z(s))-\mathcal{H}(s,x(s),\varphi_x(s))| \\
&\leqslant m_1|z(s)-x(s)|+m_2|\varphi_z(s)-\varphi_x(s)|,
\end{aligned}
$$

so that

$$
|\varphi_z(s)-\varphi_x(s)| \leqslant \frac{m_1}{1-m_2}|z(s)-x(s)|.
$$

Summing all the information up, we have

$$
\begin{aligned}
|z(t)-x(t)| &\leqslant \frac{2(1-\beta+b\beta)\varepsilon}{(2-\beta)N(\beta)}+\frac{m_1\gamma_\beta}{1-m_2}\left|z(0)-x(0)\right|+\frac{m_1\gamma_\beta}{1-m_2}\left|z(t)-x(t)\right| \\
&\quad +\frac{m_1\sigma_\beta}{1-m_2}\int_0^t\left|z(s)-x(s)\right|ds,
\end{aligned}
$$

and, then, by taking into account also that $x(0)=z(0)$, we get

$$
|z(t)-x(t)| \leqslant \frac{2(1-m_2)(1-\beta+b\beta)\varepsilon}{(1-m_2-m_1\gamma_\beta)(2-\beta)N(\beta)}+\frac{m_1\sigma_\beta}{1-m_2-m_1\gamma_\beta}\int_0^t\left|z(s)-x(s)\right|ds.
$$

By the application of Gronwall's inequality, it is deduced that

$$
|z(t)-x(t)| \leqslant \lambda_\beta \exp\frac{bm_1\sigma_\beta}{1-m_2-m_1\gamma_\beta}=c\,\varepsilon,
$$

where

$$
\lambda_\beta := \frac{2(1-m_2)(1-\beta+b\beta)\varepsilon}{(1-m_2-m_1\gamma_\beta)(2-\beta)N(\beta)},
$$

proving that the Eq (1.1) is stable in terms of Hyers-Ulam.

If we take $\xi_{\mathcal{H},\varepsilon}(t) = c\,\varepsilon$, which satisfies that $\xi_{\mathcal{H},0}$ is identically null, it is also deduced that the Eq (1.1) is generalized stable in the sense of Hyers-Ulam.

5. Stability in the sense of Hyers-Ulam-Rassias

Now, we study the stability of Eq (1.1) in the sense of Hyers-Ulam-Rassias. The approach is similar to that in the proof of Theorem 4.1, since we use again the integral formulation of the implicit fractional problem to give an estimate for the distance between an approximate and an exact solution. Note that, in Theorem 4.1, we assume existence and uniqueness conditions for the Cauchy problems (1.1) and (1.2). However, this is not essential to proceed, so that the requirements in the following result are sufficient to guarantee the existence of a least one solution to problems (1.1) and (1.2), in the terms of Theorem 3.2. Besides, an auxiliary function ψ is introduced in order to deduce the stability of the equation in the sense of Hyers-Ulam-Rassias with respect to ψ. In our procedure, Generalized Gronwall's Lemma (Lemma 2.7) is also useful to complete the proof.

Theorem 5.1. *Suppose that the conditions* (C_1) *and* (C_2) *are valid, with* $1 - m_2 - m_1\gamma_\beta > 0$. *Assume also that the following assumption holds:*

(C_3) *The function* $\psi \in C(\mathcal{J}, \mathbb{R}_+)$ *is nondecreasing and such that there exists* $\lambda_\psi > 0$ *satisfying, for each* $t \in \mathcal{J}$,

$$\int_0^t \psi(s)\,ds \leqslant \frac{\lambda_\psi(2-\beta)N(\beta) - 2 + 2\beta}{2\beta}\psi(t).$$

Under these restrictions, the problem (1.1) *is stable, in the sense of Hyers-Ulam-Rassias, with respect to* ψ.

Proof. Take an arbitrary $\varepsilon > 0$ and a function $z \in C^1(\mathcal{J}, \mathbb{R})$ satisfying the inequality

$$|{}^{CF}\mathbb{D}^\beta z(t) - \mathcal{H}(t, z(t), {}^{CF}\mathbb{D}^\beta z(t))| \leqslant \varepsilon\psi(t), \ t \in \mathcal{J}. \tag{5.1}$$

Consider the Cauchy problem

$$\begin{cases} {}^{CF}\mathbb{D}^\beta x(t) = \mathcal{H}(t, x(t), {}^{CF}\mathbb{D}^\beta x(t)), \text{ for } t \in \mathcal{J}, \\ x(0) = z(0), \end{cases}$$

and represent by $x \in C(\mathcal{J}, \mathbb{R})$ one of its solutions (there exists at least one solution under these assumptions).

By the inequality (Eq 5.1), we deduce that

$$\begin{aligned} \left| z(t) - \delta_{0,z} - \gamma_\beta\varphi_z(t) - \sigma_\beta \int_0^t \varphi_z(s)ds \right| &\leqslant \varepsilon\ {}^{CF}\mathbb{I}^\beta\psi(t) \\ &\leqslant \varepsilon\lambda_\psi\psi(t). \end{aligned}$$

Following a procedure analogous to the proof of Theorem 4.1, it is obtained, for each $t \in \mathcal{J}$, the chain of inequalities

$$\begin{aligned}|z(t)-x(t)| &= \Big|z(t)+(\delta_{0,z}-\delta_{0,x})-\delta_{0,z}+\gamma_\beta(\varphi_z(t)-\varphi_x(t))-\gamma_\beta\varphi_z(t)\\ &\quad -\sigma_\beta\int_0^t\varphi_z(s)\,ds+\sigma_\beta\int_0^t(\varphi_z(s)-\varphi_x(s))\,ds\Big|\\ &\leqslant \Big|z(t)-\delta_{0,z}-\gamma_\beta\varphi_z(t)-\sigma_\beta\int_0^t\varphi_z(s)\,ds\Big|+\gamma_\beta\Big|\varphi_z(0)-\varphi_x(0)\Big|\\ &\quad +\gamma_\beta\Big|\varphi_z(t)-\varphi_x(t)\Big|+\sigma_\beta\int_0^t\Big|\varphi_z(s)-\varphi_x(s)\Big|\,ds\\ &\leqslant \varepsilon\lambda_\psi\psi(t)+\frac{m_1\gamma_\beta}{1-m_2}|z(0)-x(0)|+\frac{m_1\gamma_\beta}{1-m_2}|z(t)-x(t)|\\ &\quad +\frac{m_1\sigma_\beta}{1-m_2}\int_0^t|z(s)-x(s)|\,ds.\end{aligned}$$

Then, due to $z(0)=x(0)$,

$$|z(t)-x(t)| \leqslant \frac{1-m_2}{1-m_2-m_1\gamma_\beta}\varepsilon\lambda_\psi\psi(t)+\frac{m_1\sigma_\beta}{1-m_2-m_1\gamma_\beta}\int_0^t\Big|z(s)-x(s)\Big|ds.$$

Applying the generalized Gronwall's inequality, we obtain

$$|z(t)-x(t)|\leqslant\frac{1-m_2}{1-m_2-m_1\gamma_\beta}\varepsilon\lambda_\psi\psi(t)\exp\left(\frac{bm_1\sigma_\beta}{1-m_2-m_1\gamma_\beta}\right)=c\,\varepsilon\,\psi(t),$$

where

$$c:=\frac{1-m_2}{1-m_2-m_1\gamma_\beta}\lambda_\psi\exp\left(\frac{bm_1\sigma_\beta}{1-m_2-m_1\gamma_\beta}\right).$$

Consequently, we deduce the stability of the Eq (1.1) in the sense of Hyers-Ulam-Rassias.

If we take $\xi_{\mathcal{H},\varepsilon}(t)=c\,\varepsilon\,\psi(t)$, which satisfies that $\xi_{\mathcal{H},0}\equiv 0$, we can also affirm the generalized stability of the Eq (1.1), in the sense of Hyers-Ulam-Rassias, with respect to ψ on $\mathcal{J}$.

6. Example

To illustrate the results obtained, we propose the following initial value problem, which is inspired in the Example given in Section 2.4.3 [26]:

$${}^{CF}\mathbb{D}^{\frac{1}{2}}x(t)=\frac{\Big(3+|x(t)|+|{}^{CF}\mathbb{D}^{\frac{1}{2}}x(t)|\Big)e^{-2t}}{(5+e^t)(1+|x(t)|+|{}^{CF}\mathbb{D}^{\frac{1}{2}}x(t)|)},\quad t\in\mathcal{J}:=\left[0,\frac{1}{2}\right],\tag{6.1}$$

$$x(0)=x_0.\tag{6.2}$$

Set

$$\mathcal{H}(t,x,z)=\frac{\Big(3+|x|+|z|\Big)e^{-2t}}{(5+e^t)(1+|x|+|z|)},\qquad\text{for }(t,x,z)\in\mathcal{J}\times\mathbb{R}\times\mathbb{R}.$$

The function $\mathcal{H}$ is obviously continuous, thus (C_1) holds.

Now, let $x,\ \hat{x},\ z,\ \hat{z} \in \mathbb{R}$, and $t \in \mathcal{J}$,

$$\begin{aligned} |\mathcal{H}(t,\hat{x},\hat{z}) - \mathcal{H}(t,x,z)| &= \frac{2e^{-2t}}{(5+e^t)}\left|\frac{\frac{1}{2}(3+|\hat{x}|+|\hat{z}|)}{1+|\hat{x}|+|\hat{z}|} - \frac{\frac{1}{2}(3+|x|+|z|)}{1+|x|+|z|}\right| \\ &\leqslant \frac{2e^{-2t}}{(5+e^t)}\Big||x|+|z|-|\hat{x}|-|\hat{z}|\Big| \\ &\leqslant \frac{1}{3}\Big(|x-\hat{x}|+|z-\hat{z}|\Big). \end{aligned}$$

Then, the assumption (C_2) holds with

$$m_1 = m_2 = \frac{1}{3}.$$

We have $\beta = \frac{1}{2}$, $N(\frac{1}{2}) = \frac{11}{12}$, $\quad \gamma_{\frac{1}{2}} = \frac{2(1-\frac{1}{2})}{(2-\frac{1}{2})N(\frac{1}{2})} = \frac{8}{11}$, $\quad \sigma_{\frac{1}{2}} = \frac{2\frac{1}{2}}{(2-\frac{1}{2})N(\frac{1}{2})} = \frac{8}{11}$, and $b = \frac{1}{2}$.

Thus, condition (3.1) is valid

$$K := \frac{m_1(2\gamma_{\frac{1}{2}} + b\sigma_{\frac{1}{2}})}{1-m_2} = \frac{10}{11} < 1.$$

Then, by Theorem 4.1, the Eq (6.1) is stable in the sense of Hyers-Ulam. Note that, from the proof of Theorem 4.1, we have

$$c\,\varepsilon = \lambda_{\frac{1}{2}} \exp\frac{bm_1\sigma_{\frac{1}{2}}}{1-m_2-m_1\gamma_{\frac{1}{2}}} = \lambda_{\frac{1}{2}} \exp\Big(\frac{2}{7}\Big),$$

where

$$\lambda_{\frac{1}{2}} := \frac{20}{7}\,\varepsilon.$$

7. Conclusions

In this paper, we have considered the problems of existence, uniqueness, and stability, from the point of view of Hyers-Ulam and Hyers-Ulam-Rassias, of the solutions to an implicit differential equation of arbitrary order $\beta \in (0,1)$ subject to initial conditions of the form (1.1) and (1.2), under the framework of Caputo-Fabrizio operators.

For the development of the main results, we first recall, in Section 2, some relevant notions and procedures concerning the Caputo-Fabrizio operators, the solvability of equations in this context, fixed point results, Gronwall-type inequalities, and the different notions of stability used in the paper (Hyers-Ulam, generalized Hyers-Ulam, Hyers-Ulam-Rassias, and generalized Hyers-Ulam-Rassias).

Section 3 is devoted to the existence results for problems (1.1) and (1.2), for which we have considered a corresponding fixed point formulation. The integral mapping defined involves, for each function x, the expression of a solution to an implicit functional equation depending on x and related to the implicit fractional problem of interest. In first place, we have derived an existence and uniqueness result based on the contraction mapping principle, and, later, by relaxing the restrictions on the constants, we obtain the existence of at least one solution to problems (1.1) and (1.2) by using Schaefer's fixed point theorem.

In Section 4, we analyze the stability of the Eq (1.1) in the sense of Hyers-Ulam by using the integral formulation of the implicit fractional problem and Gronwall's Lemma. On the other hand, in Section 5, we study the Hyers-Ulam-Rassias stability of the problem following a similar approach. However, in this case, an auxiliary function ψ is introduced in relation with the stability concept, and the sought estimate for the distance between an approximate and an exact solution is obtained by using Generalized Gronwall's Lemma (Lemma 2.7).

Finally, in Section 6, an example is presented.

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Conflict of interest

The authors declare there is no conflict of interest.

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