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*Research article*

# An efficient numerical method for a time-fractional telegraph equation

**Jian Huang***, **Zhongdi Cen and Aimin Xu**

Institute of Mathematics, Zhejiang Wanli University, Ningbo 315100, China

* **Correspondence:** Email: sword@zwu.edu.cn.

**Abstract:** In this paper a time-fractional telegraph equation is considered. First the time-fractional telegraph equation is transformed into an integral-differential equation with a weakly singular kernel. Then an integral-difference discretization scheme on a graded mesh is developed to approximate the integral-differential equation. The possible singularity of the exact solution is taken into account in the convergence analysis. It is proved that the scheme is second-order convergent for both the spatial discretization and the time discretization. Numerical experiments confirm the validity of the theoretical results.

**Keywords:** telegraph equation; caputo derivative; integro-differential equation; finite difference; singularity

## 1. Introduction

In this paper we study the following time-fractional telegraph equation (TFTE) [1–3]

$$\begin{cases} D^{\gamma}_{C,t}u(x,t) + aD^{\gamma-1}_{C,t}u(x,t) + bu(x,t) = cu_{xx}(x,t) + f(x,t), & (x,t) \in (0,1)\times(0,T], \\ u(x,0) = \phi_1(x), \quad u_t(x,0) = \phi_2(x), & x \in [0,1], \\ u(0,t) = \psi_1(t), \quad u(1,t) = \psi_2(t), & t \in [0,T], \end{cases} \tag{1.1}$$

where $1 < \gamma < 2$, $a, b, c$ and $T$ are given positive constants, $f(x,t), \phi_1(x), \phi_2(x), \psi_1(t)$ and $\psi_2(t)$ are given functions and satisfy the compatibility conditions, $D^{\gamma}_{C,t}u(x,t)$ and $D^{\gamma-1}_{C,t}u(x,t)$ are the Caputo fractional derivatives of order $\gamma$ and $\gamma-1$ respectively. The Caputo fractional derivative $D^{\gamma}_{C,t}u(x,t)$ is defined by [4, Definition 2.2]

$$D^{\gamma}_{C,t}u(x,t) = \frac{1}{\Gamma(2-\gamma)}\int_0^t (t-s)^{1-\gamma}\, u_{ss}(x,s)\mathrm{d}s.$$

The Caputo operators $D^{\gamma}_{C,t}$ and $D^{\gamma-1}_{C,t}$ have the following relationship

$$D^{\gamma}_{C,t}u(x,t) = \frac{1}{\Gamma(1-(\gamma-1))}\int_0^t (t-s)^{-(\gamma-1)}\frac{\partial}{\partial s}u_s(x,s)\mathrm{d}s = D^{\gamma-1}_{C,t}u_t(x,t).$$

Based on the above relationship, the TFTE (1.1) can be transformed into the following equivalent integral-differential equation as shown in [4, Lemma 6.2]

$$\begin{cases} u_t(x,t) + au(x,t) = a\phi_1(x) + \phi_2(x) & \\ +\frac{1}{\Gamma(\gamma-1)}\int_0^t (t-s)^{\gamma-2}\left[cu_{xx}(x,s) - bu(x,s) + f(x,s)\right]\mathrm{d}s, & (x,t)\in(0,1)\times(0,T], \\ u(x,0) = \phi_1(x), & x\in[0,1], \\ u(0,t) = \psi_1(t), \quad u(1,t) = \psi_2(t), & t\in[0,T], \end{cases} \tag{1.2}$$

which simplifies the original problem to a certain extent. For the subsequent numerical discretization and error analysis, we assume

$$\left|\frac{\partial^l f(x,t)}{\partial t^l}\right| \le C\left(1 + t^{\gamma-l}\right), \qquad l = 0,1,2. \tag{1.3}$$

The TFTE (1.1) is used to describe some phenomena such as propagation of electric signals [5], acoustic waves in porous media [6], transport of neutron in a nuclear reactor [7], and hyperbolic heat transfer [8].

There are a few numerical methods to solve the TFTE. Wang [9] developed a method in the reproducing kernel space by piecewise technique to solve the TFTE. Hosseini et al. [10] applied the radial basis functions and finite difference method to solve the TFTE. Kumar et al. [11] described a finite difference scheme for the generalized TFTE. Hashemi and Baleanu [12] utilized a combination of method of line and group preserving scheme to solve the TFTE. Wei et al. [13] discussed a fully discrete local discontinuous Galerkin finite element method for the TFTE. Sweilam [14] introduced the Sinc-Legendre collocation method for solving the TFTE. Mollahasani et al. [15] presented an operational method based on hybrid functions of Legendre polynomials and Block-Pulse-Functions for solving the TFTE. Hafez and Youssri [16] used a shifted Jacobi collocation scheme for a multidimensional TFTE. Bhrawy et al. [17] proposed an accurate and efficient spectral algorithm for the numerical solution of the two-sided space-time TFTE with three types of non-homogeneous boundary conditions. Youssri et al. [18, 19] also developed numerical spectral Legendre approaches to solve the TFTEs. Bhrawy and Zaky [20] used a method based on the Jacobi tau approximation for solving multi-term time-space TFTE. In [2, 3] B-spline collocation methods were applied to solve the TFTEs. In [21, 22] wavelet methods were used to solve the TFTEs. But these papers only study the case that the solutions of the TFTEs are sufficiently smooth. Special treatment techniques, such as graded meshes [23–27] and mapped basis functions [28] reflecting the characteristics of the singularity of the exact solution, need to be used to solve the fractional differential equations.

The aim of the present study is twofold. The first aim is to construct an integral-difference discretization scheme on a graded mesh to approximate the integral-differential equation with a weakly singular kernel transformed from the TFTE, which is a second-order convergence discretization scheme. The second aim is to take the possible singularity of the exact solution into account in the convergence analysis, where the singularity of the exact solution is reflected as

$$\left|\frac{\partial^k u(x,t)}{\partial x^k}\right| \le C, \qquad k = 0,1,\ldots,4, \tag{1.4}$$

$$\left|\frac{\partial^{l+m} u(x,t)}{\partial t^l \partial x^m}\right| \le C\left(1 + t^{\gamma-l}\right), \qquad l = 0,1,2,3, \text{ and } m = 0,1,2, \tag{1.5}$$

which can be referred in the literature [25–27, 29] for details. We show that our discretization scheme on a graded mesh is second-order convergent for both the spatial discretization and the time discretization, although the exact solution of the TFTE may have singularity. Numerical experiments confirm the effectiveness of the theoretical results, and also verify that this scheme is more accurate than the methods given in [2, 3].

**Remark 1.1** If $b, c$ are variable coefficients and $a$ is a variable coefficient with respect to $x$ , the TFTE (1.1) can also be transformed into the same integral-differential equation as given in (1.2). If $a$ is a variable coefficient with respect to $t$, we first approximate $a$ with a piecewise linear interpolation function about $t$ and then transform the approximate TFTE into an integral-differential equation as given in (1.2). If there is a nonlinear term in the TFTE, we first transform it into a linear equation by Newton iterative method and then transform it into an integral-differential equation as given in (1.2).

**Notation.** Throughout the paper, $C$ is used to indicate the positive constant independent of the mesh, and $C$ in different places can represent different values. In order to simplify the notation, the representation $g_i^j = g(x_i, t_j)$ is introduced for any function $g$ at the mesh point $(x_i, t_j) \in (0, 1) \times (0, T]$.

## 2. Discretization scheme

We developed the discretization scheme on a graded mesh $\bar{\Omega}^{N,K} \equiv \left\{\left(x_i, t_j\right) | x_i = ih, h = 1/N, 0 \leq i \leq N, 0 \leq j \leq K\right\}$ with time mesh points

$$t_j = T\left(\frac{j}{K}\right)^{\frac{2}{\gamma-1}} \tag{2.1}$$

and time sizes $\triangle t_j = t_j - t_{j-1}$ for $1 \leq j \leq K$, where the discretization parameters $N$ and $K$ are positive integers. The following integral-difference discretization scheme

$$\begin{aligned}
&\frac{U_i^j - U_i^{j-1}}{\triangle t_j} + a\frac{U_i^{j-1} + U_i^j}{2} \\
&= a\phi_{1,i} + \phi_{2,i} + \frac{1}{2\Gamma(\gamma-1)} \sum_{k=1}^{j-1} \int_{t_{k-1}}^{t_k} \left(t_{j-1} - s\right)^{\gamma-2} \left[\frac{t_k - s}{\triangle t_k}\left(c\delta_x^2 U_i^{k-1} - bU_i^{k-1} + f_i^{k-1}\right)\right. \\
&\quad \left. + \frac{s - t_{k-1}}{\triangle t_k}\left(c\delta_x^2 U_i^k - bU_i^k + f_i^k\right)\right] \mathrm{d}s + \frac{1}{2\Gamma(\gamma-1)} \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \left(t_j - s\right)^{\gamma-2} \\
&\quad \cdot \left[\frac{t_k - s}{\triangle t_k}\left(c\delta_x^2 U_i^{k-1} - bU_i^{k-1} + f_i^{k-1}\right) + \frac{s - t_{k-1}}{\triangle t_k}\left(c\delta_x^2 U_i^k - bU_i^k + f_i^k\right)\right] \mathrm{d}s
\end{aligned}$$

for $1 \leq i < N$ and $1 \leq j \leq K$ is used to approximate the integral-differential equation in (1.2), where $U_i^j$ denotes the numerical solution at the mesh point $\left(x_i, t_j\right)$ and $\delta_x^2 U_i^j = \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{h^2}$. Thus, our discretization scheme for problem (1.2) is

$$\begin{cases} \frac{U_i^j-U_i^{j-1}}{\triangle t_j}+a\frac{U_i^{j-1}+U_i^j}{2}=a\phi_{1,i}+\phi_{2,i} \\ +\frac{1}{2}\sum_{k=1}^{j-1}\left[\xi_{j-1,k}\left(c\delta_x^2U_i^{k-1}-bU_i^{k-1}+f_i^{k-1}\right)+\eta_{j-1,k}\left(c\delta_x^2U_i^k-bU_i^k+f_i^k\right)\right] \\ +\frac{1}{2}\sum_{k=1}^{j}\left[\xi_{j,k}\left(c\delta_x^2U_i^{k-1}-bU_i^{k-1}+f_i^{k-1}\right)+\eta_{j,k}\left(c\delta_x^2U_i^k-bU_i^k+f_i^k\right)\right], & 1\le i<N, 1\le j\le K, \\ U_i^0=\phi_1(x_i), & 0\le i\le N, \\ U_0^j=\psi_1(t_j),\quad U_N^j=\psi_2(t_j), & 1\le j\le K, \end{cases} \tag{2.2}$$

where

$$\xi_{j,k}=\frac{1}{\triangle t_k\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}\left(t_j-s\right)^{\gamma-2}(t_k-s)\,\mathrm{d}s,\qquad k=1,2\ldots,j, \tag{2.3}$$

$$\eta_{j,k}=\frac{1}{\triangle t_k\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}\left(t_j-s\right)^{\gamma-2}(s-t_{k-1})\,\mathrm{d}s,\qquad k=1,2\ldots,j. \tag{2.4}$$

Let $z_i^j=U_i^j-u_i^j$ for $0\le i\le N$ and $0\le j\le K$. Then, from (1.2) and (2.2) we know that the error mesh function $z$ satisfies

$$\begin{cases} \frac{z_i^j-z_i^{j-1}}{\triangle t_j}+a\frac{z_i^{j-1}+z_i^j}{2} \\ =\frac{1}{2}\sum_{k=1}^{j-1}\left[\xi_{j-1,k}\left(c\delta_x^2z_i^{k-1}-bz_i^{k-1}\right)+\eta_{j-1,k}\left(c\delta_x^2z_i^k-bz_i^k\right)\right] \\ \quad+\frac{1}{2}\sum_{k=1}^{j}\left[\xi_{j,k}\left(c\delta_x^2z_i^{k-1}-bz_i^{k-1}\right)+\eta_{j,k}\left(c\delta_x^2z_i^k-bz_i^k\right)\right]+R_i^j, & 1\le i<N, 1\le j\le K, \\ z_i^0=0, & 0\le i\le N, \\ z_0^j=z_N^j=0, & 1\le j\le K, \end{cases} \tag{2.5}$$

where

$$\begin{aligned} R_i^j \;=\;& \frac{1}{2}\left(\frac{\partial u}{\partial t}(x_i,t_{j-1})+\frac{\partial u}{\partial t}(x_i,t_j)\right)-\frac{u_i^j-u_i^{j-1}}{\triangle t_j} \\ &+\frac{1}{2\Gamma(\gamma-1)}\sum_{k=1}^{j-1}\int_{t_{k-1}}^{t_k}\left(t_{j-1}-s\right)^{\gamma-2}\left[\frac{t_k-s}{\triangle t_k}\left(c\delta_x^2u_i^{k-1}-bu_i^{k-1}+f_i^{k-1}\right)\right. \\ &\left.+\frac{s-t_{k-1}}{\triangle t_k}\left(c\delta_x^2u_i^k-bu_i^k+f_i^k\right)-\left(c\frac{\partial^2u(x_i,s)}{\partial x^2}-bu\left(x_i,s\right)+f\left(x_i,s\right)\right)\right]\mathrm{d}s \\ &+\frac{1}{2\Gamma(\gamma-1)}\sum_{k=1}^{j}\int_{t_{k-1}}^{t_k}\left(t_j-s\right)^{\gamma-2}\left[\frac{t_k-s}{\triangle t_k}\left(c\delta_x^2u_i^{k-1}-bu_i^{k-1}+f_i^{k-1}\right)\right. \\ &\left.+\frac{s-t_{k-1}}{\triangle t_k}\left(c\delta_x^2u_i^k-bu_i^k+f_i^k\right)-\left(c\frac{\partial^2u(x_i,s)}{\partial x^2}-bu\left(x_i,s\right)+f\left(x_i,s\right)\right)\right]\mathrm{d}s. \end{aligned}$$

Then, under the assumption (1.3) we can obtain

$$
\begin{aligned}
\left|R_i^j\right| \leq\ & C\int_{t_{j-1}}^{t_j}\left|\frac{\partial^3 u}{\partial s^3}(x_i,s)\right|\left(s-t_{j-1}\right)\mathrm{d}s+C\sum_{k=1}^{j-1}\int_{t_{k-1}}^{t_k}\left(t_{j-1}-s\right)^{\gamma-2}\left\{\frac{t_k-s}{\triangle t_k}(s-t_{k-1})^2\right.\\
&\cdot\left|\int_0^1\left(c\frac{\partial^4 u}{\partial x^2\partial t^2}-b\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 f}{\partial t^2}\right)(x_i,t_{k-1}+(s-t_{k-1})y)\,y\mathrm{d}y\right|\\
&\left.+\frac{s-t_{k-1}}{\triangle t_k}(t_k-s)^2\left|\int_0^1\left(c\frac{\partial^4 u}{\partial x^2\partial t^2}-b\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 f}{\partial t^2}\right)(x_i,s+(t_k-s)y)(1-y)\,\mathrm{d}y\right|\right\}\mathrm{d}s\\
&+C\sum_{k=1}^{j}\int_{t_{k-1}}^{t_k}\left(t_j-s\right)^{\gamma-2}\left\{\frac{t_k-s}{\triangle t_k}(s-t_{k-1})^2\right.\\
&\cdot\left|\int_0^1\left(c\frac{\partial^4 u}{\partial x^2\partial t^2}-b\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 f}{\partial t^2}\right)(x_i,t_{k-1}+(s-t_{k-1})y)\,y\mathrm{d}y\right|\\
&\left.+\frac{s-t_{k-1}}{\triangle t_k}(t_k-s)^2\left|\int_0^1\left(c\frac{\partial^4 u}{\partial x^2\partial t^2}-b\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 f}{\partial t^2}\right)(x_i,s+(t_k-s)y)(1-y)\,\mathrm{d}y\right|\right\}\mathrm{d}s\\
&+C\sum_{k=1}^{j-1}\int_{t_{k-1}}^{t_k}\left(t_{j-1}-s\right)^{\gamma-2}\left|\frac{t_k-s}{\triangle t_k}\left(\delta_x^2u_i^{k-1}-\frac{\partial^2 u(x_i,t_{k-1})}{\partial x^2}\right)\right.\\
&\left.+\frac{s-t_{k-1}}{\triangle t_k}\left(\delta_x^2u_i^{k}-\frac{\partial^2 u(x_i,t_{k})}{\partial x^2}\right)\right|\mathrm{d}s\\
&+C\sum_{k=1}^{j}\int_{t_{k-1}}^{t_k}\left(t_{j}-s\right)^{\gamma-2}\left|\frac{t_k-s}{\triangle t_k}\left(\delta_x^2u_i^{k-1}-\frac{\partial^2 u(x_i,t_{k-1})}{\partial x^2}\right)\right.\\
&\left.+\frac{s-t_{k-1}}{\triangle t_k}\left(\delta_x^2u_i^{k}-\frac{\partial^2 u(x_i,t_{k})}{\partial x^2}\right)\right|\mathrm{d}s\\
\leq\ & C\left(\int_{t_{j-1}}^{t_j}s^{\frac{\gamma-3}{2}}\mathrm{d}s\right)^2+C\sum_{k=1}^{j-1}\triangle t_k\int_{t_{k-1}}^{t_k}\left(t_{j-1}-s\right)^{\gamma-2}\int_{t_{k-1}}^{t_k}t^{\gamma-2}\mathrm{d}t\mathrm{d}s\\
&+C\sum_{k=1}^{j}\triangle t_k\int_{t_{k-1}}^{t_k}\left(t_{j}-s\right)^{\gamma-2}\int_{t_{k-1}}^{t_k}t^{\gamma-2}\mathrm{d}t\mathrm{d}s+Ch^2\\
\leq\ & C\left(N^{-2}+K^{-2}\right),
\end{aligned}
\tag{2.6}
$$

where we have used the linear interpolation remainder formula

$$
\begin{aligned}
\frac{t_k-t}{\triangle t_k}g_{k-1}+\frac{t-t_{k-1}}{\triangle t_k}g_k-g(t) =\ & \frac{1}{\triangle t_k}\left\{(t_k-t)(t-t_{k-1})^2\int_0^1 g''\left[t_{k-1}+(t-t_{k-1})s\right]s\mathrm{d}s\right.\\
&\left.+(t-t_{k-1})(t_k-t)^2\int_0^1 g''\left[t+(t_k-t)s\right](1-s)\,\mathrm{d}s\right\},
\end{aligned}
$$

the regularities (1.4) and (1.5), the assumption (1.3), the graded mesh (2.1), the inequality (see [30])

$$
\int_{t_{j-1}}^{t_j}s^{\alpha}\left(s-t_{j-1}\right)\mathrm{d}s\leq\frac{1}{2}\left\{\int_{t_{j-1}}^{t_j}s^{\alpha/2}\mathrm{d}s\right\}^2
$$

for $\alpha < 0$, and the following estimates

$$\begin{aligned}
&\int_{t_{j-1}}^{t_j} s^{\frac{\gamma-3}{2}}\mathrm{d}s = \frac{2}{\gamma-1}\left(t_j^{\frac{\gamma-1}{2}} - t_{j-1}^{\frac{\gamma-1}{2}}\right) \le CK^{-1},\\
&\triangle t_1 \int_0^{t_1}\left(t_j - s\right)^{\gamma-2}\int_0^{t_1} t^{\gamma-2}\mathrm{d}t\mathrm{d}s \le C\left(\triangle t_1\right)^{\gamma}\int_0^{t_1}\left(t_j - s\right)^{\gamma-2}\mathrm{d}s \le C\left(\frac{1}{K}\right)^{\frac{2\gamma}{\gamma-1}} \le CK^{-2\gamma},\\
&\sum_{k=2}^{j}\triangle t_k \int_{t_{k-1}}^{t_k}\left(t_j - s\right)^{\gamma-2}\int_{t_{k-1}}^{t_k} t^{\gamma-2}\mathrm{d}t\mathrm{d}s \le C\sum_{k=2}^{j}\left(\triangle t_k\right)^3\left(t_j - t_{k-1}\right)^{\gamma-2} t_{k-1}^{\gamma-2}\\
&= C\sum_{k=2}^{j}\left[\left(\frac{k}{K}\right)^{\frac{2}{\gamma-1}} - \left(\frac{k-1}{K}\right)^{\frac{2}{\gamma-1}}\right]^3\left[t_j - \left(\frac{k-1}{K}\right)^{\frac{2}{\gamma-1}}\right]^{\gamma-2}\left(\frac{k-1}{K}\right)^{\frac{2(\gamma-2)}{\gamma-1}}\\
&\le CK^{-3}\sum_{k=1}^{j}\left(\frac{k}{K}\right)^{\frac{6}{\gamma-1}-3}\left(t_j - \frac{k-1}{K}\right)^{\gamma-2}\left(\frac{k-1}{K}\right)^{\frac{2(\gamma-2)}{\gamma-1}}\\
&\le CK^{-2}\sum_{k=1}^{j-1}\left(t_j - \frac{k}{K}\right)^{\gamma-2}\left(\frac{k}{K}\right)^{\frac{5-\gamma}{\gamma-1}}\frac{1}{K}\\
&\le CK^{-2}\int_0^{t_j}\left(t_j - s\right)^{\gamma-2} s^{\frac{5-\gamma}{\gamma-1}}\mathrm{d}s\\
&= CK^{-2} t_j^{\gamma-1+\frac{5-\gamma}{\gamma-1}} B\left(\frac{4}{\gamma-1}, \gamma-1\right)\\
&\le CK^{-2}.
\end{aligned}$$

The following lemma gives a useful result for our convergence analysis, which is given in [31].

**Lemma 2.1** Assume that $\{\omega_k\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers satisfying

$$\omega_k \ge 0, \quad \omega_{k+1} \le \omega_k, \quad \omega_{k+1} - 2\omega_k + \omega_{k-1} \ge 0.$$

Then for any integer $K > 0$ and vector $(V_1, V_2, \ldots, V_K)^T \in \mathbb{R}^K$, the following inequality

$$\sum_{k=1}^{K}\left(\sum_{p=1}^{k}\omega_{k-p}V_p\right)V_k \ge 0$$

holds true.

Next we show that the sequence $\left\{\beta_{j,k}\right\}_{k=1}^{j}$ satisfies the conditions in Lemma 2.1 by using the technique in [32], where $\beta_{j,j-k} = \triangle t_j\left(\xi_{j,k+1} + \eta_{j,k}\right)$ with $\xi_{j,j+1} = 0$.

**Lemma 2.2** The sequence $\left\{\beta_{j,k}\right\}_{k=0}^{j}$ satisfies

$$\beta_{j,k} \ge 0, \quad \beta_{j,k+1} \le \beta_{j,k}, \quad \beta_{j,k+1} - 2\beta_{j,k} + \beta_{j,k-1} \ge 0.$$

*Proof.* From the definitions of the sequences $\left\{\xi_{j,k}\right\}_{k=1}^{j}$ and $\left\{\eta_{j,k}\right\}_{k=1}^{j}$ we have

$$\beta_{j,j-k} \quad = \quad \frac{\triangle t_j}{\triangle t_{k+1}\Gamma(\gamma-1)}\int_{t_k}^{t_{k+1}}\left(t_j - s\right)^{\gamma-2}\left(t_{k+1} - s\right)\mathrm{d}s$$

$$
\begin{aligned}
&+\frac{\triangle t_j}{\triangle t_k\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}\left(t_j-s\right)^{\gamma-2}(s-t_{k-1})\,\mathrm{d}s\\
=\;&\frac{\triangle t_j}{\triangle t_{k+1}\Gamma(\gamma-1)}\int_0^{\triangle t_{k+1}}\left(t_j-t_k-y\right)^{\gamma-2}(\triangle t_{k+1}-y)\,\mathrm{d}y\\
&+\frac{\triangle t_j}{\triangle t_k\Gamma(\gamma-2)}\int_{-\triangle t_k}^{0}\left(t_j-t_k-y\right)^{\gamma-2}(y+\triangle t_k)\,\mathrm{d}y\\
=\;&\frac{\triangle t_j}{\Gamma(\gamma-1)}\int_{-\triangle t_k}^{\triangle t_{k+1}}\left(t_j-t_k-y\right)^{\gamma-2}\left(1-\frac{|y|}{\triangle t_k\chi_{[-\triangle t_k,0]}+\triangle t_{k+1}\chi_{[0,\triangle t_{k+1}]}}\right)\mathrm{d}y \qquad (2.7)
\end{aligned}
$$

for $1 \le k < j$ and

$$\beta_{j,j-j} = \frac{1}{\Gamma(\gamma-1)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\gamma-2}\left(s-t_{j-1}\right)\mathrm{d}s, \qquad (2.8)$$

where

$$\chi_{[p,q]}(y)=\begin{cases}1, & y\in[p,q],\\ 0, & y\notin[p,q].\end{cases}$$

Obviously, from (2.7) and (2.8) we have

$$\beta_{j,j-k}\ge 0, \qquad 1\le k\le j, \qquad (2.9)$$

and

$$\beta_{j,j-(j-1)}\ge\beta_{j,j-j}. \qquad (2.10)$$

Since

$$
\begin{aligned}
&\frac{d}{dk}\left(t_j-t_k-y\right)^{\gamma-2}>0, \qquad k\ge 1 \text{ and } y\in(-\triangle t_k,\triangle t_{k+1}),\\
&\frac{d^2}{dk^2}\left(t_j-t_k-y\right)^{\gamma-2}>0, \qquad k\ge 1 \text{ and } y\in(-\triangle t_k,\triangle t_{k+1}),
\end{aligned}
$$

we can obtain

$$\beta_{j,j-k-1}\ge\beta_{j,j-k}, \quad \text{and} \quad \beta_{j,j-k-1}-2\beta_{j,j-k}+\beta_{j,j-k+1}\ge 0,$$

which imply

$$\beta_{j,k+1}\le\beta_{j,k}, \quad \text{and} \quad \beta_{j,k+1}-2\beta_{j,k}+\beta_{j,k-1}\ge 0. \qquad (2.11)$$

From (2.9)–(2.11) we know the lemma holds true. □

A modified Grönwall inequality given in [33, Lemma 3.3] also is needed in the convergence analysis of the scheme.

**Lemma 2.3** Suppose that $\alpha, C_0, T > 0$ and $d_{k,j} = C_0 \triangle t_{k+1} \left(t_j - t_k\right)^{\alpha-1}$ ($k = 0, 1, \ldots, j-1$) for $0 = t_0 < t_1 < \cdots < t_K = T$ and $j = 1, 2, \ldots, K$, where $\triangle t_{k+1} = t_{k+1} - t_k$. Assume that $g_0$ is positive and the sequence $\left\{\varphi_j\right\}$ satisfies

$$\begin{cases} \varphi_0 \leq g_0, \\ \varphi_j \leq \displaystyle\sum_{k=0}^{j-1} a_{k,j}\varphi_k + g_0, \end{cases}$$

then

$$\varphi_k \leq C g_0, \qquad j = 1, 2, \ldots, K.$$

For analyzing the stability and estimating the error for the discrete scheme (2.2) we introduce the discrete inner product and discrete $L_2$-norm $\|\cdot\|$ as follows

$$\langle v, w\rangle = h \sum_{i=0}^{N} v_i w_i \quad \text{and} \quad \|v\| = \sqrt{\langle v, v\rangle},$$

where $v$ and $w$ are two mesh functions. We also introduce the notation $\delta_x v_i = \frac{v_i - v_{i-1}}{h}$ for $1 \leq i \leq N$.

By using the technique given in [34, Theorem 3.2] we will derive the following stability result of the discrete scheme (2.2).

**Theorem 2.4** Let $\left\{U_i^j \,|0 \leq i \leq N,\ 1 \leq j \leq K\right\}$ be the solution of the discrete scheme (2.2). Then the numerical solution $U$ satisfies the following estimates

$$\left\|U^j\right\| \leq C\left(\max_{0\leq k\leq j}\left\|f^k\right\| + \|\phi_1\| + \|\phi_2\|\right), \qquad 1 \leq j \leq K,$$

where $C$ is a positive constant independent of $N$ and $K$.

*Proof.* First we rewrite (2.2) as the following equation

$$\begin{aligned} &U_i^j - U_i^{j-1} + \frac{1}{2} a \triangle t_j \left(U_i^{j-1} + U_i^j\right) = \triangle t_j \left(a\phi_{1,i} + \phi_{2,i}\right) \\ &+\frac{1}{2}\triangle t_j \sum_{k=1}^{j-1} \left[\xi_{j-1,k}\left(c\delta_x^2 U_i^{k-1} - bU_i^{k-1} + f_i^{k-1}\right) + \eta_{j-1,k}\left(c\delta_x^2 U_i^k - bU_i^k + f_i^k\right)\right] \\ &+\frac{1}{2}\triangle t_j \sum_{k=1}^{j} \left[\xi_{j,k}\left(c\delta_x^2 U_i^{k-1} - bU_i^{k-1} + f_i^{k-1}\right) + \eta_{j,k}\left(c\delta_x^2 U_i^k - bU_i^k + f_i^k\right)\right] \end{aligned} \tag{2.12}$$

for $1 \leq i < N$ and $1 \leq j \leq K$. Then, taking the inner produce of (2.12) with $U_i^j$, we can get

$$\begin{aligned} &\left\|U^j\right\|^2 - \langle U^{j-1}, U^j\rangle + \frac{1}{2} a\triangle t_j \langle U^{j-1}, U^j\rangle + \frac{1}{2} a \triangle t_j \left\|U^j\right\|^2 \\ &= \frac{1}{2}\triangle t_j \sum_{k=1}^{j-1} \left[\xi_{j-1,k}\left(\langle c\delta_x^2 U^{k-1}, U^j\rangle - \langle bU^{k-1}, U^j\rangle + \langle f^{k-1}, U^j\rangle\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+\eta_{j-1,k}\left(\langle c\delta_x^2 U^k, U^j\rangle - \langle bU^k, U^j\rangle + \langle f^k, U^j\rangle\right)\Big]\\
&+\frac{1}{2}\triangle t_j \sum_{k=1}^{j}\Big[\xi_{j,k}\left(\langle c\delta_x^2 U^{k-1}, U^j\rangle - \langle bU^{k-1}, U^j\rangle + \langle f^{k-1}, U^j\rangle\right)\\
&+\eta_{j,k}\left(\langle c\delta_x^2 U^k, U^j\rangle - \langle bU^k, U^j\rangle + \langle f^k, U^j\rangle\right)\Big] + \triangle t_j\left(a\langle \phi_1, U^j\rangle + \langle \phi_2, U^j\rangle\right).
\end{aligned}
$$

By recursion, we also can get the following equations

$$
\begin{aligned}
&\left\|U^{j-1}\right\|^2 - \langle U^{j-2}, U^{j-1}\rangle + \frac{1}{2}a\triangle t_{j-1}\langle U^{j-2}, U^{j-1}\rangle + \frac{1}{2}a\triangle t_{j-1}\left\|U^{j-1}\right\|^2\\
&= \frac{1}{2}\triangle t_{j-1}\sum_{k=1}^{j-2}\Big[\xi_{j-2,k}\left(\langle c\delta_x^2 U^{k-1}, U^{j-1}\rangle - \langle bU^{k-1}, U^{j-1}\rangle + \langle f^{k-1}, U^{j-1}\rangle\right)\\
&\quad+\eta_{j-2,k}\left(\langle c\delta_x^2 U^k, U^{j-1}\rangle - \langle bU^k, U^{j-1}\rangle + \langle f^k, U^{j-1}\rangle\right)\Big]\\
&\quad+\frac{1}{2}\triangle t_{j-1}\sum_{k=1}^{j-1}\Big[\xi_{j-1,k}\left(\langle c\delta_x^2 U^{k-1}, U^{j-1}\rangle - \langle bU^{k-1}, U^{j-1}\rangle + \langle f^{k-1}, U^{j-1}\rangle\right)\\
&\quad+\eta_{j-1,k}\left(\langle c\delta_x^2 U^k, U^{j-1}\rangle - \langle bU^k, U^{j-1}\rangle + \langle f^k, U^{j-1}\rangle\right)\Big]\\
&\quad+\triangle t_{j-1}\left(a\langle \phi_1, U^{j-1}\rangle + \langle \phi_2, U^{j-1}\rangle\right),\\
&\qquad\qquad\vdots\\
&\left\|U^2\right\|^2 - \langle U^1, U^2\rangle + \frac{1}{2}a\triangle t_2\langle U^1, U^2\rangle + \frac{1}{2}a\triangle t_2\left\|U^2\right\|^2\\
&= \frac{1}{2}\triangle t_2\sum_{k=1}^{1}\Big[\xi_{1,k}\left(\langle c\delta_x^2 U^{k-1}, U^2\rangle - \langle bU^{k-1}, U^2\rangle + \langle f^{k-1}, U^2\rangle\right)\\
&\quad+\eta_{1,k}\left(\langle c\delta_x^2 U^k, U^2\rangle - \langle bU^k, U^2\rangle + \langle f^k, U^2\rangle\right)\Big]\\
&\quad+\frac{1}{2}\triangle t_2\sum_{k=1}^{2}\Big[\xi_{2,k}\left(\langle c\delta_x^2 U^{k-1}, U^2\rangle - \langle bU^{k-1}, U^2\rangle + \langle f^{k-1}, U^2\rangle\right)\\
&\quad+\eta_{2,k}\left(\langle c\delta_x^2 U^k, U^2\rangle - \langle bU^k, U^2\rangle + \langle f^k, U^2\rangle\right)\Big] + \triangle t_2\left(a\langle \phi_1, U^2\rangle + \langle \phi_2, U^2\rangle\right),\\
&\left\|U^1\right\|^2 - \langle U^0, U^1\rangle + \frac{1}{2}a\triangle t_1\langle U^0, U^1\rangle + \frac{1}{2}a\triangle t_1\left\|U^1\right\|^2\\
&= \frac{1}{2}\triangle t_1\sum_{k=1}^{1}\Big[\xi_{1,k}\left(\langle c\delta_x^2 U^{k-1}, U^1\rangle - \langle bU^{k-1}, U^1\rangle + \langle f^{k-1}, U^1\rangle\right)\\
&\quad+\eta_{1,k}\left(\langle c\delta_x^2 U^k, U^1\rangle - \langle bU^k, U^1\rangle + \langle f^k, U^1\rangle\right)\Big] + \triangle t_1\left(a\langle \phi_1, U^1\rangle + \langle \phi_2, U^1\rangle\right).
\end{aligned}
$$

Applying the inequality

$$\langle v, w\rangle \le \|v\|\cdot\|w\| \le \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|^2$$

and the equality

$$\langle \delta_x^2 v, w\rangle = -\langle \delta_x v, \delta_x w\rangle,$$

we have

$$
\begin{aligned}
&\left\|U^j\right\|^2 - \frac{1}{2}\left\|U^j\right\|^2 - \frac{1}{2}\left\|U^{j-1}\right\|^2 + \frac{1}{2}a\triangle t_j\langle U^{j-1}, U^j\rangle + \frac{1}{2}a\triangle t_j\left\|U^j\right\|^2 \\
&\leq \frac{1}{2}\triangle t_j \sum_{k=1}^{j-1}\Big[\xi_{j-1,k}\left(-\langle c\delta_x U^{k-1}, \delta_x U^j\rangle - \langle bU^{k-1}, U^j\rangle + \langle f^{k-1}, U^j\rangle\right) \\
&\quad +\eta_{j-1,k}\left(-\langle c\delta_x U^{k}, \delta_x U^j\rangle - \langle bU^{k}, U^j\rangle + \langle f^{k}, U^j\rangle\right)\Big] \\
&\quad +\frac{1}{2}\triangle t_j \sum_{k=1}^{j}\Big[\xi_{j,k}\left(-\langle c\delta_x U^{k-1}, \delta_x U^j\rangle - \langle bU^{k-1}, U^j\rangle + \langle f^{k-1}, U^j\rangle\right) \\
&\quad +\eta_{j,k}\left(-\langle c\delta_x U^{k}, \delta_x U^j\rangle - \langle bU^{k}, U^j\rangle + \langle f^{k}, U^j\rangle\right)\Big] + \triangle t_j\left(a\langle \phi_1, U^j\rangle + \langle \phi_2, U^j\rangle\right), \\
&\left\|U^{j-1}\right\|^2 - \frac{1}{2}\left\|U^{j-1}\right\|^2 - \frac{1}{2}\left\|U^{j-2}\right\|^2 + \frac{1}{2}a\triangle t_{j-1}\langle U^{j-2}, U^{j-1}\rangle + \frac{1}{2}a\triangle t_{j-1}\left\|U^{j-1}\right\|^2 \\
&\leq \frac{1}{2}\triangle t_{j-1} \sum_{k=1}^{j-2}\Big[\xi_{j-2,k}\left(-\langle c\delta_x U^{k-1}, \delta_x U^{j-1}\rangle - \langle bU^{k-1}, U^{j-1}\rangle + \langle f^{k-1}, U^{j-1}\rangle\right) \\
&\quad +\eta_{j-2,k}\left(-\langle c\delta_x U^{k}, \delta_x U^{j-1}\rangle - \langle bU^{k}, U^{j-1}\rangle + \langle f^{k}, U^{j-1}\rangle\right)\Big] \\
&\quad +\frac{1}{2}\triangle t_{j-1} \sum_{k=1}^{j-1}\Big[\xi_{j-1,k}\left(-\langle c\delta_x U^{k-1}, \delta_x U^{j-1}\rangle - \langle bU^{k-1}, U^{j-1}\rangle + \langle f^{k-1}, U^{j-1}\rangle\right) \\
&\quad +\eta_{j-1,k}\left(-\langle c\delta_x U^{k}, \delta_x U^{j-1}\rangle - \langle bU^{k}, U^{j-1}\rangle + \langle f^{k}, U^{j-1}\rangle\right)\Big] \\
&\quad +\triangle t_{j-1}\left(a\langle \phi_1, U^{j-1}\rangle + \langle \phi_2, U^{j-1}\rangle\right), \\
&\qquad\qquad \vdots \\
&\left\|U^2\right\|^2 - \frac{1}{2}\left\|U^2\right\|^2 - \frac{1}{2}\left\|U^{1}\right\|^2 + \frac{1}{2}a\triangle t_2\langle U^{1}, U^2\rangle + \frac{1}{2}a\triangle t_2\left\|U^2\right\|^2 \\
&\leq \frac{1}{2}\triangle t_2 \sum_{k=1}^{1}\Big[\xi_{1,k}\left(-\langle c\delta_x U^{k-1}, \delta_x U^2\rangle - \langle bU^{k-1}, U^2\rangle + \langle f^{k-1}, U^2\rangle\right) \\
&\quad +\eta_{1,k}\left(-\langle c\delta_x U^{k}, \delta_x U^2\rangle - \langle bU^{k}, U^2\rangle + \langle f^{k}, U^2\rangle\right)\Big] \\
&\quad +\frac{1}{2}\triangle t_2 \sum_{k=1}^{2}\Big[\xi_{2,k}\left(-\langle c\delta_x U^{k-1}, \delta_x U^2\rangle - \langle bU^{k-1}, U^2\rangle + \langle f^{k-1}, U^2\rangle\right) \\
&\quad +\eta_{2,k}\left(-\langle c\delta_x U^{k}, \delta_x U^2\rangle - \langle bU^{k}, U^2\rangle + \langle f^{k}, U^2\rangle\right)\Big] + \triangle t_2\left(a\langle \phi_1, U^2\rangle + \langle \phi_2, U^2\rangle\right), \\
&\left\|U^1\right\|^2 - \frac{1}{2}\left\|U^1\right\|^2 - \frac{1}{2}\left\|U^{0}\right\|^2 + \frac{1}{2}a\triangle t_1\langle U^{0}, U^1\rangle + \frac{1}{2}a\triangle t_1\left\|U^1\right\|^2 \\
&= \frac{1}{2}\triangle t_1 \sum_{k=1}^{1}\Big[\xi_{1,k}\left(-\langle c\delta_x U^{k-1}, \delta_x U^1\rangle - \langle bU^{k-1}, U^1\rangle + \langle f^{k-1}, U^1\rangle\right) \\
&\quad +\eta_{1,k}\left(-\langle c\delta_x U^{k}, \delta_x U^1\rangle - \langle bU^{k}, U^1\rangle + \langle f^{k}, U^1\rangle\right)\Big] + \triangle t_1\left(a\langle \phi_1, U^1\rangle + \langle \phi_2, U^1\rangle\right).
\end{aligned}
$$

Adding up the above inequalities we can obtain

$$\begin{aligned}
\left\|U^j\right\|^2 \leq\ & 2\sum_{p=1}^{j}\sum_{k=1}^{p-1}\triangle t_p\Big[\xi_{p-1,k}\left(-\langle c\delta_x U^{k-1},\delta_x U^p\rangle-\langle bU^{k-1},U^p\rangle+\langle f^{k-1},U^p\rangle\right)\\
&+\eta_{p-1,k}\left(-\langle c\delta_x U^{k},\delta_x U^p\rangle-\langle bU^{k},U^p\rangle+\langle f^{k},U^p\rangle\right)\Big]\\
&+2\sum_{p=1}^{j}\sum_{k=1}^{p}\triangle t_p\Big[\xi_{p,k}\left(-\langle c\delta_x U^{k-1},\delta_x U^p\rangle-\langle bU^{k-1},U^p\rangle+\langle f^{k-1},U^p\rangle\right)\\
&+\eta_{p,k}\left(-\langle c\delta_x U^{k},\delta_x U^p\rangle-\langle bU^{k},U^p\rangle+\langle f^{k},U^p\rangle\right)\Big]\\
&-\sum_{p=1}^{j}\triangle t_p a\langle U^{p-1},U^p\rangle+2\sum_{p=1}^{j}\triangle t_p\left(a\langle\phi_1,U^p\rangle+\langle\phi_2,U^p\rangle\right)+\left\|U^0\right\|^2\\
\leq\ & 2\sum_{p=1}^{j}\sum_{k=1}^{p-1}\triangle t_p\left\|U^p\right\|\Big[\xi_{p-1,k}\left(b\left\|U^{k-1}\right\|+\left\|f^{k-1}\right\|\right)+\eta_{p-1,k}\left(b\left\|U^{k}\right\|+\left\|f^{k}\right\|\right)\Big]\\
&+2\sum_{p=1}^{j}\sum_{k=1}^{p}\triangle t_p\left\|U^p\right\|\Big[\xi_{p,k}\left(b\left\|U^{k-1}\right\|+\left\|f^{k-1}\right\|\right)+\eta_{p,k}\left(b\left\|U^{k}\right\|+\left\|f^{k}\right\|\right)\Big]\\
&+2\sum_{p=1}^{j}\triangle t_p\left\|U^p\right\|\left(a\left\|U^{p-1}\right\|+a\left\|\phi_1\right\|+\left\|\phi_2\right\|\right)+\left\|U^0\right\|^2,
\end{aligned}$$

where we have used Lemmas 2.1 and 2.2. From the above inequality we have

$$\begin{aligned}
\left\|U^j\right\|^2 \leq\ & C\max_{0\leq p\leq j}\left\|U^p\right\|\Bigg[\sum_{k=1}^{j-1}\left(\xi_{j-1,k+1}+\eta_{j-1,k}+\xi_{j,k+1}+\eta_{j,k}\right)\left(\left\|U^k\right\|+\max_{0\leq k\leq j}\left\|f^k\right\|\right)\\
&+\sum_{k=1}^{j-1}\triangle t_{k+1}a\left\|U^k\right\|+\left\|\phi_1\right\|+\left\|\phi_2\right\|\Bigg],
\end{aligned}\tag{2.13}$$

where $\xi_{j-1,j}=0$. For each $j$, there exists $j^*(1\leq j^*\leq j)$ such that

$$\left\|U^{j^*}\right\|=\max_{0\leq p\leq j}\left\|U^p\right\|.$$

Since $j$ in (2.13) is any integer from 0 to $N$, we have

$$\begin{aligned}
\left\|U^{j^*}\right\| \leq\ & C\Bigg[\sum_{k=1}^{j^*-1}\left(\xi_{j^*-1,k+1}+\eta_{j^*-1,k}+\xi_{j^*,k+1}+\eta_{j^*,k}\right)\left(\left\|U^k\right\|+\max_{0\leq k\leq j}\left\|f^k\right\|\right)\\
&+\sum_{k=1}^{j^*-1}\triangle t_{k+1}a\left\|U^k\right\|+\left\|\phi_1\right\|+\left\|\phi_2\right\|\Bigg].
\end{aligned}$$

Combining the above inequality with Lemma 2.2 we have

$$\left\|U^j\right\| \leq \Bigg[\sum_{k=1}^{j-1}\left(\xi_{j-1,k+1}+\eta_{j-1,k}+\xi_{j,k+1}+\eta_{j,k}\right)\left(\left\|U^k\right\|+\max_{0\leq k\leq j}\left\|f^k\right\|\right)$$

$$+\sum_{k=1}^{j-1}\triangle t_{k+1}a\left\|U^k\right\|+\|\phi_1\|+\|\phi_2\|\Bigg] \tag{2.14}$$

for $1 \le j^* \le j \le K$. Furthermore, we have

$$\begin{aligned}
\xi_{j,k+1}+\eta_{j,k} &= \frac{1}{\Gamma(\gamma+1)}\left\{\frac{1}{\Delta t_k}\left[\left(t_j-t_{k-1}\right)^{\gamma}-\left(t_j-t_k\right)^{\gamma}\right]-\frac{1}{\Delta t_{k+1}}\left[\left(t_j-t_k\right)^{\gamma}-\left(t_j-t_{k+1}\right)^{\gamma}\right]\right\}\\
&= \frac{1}{\Gamma(\gamma)}\left[\left(t_j-\theta_k\right)^{\gamma-1}-\left(t_j-\theta_{k+1}\right)^{\gamma-1}\right]\\
&\le \frac{2}{\Gamma(\gamma)}\triangle t_{k+1}\left(t_j-t_{k+1}\right)^{\gamma-2}\\
&\le \frac{2^{3-\gamma}}{\Gamma(\gamma)}\triangle t_{k+1}\left(t_j-t_k\right)^{\gamma-2},
\end{aligned} \tag{2.15}$$

where we have used the mean value theorem with $\theta_k \in (t_{k-1}, t_k)$ and the inequality $n \le 2(n-1)$ for $n \ge 2$. Combining (2.14) and (2.15) we can get

$$\begin{aligned}
\left\|U^j\right\| &\le C\left\{\sum_{k=1}^{j-1}\triangle t_{k+1}\left[1+\left(t_{j-1}-t_k\right)^{\gamma-2}+\left(t_j-t_k\right)^{\gamma-2}\right]\left(\left\|U^k\right\|+\max_{0\le k\le j}\left\|f^k\right\|\right)+\|\phi_1\|+\|\phi_2\|\right\}\\
&\le C\left(\max_{0\le k\le j}\left\|f^k\right\|+\|\phi_1\|+\|\phi_2\|\right)
\end{aligned}$$

for $1 \le j \le K$, where Lemma 2.3 has been used. This inequality implies the theorem holds true. □

Next we derive the error estimates for the integral-difference discretization scheme (2.2).

**Theorem 2.5** Under the regularity conditions (1.4) and (1.5) and the assumption (1.3), we have the following error estimates

$$\left\|U^j-u^j\right\| \le C\left(N^{-2}+K^{-2}\right), \qquad 1 \le j \le K,$$

where $C$ is a positive constant independent of $N$ and $K$.

*Proof.* Applying Theorem 2.4 to the error equation (2.5) we can derive

$$\left\|U^j\right\| \le C\max_{0\le k\le j}\left\|R^k\right\| \le C\left(N^{-2}+K^{-2}\right),$$

where we have used the estimates (2.6). From this we complete the proof. □

## 3. Numerical experiments

In this section we present some numerical results to indicate experimentally the efficiency and accuracy of the integral-difference discretization scheme. Errors and convergence rates for the integral-difference discretization scheme are presented for two examples.

**Example 4.1** We first consider the TFTE (1.1) with $a = b = 1, c = \pi, \phi_1(x) = \phi_2(x) = \psi_1(t) = 0$ and $\psi_2(t) = t^3\sin^2(1)$, where $f(x,t)$ is chosen such that the exact solution is $u(x,t) = t^3\sin^2(x)$. This equation has been tested in [2, 3].

We measure the accuracy in the discrete $L_2$-norm and $L_\infty$-norm

$$e_{L_2}^{N,K} = \max_{1\le j\le K} \left\| u^j - U^j \right\|, \qquad e_{L_\infty}^{N,K} = \max_{0\le i\le N, 0\le j\le K} \left| u_i^j - U_i^j \right|,$$

and the convergence rate

$$r_{L_2}^{N,K} = \log_2(e_{L_2}^{N,K}/e_{L_2}^{2N,2K}), \qquad r_{L_\infty}^{N,K} = \log_2(e_{L_\infty}^{N,K}/e_{L_\infty}^{2N,2K}),$$

respectively. In order to further confirm that the convergence rate in the time direction is consistent with the theoretical convergence rate, we also measure the convergence rates by fixing a large $N$ as follows

$$\bar{r}_{L_2}^{N,K} = \log_2(e_{L_2}^{N,K}/e_{L_2}^{N,2K}), \qquad \bar{r}_{L_\infty}^{N,K} = \log_2(e_{L_\infty}^{N,K}/e_{L_\infty}^{N,2K}),$$

respectively. The numerical results for Example 4.1 are tabulated in Tables 1 and 2.

**Table 1.** Error estimates $e_{L_2}^{N,K}, e_{L_\infty}^{N,K}$ and convergence rates $r_{L_2}^{N,K}, r_{L_\infty}^{N,K}$ for Example 4.1.

| $\gamma$ | norm | Number of mesh points $(K, N)$ | | | | |
|---|---|---|---|---|---|---|
| | | $(32, 32)$ | $(64, 64)$ | $(128, 128)$ | $(256, 256)$ | $(512, 512)$ |
| 1.2 | $L_2$ | 5.9593e-4 | 1.6516e-4 | 4.4234e-5 | 1.1648e-5 | 3.0354e-6 |
| | | 1.851 | 1.901 | 1.925 | 1.940 | - |
| | $L_\infty$ | 8.5299e-4 | 2.3639e-4 | 6.3289e-5 | 1.6662e-5 | 4.3412e-6 |
| | | 1.851 | 1.901 | 1.925 | 1.940 | - |
| 1.4 | $L_2$ | 1.9673e-4 | 5.1561e-5 | 1.3295e-5 | 3.3967e-6 | 8.6265e-7 |
| | | 1.932 | 1.955 | 1.969 | 1.977 | - |
| | $L_\infty$ | 2.8577e-4 | 7.4813e-5 | 1.9284e-5 | 4.9245e-6 | 1.2503e-6 |
| | | 1.934 | 1.956 | 1.969 | 1.978 | - |
| 1.6 | $L_2$ | 8.3665e-5 | 2.1305e-5 | 5.3828e-6 | 1.3545e-6 | 3.4004e-7 |
| | | 1.973 | 1.985 | 1.991 | 1.994 | - |
| | $L_\infty$ | 1.2489e-4 | 3.1809e-5 | 8.0322e-6 | 2.0206e-6 | 5.0714e-7 |
| | | 1.973 | 1.986 | 1.991 | 1.994 | - |
| 1.8 | $L_2$ | 3.8982e-5 | 9.7671e-6 | 2.4448e-6 | 6.1165e-7 | 1.5298e-7 |
| | | 1.997 | 1.998 | 1.999 | 1.999 | - |
| | $L_\infty$ | 5.6986e-5 | 1.4245e-5 | 3.5643e-6 | 8.9169e-7 | 2.2303e-7 |
| | | 2.000 | 1.999 | 1.999 | 1.999 | - |

**Table 2.** Error estimates $e_{L_2}^{N,K}, e_{L_\infty}^{N,K}$ and convergence rates $\bar{r}_{L_2}^{N,K}, \bar{r}_{L_\infty}^{N,K}$ for Example 4.1 with $N = 1024$.

| $\gamma$ | norm | Number of mesh points $K$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 32 | 64 | 128 | 256 | 512 | 1024 |
| 1.2 | $L_2$ | 6.1879e-4 | 1.7080e-4 | 4.5625e-5 | 1.1979e-5 | 3.1016e-6 | 7.8528e-7 |
| | | 1.857 | 1.904 | 1.929 | .949 | 1.982 | - |
| | $L_\infty$ | 8.8009e-4 | 2.4294e-4 | 6.4901e-5 | 1.7044e-5 | 4.4176e-6 | 1.1229e-6 |
| | | 1.857 | 1.904 | 1.929 | 1.948 | 1.976 | - |
| 1.4 | $L_2$ | 2.1850e-4 | 5.6970e-5 | 1.4630e-5 | 3.7144e-6 | 9.2607e-7 | 2.1819e-7 |
| | | 1.939 | 1.961 | 1.978 | 2.004 | 2.086 | - |
| | $L_\infty$ | 3.1067e-4 | 8.1008e-5 | 2.0808e-5 | 5.2871e-6 | 1.3225e-6 | 3.1617e-7 |
| | | 1.939 | 1.961 | 1.977 | 1.999 | 2.064 | - |
| 1.6 | $L_2$ | 1.0424e-4 | 2.6426e-5 | 6.6476e-6 | 1.6552e-6 | 3.9985e-7 | 8.5239e-8 |
| | | 1.980 | 1.991 | 2.006 | 2.050 | 2.230 | - |
| | $L_\infty$ | 1.4812e-4 | 3.7555e-5 | 9.4509e-6 | 2.3574e-6 | 5.7372e-7 | 1.2711e-7 |
| | | 1.980 | 1.990 | 2.003 | 2.039 | 2.174 | - |
| 1.8 | $L_2$ | 5.2400e-5 | 1.3135e-5 | 3.2750e-6 | ,8.0479e-7 | 1.8676e-7 | 3.8256e-8 |
| | | 1.996 | 2.004 | 2.025 | 2.107 | 2.287 | - |
| | $L_\infty$ | 7.4474e-5 | 1.8678e-5 | 4.6619e-6 | 1.1498e-6 | 2.7124e-7 | 5.5771e-8 |
| | | 1.995 | 2.002 | 2.020 | 2.084 | 2.282 | - |

**Table 3.** Error estimates $e_{L_2}^{N,K}, e_{L_\infty}^{N,K}$ and convergence rates $r_{L_2}^{N,K}, r_{L_\infty}^{N,K}$ for Example 4.2.

| $\gamma$ | norm | Number of mesh points $(K, N)$ | | | | |
|---|---|---|---|---|---|---|
| | | $(32, 32)$ | $(64, 64)$ | $(128, 128)$ | $(256, 256)$ | $(512, 512)$ |
| 1.2 | $L_2$ | 1.1821e-4 | 2.9391e-5 | 7.3519e-6 | 1.8372e-6 | 4.5927e-7 |
| | | 2.008 | 1.999 | 2.001 | 2.000 | - |
| | $L_\infty$ | 1.6019e-4 | 3.9819e-5 | 9.9670e-6 | 2.4906e-6 | 6.2257e-7 |
| | | 2.008 | 1.998 | 2.001 | 2.000 | - |
| 1.4 | $L_2$ | 6.6529e-5 | .6604e-5 | 4.1490e-6 | 1.0373e-6 | 2.5931e-7 |
| | | 2.002 | 2.001 | 2.000 | 2.000 | - |
| | $L_\infty$ | 9.1346e-5 | 2.2775e-5 | 5.6898e-6 | 1.4225e-6 | 3.5562e-7 |
| | | 2.004 | 2.001 | 2.000 | 2.000 | - |
| 1.6 | $L_2$ | 4.2340e-5 | 1.0580e-5 | 2.6446e-6 | 6.6114e-7 | 1.6528e-7 |
| | | 2.001 | 2.000 | 2.000 | 2.000 | - |
| | $L_\infty$ | 6.0023e-5 | 1.4954e-5 | 3.7358e-6 | 9.3402e-7 | 2.3349e-7 |
| | | 2.005 | 2.001 | 2.000 | 2.000 | - |
| 1.8 | $L_2$ | 2.1727e-5 | 5.4316e-6 | 1.3587e-6 | 3.3967e-7 | 8.4917e-8 |
| | | 2.000 | 1.999 | 2.000 | 2.000 | - |
| | $L_\infty$ | 3.2980e-5 | 8.2122e-6 | 2.0523e-6 | 5.1287e-7 | 1.2821e-7 |
| | | 2.006 | 2.001 | 2.001 | 2.000 | - |

**Example 4.2** We now consider the TFTE (1.1) with $a = b = c = 1, \phi_1(x) = \phi_2(x) = 0$, $\psi_1(t) = \psi_2(t) = t^\gamma$, where $f(x,t)$ is chosen such that the exact solution is $u(x,t) = t^\gamma \left(x^2 - x + 1\right)$. We also measure the accuracy in the discrete $L_2$-norm $e_{L_2}^{N,K}$, $L_\infty$-norm $e_{L_\infty}^{N,K}$ and the convergence rates $r_{L_2}^{N,K}, r_{L_\infty}^{N,K}, \bar{r}_{L_2}^{N,K}, \bar{r}_{L_\infty}^{N,K}$ as previously defined, respectively. The numerical results for Example 4.2 are tabulated in Tables 3 and 4.

**Table 4.** Error estimates $e_{L_2}^{N,K}, e_{L_\infty}^{N,K}$ and convergence rates $\bar{r}_{L_2}^{N,K}, \bar{r}_{L_\infty}^{N,K}$ for Example 4.2 with $N = 1024$.

| $\gamma$ | norm | Number of mesh points $K$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 32 | 64 | 128 | 256 | 512 | 1024 |
| 1.2 | $L_2$ | 1.1827e-4 | 2.9395e-5 | 7.3521e-6 | 1.8372e-6 | 4.5927e-7 | 1.1482e-7 |
| | | 2.008 | 1.999 | 2.001 | 2.000 | 2.000 | - |
| | $L_\infty$ | 1.6027e-4 | 3.9823e-5 | 9.9673e-6 | 2.4906e-6 | 6.2257e-7 | 1.5564e-7 |
| | | 2.009 | 1.998 | 2.001 | 2.000 | 2.000 | - |
| 1.4 | $L_2$ | 6.6566e-5 | 1.6606e-5 | 4.1491e-6 | 1.0373e-6 | 2.5931e-7 | 6.4828e-8 |
| | | 2.003 | 2.001 | 2.000 | 2.000 | 2.000 | - |
| | $L_\infty$ | 9.1393e-5 | 2.2778e-5 | 5.6900e-6 | 1.4225e-6 | 3.5562e-7 | 8.8905e-8 |
| | | 2.004 | 2.001 | 2.000 | 2.000 | 2.000 | - |
| 1.6 | $L_2$ | 4.2366e-5 | 1.0582e-5 | 2.6447e-6 | 6.6114e-7 | 1.6528e-7 | 4.1321e-8 |
| | | 2.001 | 2.000 | 2.000 | 2.000 | 2.000 | - |
| | $L_\infty$ | 6.0036e-5 | 1.4956e-5 | 3.7359e-6 | 9.3403e-7 | 2.3350e-7 | 5.8373e-8 |
| | | 2.005 | 2.001 | 2.000 | 2.000 | 2.000 | - |
| 1.8 | $L_2$ | 2.1742e-5 | 5.4325e-6 | 1.3587e-6 | ,3.3967e-7 | 8.4917e-8 | 2.1229e-8 |
| | | 2.001 | 1.999 | 2.000 | 2.000 | 2.000 | - |
| | $L_\infty$ | 3.3046e-5 | 8.2120e-6 | 2.0523e-6 | 5.1287e-7 | 1.2821e-7 | 3.2051e-8 |
| | | 2.009 | 2.001 | 2.001 | 2.000 | 2.000 | - |

Tables 1–4 show that the numerical solution of the integral-difference discretization scheme on a graded mesh converges to the exact solution with second-order accuracy for both the spatial discretization and the time discretization in the discrete $L_2$-norm and in the discrete $L_\infty$-norm, respectively. Moreover, compared with the previous methods with only first-order convergence for the time discretization, our discretization scheme improves the previous results given in [2, 3].

## 4. Conclusions and discussion

In this paper, the TFTE is transformed into an equivalent integral-differential equation with a weakly singular kernel by using the integral transformation. Then an integral-difference discretization scheme on a graded mesh is developed to approximate the integral-differential equation. The stability and convergence are proved by using the $L_2$-norm. The possible singularity of the exact solution is taken into account in the convergence analysis. It is shown that the scheme is second-order convergent for both the spatial discretization and the time discretization. The numerical experiments demonstrate the validity of our theoretical results and also verify that this scheme is more accurate than the methods given in [2, 3]. In future we will extend this method to variable-order fractional differential equations.

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## Conflict of interest

The authors declare there is no conflict of interest.

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