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Research article

Due-window assignment scheduling with past-sequence-dependent setup times

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Abstract: This article investigates the due-window assignment scheduling problem with setup times on a single machine, where setup times of jobs are past-sequence-dependent. Under common, slack and unrestricted due-window assignment methods, the goal is to determine the optimal job sequence and due-window such that the cost function (i.e., the weighted sum of earliness and tardiness, number of early and tardy jobs, due-window starting time and size) is minimized. We solve the problem optimally by introducing a polynomial time algorithm. An extension to the problem with learning and deterioration effects is also studied.

Keywords: scheduling; single-machine; due-window assignment; setup times

1. Introduction

Koulamas and Kyparisis [1] studied single machine problem with past-sequence-dependent setup times (denoted by psdst). They showed that the problem $1|psdst|\tilde{Z}$ can be solved in polynomial time, where $\tilde{Z} \in \{\tilde{C}_{\max} = \max\{\tilde{C}_l, l = 1, 2, \ldots, n\}, \sum_{l=1}^{n} \tilde{C}_l, \sum_{l=1}^{n}\sum_{j=l}^{n} |\tilde{C}_j - \tilde{C}_l|\}$ and $\tilde{C}_l$ is the completion time of job J_l. Biskup and Herrmann [2] showed that the problem $1|psdst|\sum_{l=1}^{n} \tilde{L}_l$ can be solved by the SPT (smallest processing time) rule, where $\tilde{L}_l = \tilde{C}_l - d_l$ and d_l is the due date of job $\tilde{J}_l$. Koulamas and Kyparisis [3] showed that the problem $1|psdst|\tilde{Z}$ can be solved in $O(n^2)$ time, where $\tilde{Z} \in \{\tilde{L}_{\max} = \max\{\tilde{C}_l - d_l, l = 1, 2, \ldots, n\}, \tilde{T}_{\max} = \max\{\tilde{T}_l, l = 1, 2, \ldots, n\}\}$ and $\tilde{T}_l = \max\{0, \tilde{C}_l - d_l\}$. For the objective functions $\sum_{l=1}^{n} w_l\tilde{T}_l$ (total weighted tardiness) and $\sum_{l=1}^{n} w_l\tilde{U}_l$ (weighted number of tardy jobs), they proposed dynamic programming algorithms, where w_l is the weight of job $\tilde{J}_l$. Kuo and Yang [4], Wang et al. [5], and Mani et al. [6] studied single-machine scheduling with psdst. Under learning effects (i.e., the the actual processing time of $\tilde{J}_l$ if it is scheduled in position r is $p_l^A = p_l r^{\alpha}$, where $\alpha \leq 0$ is the learning rate), Kuo and Yang [4] proved that the problem $1|p_l^A = p_l r^{\alpha}, psdst|\tilde{Z}$ can be solved in polynomial time, where $\tilde{Z} \in \{\tilde{C}_{\max}, \sum_{l=1}^{n} \tilde{C}_l, \sum_{l=1}^{n}\sum_{j=l}^{n} |\tilde{C}_j - \tilde{C}_l|, \sum_{l=1}^{n} a\tilde{E}_l + b\tilde{T}_l + \gamma d_l\}$, a, b and γ are given non-negative constants. Wang et al. [5] showed that $1|p_l^A = p_l r^{\alpha}, psdst|\tilde{Z}$ can be solved in polynomial time, where $\tilde{Z} \in \{\sum_{l=1}^{n}\tilde{C}_l^2, \sum_{l=1}^{n}\tilde{W}_l, \sum_{l=1}^{n}\sum_{j=l}^{n}|\tilde{W}_j - \tilde{W}_l|\}$ and $\tilde{W}_l = \tilde{C}_l - P_l^A$ is waiting time of job $\tilde{J}_l$. Mani et al. [6] gave a parametric analysis of normalising index on the problem $1|p_l^A = p_l r^\alpha, psdst|\sum_{l=1}^{n}\sum_{j=l}^{n}|\tilde{C}_j - \tilde{C}_l|$ for a given value of a constant learning index. Under job-dependent learning effects, Soroush [7] considered a bicriteria single machine scheduling problem with psdst. Wang et al. [8] studied common and slack due-date assignment scheduling problems with psdst. They showed that a non-regular objective function minimization remains polynomially solvable. For new trends in scheduling problems with setup times/costs, please refer to Allahverdi [9], Pei et al. [10], Muştu and Eren [11] and Weerdt et al. [12].

Recently, Wang [13] considered the due-date assignment problem, the objective is to minimize $\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d_l\right)$, where θ_l and δ_l are given non-negative constants. Under common (CON), slack (SLK) and different (unrestricted, DIF) due-date assignment methods, they showed that $1|CON/SLK/DIF, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d_l\right)$ remains polynomially solvable. The due-date assignment assumes that each job has a due-date, which is a point in time. However, in practice, the completion of a job (task) is often acceptable without penalty within a time interval, this is the due-window (DW) assignment scheduling. DW assignment has become an important topic in Just-In-Time manufacturing philosophy (see review papers Janiak et al. [14] and Rolim and Nagano [15]), i.e., the starting time and size of the DW can be adjusted to increase the flexibility of production and attractiveness to customers. For example, the process of negotiation between the producer and the customer about the delivery time of the final products, the producers are responsible for quoting the starting time of the DW and the window width to their customers. Both variables (i.e., the starting time and size of the DW) are penalised at costs that represent the industries' competitiveness (see Liman et al. [16]).

In this article, we address single-machine scheduling simultaneously with psdst and common (slack, unrestricted) due-window assignment. The purpose is to identify an optimal job sequence, starting time and size of DW, such that the generalized cost (penalty) is to be minimized. The contributions of this article are given as follows: 1) We study the DW assignment single-machine scheduling with psdst, where the objective is to minimize the generalized earliness and tardiness penalties (including earliness, tardiness, number of early and delayed, due-window starting time and size); 2) Some optimality properties are presented and showed that the problem remains polynomially solvable; 3) It is further extended the model to the case with learning and deterioration effects.

The rest of this paper is organized as follows. In Section 2, the problem is described. In Section 3, we analyze the optimal properties of the problem and present polynomial time algorithms for the methods of CONDW, SLKDW and DIFDW. In Section 4, we present extensive results to the model with learning and deterioration effects. We conclude the article in the last section.

2. Problem definition

We sate the problem under study as follows: A set of n independent and non-preemptive jobs $\widetilde{N} = \{\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_n\}$ needs to be processed on a single machine, and these jobs are available at time zero. Let $[l]$ be any job scheduled in the lth position, the psdst of job $\tilde{J}_{[l]}$ is: $s_{[l]} = \varsigma\sum_{k=1}^{l-1}p_{[k]}$, where $p_{[k]}$ is the processing time of job $\tilde{J}_{[k]}$, and $\varsigma(\varsigma > 0)$ is a given constant. The total processing requirement of job $\tilde{J}_{[l]}$ is $\varsigma\sum_{k=1}^{l-1}p_{[k]} + p_{[l]}$. The due-window of job $\tilde{J}_l$ is defined as follows: $\langle d'_l, d''_l\rangle$, where d'_l (d''_l) is the starting (finishing) time of the due-window, and $\tilde{D}_l = d''_l - d'_l$ is the due-window size of job $\tilde{J}_l$. The earliness (tardiness) of job J_l is given by $\tilde{E}_l = \max\{0, d'_l - \tilde{C}_l\}$ ($\tilde{T}_l = \max\{0, \tilde{C}_l - d''_l\}$). If job $\tilde{J}_l$ is early in a job sequence (i.e., $\tilde{C}_l < d'_l$), $\tilde{U}_l = 1$; otherwise, $\tilde{U}_l = 0$; if job J_l is tardy (i.e., $\tilde{C}_l > d''_l$), $\tilde{V}_l = 1$; otherwise, $\tilde{V}_l = 0$.

The common due-window (denoted by CONDW, see Huang et al. [17], Wang et al. [18] and Wang et al. [19]) assignment: $d'_l = d'$, and $d''_l = d''$, and $\tilde{D}_l = \tilde{D} = d'' - d'$. Our objective is to identify an optimal sequence ψ, d' and d'' to minimize

$$\tilde{G}_1(\psi, d', d'') = \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma d' + \omega \tilde{D} \right), \tag{1}$$

where a, b, θ_l, δ_l, γ and ω are given non-negative constants.

The slack due-window (denoted by SLKDW, see Wang et al. [19], Wang et al. [20], Yin et al. [21] and Yin et al. [22]) assignment: $d'_l = s_l + p_l + q'$, $d''_l = s_l + p_l + q''$, and $\tilde{D} = d''_l - d'_l = q'' - q'$, where q' and q'' are the common flow allowances (i.e., common due-window parameters) and $q^{\tilde{1}} \leq q^{\tilde{2}}$. Our objective is to identify an optimal sequence ψ, q' and q'' to minimize

$$\tilde{G}_2(\psi, q', q'') = \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma q' + \omega \tilde{D} \right). \tag{2}$$

The unrestricted due-window (denoted by DIFDW) assignment: it is assumed that there is a due-window $\langle d'_l, d''_l \rangle$ with no restrictions for job $\tilde{J}_l$. Our objective is to identify an optimal sequence ψ, d'_l and d''_l ($l = 1, 2, \ldots, n$) to minimize

$$\tilde{G}_3(\psi, d'_l, d''_l) = \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma d'_l + \omega \tilde{D}_l \right). \tag{3}$$

Denoting the above methods of the problem by

$$1|CONDW, psdst| \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma d' + \omega \tilde{D} \right), \tag{4}$$

$$1|SLKDW, psdst| \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma q' + \omega \tilde{D} \right) \tag{5}$$

and

$$1|DIFDW, psdst| \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma d'_l + \omega \tilde{D}_l \right). \tag{6}$$

Obviously, there exists an optimal sequence such that all the jobs are processed consecutively without idle time from time zero. For a given sequence $\psi = (\tilde{J}_{[1]}, \tilde{J}_{[2]}, \ldots, \tilde{J}_{[n]})$, as in Wang [13], we have

$$\tilde{C}_{[l]} = \sum_{k=1}^{l} (s_{[k]} + p_{[k]}) = \sum_{k=1}^{l} \left[1 + \varsigma(l - k)\right] p_{[k]}. \tag{7}$$

3. Main results

3.1. CONDW/SLKDW

Lemma 1. *For a given sequence ψ of the CONDW assignment method, the optimal values of d' and d'' coincide with the completion time of some job respectively, i.e., $d' = \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu]}$ $(\kappa \le \nu)$.*

Proof. Case (i): Let $\tilde{C}_{[\kappa-1]} < d' < \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu]}$, it follows that

$$\tilde{G}_1 = \sum_{l=1}^{\kappa-1} a(d' - \tilde{C}_{[l]}) + \sum_{l=\nu+1}^{n} b(\tilde{C}_{[l]} - d'') + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]} + n\gamma d' + n\omega(d'' - d').$$

If $d' = \tilde{C}_{[\kappa-1]}$ and $d'' = \tilde{C}_{[\nu]}$, we have

$$\tilde{G}_1' = \sum_{l=1}^{\kappa-1} a(\tilde{C}_{[\kappa-1]} - \tilde{C}_{[l]}) + \sum_{l=\nu+1}^{n} b(\tilde{C}_{[l]} - d'') + \sum_{l=1}^{\kappa-2} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]} + n\gamma \tilde{C}_{[\kappa-1]} + n\omega(d'' - \tilde{C}_{[\kappa-1]}).$$

If $d' = \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu]}$, we have

$$\tilde{G}_1'' = \sum_{l=1}^{\kappa-1} a(\tilde{C}_{[\kappa]} - \tilde{C}_{[l]}) + \sum_{l=\nu+1}^{n} b(\tilde{C}_{[l]} - d'') + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{j=\nu+1}^{n} \delta_{[j]} + n\gamma \tilde{C}_{[\kappa]} + n\omega(d'' - \tilde{C}_{[\kappa]}),$$

$$\begin{aligned} \tilde{G}_1 - \tilde{G}_1' &= (\kappa-1)a(d' - \tilde{C}_{[\kappa-1]}) + \theta_{[\kappa-1]} + n\gamma(d' - \tilde{C}_{[\kappa-1]}) - n\omega(d' - \tilde{C}_{[\kappa-1]}) \\ &= [(\kappa-1)a + n(\gamma - \omega)](d' - \tilde{C}_{[\kappa-1]}) + \theta_{[\kappa-1]} \end{aligned}$$

and

$$\begin{aligned} \tilde{G}_1 - \tilde{G}_1'' &= (\kappa-1)a(d' - \tilde{C}_{[\kappa]}) + n\gamma(d' - \tilde{C}_{[\kappa]}) - n\omega(d' - \tilde{C}_{[\kappa]}) \\ &= [(\kappa-1)a + n(\gamma - \omega)](d' - \tilde{C}_{[\kappa]}). \end{aligned}$$

If $(\kappa-1)a + n(\gamma - \omega) \ge 0$, then $\tilde{G}_1 \ge \tilde{G}_1'$, otherwise $\tilde{G}_1 \ge \tilde{G}_1''$. Hence, the due-window starting time d' is equal to the completion time of some job.

Case (ii): Let $d' = \tilde{C}_{[\kappa]}$ and $\tilde{C}_{[\nu-1]} < d'' < \tilde{C}_{[\nu]}$, it follows that

$$\tilde{G}_1 = \sum_{l=1}^{\kappa} a(\tilde{C}_{[\kappa]} - \tilde{C}_{[l]}) + \sum_{l=\nu}^{n} b(\tilde{C}_{[l]} - d'') + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu}^{n} \delta_{[l]} + n\gamma d' + n\omega(d'' - d').$$

When $d' = \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu-1]}$, it follows that

$$\tilde{G}_1' = \sum_{l=1}^{\kappa} a(\tilde{C}_{[\kappa]} - \tilde{C}_{[l]}) + \sum_{l=\nu}^{n} b(\tilde{C}_{[l]} - \tilde{C}_{[\nu-1]}) + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu}^{n} \delta_{[l]} + n\gamma d' + n\omega(\tilde{C}_{[\nu-1]} - d').$$

When $d' = \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu]}$, it follows that

$$\tilde{G}_1'' = \sum_{l=1}^{\kappa} a(\tilde{C}_{[\kappa]} - \tilde{C}_{[l]}) + \sum_{l=\nu}^{n} b(\tilde{C}_{[l]} - \tilde{C}_{[\nu]}) + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]} + n\gamma d' + n\omega(\tilde{C}_{[\nu]} - d'),$$

$$\tilde{G}_1 - \tilde{G}_1' = [n\omega - (n-\nu+1)b](d'' - \tilde{C}_{[\nu-1]})$$

and

$$\tilde{G}_1 - \tilde{G}_1'' = [n\omega - (n-\nu+1)b](d'' - \tilde{C}_{[\nu]}) + \delta_{[\nu]}.$$

If $n\omega - (n-\nu+1)b \geq 0$, then $\tilde{G}_1 \geq \tilde{G}_1'$, otherwise $\tilde{G}_1 \geq \tilde{G}_1''$. Thus, the due-window finishing time d'' is equal to the job completion time.

Case (iii): Similarly, let $\tilde{C}_{[\kappa-1]} < d' < \tilde{C}_{[\kappa]}$ and $\tilde{C}_{[\nu-1]} < d'' < \tilde{C}_{[\nu]}$. When d'' is shifted to the left (right) $d'' - \tilde{C}_{[\nu-1]}$ ($\tilde{C}_{[\nu]} - d''$) units of time until $d'' = \tilde{C}_{[\nu-1]}$ ($d'' = \tilde{C}_{[\nu]}$), this is Case i. When d' is shifted to the left (right) $d' - \tilde{C}_{[\kappa-1]}$ ($\tilde{C}_{[\kappa]} - d'$) units of time until $d' = \tilde{C}_{[\kappa]}$ ($d' = \tilde{C}_{[\kappa-1]}$), this is Case ii.

In summary, an optimal sequence exists in which d' and d'' are equal to the job completion time, respectively. □

Lemma 2. *For a given sequence ψ of the CONDW assignment method, the optimal values $d' = \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu]}$, where*

$$\kappa \leq \left\lceil \frac{n(\omega - \gamma)}{a} \right\rceil \tag{8}$$

and

$$\nu \geq \left\lceil \frac{n(b - \omega)}{b} \right\rceil. \tag{9}$$

Proof. This lemma can be easily proved by the standard technique of small perturbations. As given in Lemma 1, there is an sequence in which $d' = \tilde{C}_{[\kappa]}$, and $d'' = \tilde{C}_{[\nu]}$. When d' reduces $\triangle$ (job $\tilde{J}_{[\kappa]}$ is in the due-window and job $\tilde{J}_{[\kappa-1]}$ is early), the effect of moving d' is $[-(\kappa-1)a - n\gamma + n\omega]\triangle \geq 0$, it can be obtained that $\kappa \leq \frac{n(\omega-\gamma)}{a} + 1$. When d' increases $\triangle$ (job $\tilde{J}_{[\kappa]}$ is early and job $\tilde{J}_{[\kappa+1]}$ is in the due-window), the effect of moving d' is $(a\kappa + n\gamma - n\omega)\triangle + \theta_{[\kappa]} \geq 0$, thus, $\kappa \geq \frac{n(\omega-\gamma)}{a} - \frac{\theta_{[\kappa]}}{a\triangle}$. Since it is impossible to determine the value $\frac{\theta_{[\kappa]}}{a\triangle}$, it follows that $\kappa \leq \left\lceil \frac{n(\omega-\gamma)}{a} \right\rceil$.

Similarly, it follows that $\nu \geq \left\lceil \frac{n(b-\omega)}{b} \right\rceil$. □

Corollary 1. *For a given sequence ψ of the CONDW assignment method, if $\theta_l = \delta_l = 0$ ($l = 1, 2, \ldots, n$), the optimal values $d' = \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu]}$, where*

$$\kappa = \left\lceil \frac{n(\omega - \gamma)}{a} \right\rceil \tag{10}$$

and

$$\nu = \left\lceil \frac{n(b - \omega)}{b} \right\rceil. \tag{11}$$

From Lemmas 1 and 2, for the CONDW assignment method, we have

$$\tilde{G}_1(\psi, d', d'')$$

$$
\begin{aligned}
&= \sum_{l=1}^{n}\left(a\tilde{E}_{[l]} + b\tilde{T}_{[l]} + \theta_{[l]}\tilde{U}_{[l]} + \delta_{[l]}\tilde{V}_{[l]} + \gamma d' + \omega\tilde{D}\right)\\
&= \sum_{l=1}^{\kappa-1} a\tilde{E}_{[l]} + \sum_{l=\nu+1}^{n} b\tilde{T}_{[l]} + \sum_{l=1}^{\kappa-1}\theta_{[l]} + \sum_{l=\nu+1}^{n}\delta_{[l]} + n\gamma d' + n\omega\tilde{D}\\
&= \sum_{l=1}^{\kappa-1} a(\tilde{C}_{[\kappa]} - \tilde{C}_{[l]}) + \sum_{l=\nu+1}^{n} b(\tilde{C}_{[l]} - \tilde{C}_{[\nu]}) + \sum_{l=1}^{\kappa-1}\theta_{[l]} + \sum_{l=\nu+1}^{n}\delta_{[l]} + n\gamma\tilde{C}_{[\kappa]} + n\omega(\tilde{C}_{[\nu]} - \tilde{C}_{[\kappa]})\\
&= \sum_{l=1}^{\kappa-1} a\left(\sum_{k=1}^{\kappa}(1+\varsigma(\kappa-k))\,p_{[k]} - \sum_{k=1}^{l}(1+\varsigma(l-k))\,p_{[k]}\right)\\
&\quad + \sum_{l=\nu+1}^{n} b\left(\sum_{k=1}^{l}(1+\varsigma(l-k))\,p_{[k]} - \sum_{k=1}^{\nu}(1+\varsigma(\nu-k))\,p_{[k]}\right) + \sum_{l=1}^{\kappa-1}\theta_{[l]} + \sum_{l=\nu+1}^{n}\delta_{[l]}\\
&\quad + n\gamma\left(\sum_{k=1}^{\kappa}(1+\varsigma(\kappa-k))\,p_{[k]}\right) + n\omega\left(\sum_{k=1}^{\nu}(1+\varsigma(\nu-k))\,p_{[k]} - \sum_{k=1}^{\kappa}(1+b(\kappa-k))\,p_{[k]}\right)\\
&= \sum_{l=1}^{n}\Theta_l p_{[l]} + \sum_{l=1}^{\kappa-1}\theta_{[l]} + \sum_{l=\nu+1}^{n}\delta_{[l]},
\end{aligned} \tag{12}
$$

where

$$
\Theta_l = \begin{cases} a\left[(l-1)(1+\varsigma(\kappa-l)) + \frac{\varsigma(\kappa-l+1)(\kappa-l)}{2}\right] & \\ +\frac{b\varsigma(n-\nu+1)(n-\nu)}{2} + n\gamma(1+\varsigma(\kappa-l)) + n\omega\varsigma(\nu-\kappa), & l = 1,2,...,\kappa,\\ \frac{b\varsigma(n-\nu+1)(n-\nu)}{2} + n\omega(1+\varsigma(\nu-l)), & l = \kappa+1,\kappa+2,...,\nu,\\ b\left[\frac{(2+\varsigma(n-l))(n-l+1)}{2}\right], & l = \nu+1,\nu+2,...,n. \end{cases} \tag{13}
$$

Similar to the CONDW assignment method, the following results can be obtained.

Lemma 3. *For a given sequence ψ of the SLKDW assignment method, the optimal values $q' = \tilde{C}_{[\kappa-1]}$, $q'' = \tilde{C}_{[\nu-1]}$, where $\kappa \le \left\lceil\frac{n(\omega-\gamma)}{a}\right\rceil$ and $\nu \ge \left\lceil\frac{n(b-\omega)}{b}\right\rceil$ ($\kappa \le \nu$).*

Corollary 2. *For a given sequence ψ of the SLKDW assignment method, if $\theta_l = \delta_l = 0$ ($l = 1,2,\ldots,n$), the optimal values $q' = \tilde{C}_{[\kappa-1]}$ and $q'' = \tilde{C}_{[\nu-1]}$, where $\kappa = \left\lceil\frac{n(\omega-\gamma)}{a}\right\rceil$ and $\nu = \left\lceil\frac{n(b-\omega)}{b}\right\rceil$.*

From Lemma 3, it follows that

$$
\begin{aligned}
&\tilde{G}_2(\psi, q', q'')\\
&= \sum_{l=1}^{n}\left(a\tilde{E}_{[l]} + b\tilde{T}_{[j]} + \theta_{[j]}\tilde{U}_{[j]} + \delta_j\tilde{V}_{[j]} + \gamma q' + \omega\tilde{D}\right)\\
&= \sum_{l=1}^{\kappa-1} a\tilde{E}_{[l]} + \sum_{l=\nu+1}^{n} b\tilde{T}_{[l]} + \sum_{l=1}^{\kappa-1}\theta_{[l]} + \sum_{l=\nu+1}^{n}\delta_{[l]} + n\gamma q' + n\omega(q'' - q')\\
&= \sum_{l=1}^{\kappa-1} a(d_{[l]} - \tilde{C}_{[l]}) + \sum_{l=\nu+1}^{n} b(\tilde{C}_{[l]} - d_{[l]}) + \sum_{l=1}^{\kappa-1}\theta_{[l]} + \sum_{l=\nu+1}^{n}\delta_{[l]} + n\gamma\tilde{C}_{[\kappa-1]} + n\omega(\tilde{C}_{[\nu-1]} - \tilde{C}_{[\kappa-1]})\\
&= \sum_{l=1}^{\kappa-1} a(s_{[l]} + p_{[l]} + \tilde{C}_{[\kappa-1]} - \tilde{C}_{[l]}) + \sum_{l=\nu+1}^{n} b(\tilde{C}_{[l]} - s_{[l]} - p_{[l]} - \tilde{C}_{[\nu-1]}) + \sum_{l=1}^{\kappa-1}\theta_{[l]} + \sum_{l=\nu+1}^{n}\delta_{[l]}
\end{aligned}
$$

$$
\begin{aligned}
& +n\gamma\tilde{C}_{[\kappa-1]} + n\omega(\tilde{C}_{[\nu-1]} - \tilde{C}_{[\kappa-1]}) \\
= & \sum_{l=1}^{\kappa-1} a(\tilde{C}_{[\kappa-1]} - \tilde{C}_{[l-1]}) + \sum_{l=\nu+1}^{n} b(\tilde{C}_{[l-1]} - \tilde{C}_{[\nu-1]}) + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]} \\
& +n\gamma\tilde{C}_{[\kappa-1]} + n\omega(\tilde{C}_{[\nu-1]} - \tilde{C}_{[\kappa-1]}) \\
= & \sum_{l=1}^{\kappa-1} a\left(\sum_{k=1}^{\kappa-1} (1+\varsigma(\kappa-1-k))\,p_{[k]} - \sum_{k=1}^{l-1} (1+\varsigma(l-1-k))\,p_{[k]}\right) \\
& + \sum_{l=\nu+1}^{n} b\left(\sum_{k=1}^{l-1} (1+\varsigma(l-1-k))\,p_{[k]} - \sum_{k=1}^{\nu-1} (1+\varsigma(\nu-1-k))\,p_{[k]}\right) + \sum_{l=1}^{\kappa-1} \theta_{[l]} \\
& + \sum_{l=\nu+1}^{n} \delta_{[l]} + n\gamma\left(\sum_{k=1}^{\kappa-1} (1+\varsigma(\kappa-1-k))\,p_{[k]}\right) \\
& +n\omega\left(\sum_{k=1}^{\nu-1} (1+\varsigma(\nu-1-k))\,p_{[k]} - \sum_{k=1}^{\kappa-1} (1+\varsigma(\kappa-1-k))\,p_{[k]}\right) \\
= & \sum_{l=1}^{n} \Theta_l p_{[l]} + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]}, \qquad (14)
\end{aligned}
$$

where

$$
\Theta_l = \begin{cases} a\left[l(1+\varsigma(\kappa-l-1)) + \frac{\varsigma(\kappa-l-1)(\kappa-l)}{2}\right] \\ +\frac{b\varsigma(n-\nu+1)(n-\nu)}{2} + n\gamma(1+\varsigma(\kappa-l-1)) + n\omega\varsigma(\nu-\kappa), & l = 1,2,...,\kappa-1, \\ \frac{b\varsigma(n-\nu+1)(n-\nu)}{2} + n\omega(1+\varsigma(\nu-l-1)), & l = \kappa,\kappa+1,...,\nu-1, \\ b\left[\frac{(2+\varsigma(n-l-1))(n-l)}{2}\right], & l = \nu,\nu+1,...,n. \end{cases} \qquad (15)
$$

Let $x_{l,r} = 1$ if $\tilde{J}_l$ is placed in rth position, and $x_{l,r} = 0$; otherwise, If κ and ν are given, the optimal job sequence of the problems $1|CONDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d' + \omega\tilde{D}\right)$ and $1|SLKDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma q' + \omega\tilde{D}\right)$ can be obtained by solving the following **Assignment Problem** (**AP**):

$$
\text{Min} \sum_{l=1}^{n}\sum_{r=1}^{n} \Psi_{l,r} x_{l,r} \qquad (16)
$$

$$
s.t. \begin{cases} \sum_{j=1}^{n} x_{l,r} = 1, & r = 1,2,...,n, \\ \sum_{r=1}^{n} x_{l,r} = 1, & l = 1,2,...,n, \\ x_{l,r} = 0 \text{ or } 1, \end{cases} \qquad (17)
$$

where,

$$
\Psi_{l,r} = \begin{cases} \Theta_r p_l + \theta_l, & r = 1,2,...,\kappa-1, \\ \Theta_r p_l, & r = \kappa,\kappa+1,...,\nu, \\ \Theta_r p_l + \delta_l, & r = \nu+1,\nu+2,...,n, \end{cases} \qquad (18)
$$

for the CONDW assignment method, Θ_r is given by Eq (13); for the SLKDW assignment method, Θ_r is given by Eq (15).

From the above analysis, a polynomial time algorithm can be proposed to solve $1|CONDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d' + \omega\tilde{D}\right)$ and $1|SLKDW, psdst|\sum_{l=1}^{n}(a\tilde{E}_l+b\tilde{T}_l+\theta_l\tilde{U}_l+\delta_l\tilde{V}_l+\gamma q'+\omega\tilde{D})$.

Algorithm 1

Step 1. Calculate κ and ν (see Lemmas 2 and 3).

Step 2. For each pair of κ and ν, calculate $\Psi_{l,r}$ (see Eq (18)), to solve the **AP** (16)–(17). For each pair of κ and ν, a suboptimal sequence $\psi(\kappa,\nu)$ and a suboptimal objective value $\tilde{G}_1(\kappa,\nu)$ $(\tilde{G}_2(\kappa,\nu))$ can be obtained.

Step 3. The global optimal solution is the one with minimum value

$$\tilde{G}_1^*(\tilde{G}_2^*) = \min\left\{\tilde{G}_1(\kappa,\nu)(\tilde{G}_2(\kappa,\nu))|\kappa \le \left\lceil\frac{n(\omega-\gamma)}{a}\right\rceil, \nu \ge \left\lceil\frac{n(b-\omega)}{b}\right\rceil\right\}.$$

Step 4. For the CONDW assignment method, calculate $d' = \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu]}$ (such $\tilde{D} = d'' - d' = \tilde{C}_{[\nu]} - \tilde{C}_{[\kappa]}$). For the SLKDW assignment method, calculate $q' = \tilde{C}_{[\kappa-1]}$ and $q'' = \tilde{C}_{[\nu-1]}$ (such $\tilde{D} = q'' - q' = \tilde{C}_{[\nu-1]} - \tilde{C}_{[\kappa-1]}$).

Theorem 1. *Algorithm 1 solves both the problems* $1|CONDW, psdst|\sum_{l=1}^{n}(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d' + \omega\tilde{D})$ *and* $1|SLKDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma q' + \omega\tilde{D}\right)$ *in* $O(n^5)$ *time.*

Proof. The correctness of Algorithm 1 is guaranteed by Lemmas 1–3, and above analysis. Steps 1 and 4 take $O(n)$; In Step 2 for each pair of κ and ν, solving the **AP** needs $O(n^3)$; κ and ν are less than n; Step 3 takes $O(n^2)$. In summary, the overall time complexity of Algorithm 1 is $O(n^5)$. □

Now, we consider a special case, i.e., $\theta_l = \delta_l = 0$ $(l = 1, 2, \ldots, n)$, then κ and ν can be obtained by Corollary 1, we have

$$\tilde{G}_1(\psi, d', d'')(\tilde{G}_2(\psi, q', q'')) = \sum_{j=1}^{n}\left(a\tilde{E}_{[j]} + b\tilde{T}_{[j]} + \gamma d'(q') + \omega\tilde{D}\right) = \sum_{l=1}^{n}\Theta_l p_{[l]}, \tag{19}$$

where, for the CONDW assignment method, Θ_l is given by Eq (13); for the SLKDW assignment method, Θ_l is given by Eq (15).

The term (19) can be minimized by sequencing the vectors Θ_l and p_l in opposite order (i.e., the HLP rule, see Hardy et al. [23]) in $O(n\log n)$, hence, the problems $1|CONDW, psdst|\sum_{l=1}^{n}(a\tilde{E}_l + b\tilde{T}_l + \gamma d' + \omega\tilde{D})$ and $1|SLKDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \gamma q' + \omega\tilde{D}\right)$ can be solved by the following algorithm:

Algorithm 2

Step 1. Compute κ and ν (see Eqs (10) and (11)).

Step 2. By using the HLP rule to identify the optimal sequence.

Step 3. For the CONDW assignment method, calculate $d' = \tilde{C}_{[\kappa]}$ and $d'' = \tilde{C}_{[\nu]}$ (such $\tilde{D} = d'' - d' = \tilde{C}_{[\nu]} - \tilde{C}_{[\kappa]}$). For the SLKDW assignment method, calculate $q' = \tilde{C}_{[\kappa-1]}$ and $q'' = \tilde{C}_{[\nu-1]}$ (such $\tilde{D} = q'' - q' = \tilde{C}_{[\nu-1]} - \tilde{C}_{[\kappa-1]}$).

Theorem 2. *Algorithm 2 solves both the problems* $1|CONDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \gamma d' + \omega\tilde{D}\right)$ *and* $1|SLKDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \gamma q' + \omega\tilde{D}\right)$ *in* $O(n\log n)$ *time.*

3.2. DIFDW

For a given sequence ψ, the objective function of job $\tilde{J}_{[l]}$ is

$$\tilde{G}_{3,[l]} = a\max\{d'_{[l]} - \tilde{C}_{[l]}, 0\} + b\max\{\tilde{C}_{[l]} - d''_{[l]}, 0\} + \theta_{[l]}\tilde{U}_{[l]} + \delta_{[l]}\tilde{V}_{[l]} + \gamma d'_{[l]} + \omega(d''_{[l]} - d'_{[l]}). \tag{20}$$

Lemma 4. *For a given sequence ψ, there exists an optimal solution such that $d'_{[l]} \leq d''_{[l]} \leq \tilde{C}_{[l]}$.*

Proof. Note that $d'_{[l]} \leq d''_{[l]}$, then it can be divided into the following two cases.
Case (i). $d'_{[l]} \leq \tilde{C}_{[l]} \leq d''_{[l]}$, for job J_l, we have

$$\tilde{G}_{3,[l]} = \gamma d'_{[l]} + \omega(d''_{[l]} - d'_{[l]}).$$

If we move $d''_{[l]}$ to the left ensure that $d''_{[l]} = \tilde{C}_{[l]}$, we have

$$\tilde{G}_{3,[l]} = \gamma d'_{[l]} + \omega(\tilde{C}_{[l]} - d'_{[l]}) < \tilde{G}_{3,[l]},$$

it can be seen that Case (i) is not optimal.
Case (ii). $\tilde{C}_{[l]} \leq d'_{[l]} \leq d''_{[l]}$, i.e.,job $\tilde{J}_{[l]}$ is an early job, we have

$$\tilde{G}_{3,[l]} = a(d'_{[l]} - \tilde{C}_{[l]}) + \gamma d'_{[l]} + \theta_{[l]} + \omega(d''_{[l]} - d'_{[l]}).$$

Move $d'_{[l]}$ and $d''_{[l]}$ to the left ensure that $d'_{[l]} = d''_{[l]} = \tilde{C}_{[l]}$, then

$$\tilde{G}_{3,[l]} = \gamma\tilde{C}_{[l]} < \tilde{G}_{3,[l]},$$

it can be seen that case (ii) is also not optimal.
Hence, $d'_{[l]} \leq d''_{[l]} \leq \tilde{C}_{[l]}$. □

Lemma 5. *For a given sequence ψ, the optimal strategy for the DIFDW assignment is:*
Case (i). *If $\omega \geq b$, set $d'_{[l]} = d''_{[l]} = 0$ ($l = 1, 2, \ldots, n$);*
Case (ii). *If $\omega < b$ and $\gamma \geq \omega$, set $d'_{[l]} = 0, d''_{[l]} = \tilde{C}_{[l]}$ ($l = 1, 2, \ldots, n$);*
Case (iii). *If $\omega < b$ and $\gamma < \omega$, set $d'_{[l]} = d''_{[l]} = \tilde{C}_{[l]}$ ($l = 1, 2, \ldots, n$).*

Proof. For a given sequence ψ, from Lemma 4, $d'_{[l]} \leq d''_{[l]} \leq \tilde{C}_{[l]}$, then for job $\tilde{J}_{[l]}$,

$$\tilde{G}_{3,[l]} = b(\tilde{C}_{[l]} - d''_{[l]}) + \delta_{[l]} + \gamma d'_{[l]} + \omega(d''_{[l]} - d'_{[l]}) = b\tilde{C}_{[l]} + \delta_{[l]} + (\gamma - \omega)d'_{[l]} + (\omega - b)d''_{[l]}.$$

Obviously, if $\omega - b \geq 0$, then $d''_{[l]}$ should be 0, i.e., $d'_{[l]} = d''_{[l]} = 0$. If $\omega < b$ and $\gamma \geq \omega$, $d'_{[l]}$ should be 0 and $d''_{[l]}$ should be $\tilde{C}_{[l]}$. If $\omega < b$ and $\gamma < \omega$, then $d'_{[l]}$ should be $\tilde{C}_{[l]}$ and $d''_{[l]}$ should be $\tilde{C}_{[l]}$. It completes the proof. □

Lemma 6. *For the problem $1|DIFDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d'_l + \omega\tilde{D}_l\right)$, the optimal sequence can be obtained by the SPT rule, i.e., the smallest processing time first rule.*

Proof. From Lemma 5, for Case (i), if $\omega \geq b$ and $d'_{[l]} = d''_{[l]} = 0$ ($l = 1, 2, \ldots, n$), then $\tilde{E}_{[l]} = 0$, $\tilde{T}_{[l]} = \tilde{C}_{[l]}$, $\tilde{D}_{[l]} = 0$, and

$$\tilde{G}_3(\psi, d'_l, d''_l) = \sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d'_l + \omega\tilde{D}_l\right) = b\sum_{l=1}^{n}\tilde{C}_{[l]} + \sum_{l=1}^{n}\delta_{[l]}.$$

For Case (ii) in Lemma 5, if $\omega < b$, $\gamma \geq \omega$ and $d'_{[l]} = 0, d''_{[l]} = \tilde{C}_{[l]}$ $(l = 1, 2, \ldots, n)$, then $\tilde{E}_{[l]} = 0$, $\tilde{T}_{[l]} = 0$ and $\tilde{D}_{[l]} = \tilde{C}_{[l]}$, and

$$\tilde{G}_3(\psi, d'_l, d''_l) = \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d'_l + \omega\tilde{D}_l\right) = \omega \sum_{l=1}^{n} \tilde{C}_{[l]}.$$

For Case (iii) in Lemma 5, if $\omega < b$, $\gamma < \omega$ and $d'_{[l]} = d''_{[l]} = \tilde{C}_{[l]}$ $(l = 1, 2, \ldots, n)$, then $\tilde{E}_{[l]} = 0$, $\tilde{T}_{[l]} = 0$ and $\tilde{D}_{[l]} = 0$, and

$$\tilde{G}_3(\psi, d'_l, d''_l) = \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_j V_j + \gamma d'_l + \omega\tilde{D}_l\right) = \gamma \sum_{l=1}^{n} \tilde{C}_{[l]}.$$

Stem from Koulamas and Kyparisis [1], $1|psdst|\sum_{l=1}^{n}\tilde{C}_l$ can be solved by the SPT rule. Consequently, the optimal sequence for Cases (i)-(iii) is the SPT rule. It completes the proof. □

Based on the above analysis, the $1|DIFDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d'_l + \omega\tilde{D}_l\right)$ problem can be solved by the following algorithm:

Algorithm 3

Step 1. Arrange the jobs in non-decreasing order of p_l, i.e., $p_{[1]} \leq p_{[2]}, \ldots, \leq p_{[n]}$;

Step 2. Calculate $d'_{[l]}$ and $d''_{[l]}$ by Lemma 5;

Step 3. Calculate $\tilde{D}_{[l]} = d''_{[l]} - d'_{[l]}$.

Theorem 3. *Algorithm 3 solves the* $1|DIFDW, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d'_l + \omega\tilde{D}_l\right)$ *problem in* $O(n\log n)$ *time.*

3.3. A case study

In this subsection, we present a case study to illustrate the calculation steps and results of the three types of due-window assignment.

Assume a 6-job problem, where $a = 4$, $b = 7$, $\varsigma = 2$, $\gamma = 1$, $\omega = 2$, $p_1 = 6$, $p_2 = 8$, $p_3 = 9$, $p_4 = 7$, $p_5 = 4$, $p_6 = 5$, $\theta_1 = 2$, $\theta_2 = 5$, $\theta_3 = 6$, $\theta_4 = 4$, $\theta_5 = 7$, $\theta_6 = 1$, $\delta_1 = 3$, $\theta_2 = 2$, $\theta_3 = 7$, $\theta_4 = 5$, $\theta_5 = 9$, $\theta_6 = 8$.

From Algorithm 1, we have $\left\lceil \frac{n(\omega-\gamma)}{a} \right\rceil = 2$ and $\left\lceil \frac{n(b-\omega)}{b} \right\rceil = 5$. For the CONDW assignment, if $\kappa = 1, \nu = 5$, the values $\Psi_{l,r}$ are given in Table 1. By the **AP**, the sequence is $\psi(1,5) = (J_5, J_6, J_1, J_4, J_2, J_3)$ and $\tilde{G}_1(1,5) = 2026$. Similarly, for $\kappa = 1, 2, \nu = 5, 6$, the results are shown in Table 2. From Table 2, the optimal sequence is $\psi^* = (J_5, J_6, J_1, J_4, J_2, J_3)$, $\tilde{G}_1^* = 1957$, $d' = \tilde{C}_{[2]} = 13$ and $d'' = \tilde{C}_{[5]} = 52$ (such $\tilde{D} = 39$).

Table 1. Values of $\Psi_{l,r}$ for $\kappa = 1$ and $\nu = 5$.

$J_j\backslash r$	1	2	3	4	5	6
J_1	696	588	**444**	300	156	45
J_2	928	784	592	400	**208**	58
J_3	1044	882	666	450	234	**70**
J_4	812	686	518	**350**	182	54
J_5	**464**	392	296	200	104	37
J_6	580	**490**	370	250	130	**43**

Table 2. Results for the CONDW assignment.

κ	ν	$\psi(\kappa,\nu)$	$\tilde{G}_1(\kappa,\nu)$
1	5	$(J_5, J_6, J_1, J_4, J_2, J_3)$	2026
1	6	$(J_5, J_6, J_1, J_4, J_2, J_3)$	2340
2	5	$(J_5, J_6, J_1, J_4, J_2, J_3)$	1977
2	6	$(J_5, J_6, J_1, J_4, J_2, J_3)$	2315

For the SLKDW assignment, the results are shown in Table 3. From Table 3, the optimal sequence is $\psi^* = (J_5, J_6, J_1, J_4, J_2, J_3)$, $\tilde{G}_2^* = 1282$, $q' = \tilde{C}_{[1]} = 4$ and $q'' = \tilde{C}_{[4]} = 37$ (such $\tilde{D} = 33$).

Table 3. Results for the SLKDW assignment.

κ	ν	$\psi(\kappa,\nu)$	$\tilde{G}_2(\kappa,\nu)$
1	5	$(J_5, J_6, J_1, J_4, J_2, J_3)$	1307
1	6	$(J_5, J_6, J_1, J_4, J_2, J_3)$	1340
2	5	$(J_5, J_6, J_1, J_4, J_2, J_3)$	1282
2	6	$(J_5, J_6, J_1, J_4, J_2, J_3)$	1315

For the DIFDW assignment, the optimal sequence is $\psi^* = (J_5, J_6, J_1, J_4, J_2, J_3)$, $\tilde{G}_3^* = 199$, $d_5' = d_5'' = \tilde{C}_5 = 4$, $d_6' = d_6'' = \tilde{C}_6 = 13$, $d_1' = d_1'' = \tilde{C}_1 = 24$, $d_4' = d_4'' = \tilde{C}_4 = 37$, $d_2' = d_2'' = \tilde{C}_2 = 52$, $d_3' = d_3'' = \tilde{C}_3 = 69$, $\tilde{D}_5 = \tilde{D}_6 = \tilde{D}_1 = \tilde{D}_4 = \tilde{D}_2 = \tilde{D}_3 = 0$.

4. Extensions

As in Wang [24] and Yang and Kuo [25], for learning and deterioration effects, if job J_l is scheduled in rth position, its the actual processing time is $p_l^A = (p_l + \beta t_l)r^\alpha$ and $p_l^A = p_l r^{\alpha_l} + \beta t_l$, where $\beta \geq 0$ is the deterioration rate (see Gawiejnowicz [26]), $\alpha \leq 0$ ($\alpha_l \leq 0$) is the learning rate (Azzouz et al. [27], Wang et al. [28]), and t_l is the starting time of J_l. As in Cheng et al. [29], for the position-based learning and deterioration (aging) effects, if job J_l is scheduled in rth position, its the actual processing time is $p_l^A = p_l r^\alpha r^\beta$, where $-1 < \alpha < 0$ is the learning rate of job J_l, $0 < \beta_l < 1$ is the deterioration (aging) rate of job J_l. The problem with learning and deterioration effects can be denoted by

$$1|\tilde{X}, \tilde{Y}, psdst|\tilde{Z},$$

where $\tilde{X} \in \{CONDW, SLKDW, DIFDW\}$, $\tilde{Y} \in \{p_l^A = (p_l + \beta t_l)r^\alpha, p_l^A = p_l r^{\alpha_l} + \beta t_l, p_l^A = p_l r^\alpha r^\beta\}$ and $\tilde{Z} \in \{\sum_{l=1}^n \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d'/q'/d_l' + \omega\tilde{D}/\tilde{D}_l\right)\}$

4.1. *Case* $p_l^A = (p_l + \beta t_l)r^\alpha$

For $p_l^A = (p_l + \beta t_l)r^\alpha$, from Wang [13] and Wang [23], we have

$$p_{[l]}^A = p_{[l]}l^\alpha + \beta l^\alpha \left(\sum_{j=1}^{l-1} p_{j]} j^\alpha \prod_{i=j+1}^{l-1}(1 + \beta i^\alpha)\right), \tag{21}$$

hence

$$\tilde{G}_1(\psi, d', d'')(\tilde{G}_2(\psi, q', q'')) = \sum_{l=1}^{n} \Theta_l p_{[l]}^A + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]}$$
$$= \sum_{l=1}^{n} \Gamma_l p_{[l]} + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]}, \quad (22)$$

where

$$\begin{aligned}
\Gamma_1 &= \Theta_1 + \beta 2^\alpha \Theta_2 + \beta 3^\alpha (1+\beta 2^\alpha)\Theta_3 + \beta 4^\alpha (1+\beta 2^\alpha)(1+\beta 3^\alpha)\Theta_4 + \ldots + \beta n^\alpha \prod_{h=2}^{n-1}(1+\beta h^\alpha)\Theta_n \\
\Gamma_2 &= 2^\alpha \Theta_2 + \beta 2^\alpha 3^\alpha \Theta_3 + \beta 2^\alpha 4^\alpha (1+\beta 3^\alpha)\Theta_4 + \beta 2^\alpha 5^\alpha (1+\beta 3^\alpha)(1+\beta 4^\alpha)\Theta_5 + \ldots \\
&\quad + \beta 2^\alpha n^\alpha \prod_{h=3}^{n-1}(1+\beta h^\alpha)\Theta_n \\
&\ldots \\
\Gamma_{n-1} &= (n-1)^\alpha \Theta_{n-1} + \beta (n-1)^\alpha n^\alpha \Theta_n \\
\Gamma_n &= \Theta_n n^\alpha,
\end{aligned} \quad (23)$$

for the CONDW (SLKDW) assignment, Θ_l is given by (13) (Θ_l is given by Eq (15)), $l = 1, 2, \ldots, n$. Similar to Subsection 2.1, we have

Theorem 4. *Both the problems* $1|CONDW, p_l^A = (p_l + \beta t_l)r^\alpha, psdst|\sum_{l=1}^{n}(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma d' + \omega\tilde{D})$ *and* $1|SLKDW, p_l^A = (p_l + \beta t_l)r^\alpha, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma q' + \omega\tilde{D}\right)$ *can be solved in* $O(n^5)$ *time.*

Similarly, for the DIFDW assignment, minimize $\sum_{l=1}^{n}(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma d'_l + \omega\tilde{D}_l)$ is equal to minimize $\sum_{l=1}^{n} \min\{b, \gamma, \omega\}\tilde{C}_l$, we have

$$G_3(\psi, d'_l, d''_l) = \sum_{l=1}^{n} \min\{b, \gamma, \omega\}\tilde{C}_l = \sum_{l=1}^{n} \Gamma_l p_{[l]}, \quad (24)$$

where Γ_l is given by Eq (23) and $\Theta_l = \min\{b, \gamma, \omega\}(n - l + 1)(1 + \frac{\varsigma(n-l)}{2})$ (see Koulamas and Kyparisis [1]), $l = 1, 2, \ldots, n$.

Theorem 5. *The* $1|DIFDW, p_l^A = (p_l + \beta t_l)r^\alpha, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l \tilde{U}_l + \delta_l \tilde{V}_l + \gamma d'_l + \omega\tilde{D}_l\right)$ *problem can be solved in* $O(n \log n)$ *time.*

4.2. Case $p_l^A = p_l r^{\alpha_l} + \beta t_l$

For $p_l^A = p_l r^{\alpha_l} + \beta t_l$, from Yang and Kuo [24] and Subsection 3.1, we have

$$p_{[l]}^A = p_{[l]} l^{\alpha_{[l]}} + \beta p_{[l-1]}(l-1)^{\alpha_{[l-1]}} + \beta(1+\beta)p_{[l-2]}(l-2)^{\alpha_{[l-2]}} + \ldots + \beta(1+\beta)^{l-2} p_{[1]} 1^{\alpha_{[1]}}, \quad (25)$$

hence

$$\tilde{G}_1(\psi, d', d'')(\tilde{G}_2(\psi, q', q'')) = \sum_{l=1}^{n} \Theta_l p_{[l]}^A + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]}$$

$$= \sum_{l=1}^{n} \Gamma_l p_{[l]} l^{\alpha_{[l]}} + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]}, \tag{26}$$

where

$$\begin{aligned}
\Gamma_1 &= \Theta_1 + \beta\Theta_2 + \beta(1+\beta)\Theta_3 + \beta(1+\beta)^2\Theta_4 + \ldots + \beta(1+\beta)^{n-2}\Theta_n \\
\Gamma_2 &= \Theta_2 + \beta\Theta_3 + \beta(1+\beta)\Theta_4 + \beta(1+\beta)^2\Theta_5 + \ldots + \beta(1+\beta)^{n-3}\Theta_n \\
&\ldots \\
\Gamma_{n-1} &= \Theta_{n-1} + \beta\Theta_n \\
\Gamma_n &= \Theta_n,
\end{aligned} \tag{27}$$

for the CONDW assignment, Θ_l is given by Eq (13); for the SLKDW assignment, Θ_l is given by Eq (15), $l = 1, 2, \ldots, n$.

Similar to Subsection 3.1, if κ and ν are given, the optimal job sequence of $1|CONDW, p_l^A = p_l r^{\alpha_l} + \beta t_l, psdst| \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d' + \omega\tilde{D}\right)$ and $1|SLKDW, p_l^A = p_l r^{\alpha_l} + \beta t_l, psdst| \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma q' + \omega\tilde{D}\right)$ can be formulated as the following **AP**:

$$\text{Min } G = \sum_{l=1}^{n}\sum_{r=1}^{n} \Psi_{l,r} x_{l,r} \tag{28}$$

$$\text{s.t.} \qquad (17),$$

where,

$$\Psi_{l,r} = \begin{cases} \Gamma_r p_l r^{a_l} + \theta_l, & r = 1, 2, ..., \kappa - 1, \\ \Gamma_r p_l r^{a_l}, & r = \kappa, \kappa + 1, ..., \nu, \\ \Gamma_r p_l r^{a_l} + \delta_l, & r = \nu + 1, \nu + 2, ..., n, \end{cases} \tag{29}$$

Γ_r is given by Eq (27).

Theorem 6. *Both the problems* $1|CONDW, p_l^A = p_l r^{\alpha_l} + \beta t_l, psdst| \sum_{l=1}^{n} (a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d' + \omega\tilde{D})$ *and* $1|SLKDW, p_l^A = p_l r^{\alpha_l} + \beta t_l, psdst| \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma q' + \omega\tilde{D}\right)$ *can be solved in* $O(n^5)$ *time.*

For the DIFDW assignment, we have

$$\tilde{G}_3(\psi, d_l', d_l'') = \sum_{l=1}^{n} \min\{b, \gamma, \omega\}\tilde{C}_l = \sum_{l=1}^{n} \Gamma_l p_{[l]} l^{\alpha_{[l]}}, \tag{30}$$

where Γ_l is given by Eq (27) and $\Theta_l = \min\{b, \gamma, \omega\}(n - l + 1)(1 + \frac{\varsigma(n-l)}{2})$, $l = 1, 2, \ldots, n$. Similarly, the optimal job sequence of $1|DIFDW, p_l^A = p_l r^{\alpha_l} + \beta t_l, psdst| \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d_l' + \omega\tilde{D}_l\right)$ can be formulated as the following **AP**:

$$\text{Min } G = \sum_{l=1}^{n}\sum_{r=1}^{n} \Psi_{l,r} x_{l,r} \tag{31}$$

$$\text{s.t.} \qquad (17),$$

where $\Psi_{l,r} = \Gamma_r p_l r^{a_l}$, $\Theta_r = \min\{b, \gamma, \omega\}(n - r + 1)(1 + \frac{\varsigma(n-r)}{2})$, $l = 1, 2, \ldots, n$.

Theorem 7. *The* $1|DIFDW, p_l^A = p_l r^{\alpha_l} + \beta t_l, psdst| \sum_{l=1}^{n} \left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d_l' + \omega\tilde{D}_l\right)$ *problem remains polynomially solvable in* $O(n^3)$.

4.3. Case $p_l^A = p_l r^\alpha r^\beta$

For $p_l^A = p_l r^\alpha r^\beta$, from Section 2 (see Eqs (12) and (14)), we have

$$\tilde{G}_1(\psi, d', d'')(\tilde{G}_2(\psi, q', q'')) = \sum_{l=1}^{n} \Theta_l p_{[l]} l^\alpha l^\beta + \sum_{l=1}^{\kappa-1} \theta_{[l]} + \sum_{l=\nu+1}^{n} \delta_{[l]}, \quad (32)$$

where, for the CONDW assignment, Θ_l is given by Eq (13); for the SLKDW assignment, Θ_l is given by Eq (15), $l = 1, 2, \ldots, n$.

For the DIFDW assignment, we have

$$\tilde{G}_3(\psi, d_l', d_l'') = \sum_{l=1}^{n} \min\{b, \gamma, \omega\}\tilde{C}_l = \sum_{l=1}^{n} \Theta_l p_{[l]} l^\alpha l^\beta, \quad (33)$$

where $\Theta_l = \min\{b, \gamma, \omega\}(n - l + 1)(1 + \frac{\varsigma(n-l)}{2})$, $l = 1, 2, \ldots, n$.

Similar to Subsection 3.2, we have the following results.

Theorem 8. *Both the problems* $1|CONDW, p_l^A = p_l r^\alpha r^\beta, psdst|\sum_{l=1}^{n}(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d' + \omega\tilde{D})$ *and* $1|SLKDW, p_l^A = p_l r^\alpha r^\beta, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma q' + \omega\tilde{D}\right)$ *can be solved in* $O(n^5)$ *time.*

Theorem 9. *The* $1|DIFDW, p_l^A = p_l r^\alpha r^\beta, psdst|\sum_{l=1}^{n}\left(a\tilde{E}_l + b\tilde{T}_l + \theta_l\tilde{U}_l + \delta_l\tilde{V}_l + \gamma d_l' + \omega\tilde{D}_l\right)$ *problem remains polynomially solvable in* $O(n^3)$.

5. Conclusion and future research

In this article, we focused on minimizing a general non-regular performance measure with psdst. We introduced solution algorithms to three due-window assignment methods (i.e., CONDW, SLKDW and DIFDW), and we proved that the problem can be solved in polynomial time. Future research could study the problem with group technology, investigate the problem in other machine setting (e.g., flowshop or unrelated parallel machines) or explored the problem with rate-modifying activities (see Ma et al. [30], Wang and Li [31] and Wang et al. [32]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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