Research article

Bifurcation analysis and optimal control of SEIR epidemic model with saturated treatment function on the network

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Abstract: In order to study the impact of limited medical resources and population heterogeneity on disease transmission, a SEIR model based on a complex network with saturation processing function is proposed. This paper first proved that a backward bifurcation occurs under certain conditions, which means that $R_0 < 1$ is not enough to eradicate this disease from the population. However, if the direction is positive, we find that within a certain parameter range, there may be multiple equilibrium points near $R_0 = 1$. Secondly, the influence of population heterogeneity on virus transmission is analyzed, and the optimal control theory is used to further study the time-varying control of the disease. Finally, numerical simulations verify the stability of the system and the effectiveness of the optimal control strategy.

Keywords: network; bifurcation analysis; SEIR epidemic model; optimal control

1. Introduction

Many practical problems in the real world can be abstracted into complex network models for research. For example, the spread and control of epidemics, the spread of computer viruses in the Internet, and the spread of rumors can all be regarded as spreading behaviors that obey a certain law on a complex network. At present, there have been many research results on the spread and control of epidemics on complex networks. For related research reports, see references [1–23].

In classical epidemic-disease dynamics models, it is generally assumed that the rate of treatment for a disease is proportional to the number of infected persons. This means that medical resources such as drugs, vaccines, hospital beds and isolation facilities are sufficient for the epidemic. In reality, however, every community or country has adequate or limited capacity for treatment and vaccination. If too much medical resources are invested, social resources will be wasted. If fewer resources are put into care, the risk of disease outbreaks increases. Therefore, it is important to identify different capacity for treatment depending on the community or country.

In order to investigate the effect of the limited capacity for treatment on the spread of infectious disease, references [24, 25] established an infectious disease model under a discontinuous treatment strategy. Wang et al. considered the following segments of treatment functions in the literature [25]

$$h(I) = \begin{cases} kI, & 0 \leq I \leq I_0 \\ kI_0, & I > I_0 \end{cases}$$

where k is the cure rate and I_0 is the maximum capacity of the medical system. In other words, when the number of patients is small and does not exceed the maximum capacity of the medical system, the treatment rate is proportional to the number of patients. When the number of patients exceeds the maximum capacity of the medical system, the treatment rate of the disease is a constant constant. reference [26], Zhang and Liu introduced the following saturation processing functions:

$$h(I) = \frac{rI}{1 + \alpha I}$$

where r is the cure rate, α is used to describe the impact of delayed treatment of a person with an illness in a situation where medical care is limited. Related studies on the function of saturation therapy are reported in references [27–30].

This paper proposes a SEIR model with a saturated processing function based on a complex network. We will analyze the influence of the heterogeneity of the population on the spread of the virus, and use the advancement of optimal control theory to study the time-varying control of infectious diseases.

The organization structure of this article is as follows. In Section 2, we propose a network-based SEIR model with saturation processing function. In Section 3, we analyzed the dynamics of the model. In Section 4, we study the bifurcation behavior of the model. In Section 5, we study the optimal control problem. Finally, in Section 6 and Section 7, we give the results of the numerical simulation and summarize the paper.

2. The model

In this section, we propose a network-based SEIR epidemic model with saturation treatment function. Then we use the SEIR epidemic model to prove the positivity and boundedness of understanding.

In the classic compartmental model, each individual has the same probability of contact with an infected individual. However, when the population is large, it is generally believed that factors such as exposure heterogeneity should be considered. Therefore, a complicated network is added to the infectious disease model to describe contact and so on. In a complex network work, each individual corresponds to a node of the network, and the interaction between two individuals corresponds to a link between two nodes.

On the basis of reference [31], we consider the following network-based SEIR epidemic model.

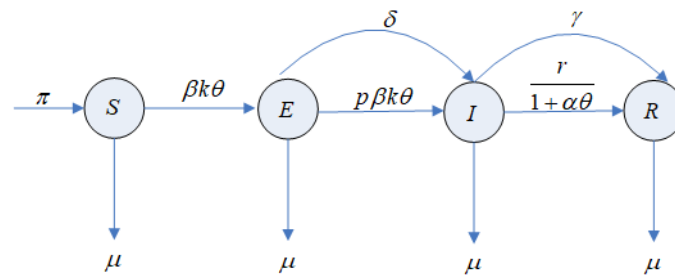The flow diagram of the state transition is depicted in Figure 1.

Figure 1. The flowchart of email virus spreading.

The dynamic equations can be written as

$$\frac{dS_k(t)}{dt} = \pi - \beta k S_k(t)\theta(t) - \mu S_k(t),$$

$$\frac{dE_k(t)}{dt} = \beta k S_k(t)\theta(t) - p\beta k E_k(t)\theta(t) - (\mu + \delta)E_k(t),$$

$$\frac{dI_k(t)}{dt} = p\beta k E_k(t)\theta(t) + \delta E_k(t) - \frac{rI_k(t)}{1 + \alpha\theta(t)} - (\mu + \gamma)I_k(t),$$

$$\frac{dR_k(t)}{dt} = \frac{rI_k(t)}{1 + \alpha\theta(t)} + \gamma I_k(t) - \mu R_k(t)$$

(2.1)

The explanation of the parameters is the same as in reference [31]. $\theta(t)$ describes a link pointing to an infected node, which satisfies the relation $\theta(t) = \frac{1}{\langle k \rangle}\sum_k kp(k)I_k(t)$, $p(k)$ is a degree distribution, $\langle k \rangle = \sum_{k=1}^{n} kp(k)$ describes the average degree. Here, we assume that the connectivities of nodes in network at each time are uncorrelated.

$\frac{rI_k(t)}{1+\alpha\theta(t)}$ represents the recovery of the k-th infection group after treatment.

$g(\theta) = \frac{1}{\langle k \rangle}\sum_{h=1}^{n} hP(h)\frac{rI_h(t)}{1+\alpha\theta(t)} = \frac{r\theta(t)}{1+\alpha\theta(t)}$ represents the probability that a given edge is connected to an infected node that is recovering through treatment. Similar to the processing function given in reference [26], g is a saturation processing function.

In the following cases, we make the following assumptions: (i) all new births are susceptible; (ii) the total number of nodes is constant, and the number of deaths is equal to the number of births, so $\pi = \mu$; (iii) assume that the degree of each node is time-invariant.

N_k is a constant, stands for the number of nodes with degree k. Then $N_k = S_k + E_k + I_k + R_k, k = 1, 2, \triangle\triangle\triangle, n$ and $\sum_k N_k = N$.

For the practice, the initial condition for model (2.1) satisfy:

$$\begin{cases} 0 \leq S_k(0), E_k(0), I_k(0), R_k(0), \leq N_k, \\ S_k(0) + E_k(0) + I_k(0) + R_k(0) = N_k, \\ k = 1, 2, \cdots, n \end{cases}$$

(2.2)

The probability $0 \leq \theta(t) \leq 1$ describes a link pointing to an infected host, which satisfies the relation

$$\theta(t) = \frac{1}{\langle k \rangle}\sum_k kP(k)I_k(t),$$

(2.3)

At t time, the global ensity is

$$S(t) = \sum_{k=1}^{n} P(k)S_k(t), \quad E(t) = \sum_{k=1}^{n} P(k)E_k(t), \quad I(t) = \sum_{k=1}^{n} P(k)I_k(t), \quad R(t) = \sum_{k=1}^{n} P(k)R_k(t)$$

Because

$$\frac{d(S_k(t) + E_k(t) + I_k(t) + R_k(t))}{dt} = 0$$

For the sake of convenience. it is assumed that the system has been normalized, so there are $S_k(t) + E_k(t) + I_k(t) + R_k(t) = 1$, So the positive invariant set of system (3.1) is

$$\Omega = \{(S_k(t), E_k(t), I_k(t), R_k(t)) : S_k(t) > 0, E_k(t) \geq 0, I_k(t)) \geq 0,$$
$$R_k(t) \geq 0, S_k(t) + E_k(t) + I_k(t) + R_k(t) = 1\}$$

3. Global behavior of the model

3.1. Equilibria and basic reproduction number

Lemma 1. *Suppose that $S_k(t), E_k(t), I_k(t), R_k(t)$ is a solution of model* (2.1) *satisfying initial conditions of Eq.(2.2). Then $\Omega = \{(S_1, E_1, I_1, R_1 \cdots, S_n, E_n, I_n, R_n) : S_k \geq 0, E_k \geq 0, I_k \geq 0, R_k \geq 0, S_k + E_k + I_k + R_k = 1, k = 1, 2, \cdots, n\}$ is a positively invariant for model* (2.1).

Proof. First, we proved that $E_k(t) \geq 0$ for any $t > 0$. If not, as $E_k(0) > 0$, there exist $k_0 \in \{1, 2, \Delta\Delta\Delta, n\}$ and t^* such that $t^* = \inf\{t | E_{k_0}(t) = 0, \dot{E}_{k_0}(t) < 0\}$. On the hand, according to the definition of t^*, we have $E_{k_0}(t^*) = 0, \dot{E}_{k_0}(t^*) < 0$ and $E_{k_0}(t) > 0$ for $0 \leq t < t^*$. From the second equation of model (2.1) , we obtain $\dot{E}_{k_0}(t^*) = \beta k_0 \lambda S_{k_0}(t^*)\theta(t^*) < 0$, which leads to $S_{k_0}(t^*) < 0$. On the other hand, in the time interval $[0, t^*]$, from the first equation of model (3.1) , we have $\dot{S}_{k_0}(t) \geq -\beta k S_{k_0}(t)\theta(t) - \mu S_{k_0}(t)$, which implies $S_{k_0}(t) \geq S_{k_0}(0)e^{-\beta k\theta(t)-\mu} \geq 0$ for $0 \leq t \leq t^*$. In particular, it follows $S_{k_0}(t^*) \geq 0$. It is a contradiction, so $E_k(t) \geq 0$ for any $t > 0$. Using the same method, we can verify $S_k(t) \geq 0, I_k(t) \geq 0$ and $R_k(t) \geq 0$ for any $t > 0$.

The proof is complete. $\qquad\square$

Theorem 1. *Basic reproduction number of system* (2.1) *is $R_0 = \frac{\beta\delta}{(\mu+\delta)(r+\mu+\gamma)} \frac{\langle k^2 \rangle}{\langle k \rangle}$. If $R_0 < 1$, the system* (2.1) *has only virus-free quilibrium ; if $R_0 > 1$, then system* (2.1) *has endemic equilibrium.*

Proof. Assuming that $E = (S_1, E_1, I_1, R_1, \cdots, S_n, E_n, I_n, R_n)$ is the unique balance of system (2.1), then E should satisfy

The equilibria of system (2.1) are determined by setting

$$\begin{cases} \beta k S_k \theta - p\beta k E_k \theta - (\mu + \delta)E_k = 0, \\ p\beta k E_k \theta + \delta E_k - \dfrac{rI_k}{1 + \alpha\theta} - (\mu + \gamma)I_k = 0, \\ \dfrac{rI_k}{1 + \alpha\theta} + \gamma I_k - \mu R_k = 0, \\ S_k + E_k + I_k + R_k = 1, k = 1, 2, \cdots, n \end{cases} \qquad (3.1)$$

After calculation, there is

$$
\begin{cases}
S_k = \dfrac{1}{\beta k\theta}(p\beta k\theta + \mu + \delta)\dfrac{1}{p\beta k\theta + \delta}(\dfrac{r}{1+\alpha\theta} + \mu + \gamma)I_k, \\[2mm]
E_k = \dfrac{1}{p\beta k\theta + \delta}(\dfrac{r}{1+\alpha\theta} + \mu + \gamma)I_k, \\[2mm]
R_k = \dfrac{1}{\mu}(\dfrac{r}{1+\alpha\theta} + \gamma)I_k, \\[2mm]
\{\dfrac{p\beta k\theta + \mu + \delta}{\beta k\theta(p\beta k\theta + \delta)}(\dfrac{r}{1+\alpha\theta} + \mu + \gamma) + \dfrac{1}{p\beta k\theta + \delta}(\dfrac{r}{1+\alpha\theta} + \mu + \gamma) \\[2mm]
\qquad + 1 + \dfrac{1}{\mu}(\dfrac{r}{1+\alpha\theta} + \gamma)\}I_k = 1
\end{cases}
\tag{3.2}
$$

This implies that

$$
I_k = \frac{\beta k\theta}{\frac{p\beta k\theta+\mu+\delta}{p\beta k\theta+\delta}(\frac{r}{1+\alpha\theta} + \mu + \gamma) + \frac{\beta k\theta}{p\beta k\theta+\delta}(\frac{r}{1+\alpha\theta} + \mu + \gamma) + \beta k\theta + \frac{\beta k\theta}{\mu}(\frac{r}{1+\alpha\theta} + \gamma)}
\tag{3.3}
$$

Substituting I_k into system (2.3), we have

$$
\begin{aligned}
\theta &= \frac{1}{\langle k\rangle}\sum_{h=1}^{n} hP(h)I_h \\
&= \frac{1}{\langle k\rangle}\sum_{h=1}^{n} hP(h)\frac{\beta h\theta}{\frac{p\beta h\theta+\mu+\delta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta} + \mu + \gamma) + \frac{\beta h\theta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta} + \mu + \gamma) + \beta h\theta + \frac{\beta h\theta}{\mu}(\frac{r}{1+\alpha\theta} + \gamma)}
\end{aligned}
\tag{3.4}
$$

$$
\theta[1 - \frac{1}{\langle k\rangle}\sum_{h=1}^{n}\frac{\beta h^2 P(h)}{\frac{p\beta h\theta+\mu+\delta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+\frac{\beta h\theta}{p\beta h\Theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+\beta h\theta+\frac{\beta h\theta}{\mu}(\frac{r}{1+\alpha\theta}+\gamma)}] = 0
$$

Define

$$
F(\theta) = 1 - \frac{1}{\langle k\rangle}\sum_{h=1}^{n}\frac{\beta h^2 P(h)}{\frac{p\beta h\Theta+\mu+\delta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+\frac{\beta h\theta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+\beta h\theta+\frac{\beta h\theta}{\mu}(\frac{r}{1+\alpha\theta}+\gamma)}
$$

Because

$$
F(1) > 1 - \frac{1}{\langle k\rangle}\sum_{h=1}^{n}\frac{\beta h^2 P(h)}{\beta h} = 0
$$

and F is continuous on $[0, 1]$. According to the intermediate value theorem, if $F(0) < 0$, then $F(\theta) = 0$ has a positive solution. While

$$
F(0) = 1 - \frac{1}{\langle k\rangle}\sum_{h=1}^{n}\frac{\beta h^2 P(h)}{\frac{\mu+\delta}{\delta}(r+\mu+\gamma)} < 0
$$

namely

$$
\frac{\beta\delta}{(\mu+\delta)(r+\mu+\gamma)}\frac{\langle k^2\rangle}{\langle k\rangle} > 1
$$

Then let $R_0 = \frac{\beta\delta}{(\mu+\delta)(r+\mu+\gamma)}\frac{\langle k^2\rangle}{\langle k\rangle}$. If $R_0 < 1$, the system (2.1) has only disease-free quilibrium E_0; if $R_0 > 1$, then system (1) has endemic equilibrium. $\qquad\square$

3.2. Stability of disease-free equilibrium

Remark 1. *One can also obtain the same basic reproduction number by using the method of second generation matrix [32], where indicates that if $R_0 < 1$, then the disease-free equilibrium E_0 of model (2.1) is locally asymptotically stable; if $R_0 > 1$, then E_0 is unstable.*

Theorem 2. *For system (2.1), if $R_0 < 1$, then the free quilibrium E_0 is locally asymptotically stable.*

Proof. 1. Since $S_k + E_k + I_k + R_k = 1$ system (2.1) is converted into an equivalent system:

$$\frac{dS_k(t)}{dt} = \pi - \beta k S_k(t)\theta(t) - \mu S_k(t),$$
$$\frac{dE_k(t)}{dt} = \beta k S_k(t)\theta(t) - p\beta k E_k(t)\theta(t) - (\mu + \delta)E_k(t), \tag{3.5}$$
$$\frac{dI_k(t)}{dt} = p\beta k E_k(t)\theta(t) + \delta E_k(t) - \frac{rI_k(t)}{1 + \alpha\theta(t)} - (\mu + \gamma)I_k(t)$$

The Jacobian matrix of model (3.5) at disease-free equilibrium $J_{2n \times 2n}$ is given by:

$$J_{E_0} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{31} & J_{33} \end{pmatrix}$$

and

$$J_{11} = \begin{pmatrix} -\mu & 0 & \cdots & 0 \\ 0 & -\mu & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\mu \end{pmatrix}$$

$$J_{22} = \begin{pmatrix} -(\mu + \delta) & 0 & \cdots & 0 \\ 0 & -(\mu + \delta) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -(\mu + \delta) \end{pmatrix}$$

$$J_{33} = \begin{pmatrix} -r - (\mu + \gamma) & 0 & \cdots & 0 \\ 0 & -r - (\mu + \gamma) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -r - (\mu + \gamma) \end{pmatrix}$$

$$J_{12} = J_{21} = J_{31} == \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$J_{13} = \begin{pmatrix} -\beta \cdot 1\frac{1 \cdot p(1)}{\langle k \rangle} & -\beta \cdot 1\frac{2 \cdot p(2)}{\langle k \rangle} & \cdots & -\beta \cdot 1\frac{n \cdot p(n)}{\langle k \rangle} \\ -\beta \cdot 2\frac{1 \cdot p(1)}{\langle k \rangle} & -\beta \cdot 2\frac{2 \cdot p(2)}{\langle k \rangle} & \cdots & -\beta \cdot 2\frac{n \cdot p(n)}{\langle k \rangle} \\ \cdots & \cdots & \cdots & \cdots \\ -\beta \cdot n\frac{1 \cdot p(1)}{\langle k \rangle} & -\beta \cdot n\frac{2 \cdot p(2)}{\langle k \rangle} & \cdots & -\beta \cdot n\frac{n \cdot p(n)}{\langle k \rangle} \end{pmatrix}$$

$$J_{23} = \begin{pmatrix} \beta \cdot 1\frac{1 \cdot p(1)}{\langle k \rangle} & \beta \cdot 1\frac{2 \cdot p(2)}{\langle k \rangle} & \cdots & \beta \cdot 1\frac{n \cdot p(n)}{\langle k \rangle} \\ \beta \cdot 2\frac{1 \cdot p(1)}{\langle k \rangle} & \beta \cdot 2\frac{2 \cdot p(2)}{\langle k \rangle} & \cdots & \beta \cdot 2\frac{n \cdot p(n)}{\langle k \rangle} \\ \cdots & \cdots & \cdots & \cdots \\ \beta \cdot n\frac{1 \cdot p(1)}{\langle k \rangle} & \beta \cdot n\frac{2 \cdot p(2)}{\langle k \rangle} & \cdots & \beta \cdot n\frac{n \cdot p(n)}{\langle k \rangle} \end{pmatrix}$$

The characteristic equation of the disease-free equilibrium is

$$(\lambda + \mu)^n (\lambda + \mu + \delta)^{n-1} (\lambda + r + \mu + \gamma)^{n-1} [(\lambda + \mu + \delta)(\lambda + r + \mu + \gamma) - \delta\beta\frac{\langle k^2 \rangle}{\langle k \rangle}] = 0$$

The eigenvalues of J_{E_0} are all negative when $R_0 < 1$, the disease-free equilibrium E_0 of model (2.1) is locally asymptotically stable when $R_0 < 1$. □

Theorem 3. *For system* (2.1), *Denote* $\hat{R}_0 = \frac{\beta\langle k^2 \rangle}{(\frac{r}{1+\alpha} + \mu + \gamma)\langle k \rangle}$. *If* $\hat{R}_0 < 1$, *then the free quilibrium* E_0 *is globally asymptotically stable.*

Proof. Constructing a lyapunov function:

$$V = \sum_{k=1}^n \frac{kP(k)}{\langle k \rangle}\{S_k(t) - 1 - lnS_k(t) + E_k(t) + I_k(t)\}$$

Deriving the V function along system (2.1),

$$\begin{aligned} \frac{dV}{dt}\Big|_{(2.1)} &= \sum_{k=1}^n \frac{kP(k)}{\langle k \rangle}\{S'_k(t) - \frac{S'_k(t)}{S_k(t)} + E'_k(t) + I'_k(t)\} \\ &= \sum_{k=1}^n \frac{kP(k)}{\langle k \rangle}\{(1 - \frac{1}{S_k(t)})\{\pi - \beta kS_k(t)\theta(t) - \mu S_k(t) \\ &\quad + \beta kS_k(t)\theta(t) - p\beta kE_k(t)\theta(t) - (\mu + \delta)E_k(t) \\ &\quad + p\beta kE_k(t)\theta(t) + \delta E_k(t) - \frac{rI_k(t)}{1 + \alpha\theta(t)} - (\mu + \gamma)I_k(t)\} \\ &= \sum_{k=1}^n \frac{kP(k)}{\langle k \rangle}\{\pi - \beta kS_k(t)\theta(t) - \mu S_k(t) - \frac{\pi}{S_k(t)} + \beta k\theta(t) + \mu \\ &\quad + \beta kS_k(t)\theta(t) - p\beta kE_k(t)\theta(t) - (\mu + \delta)E_k(t) \\ &\quad + p\beta kE_k(t)\theta(t) + \delta E_k(t) - \frac{rI_k(t)}{1 + \alpha\theta(t)} - (\mu + \gamma)I_k(t)\} \\ &= \sum_{k=1}^n \frac{kP(k)}{\langle k \rangle}\pi\{1 - \frac{\mu S_k(t)}{\pi} - \frac{1}{S_k(t)} + \frac{\mu}{\pi}\} \\ &\quad + \sum_{k=1}^n \frac{kP(k)}{\langle k \rangle}\{\beta k\theta(t) - \frac{rI_k(t)}{1 + \alpha\theta(t)} - \mu E_k(t) - (\mu + \gamma)I_k(t)\} \end{aligned}$$

We now claim that if $0 < S_k(t) < 1$, the first term is negative. In fact, because of $\pi = \mu$, we have

$$1 - \frac{\mu S_k(t)}{\pi} - \frac{1}{S_k(t)} + \frac{\mu}{\pi} = [1 - S_k(t)][1 - \frac{1}{S_k(t)}] < 0$$

This means

$$\frac{dV}{dt}\Big|_{(2.1)} \le \sum_k \frac{kP(k)}{\langle k \rangle}\{\beta k\theta(t) - \frac{rI_k(t)}{1 + \alpha\theta(t)} - (\mu + \gamma)I_k(t)\}$$

$$= \frac{\beta\langle k^2 \rangle}{\langle k \rangle}\theta(t) - \frac{r\theta(t)}{1 + \alpha\theta(t)} - (\mu + \gamma)\theta(t)$$

Because $0 < I_k(t) < 1$ for all k, we know that $0 < \theta(t) < 1$. Therefore

$$\frac{dV}{dt}\Big|_{(2.1)} \le [\frac{\beta\langle k^2 \rangle}{\langle k \rangle} - \frac{r}{1 + \alpha} - (\mu + \gamma)]\theta(t)$$

$$= (\frac{r}{1 + \alpha} + \mu + \gamma))(\hat{R}_0 - 1)\theta(t)$$

If $\hat{R}_0 < 1$, $\frac{dV}{dt}\Big|_{(2.1)} \le 0$. The equation holds if and only if $I_k(t) = 0$. For the limit system $E'_k(t) = -(\mu + \delta)E_k(t)$, it is easy to see that $\lim_{t \to +\infty} E_k(t) = 0$, For the limit system $R'_k(t) = -\mu R_k(t)$, it is easy to see that $\lim_{t \to +\infty} R_k(t) = 0$. Finally, from $S_k(t) + E_k(t) + I_k(t) + R_k(t) = 1$, we get $\lim_{t \to +\infty} S_k(t) = 1$. According to the LaSalle invariant principle, E_0 is globally attractive. Combining the locally asymptotically stability, we conclued that the free quilibrium E_0 is globally asymptotically stable. □

We noticed that in Theorem 3, the disease-free equilibrium E_0 is globally asymptotically stable only when $\hat{R}_0 < 1$. In addition, you can see that $R_0 < \hat{R}_0$. This prompts us to think about whether the disease can persist even if $R_0 < 1$.

4. Bifurcation analysis

In this section, we will determine whether there is a backward bifurcation at $R_0 = 1$. More accurately, we derive the condition for determining the bifurcation direction at $R_0 = 1$. let us find endemic equilibria. At an endemic equilibrium, we have $I_k > 0$ for all $k = 1, 2, ..., n$, which means $\theta > 0$. From $F(\theta) = 0$, we can see that the unique equilibrium should satisfy the following equation :

$$\frac{1}{\langle k \rangle} \sum_{h=1}^{n} \frac{\beta h^2 P(h)}{\frac{p\beta h\theta + \mu + \delta}{p\beta h\theta + \delta}(\frac{r}{1+\alpha\theta} + \mu + \gamma) + \frac{\beta h\theta}{p\beta h\theta + \delta}(\frac{r}{1+\alpha\theta} + \mu + \gamma) + \beta h\theta + \frac{\beta h\theta}{\mu}(\frac{r}{1+\alpha\theta} + \gamma)} = 1 \qquad (4.1)$$

The denominator and numerator are multiplied by the quantity $\frac{\delta\langle k^2 \rangle}{(\mu+\delta)(r+\mu+\gamma+\mu_d)\langle k \rangle}$, which can be expressed by R_0 and θ as follows:

$$\frac{1}{\langle k \rangle} \sum_{h=1}^{n} \frac{R_0 h^2 P(h)}{\frac{\delta\langle k^2 \rangle}{(\mu+\delta)(r+\mu+\gamma)\langle k \rangle}\frac{p\beta h\theta + \mu + \delta}{p\beta h\theta + \delta}(\frac{r}{1+\alpha\theta} + \mu + \gamma) + \frac{R_0 h\theta}{p\beta h\theta + \delta}(\frac{r}{1+\alpha\theta} + \mu + \gamma + \mu_d) + R_0 h\theta + \frac{R_0 h\theta}{\mu}(\frac{r}{1+\alpha\theta} + \gamma)} = 1$$

If the local θ is a function of R_0, the sign of the bifurcation direction is the slope at $(R_0, \theta) = (1, 0)$ (cf. Figure 2). More specifically, if the derivative is positive at the critical value $(R_0, \theta) = (1, 0)$, that is

$$\frac{\partial\theta}{\partial R_0}\Big|_{(R_0,\theta)=(1,0)} > 0$$

the endemic equilibrium curve diverges forward. Conversely, if the derivative is negative at the critical value $(R_0, \theta) = (1, 0)$, that is

$$\frac{\partial\theta}{\partial R_0}\Big|_{(R_0,\theta)=(1,0)} < 0$$
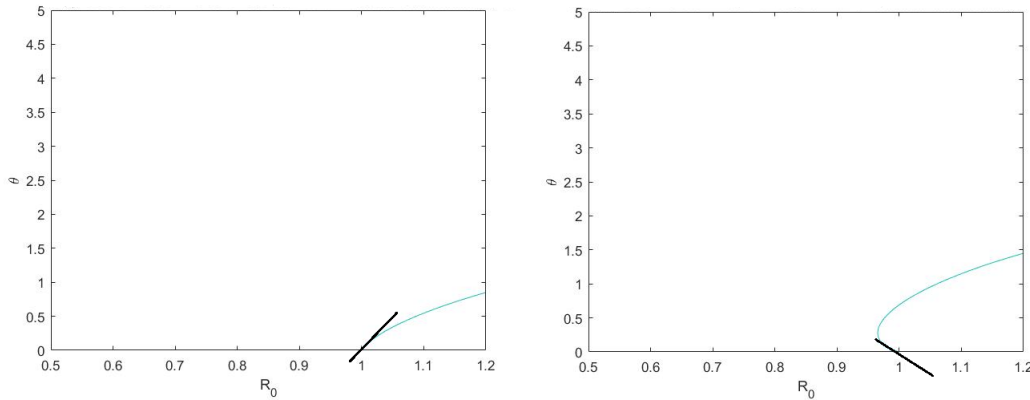
Figure 2. The left figure shows the forward bifurcation at $R_0 = 1$, where the derivative at $(R_0, \theta) = (1, 0)$ is positive. The figure on the right shows the backward bifurcation at $R_0 = 1$, where the derivative at $(R_0, \theta) = (1, 0)$ is negative.

the local equilibrium curve diverges backward.

$$\frac{1}{\langle k \rangle} \sum_{h=1}^{n} \frac{I_1 - I_2}{\{\frac{\delta \langle k^2 \rangle}{(\mu+\delta)(r+\mu+\gamma)\langle k \rangle} \frac{p\beta h\theta+\mu+\delta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+\frac{R_0 h\theta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+R_0 h\theta+\frac{R_0 h\theta}{\mu}(\frac{r}{1+\alpha\theta}+\gamma)\}^2} = 0$$

$$I_1 = h^2 P(h)\{\frac{\delta \langle k^2 \rangle}{(\mu+\delta)(r+\mu+\gamma)\langle k \rangle} \frac{p\beta h\theta+\mu+\delta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)$$
$$+ \frac{R_0 h\theta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+R_0 h\theta+\frac{R_0 h\theta}{\mu}(\frac{r}{1+\alpha\theta}+\gamma)\}$$

where

$$I_1|_{\substack{R_0=1 \\ \theta=0}} = h^2 P(h)\frac{\langle k^2 \rangle}{\langle k \rangle}$$

where

$$I_2 = R_0 h^2 P(h)\{\frac{\delta \langle k^2 \rangle}{(\mu+\delta)(r+\mu+\gamma)\langle k \rangle}[\frac{-\mu p\beta h}{(p\beta h\theta+\delta)^2}(\frac{r}{1+\alpha\theta}+\mu+\gamma)$$
$$+ \frac{p\beta h\theta+\mu+\delta}{p\beta h\theta+\delta}\frac{-r\alpha}{(1+\alpha\theta)^2}]\frac{\partial\theta}{\partial R_0} + \frac{h\theta}{p\beta h\theta+\delta}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+$$
$$[\frac{R_0 h\delta}{(p\beta h\theta+\delta)^2}(\frac{r}{1+\alpha\theta}+\mu+\gamma)+\frac{R_0 h\theta}{p\beta h\theta+\delta}\frac{-r\alpha}{(1+\alpha\theta)^2}]\frac{\partial\theta}{\partial R_0}+h\theta$$
$$+ R_0 h\frac{\partial\theta}{\partial R_0}+\frac{h\theta}{\mu}(\frac{r}{1+\alpha\theta}+\gamma)+[\frac{R_0 h}{\mu}(\frac{r}{1+\alpha\theta}+\gamma)+\frac{R_0 h\theta}{\mu}\frac{-r\alpha}{(1+\alpha\theta)^2}]\frac{\partial\theta}{\partial R_0}\}$$

$$I_2|_{\substack{R_0=1 \\ \theta=0}} = h^2 P(h)\{-\frac{hp\beta\mu\langle k^2 \rangle}{\delta(\mu+\delta)\langle k \rangle}-\frac{r\alpha\langle k^2 \rangle}{(r+\mu+\gamma)\langle k \rangle}+\frac{h}{\delta}(r+\mu+\gamma)+h+\frac{h}{\mu}(r+\gamma)\}\frac{\partial\theta}{\partial R_0}$$

Substituting $R_0 = 1$ amd $\theta = 0$ into Eq (4.1), we have

$$\frac{1}{\langle k \rangle} \sum_{h=1}^{n} h^2 P(h)\frac{\frac{\langle k^2 \rangle}{\langle k \rangle}-\{-\frac{hp\beta\mu\langle k^2 \rangle}{\delta(\mu+\delta)\langle k \rangle}-\frac{r\alpha\langle k^2 \rangle}{(r+\mu+\gamma)\langle k \rangle}+\frac{h}{\delta}(r+\mu+\gamma)+h+\frac{h}{\mu}(r+\gamma)\}\frac{\partial\theta}{\partial R_0}}{(\frac{\langle k^2 \rangle}{\langle k \rangle})^2} = 0$$

This yields

$$\frac{1}{\langle k\rangle}\sum_{h=1}^{n}h^2 P(h)\frac{\{-\frac{hp\beta\mu\langle k^2\rangle}{\delta(\mu+\delta)\langle k\rangle}-\frac{r\alpha\langle k^2\rangle}{(r+\mu+\gamma)\langle k\rangle}+\frac{h}{\delta}(r+\mu+\gamma)+h+\frac{h}{\mu}(r+\gamma)\}\frac{\partial\theta}{\partial R_0}}{(\frac{\langle k^2\rangle}{\langle k\rangle})^2}=1$$

From this we have

$$\{-\frac{p\beta\mu\langle k^3\rangle}{\delta(\mu+\delta)\langle k^2\rangle}-\frac{r\alpha}{r+\mu+\gamma}+\frac{r+\mu+\gamma}{\delta}\frac{\langle k\rangle\langle k^3\rangle}{\langle k^2\rangle^2}+\frac{\langle k\rangle\langle k^3\rangle}{\langle k^2\rangle^2}+\frac{r+\gamma}{\mu}\frac{\langle k\rangle\langle k^3\rangle}{\langle k^2\rangle}\}\frac{\partial\theta}{\partial R_0}\Big|_{\substack{R_0=1\\\theta=0}}=1 \qquad (4.2)$$

We can from Eq (4.2) that

$$\frac{\partial\theta}{\partial R_0}\Big|_{(R_0,\theta)=(1,0)}<0$$

$$\Leftrightarrow \alpha>\frac{r+\mu+\gamma}{r}[\frac{r+\mu+\gamma}{\delta}+1+\frac{r+\gamma}{\mu}-\frac{p\beta\mu\langle k^2\rangle}{\delta(\mu+\delta)\langle k\rangle}]\frac{\langle k\rangle\langle k^3\rangle}{\langle k^2\rangle^2}$$

To conclude, we have the following theorem:

Theorem 4. *System (2.1) has backward bifurcation at $R_0=1$ if and only if*

$$\alpha>\frac{r+\mu+\gamma}{r}[\frac{r+\mu+\gamma}{\delta}+1+\frac{r+\gamma}{\mu}-\frac{p\beta\mu\langle k^2\rangle}{\delta(\mu+\delta)\langle k\rangle}]\frac{\langle k\rangle\langle k^3\rangle}{\langle k^2\rangle^2} \qquad (4.3)$$

where $\langle k^2\rangle=\sum_{h=1}^{n}h^2 P(h)$, $\langle k^3\rangle=\sum_{h=1}^{n}h^3 P(h)$.

It can be seen from Theorem 4 that the nonlinear processing function does play a key role in causing backward bifurcation. More precisely, if α is large enough to satisfy condition (4.3), backward bifurcation will occur. Otherwise, when this effect is weak, there is no backward bifurcation.

5. Optimal quarantine control

In order to achieve the control goal and reduce the control cost, the optimal control theory is a feasible method.

Due to practical needs, we define a limited terminal time. Then, model (2.1) is rewritten as

$$\begin{aligned}
\frac{dS_k(t)}{dt} &= \pi-\beta k S_k(t)\theta(t)-\mu S_k(t),\\
\frac{dE_k(t)}{dt} &= \beta k S_k(t)\theta(t)-p\beta k E_k(t)\theta(t)-(\mu+\delta)E_k(t),\\
\frac{dI_k(t)}{dt} &= p\beta k E_k(t)\theta(t)+\delta E_k(t)-\frac{r_k(t)I_k(t)}{1+\alpha\theta(t)}-(\mu+\gamma)I_k(t),\\
\frac{dR_k(t)}{dt} &= \frac{r_k(t)I_k(t)}{1+\alpha\theta(t)}+\gamma I_k(t)-\mu R_k(t)
\end{aligned} \qquad (5.1)$$

The objective function is set as

$$J(r_k(t)) = \int_0^T \sum_{k=1}^n [I_k(t) + \frac{1}{2} A_k r_k^2(t)] dt, \tag{5.2}$$

with Lagrangian

$$L(X(t), r_k(t)) = \sum_{k=1}^n [I_k(t) + \frac{1}{2} A_k r_k^2(t)] \tag{5.3}$$

Our objective is to find a optimal control $r_k^*(t)$ such that

$$J(r_k^*(t)) = \min_U \int_0^T L(X(t), r_k(t)) dt \tag{5.4}$$

where $U = \{r_k, 0 \leq r_k(t) \leq 1, t \in [0, T]\}$ is the control set. Next we will analyze the optimal control problem.

5.1. The existence of optimal control solution

In order to obtain the analytic solution of the optimal control, we need to prove the existence of the optimal control first. The following lemma comes from reference [33].

Lemma 2. *If the following five conditions can be met simultaneously:*
(C1) U is closed and convex.
(C2) There is $r \in U$ such that the constraint dynamical system is solvable.
(C3) $F(X(t), r_k(t))$ is bounded by a linear function in X.
(C4) $L(X(t), r_k(t))$ is concave on U.
(C5) $L(X(t), r_k(t)) \geq c_1 \|r\|_2^\varphi + C_2$ for some $\varphi > 1$, $c_1 > 0$ and c_2.
Then the optimal control problem has an optimal solution.

Theorem 5. *There exists an optimal solution $r^*(t)$ stastify the control system.*

Proof. Let us show that the five conditions in Lemma 3 hold true.

i) It is easy to prove the control set U is closed and convex. Suppose r is a limit point of U, there exists a sequence of points $\{r_n\}_{n=1}^\infty$, we have $r \in (L^2[0, T])^n$. The closedness of U follows from $0 \leq r = \lim_{n \to \infty} r_n \leq 1$.
   Let $r_1, r_2 \in U$, $\eta \in (0, 1)$, we have $0 \leq (1 - \eta)r_1 + \eta r_2 \leq 1 \in (L^2[0, T])^n$ as $(L^2[0, T])^n$ is a real vector space.

ii) For any control variable $r \in U$, the solution of system (12) obviously exists following from the Continuation Theorem for Differential Systems [34].

iii) Let the function $F(X(t), r_k(t))$ resprensts the right side of model (5.1), F is continuous, bounded and can be written as a linear function of X in three state.

$$-\beta k S_k(t) - \mu S_k(t) \leq -\beta k S_k(t)\theta(t) - \mu S_k(t) \leq \dot{S}_k(t) \leq \pi,$$

$$-p\beta k E_k(t) - (\mu + \delta)E_k(t) \leq -p\beta k E_k(t)\theta(t) - (\mu + \delta)E_k(t) \leq \dot{E}_k(t) \leq \beta k S_k(t),$$

$$-r_k(t)I_k(t) - (\mu + \gamma)I_k(t) \le -\frac{r_k(t)I_k(t)}{1 + \alpha\theta(t)} - (\mu + \gamma)I_k(t) \le \dot{I}_k(t) \le p\beta k E_k(t) + (\mu + \delta)E_k(t),$$

$$-\mu R_k(t) \le \dot{R}_k(t) \le \frac{r_k(t)I_k(t)}{1 + \alpha\theta(t)} + \gamma I_k(t) \le r_k(t)I_k(t).$$

iv) $L(X(t), r_k(t))$ is concave on U. We calculate the function $\frac{\partial^2 L}{\partial r^2} = r_k(t) \ge 0$, so the function L is concave on U.

v) Since $I_k(t) \ge 0$, $L \ge \sum_{k=1}^{n} \frac{1}{2} A_k r_k^2(t) \ge \frac{1}{2} min\{A_k\} \|r\|_2^2$. There exsits $\varphi = 2$, $C_1 = \frac{1}{2} min\{A_k\}$ and $C_2 = 0$ such that $L(X(t), r_k(t)) \ge c_1 \|r\|_2^\varphi + C_2$.

$\square$

5.2. Solution to the optimal control problem

In this section, we will deal with the OCP based on the Pontryagin Maximum Principle [35]. Define the Hamiltonian H as

$$H = \sum_{k=1}^{n}[I_k(t) + \frac{1}{2}A_k r_k^2(t)] + \sum_{k=1}^{n}[\lambda_{1k}(t)\dot{S}_k(t) + \lambda_{2k}(t)\dot{E}_k(t) + \lambda_{3k}(t)\dot{I}_k(t) + \lambda_{4k}(t)\dot{R}_k(t)]$$

where $\lambda_{1k}(t), \lambda_{2k}(t), \lambda_{3k}(t), \lambda_{4k}(t)$ are the adjoint variables to be determined later.

Theorem 6. *Let $S_k^*(t)$, $E_k^*(t)$, $I_k^*(t)$, $R_k^*(t), k = 1, 2, \cdots, n$ be the optimal state solutions of dynamic model (5.1) related to the optimal control $r^*(t) = (r_1^*(t), r_2^*(t), \cdots, r_n^*(t))$. Let $\theta^*(t) = \frac{\sum_{k=1}^{n} kp(k)I_k^*(t)}{\langle k \rangle}$. And there exist adjoint varibles $\lambda_{1k}(t), \lambda_{2k}(t), \lambda_{3k}(t), \lambda_{4k}(t)$ that satisfy*

$$\begin{cases} \dot{\lambda}_{1k}(t) = \beta k \theta^*(t)[\lambda_{1k}(t) - \lambda_{2k}(t)] + \mu\lambda_{1k}(t), \\ \dot{\lambda}_{2k}(t) = p\beta k\theta^*(t)[\lambda_{2k}(t) - \lambda_{3k}(t)] + \delta[\lambda_{2k}(t) - \lambda_{3k}(t)] + \mu\lambda_{2k}(t), \\ \dot{\lambda}_{3k}(t) = -1 + \frac{1}{\langle k \rangle}\beta k P(k)\sum_{k=1}^{n} kS_k^*(t)(\lambda_{1k}(t) - \lambda_{2k}(t)) \\ \qquad + \frac{1}{\langle k \rangle}p\beta k P(k)\sum_{k=1}^{n} kE_k^*(t)(\lambda_{2k}(t) - \lambda_{3k}(t)) + \frac{r_k^*(t)}{1 + \alpha\theta^*(t)}(\lambda_{3k}(t) - \lambda_{4k}(t)) \\ \qquad - \frac{r_k^*(t)\alpha k P(k)}{(1 + \alpha\theta^*(t))^2\langle k \rangle}\sum_{k=1}^{n} I_k^*(t)(\lambda_{3k}(t) - \lambda_{4k}(t)) + (\mu + \gamma)\lambda_{3k}(t) - \gamma\lambda_{4k}(t), \\ \dot{\lambda}_{4k}(t) = \mu\lambda_{4k}(t) \end{cases}$$

with transversality condition

$$\lambda_{1k}(T) = \lambda_{2k}(T) = \lambda_{3k}(T) = \lambda_{4k}(T) = 0, k = 1, 2, \cdots, n$$

In addition, the optimal control $r_k^*(t)$ is given by

$$r_k^*(t) = \min\left\{\max\left(0, \frac{1}{B_k}\{\frac{I_k^*(t)}{1 + \alpha\theta^*(t)}[\lambda_{3k}(t) - \lambda_{4k}(t)]\}\right), 1\right\}.$$

Proof. According to the Pontryagin Maximum Principle [cankaowenxian] with the Hamiltonianfunction, the adjoint equations can be determined by the following equations

$$
\begin{aligned}
\frac{d\lambda_{1k}(t)}{dt} &= -\frac{\partial H}{\partial S_k}\bigg|_{S_k(t)=S_k^*(t),E_k(t)=E_k^*(t),I_k(t)=I_k^*(t),R_k(t)=R_k^*(t)} \\
&= \beta k\theta^*(t)[\lambda_{1k}(t) - \lambda_{2k}(t)] + \mu\lambda_{1k}(t), \\
\frac{d\lambda_{2k}(t)}{dt} &= -\frac{\partial H}{\partial E_k}\bigg|_{S_k(t)=S_k^*(t),E_k(t)=E_k^*(t),I_k(t)=I_k^*(t),R_k(t)=R_k^*(t)} \\
&= p\beta k\theta^*(t)[\lambda_{2k}(t) - \lambda_{3k}(t)] + \delta[\lambda_{2k}(t) - \lambda_{3k}(t)] + \mu\lambda_{2k}(t), \\
\frac{d\lambda_{3k}(t)}{dt} &= -\frac{\partial H}{\partial I_k}\bigg|_{S_k(t)=S_k^*(t),E_k(t)=E_k^*(t),I_k(t)=I_k^*(t),R_k(t)=R_k^*(t)} \\
&= -1 + \frac{1}{\langle k \rangle}\beta k P(k) \sum_{k=1}^{n} k S_k^*(t)(\lambda_{1k}(t) - \lambda_{2k}(t)) \\
&\quad + \frac{1}{\langle k \rangle}p\beta k P(k) \sum_{k=1}^{n} k E_k^*(t)(\lambda_{2k}(t) - \lambda_{3k}(t)) + \frac{r_k^*(t)}{1 + \alpha\theta^*(t)}(\lambda_{3k}(t) - \lambda_{4k}(t)) \\
&\quad - \frac{r_k^*(t)\alpha k P(k)}{(1 + \alpha\theta^*(t))^2\langle k \rangle} \sum_{k=1}^{n} I_k^*(t)(\lambda_{3k}(t) - \lambda_{4k}(t)) + (\mu + \gamma + \mu_d)\lambda_{3k}(t) - \gamma\lambda_{4k}(t), \\
\frac{d\lambda_{4k}(t)}{dt} &= -\frac{\partial H}{\partial R_k}\bigg|_{S_k(t)=S_k^*(t),E_k(t)=E_k^*(t),I_k(t)=I_k^*(t),R_k(t)=R_k^*(t)} \\
&= \mu\lambda_{4k}(t)
\end{aligned}
\tag{5.5}
$$

Furthermore, by the necessary condition, we have

$$
\frac{\partial H}{\partial r_k}\bigg|_{S_k(t)=S_k^*(t),E_k(t)=E_k^*(t),I_k(t)=I_k^*(t),R_k(t)=R_k^*(t)} = 0
\tag{5.6}
$$

$$
B_k r_k^*(t) + \frac{I_k^*(t)}{1 + \alpha\theta^*(t)}[\lambda_{4k}(t) - \lambda_{3k}(t)] = 0
\tag{5.7}
$$

$$
r_k^*(t) = \frac{1}{B_k}\left\{\frac{I_k^*(t)}{1 + \alpha\theta^*(t)}[\lambda_{3k}(t) - \lambda_{4k}(t)]\right\}.
\tag{5.8}
$$

So the optimal control problem can be determined by the following system:

$$
\begin{cases}
\dfrac{dS_k^*(t)}{dt} = \pi - \beta k S_k^*(t)\theta^*(t) - \mu S_k^*(t), \\[2mm]
\dfrac{dE_k^*(t)}{dt} = \beta k S_k^*(t)\theta^*(t) - p\beta k E_k^*(t)\theta^*(t) - (\mu + \delta)E_k^*(t), \\[2mm]
\dfrac{dI_k^*(t)}{dt} = p\beta k E_k^*(t)\theta^*(t) + \delta E_k^*(t) - \dfrac{r_k^*(t)I_k(t)}{1 + \alpha\theta^*(t)} - (\mu + \gamma)I_k^*(t), \\[2mm]
\dfrac{dR_k^*(t)}{dt} = \dfrac{r_k^*(t)I_k(t)}{1 + \alpha\theta^*(t)} + \gamma I_k^*(t) - \mu R_k^*(t) \\[2mm]
\dot{\lambda}_{1k}(t) = \beta k\theta^*(t)[\lambda_{1k}(t) - \lambda_{2k}(t)] + \mu\lambda_{1k}(t), \\[2mm]
\dot{\lambda}_{2k}(t) = p\beta k\theta^*(t)[\lambda_{2k}(t) - \lambda_{3k}(t)] + \delta[\lambda_{2k}(t) - \lambda_{3k}(t)] + \mu\lambda_{2k}(t), \\[2mm]
\dot{\lambda}_{3k}(t) = -1 + \dfrac{1}{\langle k \rangle}\beta k P(k)\displaystyle\sum_{k=1}^{n} k S_k^*(t)(\lambda_{1k}(t) - \lambda_{2k}(t)) \\[4mm]
\qquad\quad + \dfrac{1}{\langle k \rangle}p\beta k P(k)\displaystyle\sum_{k=1}^{n} k E_k^*(t)(\lambda_{2k}(t) - \lambda_{3k}(t)) + \dfrac{r_k^*(t)}{1 + \alpha\theta^*(t)}(\lambda_{3k}(t) - \lambda_{4k}(t)) \\[4mm]
\qquad\quad - \dfrac{r_k^*(t)\alpha k P(k)}{(1 + \alpha\theta^*(t))^2\langle k \rangle}\displaystyle\sum_{k=1}^{n} I_k^*(t)(\lambda_{3k}(t) - \lambda_{4k}(t)) + (\mu + \gamma)\lambda_{3k}(t) - \gamma\lambda_{4k}(t), \\[4mm]
\dot{\lambda}_{4k}(t) = \mu\lambda_{4k}(t), \\[2mm]
r_k^*(t) = \min\left\{\max\left(0, \dfrac{1}{B_k}\left\{\dfrac{I_k^*(t)}{1 + \alpha\theta^*(t)}[\lambda_{3k}(t) - \lambda_{4k}(t)]\right\}\right), 1\right\}.
\end{cases}
\tag{5.9}
$$

with initial condition

$$
\begin{cases}
0 < S_k^*(0) < 1, \\
0 < I_k^*(0) < 1, \\
E_k^*(0) = 0, \\
R_k^*(0) = 0, \\
S_k^*(0) + I_k^*(0) = 1,
\end{cases}
\tag{5.10}
$$

and transversality condition

$$
\lambda_{1k}(T) = \lambda_{2k}(T) = \lambda_{3k}(T) = \lambda_{4k}(T) = 0, k = 1, 2, \cdots, n
\tag{5.11}
$$

$\square$

6. Stochastic and numerical simulations

Owing to our models are network-based models, it is indispensable to carry out stochastic simulations as well as numerical simulations. The validity of the model can be better verified by means of combining numerical simulations of differential system and node-based stochastic simulations.

Table 1. The parameter value.

Parameter	Value
π	0.1
δ	0.8
μ	0.01
γ	0.001
p	0.15
β	0.08

6.1. Experimental setup

In the following examples, numerical simulations is implementedby using the classic BA scale-free network model generation algorithm. We consider scale-free networks with degree distribution $P(k) = \eta k^{-2.5}$ for $h = 1, 2, ..., 100$, where the constant η is chosen to maintain $\sum_{k=1}^{100} p(k) = 1$. Under the aforementioned conditions, we can obtain $\langle k \rangle = 1.7995$, $\langle k^2 \rangle = 13.8643$, $\langle k^3 \rangle = 500.7832$.

Optimality system (5.9) is solved by using the forward-backward Runge-Kutta fourth order method [36].

he setting of the parameters of the models are summarized in Table 1. To note that the parameters can be changed according to various application scenarios.

6.2. Experimental results

Example 1. The first example is used to verify the threshold and stability results of the model. For the network, in the case of Figure 3 (a), (b), (c), we draw the trajectory of the average number of infected $\sum_{k=1}^{n} p(k)I_k^*$ when $R_0 < 1$, $R_0 = 1$, and $R_0 > 1$.
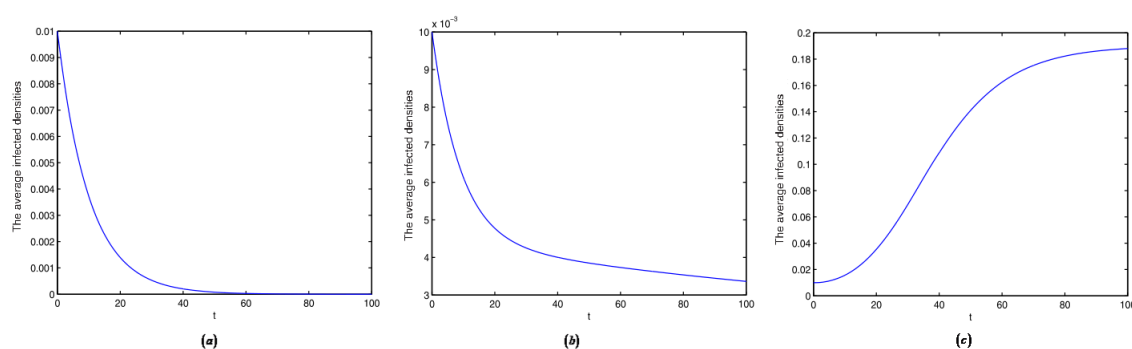

Figure 3. The average infected densities $\sum_{k=1}^{n} p(k)I_k^*$ with respect to (a) $R_0 < 1$, (b) $R_0 = 1$, (c) $R_0 > 1$.

Example 2. In the second example, we study the case of forward bifurcation. We choose parameter $\alpha = 22, 25, 28, 31$ to describe the bifurcation diagram in Figure 3, showing a positive bifurcation when

$R_0 = 1$. An interesting observation is that the local balance curve has an "S-shaped" α value. In this case, there is backward bifurcation from an endemic equilibrium at a certain value of R_0, which leads to the existence of multiple characteristic balances.
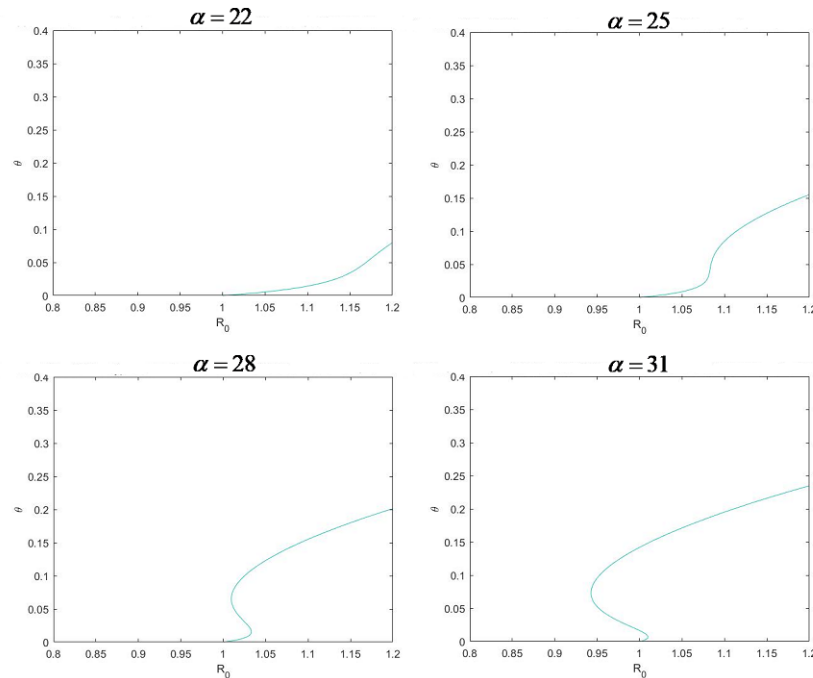

Figure 4. Bifurcation diagrams in the (R_0, θ)-plane with different values of α. Forward bifurcation occurs at $R_0 = 1$ for each panel.

Example 3. The third example shows the effectiveness of the optimal control strategy. We choose parameter $\alpha = 31$. The initial conditions are $S_k(0) = 0.9$, $E_k(0) = 0$, $I_k(0) = 0.1$, $R_k(0) = 0$. For the optimal control problems, the weights parameters are set as $A_k = 0.8$, $T = 10$. Through strategy simulation, the mean value of optimal control is obtained $\langle r \rangle = \frac{1}{T} \int_0^T r_k^*(t)dt$. The average value of the control measures is shown in the middle column of Figure 4. The left column shows the average number of infections for different control strategies, and the right column shows the cost. It can be seen from Figure 3 that in each case, the optimal control effect has reached the maximum control effect, but the cost is lower.

7. Conclusions

In order to understand the impact of limited medical resources and population heterogeneity on disease transmission, this paper proposes a SEIR model with a saturated processing function based on a complex network. The model proved that there is a threshold R_0, which determines the stability of the disease-free equilibrium point. In addition, we also proved that there is a backward branch at $R_0 = 1$. In this case, people cannot eradicate the disease unless the value of R_0 decreases such that $R_0 < \hat{R}_0 < 1$. In summary, in order to control or prevent infectious diseases, our research results show that if a backward bifurcation occurs at $R_0 = 1$, then we need a stronger condition to eliminate the
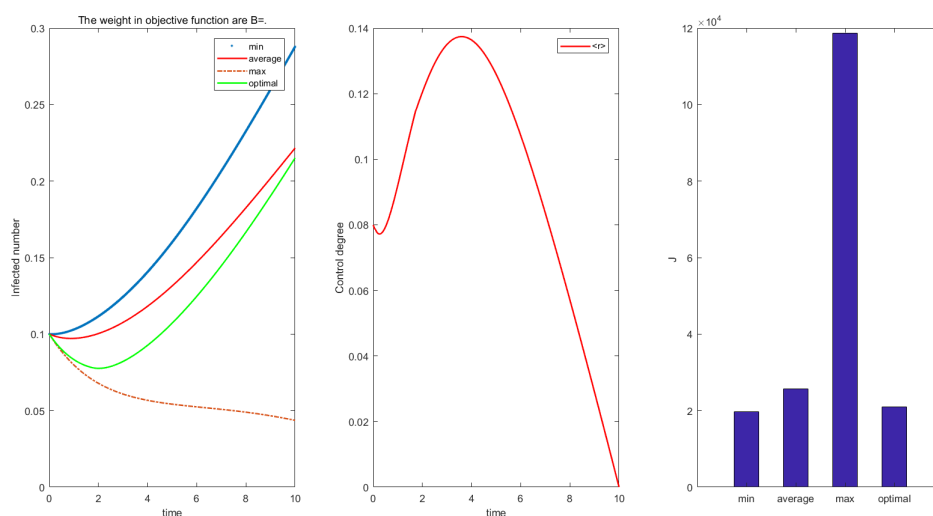
Figure 5. Infected number, control degree and cost.

disease. However, if a positive bifurcation occurs at $R_0 = 1$, the initial infectious invasion may need to be controlled at a low level, so that disease or death or close to a low endemic state of stability.

For the SEIR model with saturation processing function, in order to formulate effective countermeasures, two methods of constant control and time-varying control are introduced to control the model. For the optimal defense situation, the optimal control problem is discussed, including the existence and uniqueness of the optimal control and its solution. Finally, we performed some numerical simulations to illustrate the theoretical results. Optimal control does achieve a balance between control objectives and control costs.

Acknowledgments

Conflict of interest

The authors declare there is no conflict of interest in this paper.

References

1. R. Pastor-Satorras, A. Vespignani, Epidemic spreading in scale-free networks, *Phys. Rev. Lett.*, **86** (2001), 3200–3203. doi: 10.1103/PhysRevLett.86.3200.

2. R. Pastor-Satorras, A. Vespignani, Epidemic dynamics and endemic states in complex networks, *Phys. Rev. E*, **63** (2001), 066117. doi: 10.1103/PhysRevE.63.066117.

3. Y. Moreno, R. Pastor-Satorras, A. Vespignani, Epidemic outbreaks in complex heterogeneous networks, *Eur. Phys. J. B*, **26** (2002), 521–529. doi: 10.1140/epjb/e20020122.

4. C. Li, C. Tsai, S. Yang, Analysis of epidemic spreading of an SIRS model in complex heterogeneous networks, *Commun. Nonlinear Sci.*, **19** (2014), 1042–1054. doi: 10.1016/j.cnsns.2013.08.033.

5. S. Huang, F. Chen, L. Chen, Global dynamics of a network-based SIQRS epidemic model with demographics and vaccination, *Commun. Nonlinear Sci.*, **43** (2017), 296–310. doi: 10.1016/j.cnsns.2016.07.014.

6. T. Li, Y. Wang, Z. Guan, Spreading dynamics of a SIQRS epidemic model on scale-free networks, *Commun. Nonlinear Sci.*, **19** (2014), 686–692. doi: 10.1016/j.cnsns.2013.07.010.

7. J. Juang, Y. H. Liang, Analysis of a general SIS model with infective vectors on the complex networks, *Physica A*, **437** (2015), 382–395. doi: 10.1016/j.physa.2015.06.006.

8. Y. Wang, Z. Jin, Z. Yang, Z. Zhang, T. Zhou, G. Sun, Global analysis of an SIS model with an infective vector on complex networks, *Nonlinear Anal. Real World Appl.*, **13** (2012), 543–557. doi: 10.1016/j.nonrwa.2011.07.033.

9. M. Yang, G. Chen, X. Fu, A modified SIS model with an infective medium on complex networks and its global stability, *Phys. A*, **390** (2011), 2408–2413. doi: 10.1016/j.physa.2011.02.007.

10. Q. Wu, X. Fu, M. Yang, Epidemic thresholds in a heterogenous population with competing strains, *Chinese Phys. B*, **20** (2011), 046401. doi: 10.1088/1674-1056/20/4/046401.

11. Q. Wu, M. Small, H. Liu, Superinfection behaviors on scale-free networks with competing strains, *J. Nonlinear Sci.*, **23** (2013), 113–127. doi: 10.1007/s00332-012-9146-1.

12. J. Yang, C. H. Li, Dynamics of a competing two-strain SIS epidemic model on complex networks with a saturating incidence rate, *J. Phys. A*, **49** (2016), 215601. doi: 10.1088/1751-8113/49/21/215601.

13. L. J. Chen, J. T. Sun, Global stability and optimal control of an SIRS epidemic model on heterogeneous networks, *Phys. A*, **410** (2014), 196–204. doi: 10.1016/j.physa.2014.05.034.

14. L. J. Chen, J. T. Sun, Optimal vaccination and treatment of an epidemic network model, *Phys. Lett. A*, **378** (2014), 3028–3036. doi: 10.1016/j.physleta.2014.09.002.

15. D. G. Xu, X. Y. Xu, Y. F. Xie, C. H. Yang, Optimal control of an SIVRS epidemic spreading model with virus variation based on complex networks, *Commun. Nonlinear Sci.*, **48** (2017), 200–210. doi: 10.1016/j.cnsns.2016.12.025.

16. N. Jia, L. Ding, Y. J. Liu, P. Hu, Global stability and optimal control of epidemic spreading on multiplex networks with nonlinear mutual interaction, *Phys. A*, **502** (2018), 93–105. doi: 10.1016/j.physa.2018.02.056.

17. K. Li, G. Zhu, Z. Ma, L. Chen, Dynamic stability of an SIQS epidemic network and its optimal control, *Commun. Nonlinear Sci.*, **66** (2019), 84–95. doi: 10.1016/j.cnsns.2018.06.020.

18. L. J. Chen, S. Y. Huang, F. D. Chen, M. J. Fu, The bifurcation analysis and optimal feedback mechanism for an SIS epidmic model on networks, *Adv. Differ. Equations*, **529** (2019), 1–13. doi: 10.1186/s13662-019-2460-2.

19. C. J. Xu, M. X. Liao, P. L. Li, Y. Guo, Q. M. Xiao, S. Yuan, Influence of multiple time delays on bifurcation of fractional-order neural networks, *Appl. Math. Comput.*, **361** (2019), 565–582. doi: 10.1016/j.amc.2019.05.057.

20. Y. K. Xie, Z. Wang, J. W. Lu, Y. X. Li, Stability analysis and control strategies for a new SIS epidemic model in heterogeneous networks, *Math. Comput.*, **383** (2020), 125381. doi: 10.1016/j.amc.2020.125381.

21. Y. K. Xie, Z. Wang, Transmission dynamics, global stability and control strategies of a modified SIS epidemic model on complex networks with an infective medium, *Math. Comput. Simul.*, **188** (2021), 23–34. doi: 10.1016/j.matcom.2021.03.029.

22. C. J. Xu, Z. X. Lin, M. X. Liao, P. L. Li, Q. M. Xiao, S. Yuan, Fractional-order bidirectional associate memory (BAM) neural networks with multiple delays: The case of Hopf bifurcation, *Math. Comput. Simulat.*, **182** (2021), 471–494. doi: 10.1016/j.matcom.2020.11.023.

23. C. J. Xu, Z. X. Lin, L. Y. Yao, C. Aouiti, Further exploration on bifurcation of fractional-order six-neuron bi-directional associative memory neural networks with multi-delays, *Appl. Math. Comput.*, **410** (2021), 126458. doi: 10.1016/j.amc.2021.126458.

24. W. D. Wang, S. G. Ruan, Bifurcation in epidemic model with constant removal rate infectives, *J. Math. Anal. Appl.*, **291** (2015), 775–793. doi: 10.1016/j.jmaa.2003.11.043.

25. W. D. Wang, Backward Bifurcation of An Epidemic Model with Treatment, *Math. Biosci.*, **201** (2006), 58–71. doi: 10.1016/j.mbs.2005.12.022.

26. X. Zhang, X. Liu, Backward bifurcation of an epidemic model with saturated treatment function, *J. Math. Anal. Appl.*, **348** (2008), 433–443. doi: 10.1016/j.jmaa.2008.07.042.

27. J. Wang, S. Liu, B. Zhang, Y. Takeuchi, Qualitative and bifurcation analysis using an SIR model with a saturated treatment function, *Math. Comput. Model.*, **55** (2012), 710–722. doi: 10.1016/j.mcm.2011.08.045.

28. J. Wei, J. Cui, Dynamics of SIS epidemic model with the standard incidence rate and saturated treatment function, *Int. J. Biomath.*, **5** (2012),1260003. doi: 10.1142/S1793524512600030.

29. J. Cui, X. Mu, H. Wan, Saturation recovery leads to multiple endemic equilibria and backward bifurcation, *J. Theor. Biol.*, **254** (2008), 275–283. doi: 10.1016/j.jtbi.2008.05.015.

30. L. H. Zhou, M. Fan, Dynamics of an SIR epidemic model with limited medical resources revisited, *Nonlinear Anal. Real World Appl.*, **13** (2012), 312–324. doi: 10.1016/j.nonrwa.2011.07.036.

31. I. M. Wangari, S. Davis, L. Stonea, Backward bifurcation in epidemic models: Problems arising with aggregated bifurcation parameters, *Appl. Math. Model.*, **40** (2016), 1669–1675. doi: 10.1016/j.apm.2015.07.022.

32. P. Van Den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.*, **180** (2002), 29–48. doi: 10.1016/S0025-5564(02)00108-6.

33. F. D. Sahneh, C. Scoglio, P. V. Mieghem, Generalized epidemic mean-field model for spreading processes over multilayer complex networks, *IEEE/ACM Trans. Netw.*, **21** (2013), 1609–1620. doi: 10.1109/TNET.2013.2239658.

34. R. C. Robinson, *An Introduction to Dynamical Systems: Continuous and Discrete*, American Mathematical Society, 2012.

35. D. E. Kirk, *Optimal Control Theory: An Introduction*, Dover Publications, 2004.

36. B. Buonomo, D. Lacitignola, C. Vargas-De-León, Qualitative analysis and optimal control of an epidemic model with vaccination and treatment, *Math. Comput. Simulat.*, **100** (2014), 88–102. doi: 10.1016/j.matcom.2013.11.005.