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Research article

Third-order neutral differential equations of the mixed type: Oscillatory and asymptotic behavior

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Abstract: In this work, by using both the comparison technique with first-order differential inequalities and the Riccati transformation, we extend this development to a class of third-order neutral differential equations of the mixed type. We present new criteria for oscillation of all solutions, which improve and extend some existing ones in the literature. In addition, we provide an example to illustrate our results.

Keywords: thrid-order differential equations; delay; mixed-neutral term; oscillation criteria

1. Introduction

This work is concerned with the oscillatory properties of the third-order half-linear delay differential equation (DDE) with mixed neutral term

$$\left(m\left(u\right)\left(y''\left(u\right)\right)^{\kappa}\right)' + q\left(u\right)x^{\kappa}\left(\theta\left(u\right)\right) = 0,\ u \geq u_0, \tag{1.1}$$

where κ is a quotient of add positive integers and

$$y(u) := x(u) + \rho_1(u)\, x(\tau_1(u)) + \rho_2(u)\, x(\tau_2(u)). \tag{1.2}$$

Throughout this work, we assume the following hypotheses:

($\mathbf{I}_1$) $m, q \in C([u_0, \infty), (0, \infty))$ and
$$\int_{u_0}^{\infty} m^{-1/\kappa}(s)\, \mathrm{d}s = \infty;$$
($\mathbf{I}_2$) $\rho_i \in C([u_0, \infty), [0, \infty)), \rho_i(u) \leq \rho_i < \infty$ such that ρ_i are positive constants for $i = 1, 2$;
($\mathbf{I}_3$) $\theta, \tau_i \in C^1([u_0, \infty), \mathbb{R}), \theta(u) < u, \tau_1(u) \leq u, \tau_2(u) > u, \tau_i'(u) \leq \tau_i, \theta \circ \tau_i = \tau_i \circ \theta$ and $\lim_{u\to\infty} \theta(u) = \lim_{u\to\infty} \tau_i(u) = \infty$ for $i = 1, 2$.

A function x is a solution of (1.1) if $x \in C([u_x, \infty), \mathbb{R}))$ that satisfies (1.1) on $[u_x, \infty)$ with $u_x \geq u_0$, and satisfies the property $m(u)(y''(u))^{\kappa} \in C^1([u_x, \infty), \mathbb{R}))$. We focus on the nontrivial solutions of (1.1) existing on some half-line $[u_x, \infty)$ and satisfying the condition $\sup\{|x(u)| : u_* \leq u < \infty\} > 0$ for any $u_* \geq u_x$. A solution x of (1.1) is said to be *oscillatory* if it has arbitrary large zeros; otherwise, it is said to be *non-oscillatory.*

Functional differential equations have received wide attention by researchers as a result of their many applications in various fields of the real world, for example, technology and natural sciences [1]. In particular, the half-linear differential equations arise in the study of p-Laplace equations, porous medium problems, chemotaxis models, and so forth; see, for instance, the papers [2–5] for more details.

In the past few years, researchers have focused their effort on studying the oscillatory or non-oscillatory behavior of differential equations of different orders, by using various techniques. It has been noted the emergence of many studies interested in finding criteria of oscillation of second-order differential equations see [6–9], while interest in third-order differential equations is much less specially third-order differential equations of neutral type with a delayed or advanced argument; so, we find only a few references like [10–13]. In addition to the scarcity of literature on oscillation results for third order neutral differential equations with mixed arguments.

On the other hand, it was found that most of the studies devoted their studies to establishing conditions which ensure that the solutions to these equations are oscillatory or tend to zero, we mention here the papers [14–20]. Then some papers have appeared that sought to improve the previous conditions by creating conditions that ensure oscillation of all solutions, see [21–23].

Regarding the oscillatory behavior of differential equations with mixed arguments, we refer the reader to [24–26] and the references cited therein.

Next, we review some of the related results from the literature that were motivation for this work.

Han et al. in [27] established some oscillation criteria of

$$(m(u)(y''(u)))' + q_1(u)\, x(\theta_1(u)) + q_2(u)\, x(\theta_2(u)) = 0,$$

where (1.2) holds.

In reference [24], Baculikova and Dzurina gave some sufficient conditions for oscillation of the third-order DDE

$$(m(u)(x'(u)))'' + q(u)\, f(x(\tau(u))) + \rho(u)\, h(x(\theta(u))) = 0,$$

where $\tau(u) \leq u$ and $\theta(u) \geq u$. Further, Thandapani and Rama [26] established some oscillation theorems for DDE

$$(m(y''(u)))' + q_1(u)\,x(\theta_1(u)) + q_2(u)\,x(\theta_2(u)) = 0,$$

where (1.2) holds.

Grace [25] obtained some oscillation theorems for the odd order mixed neutral DDE

$$(x(u) + \rho_1 x(u-\tau_1) + \rho_2 x(u+\tau_2))^{(n)} + q_1(u-\theta_1) + q_1(u+\theta_2) = 0,$$

where $n \geq 1$.

Moaaz et al. [28] established sufficient conditions to ensure that the solutions of canonical equations

$$(m(u)(y''(u))^{\kappa})' + q_1(u)\,f(x(\theta_1(u))) + \rho(u)\,h(x(\theta_2(u))) = 0$$

are oscillatory or tend to zero where (1.2) holds.

In this paper, we aim to study the oscillatory behavior of a class of third-order DDE with neutral term. We obtain sufficient conditions that guarantee the oscillation of all solutions by using both Riccati substitution and comparison techniques. Our results extend and complement some of the relevant results that were recently published in the literature.

The following lemma will help us to prove our main results:

Lemma 1.1. *[29, Lemma 4] Assume that $g(u)$, $g'(u)$ and $g''(u)$ are positive functions; furthermore, $g'''(u)$ is negative on (u_0,∞). Then,*

$$\frac{g(u)}{g'(u)} \geq \frac{k}{2}u,$$

for some $k \in (0,1)$.

Lemma 1.2. *[30, Lemma 2.3] Let $G(v) = Cv - Dv^{\kappa+1/\kappa}$ where $C, D > 0$ and κ be a ratio of two odd positive integers. Then G has the maximum value on $\mathbb{R}$ at $v^* = (\kappa C/(\kappa+1)D)^{\alpha}$ such that*

$$\max_{v\in\mathbb{R}} G(v) = G(v^*) = \frac{\kappa^{\kappa}}{(\kappa+1)^{\kappa+1}}\frac{C^{\kappa+1}}{D^{\kappa}}. \tag{1.3}$$

2. Main results

For the sake of brevity, we will define the operator $m(y'')^{\kappa} = Ly(u)$ and adopt the following notation:

$$\begin{aligned}
\widehat{\rho}(u) &= \min\{\rho_1(u), \rho_2(u)\};\\
\widehat{q}(u) &= \min\{q(u), q(\tau_1(u)), q(\tau_2(u))\},
\end{aligned}$$

and

$$\widetilde{\eta}(v,w) = \int_w^v \left(\int_h^v \frac{1}{m^{1/\kappa}(s)}\mathrm{d}s\right)\mathrm{d}h.$$

Lemma 2.1. *[19] Assume that $a_1, a_2, a_3 \in [0,\infty)$ and $\gamma > 0$. Then*

$$(a_1 + a_2 + a_3)^{\gamma} \leq \lambda^2\left(a_1^{\gamma} + a_2^{\gamma} + \frac{1}{\lambda}a_3^{\gamma}\right). \tag{2.1}$$

where

$$\lambda := \begin{cases} 1 & \text{if } \gamma \leq 1\\ 2^{\gamma-1} & \text{if } \gamma > 1. \end{cases}$$

Proof. Combining Lemma 1 with Lemma 2 in [19], the proof is straightforward .

Lemma 2.2. *[31] Let $x > 0$ be a solution of Eq (1.1). Then y has only one of the following cases:*

($\boldsymbol{i}$) *The functions $y(u)$, $y'(u)$, $y''(u)$ are positive;*

($\boldsymbol{ii}$) *The functions $y(u)$ and $y''(u)$ are positive and $y'(u)$ is negative.*

Theorem 2.3. *Assume that x is a positive solution of Eq (1.1) and $\theta(u) < \tau_1(u)$ and $\tau_1^{-1}(\phi(u)) < u$. If there exist functions $\phi, \delta \in C([u_0, \infty), (0, \infty))$ satisfying $\theta(u) < \phi(u)$ such that*

$$\limsup_{u\to\infty} \int_{u_1}^{u} \left(\frac{k^\kappa \delta(s) \widehat{q}(s) \theta^\kappa(s)}{2^\kappa \lambda^2} - \frac{1}{(\kappa+1)^{\kappa+1}} \left(\frac{1}{\lambda} + \frac{\rho_1^\kappa}{\tau_1} + \frac{\rho_2^\kappa}{\tau_2} \right) \frac{(\delta'(s))^{\kappa+1} m(\theta(s))}{(\delta(s)\theta'(s))^\kappa} \right) \mathrm{d}s = \infty, \tag{2.2}$$

and the first-order DDE

$$F'(u) + \left(\frac{\lambda \tau_1 \tau_2}{\tau_1 \tau_2 + \lambda \tau_2 \rho_1^\kappa + \lambda \tau_1 \rho_2^\kappa} \right) \widehat{q}(u) \widetilde{\eta}^\kappa(\phi(u), \theta(u)) F\left(\tau_1^{-1}(\phi(u))\right) = 0 \tag{2.3}$$

is oscillatory, then every solution of Eq (1.1) oscillates.

Proof. Assume that $x > 0$ is a solution of Eq (1.1) on $[u_0, \infty)$. Thus there is a $u_1 \geq u_0$ with $x(\tau(u))$ and $x(\theta(u))$ are positive functions for all $u \geq u_1$. From the corresponding function $y(u)$, we get

$$y^\kappa(\theta(u)) = \left[x(\theta(u)) + (\rho_1(\theta(u)) x(\tau_1(\theta(u))) + \rho_2(\theta(u)) x(\tau_2(\theta(u)))) \right]^\kappa. \tag{2.4}$$

By Lemma 2.2, Eq (2.4) becomes

$$y^\kappa(\theta(u)) \leq \lambda^2 \left(\frac{1}{\lambda} x^\kappa(\theta(u)) + \rho_1^\kappa(\theta(u)) x^\kappa(\tau_1(\theta(u))) + \rho_2^\kappa(\theta(u)) x^\kappa(\tau_2(\theta(u))) \right).$$

From Eq (1.1) and ($\theta \circ \tau_i = \tau_i \circ \theta$), we get

$$\begin{aligned} 0 &= \frac{\rho_1^\kappa}{\tau_1'(u)} (Ly(\tau_1(u)))' + \rho_1^\kappa q(\tau_1(u)) x^\kappa(\theta(\tau_1(u))) \\ &\geq \frac{\rho_1^\kappa}{\tau_1} (Ly(\tau_1(u)))' + \rho_1^\kappa q(\tau_1(u)) x^\kappa(\tau_1(\theta(u))). \end{aligned} \tag{2.5}$$

And

$$\begin{aligned} 0 &= \frac{\rho_2^\kappa}{\tau_2'(u)} (Ly(\tau_2(u)))' + \rho_2^\kappa q(\tau_2(u)) x^\kappa(\theta(\tau_2(u))) \\ &\geq \frac{\rho_2^\kappa}{\tau_2} (Ly(\tau_2(u)))' + \rho_2^\kappa q(\tau_2(u)) x^\kappa(\tau_2(\theta(u))). \end{aligned} \tag{2.6}$$

Combining Eqs (1.1), (2.5) and (2.6), we obtain

$$0 \geq \left(\frac{1}{\lambda} Ly(u) + \frac{\rho_1^\kappa}{\tau_1} Ly(\tau_1(u)) + \frac{\rho_2^\kappa}{\tau_2} Ly(\tau_2(u)) \right)'$$

$$+\widehat{q}(u)\left(\frac{1}{\lambda}x^{\kappa}(\theta(u))+\rho_1^{\kappa}x^{\kappa}(\tau_1(\theta(u)))+\rho_2^{\kappa}x^{\kappa}(\tau_2(\theta(u)))\right).$$

Hence,

$$0\geq\left(\frac{1}{\lambda}Ly(u)+\frac{\rho_1^{\kappa}}{\tau_1}Ly(\tau_1(u))+\frac{\rho_2^{\kappa}}{\tau_2}Ly(\tau_2(u))\right)'+\frac{1}{\lambda^2}\widehat{q}(u)\,y^{\kappa}(\theta(u)). \tag{2.7}$$

By Lemma 2.2, $y(u)$ has two possible cases. First assume $y(u)$ has the property (**i**), we set

$$\varpi_1(u)=\delta(u)\frac{Ly(u)}{(y'(\theta(u)))^{\kappa}}. \tag{2.8}$$

Hence, by differentiating Eq (2.8), we get

$$\varpi_1'(u)=\delta'(u)\frac{Ly(u)}{(y'(\theta(u)))^{\kappa}}+\delta(u)\frac{(Ly(u))'}{(y'(\theta(u)))^{\kappa}}-\frac{\kappa\delta(u)\,Ly(u)\,y''(\theta(u))\,\theta'(u)}{(y'(\theta(u)))^{\kappa+1}}. \tag{2.9}$$

From the fact that $Ly(u)\leq 0$ and $\theta(u)<u$, we get

$$Ly(u)\leq Ly(\theta(u)).$$

Above inequality together with Eqs (2.8) and (2.9) becomes

$$\varpi_1'(u)\leq\delta'(u)\frac{\varpi_1(u)}{\delta(u)}+\delta(u)\frac{(Ly(u))'}{(y'(\theta(u)))^{\kappa}}-\frac{\kappa\theta'(u)}{\delta^{\frac{1}{\kappa}}(u)\,m^{\frac{1}{\kappa}}(\theta(u))}\varpi_1^{\frac{\kappa+1}{\kappa}}(u).$$

Using Lemma 1.2 with $V=\varpi_1(u)$, $C=\frac{\delta'(u)}{\delta(u)}$, $D=\frac{\kappa\theta'(u)}{\delta^{\frac{1}{\kappa}}(u)m^{\frac{1}{\kappa}}(\theta(u))}$, we obtain

$$\varpi_1'(u)\leq\delta(u)\frac{(Ly(u))'}{(y'(\theta(u)))^{\kappa}}+(\kappa+1)^{-(\kappa+1)}\frac{(\delta'(u))^{\kappa+1}m(\theta(u))}{(\delta(u)\,\theta'(u))^{\kappa}}. \tag{2.10}$$

Further, we set

$$\varpi_2(u)=\delta(u)\frac{Ly(\tau_1(u))}{(y'(\theta(u)))^{\kappa}}.$$

By differentiating $\varpi_2(u)$ and using $\theta(u)<\tau_1(u)$, we find

$$\varpi_2'(u)\leq\delta'(u)\frac{\varpi_2(u)}{\delta(u)}+\delta(u)\frac{(Ly(\tau_1(u)))'}{(y'(\theta(u)))^{\kappa}}-\frac{\kappa\theta'(u)}{\delta^{\frac{1}{\kappa}}(u)\,m^{\frac{1}{\kappa}}(\theta(u))}\varpi_2^{\frac{\kappa+1}{\kappa}}(u).$$

Using Lemma 1.2 with $V=\varpi_2(u)$, $C=\frac{\delta'(u)}{\delta(u)}$ and $D=\frac{\kappa\theta'(u)}{\delta^{\frac{1}{\kappa}}(u)m^{\frac{1}{\kappa}}(\theta(u))}$, we get

$$\varpi_2'(u)\leq\delta(u)\frac{(Ly(\tau_1(u)))'}{(y'(\theta(u)))^{\kappa}}+(\kappa+1)^{-(\kappa+1)}\frac{(\delta'(u))^{\kappa+1}m(\theta(u))}{(\delta(u)\,\theta'(u))^{\kappa}}. \tag{2.11}$$

Now, we set another positive function

$$\varpi_3(u)=\delta(u)\frac{Ly(\tau_2(u))}{(y'(\theta(u)))^{\kappa}}. \tag{2.12}$$

By differentiating Eq (2.12), and similar to Eq (2.11), we get

$$\varpi_3'(u) \leq \delta(u)\frac{(Ly(\tau_2(u)))'}{(y'(\theta(u)))^{\kappa}} + (\kappa+1)^{-(\kappa+1)}\frac{(\delta'(u))^{\kappa+1}m(\theta(u))}{(\delta(u)\theta'(u))^{\kappa}}. \tag{2.13}$$

From Eqs (2.10), (2.11) and (2.13), we get

$$\begin{aligned}\frac{1}{\lambda}\varpi_1'(u) + \frac{\rho_1^{\kappa}}{\tau_1}\varpi_2'(u) + \frac{\rho_2^{\kappa}}{\tau_2}\varpi_3'(u) &\leq \delta(u)\left(\frac{\frac{1}{\lambda}(Ly(u))' + \frac{\rho_1^{\kappa}}{\tau_1}(Ly(\tau_1(u)))' + \frac{\rho_2^{\kappa}}{\tau_2}(Ly(\tau_2(u)))'}{(y'(\theta(u)))^{\kappa}}\right) \\ &\quad + \frac{1}{(\kappa+1)^{\kappa+1}}\left(\frac{1}{\lambda} + \frac{\rho_1^{\kappa}}{\tau_1} + \frac{\rho_2^{\kappa}}{\tau_2}\right)\frac{(\delta'(u))^{\kappa+1}m(\theta(u))}{(\delta(u)\theta'(u))^{\kappa}}.\end{aligned}$$

Using Eq (2.7), one obtains

$$\begin{aligned}\frac{1}{\lambda}\varpi_1'(u) + \frac{\rho_1^{\kappa}}{\tau_1}\varpi_2'(u) + \frac{\rho_2^{\kappa}}{\tau_2}\varpi_3'(u) &\leq -\frac{\delta(u)}{\lambda^2}\widehat{q}(u)\left(\frac{y^{\kappa}(\theta(u))}{(y'(\theta(u)))^{\kappa}}\right) \\ &\quad + (\kappa+1)^{-(\kappa+1)}\left(\frac{1}{\lambda} + \frac{\rho_1^{\kappa}}{\tau_1} + \frac{\rho_2^{\kappa}}{\tau_2}\right)\frac{(\delta'(u))^{\kappa+1}m(\theta(u))}{(\delta(u)\theta'(u))^{\kappa}}.\end{aligned} \tag{2.14}$$

Using Lemma 1.1, we have

$$\frac{y^{\kappa}(\theta(u))}{(y'(\theta(u)))^{\kappa}} \geq \frac{k^{\kappa}}{2^{\kappa}}\theta^{\kappa}(u).$$

Combining the last inequality with Eq (2.14), we find

$$\begin{aligned}&\frac{1}{\lambda}\varpi_1'(u) + \frac{\rho_1^{\kappa}}{\tau_1}\varpi_2'(u) + \frac{\rho_2^{\kappa}}{\tau_2}\varpi_3'(u) \\ \leq\ & -\frac{k^{\kappa}\delta(u)\widehat{q}(u)\theta^{\kappa}(u)}{2^{\kappa}\lambda^2} + (\kappa+1)^{-(\kappa+1)}\left(\frac{1}{\lambda} + \frac{\rho_1^{\kappa}}{\tau_1} + \frac{\rho_2^{\kappa}}{\tau_2}\right)\frac{(\delta'(u))^{\kappa+1}m(\theta(u))}{(\delta(u)\theta'(u))^{\kappa}}.\end{aligned} \tag{2.15}$$

Integrating Eq (2.15) from u_1 to u , we conclude

$$\begin{aligned}&\int_{u_1}^{u}\left(\frac{k^{\kappa}\delta(s)\widehat{q}(s)\theta^{\kappa}(s)}{2^{\kappa}\lambda^2} - (\kappa+1)^{-(\kappa+1)}\left(\frac{1}{\lambda} + \frac{\rho_1^{\kappa}}{\tau_1} + \frac{\rho_2^{\kappa}}{\tau_2}\right)\frac{(\delta'(s))^{\kappa+1}m(\theta(s))}{(\delta(s)\theta'(s))^{\kappa}}\right)\mathrm{d}s \\ \leq\ & \frac{1}{\lambda}\varpi_1'(u_1) + \frac{\rho_1^{\kappa}}{\tau_1}\varpi_2'(u_1) + \frac{\rho_2^{\kappa}}{\tau_2}\varpi_3'(u_1).\end{aligned}$$

Now, assume that $y(u)$ has the property (**ii**), since $Ly(u)$ is nonincreasing, we have

$$-y'(h) \geq \int_h^v \frac{1}{m^{1/\kappa}(s)}L^{1/\kappa}y(s)\,\mathrm{d}s \geq L^{1/\kappa}y(v)\int_h^v \frac{1}{m^{1/\kappa}(s)}\mathrm{d}s, \text{ for } h \leq v. \tag{2.16}$$

Integrating Eq (2.16) from w to v, we see that

$$y(w) \geq L^{1/\kappa}y(v)\int_w^v\left(\int_h^v \frac{1}{m^{1/\kappa}(s)}\mathrm{d}s\right)\mathrm{d}h.$$

This yields

$$y(w) \geq \widetilde{\eta}(v,w)\,L^{1/\kappa}y(v). \tag{2.17}$$

But $w = \theta(u), v(u) = \phi(u)$, we obtain

$$y(\theta(u)) \geq \widetilde{\eta}(\phi(u), \theta(u)) L^{1/\kappa} y(\phi(u)). \tag{2.18}$$

Define the function F by

$$F(u) = \frac{1}{\lambda} Ly(u) + \frac{\rho_1^\kappa}{\tau_1} Ly(\tau_1(u)) + \frac{\rho_2^\kappa}{\tau_2} Ly(\tau_2(u)).$$

In view of Eq (1.1) and $\tau_1 \leq u \leq \tau_2$, we get

$$F(u) \leq Ly(\tau_1(u)) \left(\frac{1}{\lambda} + \frac{\rho_1^\kappa}{\tau_1} + \frac{\rho_2^\kappa}{\tau_2} \right),$$

that is,

$$Ly(\phi(u)) \geq F\left(\tau_1^{-1}(\phi(u))\right) \left(\frac{\lambda \tau_1 \tau_2}{\tau_1 \tau_2 + \lambda \tau_2 \rho_1^\kappa + \lambda \tau_1 \rho_2^\kappa} \right). \tag{2.19}$$

Substituting Eqs (2.18) and (2.19) in Eq (2.7) yields that

$$F'(u) + \left(\frac{\lambda \tau_1 \tau_2}{\tau_1 \tau_2 + \lambda \tau_2 \rho_1^\kappa + \lambda \tau_1 \rho_2^\kappa} \right) \widehat{q}(u) \widetilde{\eta}^\kappa(\phi(u), \theta(u)) F\left(\tau_1^{-1}(\phi(u))\right) \leq 0.$$

According to [32, Theorem 1], the delay differential equation (2.3) also has a positive solution, which is a contradiction.

Corollary 1. *Let x be a solution of (1.1) and positive eventually, $\theta(u) < \tau_1(u)$ and $\tau_1^{-1}(\phi(u)) < u$. If there exist functions $\phi, \delta \in C([u_0, \infty), (0, \infty))$ satisfying $\theta(u) < \phi(u)$ such that Eq (2.2) holds and*

$$\liminf_{u \to \infty} \int_{\tau_1^{-1}(\phi(u))}^{u} \widehat{q}(s) \widetilde{\eta}^\kappa(\phi(s), \theta(s)) \, \mathrm{d}s > \left(\frac{\tau_1 \tau_2 + \lambda \tau_2 \rho_1^\kappa + \lambda \tau_1 \rho_2^\kappa}{\lambda \tau_1 \tau_2} \right), \tag{2.20}$$

then every solution of Eq (1.1) oscillates.

Proof. Assume that x is a solution of Eq (1.1) and x is positive eventually. Thus there is a $u_1 \geq u_0$ with $x(\tau(u)) > 0$ and $x(\theta(u)) > 0$ for all $u \geq u_1$. When $y(u)$ satisfies the property (**i**), the proof is similar to that of Theorem 2.3, and hence is omitted and when $y(u)$ satisfies property (**ii**), it is well known from [33] that Eq (2.20) implies oscillation of Eq (2.3).

Example 2.4. Consider the third-order differential equation of mixed type

$$\left(x(u) + \frac{1}{3} x\left(\frac{1}{3}u\right) + \frac{1}{3} x(2t) \right)''' + \frac{q_0}{u^3} x\left(\frac{1}{2}u\right) = 0, \tag{2.21}$$

where $q_0 > 0$. Note that, $\rho_1 = \rho_2 = \frac{1}{3}$, $\tau_1(u) = \frac{1}{3}u$, $\tau_2(u) = 2t$, $\tau_1 = \tau_2 = 2, \widehat{q}(u) = \frac{q_0}{8u^3}$ and $\theta(u) = \frac{1}{2}u$. By choosing $\delta(u) = u$, it is easy to see that Eq (2.2) becomes

$$\liminf_{u \to \infty} \int_{u_1}^{u} \left(\frac{q_0}{32s} - \frac{1}{s}\left(\frac{17}{24}\right) \right) \mathrm{d}s = \infty.$$

Thus, it is validated when

$$q_0 > 22.66.$$

Now, set $\phi(u) = \frac{3}{4}u$, we get $\tau_1^{-1}(\phi(u)) = 3u$ and $\widetilde{\eta}(\phi(u), \theta(u)) = \frac{u^2}{32}$, moreover Eq (2.20) yields

$$\liminf_{u\to\infty} \int_{3u}^{u} \frac{q_0}{8s^3}\frac{s^2}{32}\mathrm{d}s > \left(\frac{\tau_1\tau_2 + \lambda\tau_2\rho_1^\kappa + \lambda\tau_1\rho_2^\kappa}{\mu\tau_1\tau_2}\right)$$

or

$$q_0 > \frac{3(256)}{2\ln\frac{4}{3}}.$$

Then, Eq (1.1) is oscillatory if $q_0 > \frac{3(256)}{2\ln\frac{4}{3}}$.

3. Conclusions

It is easy to notice the great development in the study of oscillation of the delay differential equations in recent times. In this paper, we established the oscillation criteria for a class of third-order delay differential equations. By using comparison principles and Riccati transformation, we presented some sufficient conditions which ensure that every solution of Eq (1.1) is oscillatory. The approach used does not need to be restricted by the condition $0 < \rho_i(u) < 1$, unlike many previous work. For interested researchers, results presented in this paper may be extended to more general equations than Eq (1.1). Another interesting problem for further research is to obtain new criteria for oscillatory solutions for Eq (1.1) without requiring $\theta \circ \tau_i = \tau_i \circ \theta$.

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Conflict of interest

There are no competing interests.

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