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Research article

Bifurcations in discontinuous mathematical models with control strategy for a species

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Abstract: In this paper a preliminary mathematical model is proposed, by means of a system of ordinary differential equations, for the growth of a species. In this case, the species does not interact with another species and is divided into two stages, those that have or have not reached reproductive maturity, with natural and control mortality for both stages. When performing a qualitative analysis to determine conditions in the parameters that allow the extinction or preservation of the species, a modification is made to the model when only control is assumed for each of the stages if the number of species in that stage is above a critical value. These studies are carried out by bifurcation analysis with respect to two parameters: control for each stage and their critical values. It is concluded that for certain conditions in their parameters, the dynamics in each of the controlled stages converge to their critical values.

Keywords: filippov systems; pseudo-equilibrium; growth threshold; stability; control parameter

1. Introduction

A great variety of phenomena in nature are modeled through differential equations systems of the form

$$\dot{w} = f(w, \alpha), \tag{1.1}$$

where $f : \Omega \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous and differentiable vector field, $w \in \mathbb{R}^n$ is a vector of states and $\alpha \in \mathbb{R}^m$ is a vector of parameters. In particular, the systems (1.1) are used to explain the dynamics of species that inhabit the same environment to determine whether or not they become extinct over time [1–4].

For example, if we consider a species $v(t) \geq 0$ divided into two stages, those that have or have not reached reproductive maturity, in a controlled environment, with carrying capacity $K > 0$, and which does not interact with other species to subsist in an environment. As shown in Figure 1, if $u_{1,2} \geq 0$ are mortality parameters due to control, chemical or biological, for both stages, a simple model describing the dynamics of the species is given by:

$$\begin{cases} \dot{x} &= ay - (d_1 + u_1)x, \\ \dot{y} &= \dfrac{bx}{c+x}\left(1 - \dfrac{y}{K}\right) - (a + d_2 + u_2)y, \end{cases} \tag{1.2}$$

where $x(t) \geq 0$ describes the reproductive maturity stage of the species, $y(t) \geq 0$ the stage of those that have not yet reached maturity, $a > 0$ the rate at which the species reaches reproductive maturity, $b > 0$ the maximun birth rate for each adult in a unit of time, $c > 0$ is an auxiliary (half-saturation) parameter that affects the hyperbolic function of the per capita adult birth rate curve, and $d_{1,2} > 0$ the natural mortality rates.

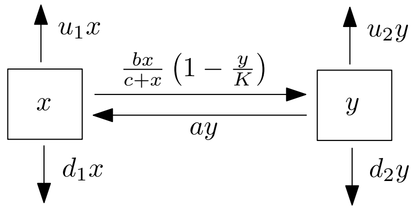


Figure 1. Model construction (1.2).

The model (1.2) is used to analyze the dynamics of the states of a species and determine conditions in the parameters for the extinction of the species when the control strategy is activated at all times. For example, the model (1.2) has been used to describe the growth of the Aedes aegypti mosquito, divided into immature stages (egg, larva and pupae) $y(t)$ and adult mosquitoes $x(t)$, and determine the conditions in the control parameter in some of its stages that allow the extinction of the mosquito and thus prevent the spread of dengue to the population [5, 6].

However, many ecological phenomena are modeled by discontinuous dynamical systems, called Filippov systems [7], whose movement is characterized by periods of smooth evolution that are interrupted by instantaneous events. For example, if a species is above a threshold value, then control measures need to be applied, biologically or chemically, in order to keep the number of the species below the threshold [8–12]. However, traditional analysis and bifurcations in dynamical systems has focused on continuous problems whose instantaneous events do not occur, leaving aside those systems that are discontinuous.

The planar Filippov systems are represented by the system of differential equations of the form

$$\dot{w} = \begin{cases} f_1(w, \alpha), & w \in S_1 \subset \mathbb{R}^2, \\ f_2(w, \alpha), & w \in S_2 \subset \mathbb{R}^2, \end{cases} \tag{1.3}$$

where S_1 and S_2 are open sets, separated by a differentiable curve Σ, and $f_{1,2} : S_{1,2} \subset \mathbb{R}^2 \times \mathbb{R}^m \rightarrow \mathbb{R}^2$ are continuous and differentiable functions.

In addition to generic bifurcations in continuous dynamical systems [1], the planar Filippov systems (1.3) could present bifurcations sliding, where variations in the bifurcation parameter cause alterations in Σ. All possible bifurcations in planar Filippov systems (1.3) were listed by Kuznetsov [13]. Similarly, for planar Filippov systems (1.3) whose bifurcation analysis is not immediately determined, the software created by Dercole and Kuznetsov computationally determines the possible bifurcation cases [14, 15] .

In particular, if the control strategy cannot be maintained all times, due to economic or natural issues for example, if only the control parameter $u_1 > 0$, which represents the mortality rate of the species $x(t)$ when applying a chemical or biological control, is considered, if the amount of the species at stage $x(t)$ is above a critical value P_x, then the model (1.2) is modified by:

$$\left\{\begin{array}{lcl} \dot{x} & = & ay - [d_1 + g(x)]x, \\ \dot{y} & = & \dfrac{bx}{c+x}\left(1 - \dfrac{y}{K}\right) - (a + d_2)y, \end{array}\right. \tag{1.4}$$

with

$$g(x) = \left\{\begin{array}{ll} u_1, & \text{if} \quad x > P_x, \\ 0, & \text{if} \quad x < P_x. \end{array}\right.$$

On the other hand, if the objective is to control the species in the non-reproductive state $y(t)$ when it is above a value P_y, model (1.2) takes the following form,

$$\left\{\begin{array}{lcl} \dot{x} & = & ay - d_1 x, \\ \dot{y} & = & \dfrac{bx}{c+x}\left(1 - \dfrac{y}{K}\right) - [a + d_2 + h(y)]y, \end{array}\right. \tag{1.5}$$

with

$$h(y) = \left\{\begin{array}{ll} u_2, & \text{if} \quad y > P_y, \\ 0, & \text{if} \quad y < P_y. \end{array}\right.$$

Therefore, the goal of this paper is to determine all possible dynamics for the discontinuous models (1.4) and (1.5). To provide the necessary background, some elements such as tangent points, pseudo-equilibrium and sliding segment used in the Filippov systems are described in Section 2. In Section 3, a global qualitative analysis is performed to identify conditions in the parameters that allow the extinction or conservation of the species in the long term for the preliminary model (1.2) by adding the control parameters $u_1 > 0$ and $u_2 > 0$ for $x(t) > 0$ and $y(t) > 0$, respectively. In Sections 4 and 5, a qualitative and bifurcation analysis is performed for the model (1.4) and (1.5), respectively, with respect to two parameters: control and critical value.

2. Basic notions of Filippov system

Let X and Y be vector fields of class C^r, with $r > 1$, in an open set $\Omega \subset \mathbb{R}^2$ such that $(0, 0) \in \Omega$. Let $f : \Omega \to \mathbb{R}$ be a function of class C^r, $r > 1$, such that $\text{grad} f(x, y) \neq 0$ for all $(x, y) \in \Omega$ and

$$\Sigma = \{(x, y) \in \Omega : f(x, y) = 0\}$$

an open and differentiable dividing curve which divides U into two open regions

$$\Sigma^+ = \{(x, y) \in \Omega : f(x, y) > 0\}, \quad \Sigma^- = \{(x, y) \in \Omega : f(x, y) < 0\},$$

with $\overline{\Sigma^+}$ and $\overline{\Sigma^-}$ their closures.

According to [12, 16], a Filippov planar system $Z = (X, Y)$ is a vector field defined by

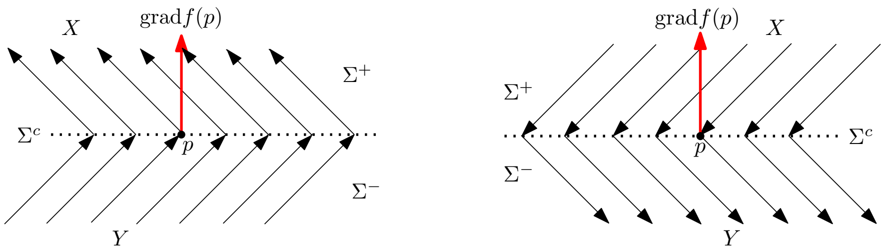
$$Z(x,y) = \begin{cases} X(x,y), & (x,y) \in \Sigma^+ \\ Y(x,y), & (x,y) \in \Sigma^-, \end{cases}$$

where X and Y are of class C^r, $r > 1$, in $\overline{\Sigma^+}$ and $\overline{\Sigma^-}$, respectively.

In order to establish the dynamics given by the Filippov planar system $Z = (X, Y)$ on U, we need to denote the local trajectory $\varphi_Z(t, p)$ for a initial point $p \in U$. For this purpose, it is important to determine whether point p belongs to Σ^+, Σ or Σ^-.

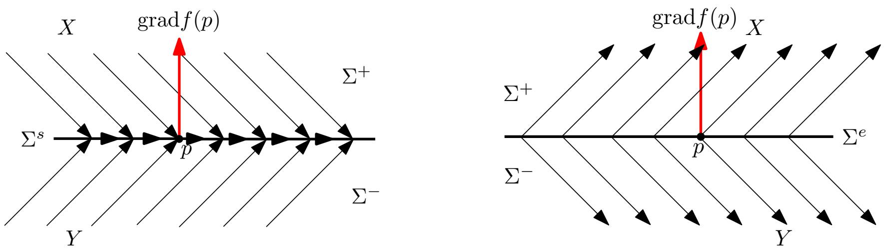
If $p \in \Sigma^+$ or $p \in \Sigma^-$, the local trajectory in $Z = (X, Y)$, with initial point in p, is defined by trajectory in the vector fields X or Y, respectively. However, a trajectory must also be defined for the initial points $p \in \Sigma$. To do this, considering $Xf(p) = \langle X(p), \mathrm{grad} f(p)\rangle$ and $Yf(p) = \langle Y(p), \mathrm{grad} f(p)\rangle$, Σ is divided into three disjoint regions given by:

- Crossing region: $\Sigma^c = \{p \in \Sigma : Xf(p) \cdot Yf(p) > 0\}$ as seen in Figure 2,
- Sliding region: $\Sigma^s = \{p \in \Sigma : Xf(p) < 0, Yf(p) > 0\}$ represented by Figure 3(a),
- Escaping region: $\Sigma^e = \{p \in \Sigma : Xf(p) > 0, Yf(p) < 0\}$ represented by Figure 3(b),



(a) $Xf(p), Yf(p) > 0$. (b) $Xf(p), Yf(p) < 0$.

Figure 2. Crossing region Σ^c.



(a) Sliding region. (b) Escaping region.

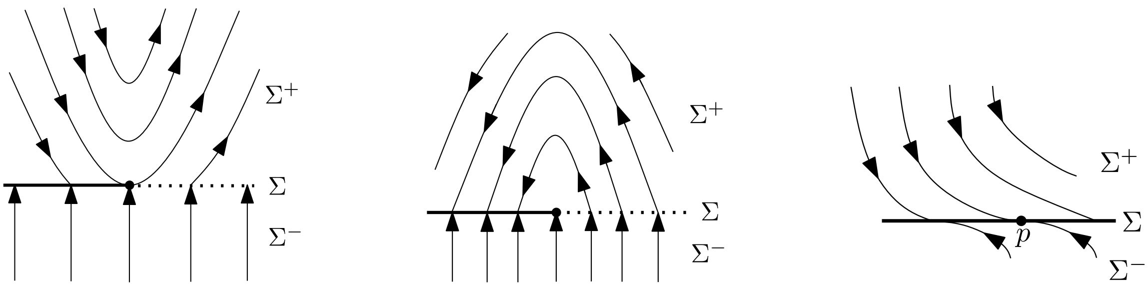
Figure 3. Regions Σ^s and Σ^e.

If the boundary of the regions Σ^c, Σ^s or Σ^e are denoted by $\partial\Sigma c$, $\partial\Sigma s$ and $\partial\Sigma e$, respectively, the points $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$, that is, $p \in \Sigma$ such that $Xf(p) = 0$ or $Yf(p) = 0$, is called *tangency point*, and it can be classified as:

- *quadratic* if $Xf(p) = 0$ and $X^2 f(p) = \langle X(p), \mathrm{grad} Xf(p)\rangle \neq 0$, or $Yf(p) = 0$ and $Y^2 f(p) = \langle Y(p), \mathrm{grad} Yf(p)\rangle \neq 0$. A quadratic tangency $p \in \Sigma$ is *regular* if $Xf(p) = 0$, $X^2 f(p) \neq 0$ and

$Yf(p) \neq 0$; or $Yf(p) = 0$, $Y^2 f(p) \neq 0$ and $Xf(p) \neq 0$. For the first case, a regular quadratic tangency is *visible* if $X^2 f(p) > 0$ and *invisible* if $X^2 f(p) < 0$ as seen in Figure 4(a). For the second case, $p \in \Sigma$ is visible if $Y^2 f(p) < 0$ and invisible if $Y^2 f(p) > 0$ as seen in Figure 4(b).

- *cubic* if $Xf(p) = X^2 f(p) = 0$ and $X^3 f(p) = \left\langle X(p), \mathrm{grad} X^2 f(p) \right\rangle \neq 0$ or $Yf(p) = Y^2 f(p) = 0$ and $Y^3 f(p) = \left\langle Y(p), \mathrm{grad} Y^2 f(p) \right\rangle \neq 0$, as seen in Figure 4(c).



(a) Regular visible quadratic (b) Regular invisible quadratic (c) Cubic

Figure 4. Example of tangency points in $Z = (X, Y)$.

We will now define the trajectory for a initial point p in Σ^c, Σ^s or Σ^e. As observed in Figure 2, in Σ^c, since both vector fields point either towards Σ^+ or Σ^-, it is enough to match the trajectories of X and Y through that point. According to Filippov's method [13, 16, 17], the trajectory in Σ^s or Σ^e is given by a convex combination of the vector fields X and Y tangent to Σ, that is,

$$Z^s(p) = \lambda(p)X(p) + (1 - \lambda(p))Y(p).$$

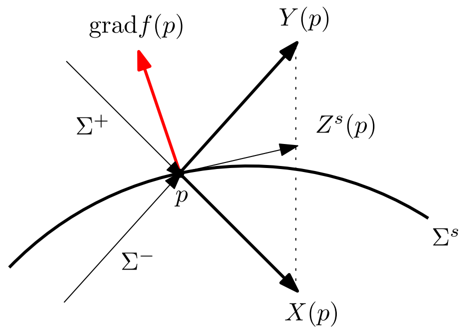


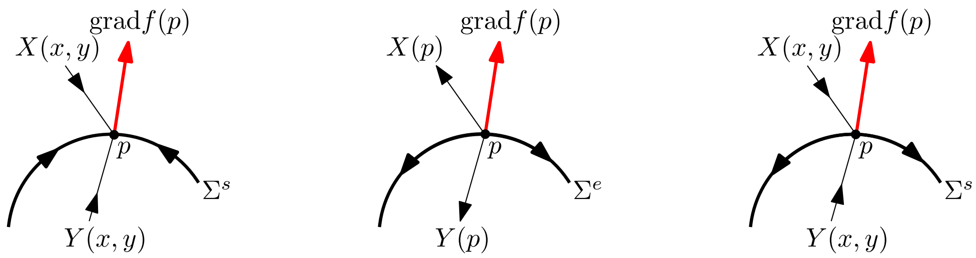
Figure 5. Construction of trajectories $Z^s(p)$.

In view of the Figure 5, the *sliding vector field* Z^s is given by

$$Z^s(p) = \frac{1}{Yf(p) - Xf(p)}(Yf(p)X(p) - Xf(p)Y(p)), \tag{2.1}$$

defined in $\Sigma^e \cup \Sigma^s$. For $p \in \Sigma^e \cup \Sigma^s$, the local trajectory of p is given by this vector field.

In $Z = (X, Y)$ the point $p \in \Sigma^s \cup \Sigma^e$ is called *pseudo-equilibrium* if $Z^s(p) = 0$, which is further classified as: *stable pseudo-node* if $p \in \Sigma^s$ and $(Z^s)'(p) < 0$ as shown in Figure 6(a), *unstable pseudo-node* $p \in \Sigma^e$ and $(Z^s)'(p) > 0$ as shown in Figure 6(b) and, *pseudo-saddle* if $p \in \Sigma^s$ and $(Z^s)'(p) > 0$ as shown in Figure 6(c), or $p \in \Sigma^e$ and $(Z^s)'(p) < 0$.



(a) Stable pseudo-node (b) Unstable pseudo-node (c) Pseudo-saddle

Figure 6. Examples of pseudo-equilibrium in $Z = (X, Y)$.

Keeping in mind this background, the trajectory over the vector field of $Z = (X, Y)$ is defined as follows.

Definition 1. *Let φ_X and φ_Y the trajectories in the vector fields X and Y defined for for $t \subset I \in \mathbb{R}$, respectively. The local trajectory φ_Z in $Z = (X, Y)$ through a point p is defined as follows:*

- *For $p \in \Sigma^+$ or $p \in \Sigma^+$ such that $X(p) \neq 0$ or $Y(p) \neq 0$ respectively, the trajectory is given by $\varphi_Z(t, p) = \varphi_X(t, p)$ or $\varphi_Z(t, p) = \varphi_X(t, p)$ respectively, for $t \subset I \in \mathbb{R}$.*
- *For $p \in \Sigma^c$ such that $Xf(p), Yf(p) > 0$, as shown in Figure 2(a), and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t, p) = \varphi_Y(t, p)$ for $t \subset I \cap \{t \leq 0\}$ and $\varphi_Z(t, p) = \varphi_X(t, p)$ for $t \subset I \cap \{t \geq 0\}$. For the case $Xf(p), Yf(p) < 0$, as shown in Figure 2(a), the trajectory is defined as $\varphi_Z(t, p) = \varphi_Y(t, p)$ for $t \subset I \cap \{t \geq 0\}$ and $\varphi_Z(t, p) = \varphi_X(t, p)$ for $t \subset I \cap \{t \leq 0\}$.*
- *For $p \in \Sigma^e \cup \Sigma^s$ such that $Z^s(p) \neq 0$, the trajectory is given by $\varphi_Z(t, p) = \varphi_{Z^s}(t, p)$ for $t \in I \subset \mathbb{R}$, where Z^s is the sliding vector field given in (2.1).*
- *For $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$ such that the definitions of trajectories for points in Σ in both sides of p can be extended to p and coincide, the trajectory through p is this trajectory. We will call these points regular tangency points.*
- *For any other point $\varphi_Z(t, p) = \{p\}$ for all $t \in I \subset \mathbb{R}$. This is the case of the tangency points in Σ which are not regular and which will be called singular tangency points and the critical points of X in Σ^+, Y in Σ^- and Z^s in $\Sigma^e \cup \Sigma^s$.*

With the basic notions for Filippov systems, we can perform the qualitative analysis for the systems (1.4) and (1.5). However, a qualitative analysis will be performed on system (1.2), and its results will be used to describe the dynamics of systems (1.4) and (1.5).

3. Preliminary mathematical model

Asume that $v(t) \geq 0$ is a quantity of a species at time $t \geq 0$, from which it does not interact with any other species to subsist an environment, and divided into two stages: those that have reached reproductive maturity $x(t) \geq 0$ and those that have not $y(t) \geq 0$, whose dynamics is described by the system (1.2).

Before performing a qualitative analysis of the system (1.2) it must be verified that it is mathematically well-posed and its trajectories remain in the region of biological sense:

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{aK}{d_1 + u_1}, 0 \leq y \leq K\right\}.$$

Lemma 1. *For an arbitrary initial condition* $(x(0), y(0)) \in \Omega$*, the system (1.2) has a unique solution. Moreover, the set* Ω *is invariant over the vector field of the system (1.2).*

Proof. Since the vector field of the system (1.2) is continuously differentiable, by the Existence and Uniqueness Theorem [18], the uniqueness of the solution is guaranteed. On the other hand, it must be guaranteed that the change in the solutions of the model on the boundary of Ω remain in Ω. Indeed, if $x = 0$ then $\dot{x} = ay \geq 0$ for all $0 \leq y \leq K$. Similarly, if $y = 0$ then $\dot{y} = \frac{bx}{x+c} \geq 0$ for all $0 \leq x \leq \frac{aK}{d_1+u_1}$. If $y = K$, then $\dot{y} = -(a + d_2 + u_2)y \leq 0$ for all $0 \leq y \leq K$. Analogously, if $x = \frac{aK}{d_1+u_1}$ then $\dot{x} = a(y - K) \leq 0$ for all $0 \leq y \leq K$. This shows that the trajectories over the vector field cannot cross the boundary of Ω. Then, Ω is invariant. □

3.1. Qualitative analysis

Before calculating and determining the local and global stability of the possible equilibrium points of the system (1.2), the following result shows that the system (1.2) has no limit cycles in Ω.

Lemma 2. *The system (1.2) has no limit cycles in* Ω*.*

Proof. If

$$\begin{aligned} f(x, y, u_1) &= ay - (d_1 + u_1)x, \\ g(x, y, u_2) &= \frac{bx}{x + c}\left(1 - \frac{y}{K}\right) - (a + d_2 + u_2)y, \end{aligned} \tag{3.1}$$

then,

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\left[a + d_1 + d_2 + u_1 + u_2 + \frac{xb}{K(x + c)}\right] < 0,$$

for all $(x, y) \in \Omega$ and $u_{1,2} \geq 0$. By Poincaré-Bendixson's criterion [18], the system (1.2) has no limit cycles in Ω. □

On the other hand, the equilibrium points of the system (1.2) are given by

$$\begin{aligned} &P_0 = (0, 0), \\ &P_{(u_1,u_2)} = \left(\frac{K[\phi(u_1, u_2) - 1]}{(d_1 + u_1)[b + K(a + d_2 + u_2)]}, \frac{K[\phi(u_1, u_2) - 1]}{a[b + K(a + d_2 + u_2)]}\right), \end{aligned}$$

where

$$\phi(u_1, u_2) = \frac{ab}{c(d_1 + u_1)(a + d_2 + u_2)}, \tag{3.2}$$

is the species growth threshold and indicates the average number of species that have reached reproductive maturity produced by a species. In this case, if $\phi(u_1, u_2) > 1$ then $P_{(u_1,u_2)} \in \Omega$. If $\phi(u_1, u_2) = 1$ then $P_0 = P_{(u_1,u_2)} \in \Omega$. However, for $\phi(u_1, u_2) < 1$ then $P_{(u_1,u_2)} \notin \Omega$.

The Jacobian matrix $J(x, y)$ of the system (1.2) evaluated at P_0 is given by:

$$J(P_0) = \begin{bmatrix} -(d_1 + u_1) & a \\ \dfrac{b}{c} & -(a + d_2 + u_2) \end{bmatrix},$$

with

$$\begin{aligned} \mathrm{tr}J(P_0) &= -(a + d_1 + d_2 + u_1 + u_2) < 0, \\ \mathrm{det}J(P_0) &= (d_1 + u_1)(ac + cd_2 + cu_2)[1 - \phi(u_1, u_2)]. \end{aligned}$$

If $\phi(u_1, u_2) > 1$ then $\mathrm{det}J(P_0) < 0$ and thus, by the Grobman-Hartman Theorem [18], P_0 is locally unstable. However, if $\phi(u_1, u_2) < 1$, that is, $P_{(u_1,u_2)} \notin \Omega$, the equilibrium P_0 is locally stable, and corresponds to a node since

$$\Delta = [\mathrm{tra}J(P_0)]^2 - 4\mathrm{det}J(P_0) = \frac{4b}{c} + [a - (d_1 - d_2 + u_1 - u_2)]^2 > 0. \tag{3.3}$$

Moreover, if $\phi(u_1, u_2) < 1$, and since Ω is invariant and has no limit cycles over the system (1.2), by the Poincaré-Bendixson Theorem [18], every trajectory in the system (1.2) converges to the equilibrium P_0, which guarantees its global stability.

On the other hand, if $\phi(u_1, u_2) > 1$, when evaluating the Jacobian matrix $J(x, y)$ of the system (1.2) around the equilibrium $P_{(u_1,u_2)}$,

$$J\left(P_{(u_1,u_2)}\right) = \begin{bmatrix} -(d_1 + u_1) & a \\ \dfrac{c(a + d_2 + u_2)(d_1 + u_1)^2[aK + b + K(d_2 + u_2)]}{ab(aK + cd_1 + cu_2)} & -\dfrac{a[aK + b + K(d_2 + u_2)]}{aK + c(d_1 + u_1)} \end{bmatrix},$$

we have to

$$\begin{aligned} \mathrm{tr}J\left(P_{(u_1,u_2)}\right) &= -\left\{d_1 + u_1 + \frac{a[aK + b + K(d_2 + u_2)]}{aK + c(d_1 + u_1)}\right\} < 0, \\ \mathrm{det}J\left(P_{(u_1,u_2)}\right) &= \frac{(d_1 + u_1)^2[b + K(a + d_2 + u_2)]^2[\phi(u_1, u_2) - 1]}{b[aK + c(d_1 + u_1)]} > 0, \end{aligned}$$

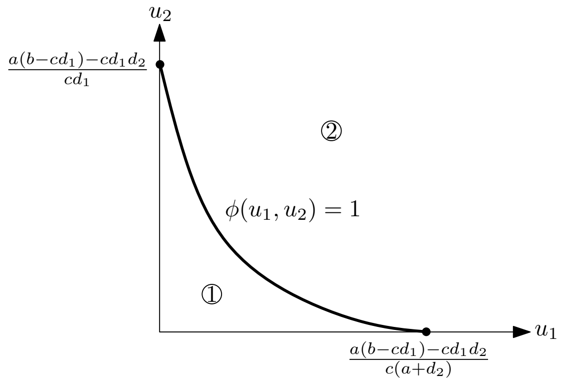
and

$$\triangle = [\mathrm{tr}J\left(P_{(u_1,u_2)}\right)]^2 - 4\mathrm{det}J\left(P_{(u_1,u_2)}\right) > 0.$$

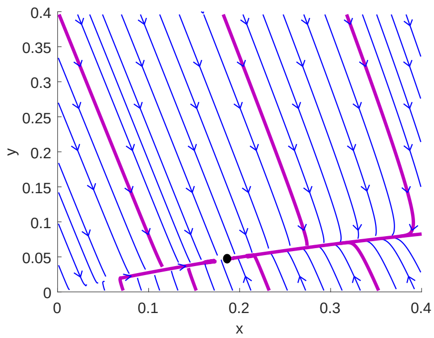
Therefore, $P_{(u_1,u_2)}$ is locally a stable node. Moreover, since Ω is invariant and has no limit cycles, by the Poincaré - Bendixson Theorem, the trajectories of the system (1.2), regardless of the initial condition $(x(0), y(0)) \in \Omega$, converge to the equilibrium $P_{(u_1,u_2)}$, so $P_{(u_1,u_2)}$ is globally asymptotically stable. Therefore the following result has been proved.

Theorem 1. *If $\phi(u_1, u_2) < 1$ then the equilibrium $P_{(u_1,u_2)} \notin \Omega$ and P_0 is globally a stable node over the system (1.2). If $\phi(u_1, u_2) > 1$, the equilibrium P_0 is unstable and $P_{(u_1,u_2)}$ is a globally stable node over the system (1.2).*

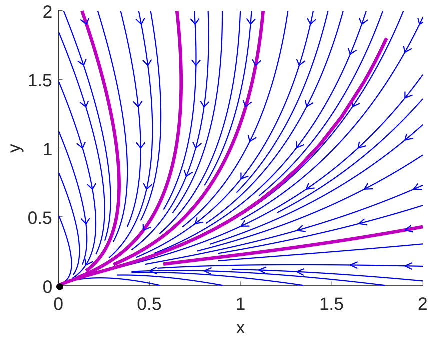
In Figure 7 we observe all possible dynamics of the system (1.2) with respect to the bifurcation curve $\phi(u_1, u_2) = 1$. Region 1 shows the global stability of the equilibrium $P_{(u_1,u_2)}$. Region 2 shows that all solutions of the system (1.2) converge to the origin P_0.



(a) Bifurcation curves in the plane (u_1, u_2)



(b) Region 1



(c) Region 2

Figure 7. Bifurcation diagram of the system (1.2) and phase portraits characterizing each region, with fixed parameters $a = b = c = K = 1$, $d_1 = d_2 = 0.2$ and $u_2 = 2$. Region 1: $u_1 = 0.05$. Region 2: $u_1 = 2$.

4. Control at reproductive maturity stage $x(t)$

According to the hypothesis stated in Section 3, suppose that only the dynamics in $x(t)$ can be controlled when it is above a critical value $P_x \leq \frac{aK}{d_1+u_1}$. In this case, the dynamics of $x(t)$ and $y(t)$ is described by the system (1.4), equivalent to:

$$Z_1(x,y) = \begin{cases} X_1(x,y) = \begin{pmatrix} ay - (d_1+u_1)x \\ \dfrac{bx}{x+c}\left(1-\dfrac{y}{K}\right) - (a+d_2)y \end{pmatrix}, & x > P_x \\ X_2(x,y) = \begin{pmatrix} ay - d_1 x \\ \dfrac{bx}{x+c}\left(1-\dfrac{y}{K}\right) - (a+d_2)y \end{pmatrix}, & x < P_x \end{cases} \tag{4.1}$$

where

$$\begin{aligned} \Sigma_1^+ &= \left\{(x,y) \in \mathbb{R}^2 : P_x < x \leq \frac{aK}{d_1+u_1}, 0 \leq y \leq K\right\}, \\ \Sigma_1 &= \{(P_x, y) \in \mathbb{R}^2 : 0 \leq y \leq K\}, \\ \Sigma_1^- &= \{(x,y) \in \mathbb{R}^2 : 0 \leq x < P_x, 0 \leq y \leq K\}. \end{aligned}$$

4.1. Qualitative analysis

To perform a qualitative analysis of the Filippov system (4.1) we must calculate the equilibrium points for the X_1 and X_2 fields, the Σ_1^s, Σ_1^e, Σ_1^c regions and the Z_1^s sliding segment. Indeed, as observed in Section 3, the equilibrium point over the vector field X_1 are given by

$$P_{(u_1,0)} = \left(\frac{K[\phi(u_1,0) - 1]}{(d_1 + u_1)[b + K(a + d_2)]}, \frac{K[\phi(u_1,0) - 1]}{a[b + K(a + d_2)]} \right), \tag{4.2}$$

and the equilibrium points of the field X_2 are

$$\begin{aligned} P_0 &= (0,0), \\ P_{(0,0)} &= \left(\frac{K[\phi(0,0) - 1]}{d_1[b + K(a + d_2)]}, \frac{K[\phi(0,0) - 1]}{a[b + K(a + d_2)]} \right), \end{aligned} \tag{4.3}$$

with $\phi(u_1, 0)$ and $\phi(0, 0)$ as described in system (3.2), for $u_2 = 0$ and $u_1 = u_2 = 0$, respectively.

Since $(d_1 + u_1)(a + d_2) > d_1(a + d_2)$, if $\phi(u_1, 0) > 1$, then $\phi(0, 0) > 1$, and so $P_{(0,0)}, P_{(u_1,0)} \in \Omega$. However, if $\phi(0, 0) < 1$ it follows that $P_{(0,0)}, P_{(u_1,0)} \notin \Omega$.

On the other hand, if we consider $f_1(x, y) = x - P_x$, then for all $p \in \Sigma_1$,

$$\begin{aligned} X_1 f_1(p) &= \langle X_1(p), \mathrm{grad} f_1(p) \rangle = ay - (d_1 + u_1)P_x, \\ X_2 f_1(p) &= \langle X_2(p), \mathrm{grad} f_1(p) \rangle = ay - d_2 P_x, \end{aligned}$$

such that $X_1 f_1(p) \cdot X_2 f_1(p) < 0$ it follows that:

- $ay - (d_1 + u_1)P_x < 0$ and $ay - d_1 P_x > 0$, that is,

$$\frac{d_1 P_x}{a} < y < \frac{(d_1 + u_1)P_x}{a},$$

or

- $ay - (d_1 + u_1)P_x > 0$ and $ay - d_1 P_x < 0$, equivalent to,

$$\frac{(d_1 + u_1)P_x}{a} < y < \frac{d_1 P_x}{a}.$$

Since $\frac{d_1}{a} < \frac{d_1+u_1}{a}$, and by considering $\frac{(d_1+u_1)P_x}{a} < K$, then:

$$\begin{aligned} \Sigma_1^s &= \left\{ (P_x, y) \in \mathbb{R}^2 : \frac{d_1 P_x}{a} < y < \frac{(d_1 + u_1)P_x}{a} \right\}, \\ \Sigma_1^c &= \left\{ (P_x, y) \in \mathbb{R}^2 : y < \frac{d_1 P_x}{a} \text{ o } \frac{(d_1 + u_1)P_x}{a} < y \right\}, \\ \Sigma_1^e &= \emptyset. \end{aligned}$$

The vector field of the sliding segment $Z_1^s(p) = \left(0, Z_{1y}^s(p)\right)^T$, with $p \in \Sigma_1^s$, is given by

$$Z_1^s(p) = \left(\begin{array}{c} 0 \\ \dfrac{bKP_x - y[P_x(aK + b + d_2K) + cK(a + d_2)]}{K(P_x + c)} \end{array} \right),$$

with pseudo-equilibrium

$$PN_1 = \left(P_x, \frac{bKP_x}{P_x(aK + b + d_2K) + cK(a + d_2)}\right) \in \Sigma_1^s. \tag{4.4}$$

Note that PN_1 corresponds to a stable pseudo-node because $Z_{1y}^s > 0$ if $y < PN_1$ and $Z_{1y}^s < 0$ if $y > PN_1$.

The sliding segment Σ_1^s has two tangent points given by:

$$T_1^1 = \left(P_x, \frac{d_1 P_x}{a}\right) \text{ and } T_1^2 = \left(P_x, \frac{(d_1 + u_1)P_x}{a}\right). \tag{4.5}$$

The tangent point T_1^1 is visible if $X_2^2 f_1(T_1^1) = \left\langle X_2(T_1^1), \mathrm{grad} X_2 f_1(T_1^1)\right\rangle < 0$, that is,

$$P_x > \frac{K[\phi(0,0) - 1]}{d_1[b + K(a + d_2)]},$$

and invisible if

$$P_x < \frac{K[\phi(0,0) - 1]}{d_1[b + K(a + d_2)]}.$$

Similarly, the tangent point T_1^2 is visible if $X_1^2 f_1(T_1^2) = \left\langle X_1(T_1^2), \mathrm{grad} X_1 f_1(T_1^2)\right\rangle > 0$, that is,

$$P_x < \frac{K[\phi(u_1, 0) - 1]}{(d_1 + u_1)[b + K(a + d_2)]},$$

and invisible if

$$P_x > \frac{K[\phi(u_1, 0) - 1]}{(d_1 + u_1)[b + K(a + d_2)]}.$$

On the other hand, if $\phi(u_1, 0) > 1$ we have that the equilibrium $P_{(u_1,0)}$, $P_{(0,0)}$ and the pseudo-equilibrium PN_1 do not exist simultaneously in the phase portrait of the Filippov system (4.1) as shown in the following result:

Lemma 3. *If $\phi(u_1, 0) > 1$, the equilibrium $P_{(u_1,0)}$, $P_{(0,0)}$ and the pseudo-equilibrium PN_1 do not coexist in the Filippov system* (4.1).

Proof. If the pseudo-node PN_1 exists, that is $PN_1 \in \overline{T_1^1 T_1^2} \equiv \Sigma_1^s$, of systems (4.4) and (4.5) it follows that:

$$\frac{K[\phi(u_1, 0) - 1]}{(d_1 + u_1)[b + K(a + d_2)]} < P_x < \frac{K[\phi(0, 0) - 1]}{d_1[b + K(a + d_2)]}. \tag{4.6}$$

Therefore, if $\frac{K[\phi(u_1,0)-1]}{(d_1+u_1)[b+K(a+d_2)]} < P_x$ then $P_{(u_1,0)} \notin \Sigma_1^+$, that is, $P_{(u_1,0)}$ is not defined in the Filippov system (4.1). Analogously, if $P_x < \frac{K[\phi(0,0)-1]}{d_1[b+K(a+d_2)]}$ then $P_{(0,0)} \notin \Sigma_1^-$.

On the other hand, if the equilibrium $P_{(0,0)} \in \Sigma_1^-$, that is $\frac{K[\phi(0,0)-1]}{d_1[b+K(a+d_2)]} < P_x$, from system (4.6) we have that $PN_1 \notin \Sigma_1^s$. Analogously, if $P_{(u_1,0)} \in \Sigma^+$, that is $P_x < \frac{K[\phi(u_1,0)-1]}{(d_1+u_1)[b+K(a+d_2)]}$, then $PN_1 \notin \Sigma_1^s$.

It remains to verify that $P_{(u_1,0)}$ and $P_{(0,0)}$ do not coexist. Indeed, since

$$\frac{K[\phi(u_1, 0) - 1]}{(d_1 + u_1)[b + K(a + d_2)]} < \frac{K[\phi(0, 0) - 1]}{d_1[b + K(a + d_2)]},$$

if $P_{(0,0)} \in \Sigma_1^-$, that is, $\frac{K[\phi(0,0)-1]}{d_1[b+K(a+d_2)]} < P_x$, then $\frac{K[\phi(u_1,0)-1]}{(d_1+u_1)[b+K(a+d_2)]} < P_x$ and so $P_{(u_1,0)} \notin \Sigma_1^+$. Similarly, if $P_{(u_1,0)} \in \Sigma_1^+$ then $P_{(0,0)} \notin \Sigma_1^-$. □

If $\phi(0,0) < 1$, then the pseudo-equilibrium $PN_1 \notin \Sigma_1^s$. Consequently, and as stated in Theorem 1, the equilibrium P_0 is locally asymptotically stable in the Filippov system (4.1) for all $u_1, P_x > 0$.

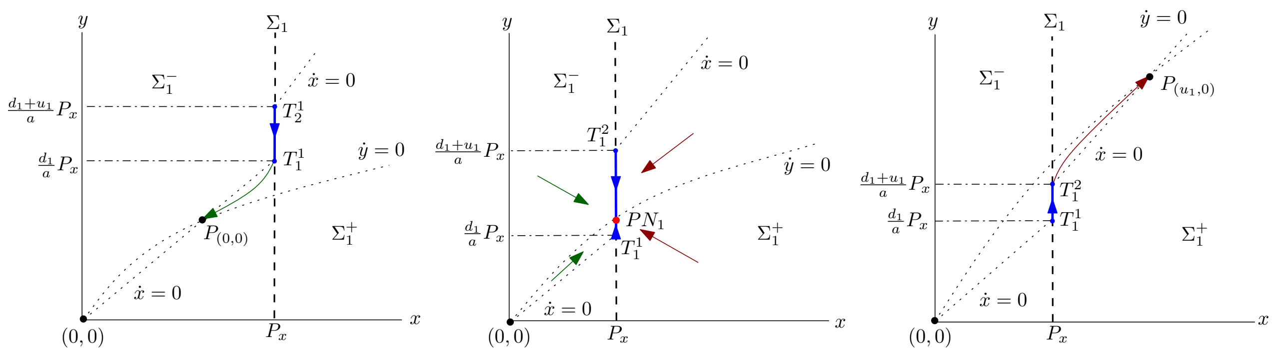
Lemma 4. *If $\phi(0,0) < 1$, the pseudo-equilibrium PN_1 does not belong to the Filippov system* (4.1). *That is, the equilibrium P_0 is locally a stable node.*

Proof. If the pseudo-node PN_1 exists, then

$$\frac{K[\phi(u_1,0)-1]}{(d_1+u_1)[b+K(a+d_2)]} < P_x < \frac{K[\phi(0,0)-1]}{d_1[b+K(a+d_2)]}.$$

Since $\phi(0,0) < 1$ it follows that $P_x < 0$ is contradictory. □

Note that if $PN \in \Sigma_1^s$ then $\overrightarrow{T_1^1 PN_1}$ and $\overrightarrow{T_1^2 PN_1}$. However, if $P_{(0,0)} \in \Sigma_1^-$ then $\overrightarrow{T_1^2 T_1^1} \equiv \Sigma_1^s$ and $\overrightarrow{T_1^1 T_1^2} \equiv \Sigma_1^s$ if $P_{(u_1,0)} \in \Sigma^+$. In addition, and as observed in Figure 8, the intersection of the nullclines of the Filippov system (4.1) over each region $\Sigma_1^\pm$ with respect to Σ_1 determines the formation of $P_{(0,0)}$, PN_1 or $P_{(u_1,0)}$ in the Filippov system (4.1).



(a) Equilibrium $P_{(0,0)}$ (b) Pseudo-equilibrium PN_1 (c) Equilibrium $P_{(u_1,0)}$

Figure 8. Nullclines $\dot{x} = 0$, $\dot{y} = 0$ and slidind segment Σ_1^s of the Filippov system (4.1).

In case the pseudo-equilibrium $PN_1 \in \Sigma_1^s$, that is, if the parameters of the Filippov system (4.1) satisfies (4.6), it follows that the trajectories of the Filippov system converge to PN_1, and thus PN_1 is globally asymptotically stable shown in the following result, whose proof was inspired by [19, 20].

Theorem 2. *If $PN_1 \in \Sigma_1^s$, then PN_1 is globally asymptotically stable.*

Proof. Since PN_1 is locally stable, P_0 is unstable and the interior equilibriums $P_{(0,0)}, P_{(u_1,0)} \notin \Sigma_1^\pm$, to guarantee the global asymptotic stability of PN_1 three conditions must be proved:

1) *There are no limit cycles in $\Sigma_1^\pm$*: By Lemma 2, with $u_2 = 0$, it guarantees the nonexistence of limit cycles in $\Sigma_1^\pm$.

2) *There is no closed orbit for system (4.1) which contains a part of Σ_1^s*: Since the pseudo-equilibrium PN is stable in Σ_1^s, there is not closed orbit for system (4.1) containing a part of Σ_1^s.

3) *There is no closed trajectory which contains $\Sigma_1^\pm$ and Σ_1^s*: Indeed, if it is assumed that there is a closed orbit Γ, with period T, of Filippov system (4.1), which passes through Σ_1 and encloses Σ_1^s, let P and Q as the intersection points of Γ and Σ_1, respectively, as observed in Figure 9. Similarly,

let $P_1 = P + a_1(\theta)$ and $Q_1 = Q - a_2(\theta)$, with intersection points of Γ and the line $x = P_x - \theta$, respectively, and $P_2 = P + b_1(\theta)$ and $Q_2 = Q - b_2(\theta)$ as the intersection points of Γ and $x = P_x + \theta$, where $\theta > 0$ is sufficiently small. Moreover, $a_{1,2}(\theta)$ and $b_{1,2}(\theta)$ are continuous with respect θ and $a_{1,2}(\theta), b_{1,2}(\theta) \to 0$ when $\theta \to 0$.

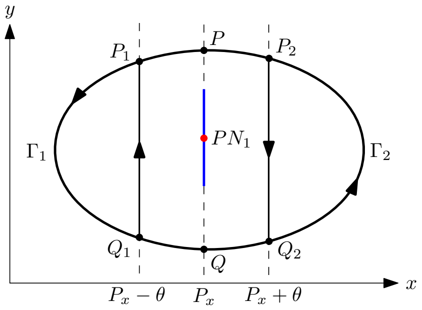


Figure 9. Limit cycle Γ in Filippov System (4.1).

As shown in Figure 9, the Γ_1 and $\overrightarrow{Q_1P_1}$ is locate in the region Σ_1^-. Similarly, the Γ_2 and $\overrightarrow{P_2Q_2}$ is locate in the region Σ_1^+. Furthermore, the dynamics of the disease Filippov system in region Σ_1^+ are represented by (3.1) with $u_2 = 0$. If $\partial\Sigma_1^+$ denote the boundary of Σ_1^+, by using Green's Theorem [18], we have

$$\begin{aligned}
\iint_{\Sigma_1^+}\left(\frac{\partial f}{\partial x}(x,y,u_1)+\frac{\partial g}{\partial y}(x,y,0)\right)\mathrm{d}\sigma &= \iint_{\Sigma_1^+}\frac{\partial f}{\partial x}(x,y,u_1)\mathrm{d}\sigma+\iint_{\Sigma_1^+}\frac{\partial g}{\partial y}(x,y,0)\mathrm{d}\sigma \\
&= \oint_{\partial\Sigma_1^+} f(x,y,u_1)\mathrm{d}y-\oint_{\partial\Sigma_1^+} g(x,y,0)\mathrm{d}x \\
&= \left(\int_{\Gamma_2} f(x,y,u_1)\mathrm{d}y+\int_{\overrightarrow{P_2Q_2}} f(x,y,u_1)\mathrm{d}y\right) \\
&- \left(\int_{\Gamma_2} g(x,y,0)\mathrm{d}x+\int_{\overrightarrow{P_2Q_2}} g(x,y,0)\mathrm{d}x\right) \\
&= \int_{\overrightarrow{P_2Q_2}} f(x,y,u_1)\mathrm{d}y,
\end{aligned} \tag{4.7}$$

where $\mathrm{d}x = f(x,y,u_1)\mathrm{d}t$, $\mathrm{d}y = g(x,y,0)\mathrm{d}t$, and there is no change of x in $\overrightarrow{P_2Q_2}$, then

$$\int_{\overrightarrow{P_2Q_2}} g(x,y,0)\mathrm{d}x = \int_{P_x+\theta}^{P_x+\theta} g(x,y,0)\mathrm{d}x = 0.$$

Similarly, the dynamics in Σ_1^- is represented by (3.1) with $u_1 = u_2 = 0$. By Green's Theorem,

$$\iint_{\Sigma_1^-}\left(\frac{\partial f}{\partial x}(x,y,0)+\frac{\partial g}{\partial y}(x,y,0)\right)\mathrm{d}\sigma = \int_{\overrightarrow{Q_1P_1}} f(x,y,0)\mathrm{d}y \tag{4.8}$$

Suppose that $\Sigma_{10}^- \subset \Sigma_1^-$ and

$$\epsilon = \iint_{\Sigma_{10}^-}\left(\frac{\partial f}{\partial x}(x,y,0)+\frac{\partial g}{\partial y}(x,y,0)\right)\mathrm{d}\sigma = \oint_{\partial\Sigma_{10}^+}(f(x,y,0)\mathrm{d}y - g(x,y,0)\mathrm{d}y) > 0. \tag{4.9}$$

From Lemma 2, and based on (4.9), we have

$$0 < \epsilon < \int_{\overrightarrow{P_2Q_2}} f(x, y, u_1)\mathrm{d}y + \int_{\overrightarrow{Q_1P_1}} f(x, y, 0)\mathrm{d}y. \tag{4.10}$$

As observed in Figure 9, if $\theta \to 0$ in the sum of (4.7) and (4.8) we have that

$$\begin{aligned} &\lim_{\theta\to 0}\left(\int_{\overrightarrow{P_2Q_2}} f(x, y, u_1)\mathrm{d}y + \int_{\overrightarrow{Q_1P_1}} f(x, y, 0)\mathrm{d}y\right) \\ = \;&\lim_{\theta\to 0}\left\{\int_{Q-b_2(\theta)}^{P+b_1(\theta)} [ay - (d_1 + u_1)P_x]\mathrm{d}y - \int_{Q-a_2(\theta)}^{P+a_1(\theta)} (ay - d_1P_x)\mathrm{d}y\right\} = u_1(Q - P) < 0. \end{aligned} \tag{4.11}$$

Then (4.10) holds, which contradicts to (4.11). Thus there is no closed orbit containing Σ_1^s and PN_1. Therefore, PN_1 is globally asymptotically stable.

□

Similarly, if the equilibrium $P_{(u_1,0)} \in \Sigma_1^+$, by Theorem (4.4) and Lemma 3 it follows that the trajectories of the Filippov system (4.1) converge to $P_{(u_1,0)}$. Therefore, the following result shows that the equilibrium $P_{(u_1,0)}$ is globally asymptotically stable.

Theorem 3. *If $P_{(u_1,0)} \in \Sigma_1^+$, then $P_{(u_1,0)}$ is globally asymptotically stable.*

Proof. Since $PN_1 \notin \Sigma_1^s$, $P_{(0,0)} \notin \Sigma_1^-$, P_0 is unstable and $P_{(u_1,0)} \in \Sigma_1^+$ is a locally stable node, to determine that $P_{(u_1,0)}$ is globally asymptotically stable three conditions must be proved:

1) *There are no limit cycles in $\Sigma_1^\pm$*: By Lemma 2, with $u_2 = 0$, it guarantees the nonexistence of limit cycles in $\Sigma_1^\pm$.

2) *There is no closed orbit for system (4.1) which contains a part of Σ_1^s*: As observed in Figure 8(c), any trajectory with initial condition on Σ_1^s must slide on Σ_1^s, reach T_1^2 and escape to Σ_1^+. Therefore, we will prove that the solution of system (4.1) starting from the tangent point T_1^2 cannot enter the Σ_1^s again. Indeed, if the trajectory starting at T_1^2, encircles $P_{(u_1,0)}$ and intercepts with T_1^2 forming a periodical orbit Γ, then any trajectory with initial condition outside Γ will not be able to cross Γ, and certainly cannot tend to the endemic equilibrium $P_{(u_1,0)}$, which contradicts the stability of $P_{(u_1,0)}$. Similarly, if the trajectory starting at T_1^2 and encircling the equilibrium $P_{(u_1,0)}$, intercepts Σ_1^s at some point other than T_1^2, then $P_{(u_1,0)}$ must be unstable, which also contradicts to the statement that $P_{(u_1,0)}$ is a stable equilibrium. Therefore, there is no closed orbit of Filippov system (4.1) containing part of Σ_1^s.

3) *There is no closed trajectory which contains $\Sigma_1^\pm$ and Σ_1^s*: This step is similar to that of Theorem 2, and we omit it here for brevity.

□

If $\phi(u_1, 0) > 1$ and the equilibrium $P_{(0,0)} \in \Sigma_1^-$, by Theorem 1 and Lemma 3, the trajectories of the Filippov system (4.1) converge to $P_{(0,0)}$. Similarly, if the condition shown in Lemma 4 is satisfied, then the equilibrium P_0 is globally asymptotically stable, shown in the following result.

Theorem 4. *If $\phi(u_1, 0) > 1$ and $P_{(0,0)} \in \Sigma_1^-$, then $P_{(0,0)}$ is globally asymptotically stable. If $\phi(0, 0) < 1$, then P_0 is globally asymptotically stable.*

Proof. A similar procedure to that of Theorem 3 can be used to prove this theorem, and we omit it here for brevity. □

4.2. Bifurcation diagram

Finally, we analyze the cases in which the parameters u_1 and P_x can modify the phase diagrams of the Filippov system (4.1) using the results found by the qualitative analysis presented in Section 4.1.

For the case where $\phi(0, 0) < 1$, and according to Lemma 4, the Filippov system (4.1) has a globally asymptotically stable equilibrium P_0 for all $u_1 > 0$ and $0 < P_x \le \frac{aK}{d_1+u_1}$ as observed in Figure 10.

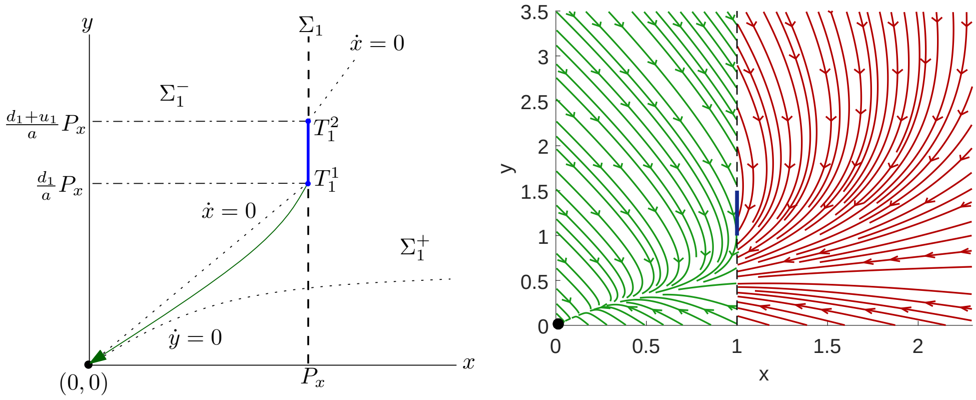


Figure 10. Nullclines $\dot{x} = 0$ and $\dot{y} = 0$, slidind segment Σ_1^s and phase portrait of the Filippov system (4.1) with parameters: $a = c = d_1 = d_2 = 1$, $K = 3.5$, $b = 2$, $u_1 = 0.5$ and $P_x = 1$.

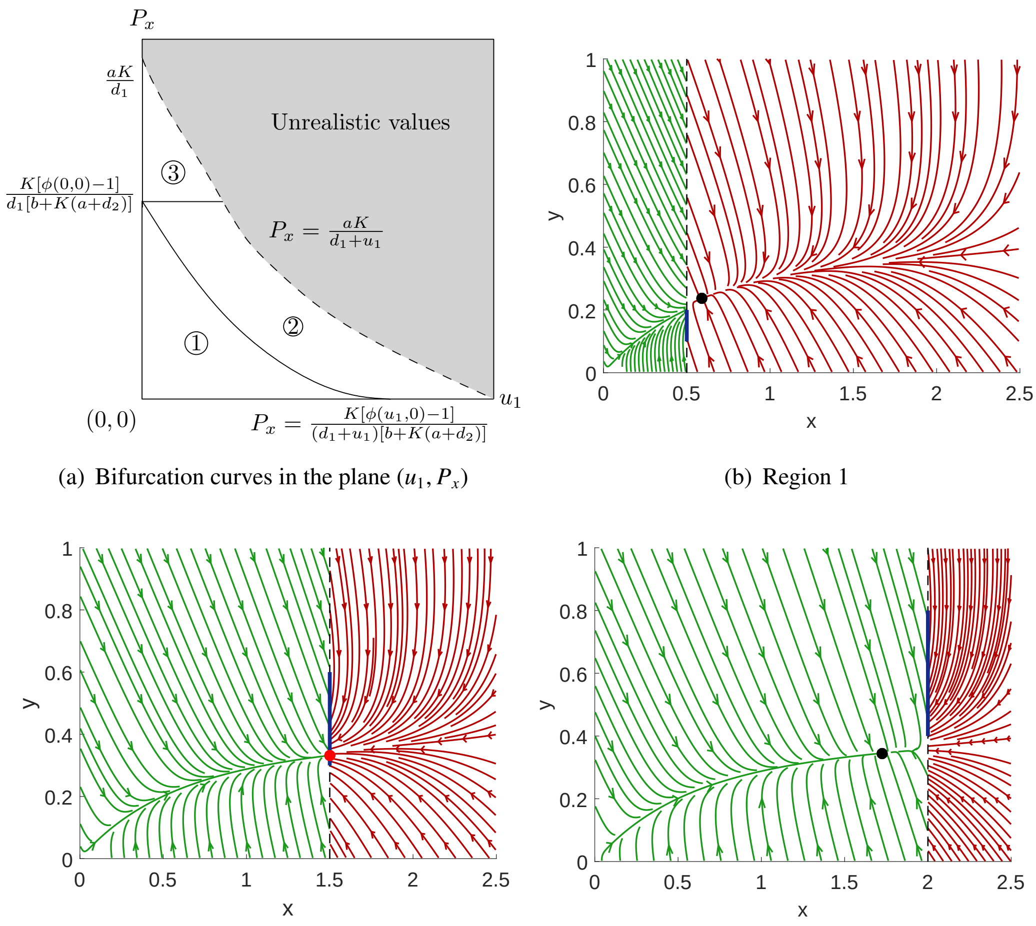
For the case where $\phi(u_1, 0) > 1$, the collision of PN_1 with T_1^1 or with T_1^2 are represented, respectively, by the curves

$$P_x = \frac{K[\phi(0,0) - 1]}{d_1[b + K(a + d_2)]} \quad \text{and} \quad P_x = \frac{K[\phi(u_1,0) - 1]}{(d_1 + u_1)[b + K(a + d_2)]}, \tag{4.12}$$

which generates three bifurcation regions as seen in Figure 11(a). The curves (4.12) intersect at the point $\left(0, \frac{K[\phi(0,0)-1]}{d_1[b+K(a+d_2)]}\right)$ and form a collision between T_1^1 and T_1^2 with PN_1 for which the sliding segment does not exist.

In this case, the pseudo-node PN_1 exists only in Region 2, as shown in Figure 11(c). Analogously, the point T_1^1 is visible in Region 3, as shown in Figure 11(d), and invisible in Regions 1 and 2, as shown in Figure 11(b),(c). T_1^2 is visible in Region 1 and invisible in Regions 2 and 3.

On the other hand, $P_{(0,0)}$ exists in the Filippov system (4.1) if $\frac{K[\phi(0,0)-1]}{d_1[b+K(a+d_2)]} < P_x$. Therefore, $P_{(0,0)}$ exists in Region 3 and does not exist in Regions 1 and 2. Analogously, $P_{(u_1,0)}$ exists only in Region 1 since $P_x < \frac{K[\phi(u_1,0)-1]}{(d_1+u_1)[b+K(a+d_2)]}$. The phase portraits for Regions 1–4 of the Filippov system (4.1) are observed in Figure 11.



(a) Bifurcation curves in the plane (u_1, P_x)

(b) Region 1

(c) Region 2

(d) Region 3

Figure 11. Bifurcation diagram of the Filippov system (4.1) in the plane (u_1, P_x), and phase portraits characterizing each region, with fixed parameters: $a = b = c = K = 1$ and $d_1 = d_2 = u_1 = 0.2$. Black point: $P_{(0,0)}$ or $P_{(u_1,0)}$. Red point: PN_1. Blue line: sliding segment.

5. Control in the non-reproductive stage $y(t)$

If the stage $y(t)$ is controlled when it is above a critical value $P_y \leq K$, then the change in $x(t)$ and $y(t)$ with respect to time $t \geq 0$ is given by the system (1.5), equivalent to:

$$Z_2(x,y) = \begin{cases} X_3(x,y) = \begin{pmatrix} ay - d_1 x \\ \dfrac{bx}{x+c}\left(1 - \dfrac{y}{K}\right) - (a + d_2 + u_2)y \end{pmatrix}, & y > P_y \\ X_2(x,y) = \begin{pmatrix} ay - d_1 x \\ \dfrac{bx}{x+c}\left(1 - \dfrac{y}{K}\right) - (a + d_2)y \end{pmatrix}, & y < P_y \end{cases} \tag{5.1}$$

where

$$\begin{aligned}\Sigma_2^+ &= \left\{(x,y)\in\mathbb{R}^2 : 0\le x\le \frac{aK}{d_1}, P_y < y \le K\right\},\\ \Sigma_2 &= \left\{(x,P_y)\in\mathbb{R}^2 : 0\le x\le \frac{aK}{d_1}\right\},\\ \Sigma_2^- &= \left\{(x,y)\in\mathbb{R}^2 : 0\le x\le \frac{aK}{d_1}, 0\le y < P_y\right\}.\end{aligned}$$

5.1. Qualitative analysis

To perform a qualitative analysis of the Filippov system (5.1) we must calculate the equilibrium points for the X_2 and X_3 fields, the Σ_2^s, Σ_2^e, Σ_2^c regions and the Z_2^s sliding segment. Indeed, the equilibrium point over the vector field X_3 is

$$P_{(0,u_2)} = \left(\frac{K[\phi(0,u_2)-1]}{d_1[b+K(a+d_2+u_2)]}, \frac{K[\phi(0,u_2)-1]}{a[b+K(a+d_2+u_2)]}\right),$$

with $\phi(0,u_2)$ as described in (3.2). The equilibrium points of the X_2 field are given by (4.3).

Note that for $\phi(0,u_2) > 1$, and given that $(d_1+u_2)(ac+cd_2) > d_1(ac+cd_2)$, we have that $\phi(0,0) > 1$ and so $P_{(0,0)}, P_{(0,u_2)} \in \Omega$. However, if $\phi(0,0) < 1$ it follows that $P_{(0,0)}, P_{(0,u_2)} \notin \Omega$.

On the other hand, since $f_2(x,y) = y - P_y$, then for all $p\in\Sigma_2$ we have that:

$$\begin{aligned}X_3 f_2(p) &= \langle X_3(p), \mathrm{grad} f_2(p)\rangle\\ &= -\frac{x\left\{P_y[b+K(a+d_2+u_2)] - bK\right\} + cKP_y(a+d_2+u_2)}{K(x+c)},\\ X_2 f_2(p) &= \langle X_2(p), \mathrm{grad} f_2(p)\rangle\\ &= -\frac{x\left\{P_y[b+K(a+d_2)] - bK\right\} + cKP_y(a+d_2)}{K(x+c)}.\end{aligned}$$

Therefore, the tangent points of the Filippov system (5.1) are given by:

$$T_2^1 = \left(\frac{cKP_y(a+d_2)}{bK-P_y[b+K(a+d_2)]}, P_y\right) \quad\text{and}\quad T_2^2 = \left(\frac{cKP_y(a+d_2+u_2)}{bK-P_y[b+K(a+d_2+u_2)]}, P_y\right). \tag{5.2}$$

Since $bK - P_y[b+K(a+d_2+u_2)] < bK - P_y[b+K(a+d_2)]$, if $0 < bK - P_y[b+K(a+d_2+u_2)] < bK - P_y[b+K(a+d_2)]$ then T_1^2 and T_2^2 exist, with $T_2^1 < T_2^2$. However, if $bK-P_y[b+K(a+d_2+u_2)] < 0 < bK-P_y[b+K(a+d_2)]$ then T_2^2 does not exist, if $bK-P_y[b+K(a+d_2+u_2)] < bK-P_y[b+K(a+d_2)] < 0$ there are no tangent points in the Filippov system (5.1).

Therefore, if T_2^1, T_2^2 exist, and by supposing that $\frac{cKP_y(a+d_2+u_2)}{bK-P_y[b+K(a+d_2+u_2)]} < \frac{aK}{d_1}$, then

$$\begin{aligned}\Sigma_2^s &= \left\{(x,P_y)\in\mathbb{R}^2 : \frac{cKP_y(a+d_2+u_2)}{bK-P_y[b+K(a+d_2+u_2)]} < x < \frac{cKP_y(a+d_2+u_2)}{bK-P_y[b+K(a+d_2+u_2)]}\right\},\\ \Sigma_2^c &= \left\{(x,P_y)\in\mathbb{R}^2 : x < \frac{cKP_y(a+d_2)}{bK-P_y[b+K(a+d_2)]} \text{ o } \frac{cKP_y(a+d_2+u_2)}{bK-P_y[b+K(a+d_2+u_2)]} < x\right\},\\ \Sigma_2^e &= \emptyset.\end{aligned}$$

If T_2^2 does not exist, and by assuming that $\frac{cKP_y(a+d_2)}{bK-P_y[b+K(a+d_2)]} < \frac{aK}{d_1}$, then:

$$\begin{aligned} \Sigma_2^s &= \left\{(x, P_y) \in \mathbb{R}^2 : \frac{cKP_y(a+d_2)}{bK - P_y[b+K(a+d_2)]} < x\right\}, \\ \Sigma_2^c &= \left\{(x, P_y) \in \mathbb{R}^2 : x < \frac{cKP_y(a+d_2)}{bK - P_y[b+K(a+d_2)]}\right\}, \\ \Sigma_2^e &= \emptyset. \end{aligned}$$

For the case where there are no tangent points, then:

$$\Sigma_2^c = \Sigma_2 \quad \text{and} \quad \Sigma_2^s = \Sigma_2^e = \emptyset.$$

The tangent point T_1^2 is visible if $X_2^2 f_2(T_1^2) = \left\langle X_2^2(T_1^2), \mathrm{grad} X_2^2 f(T_1^2)\right\rangle < 0$, that is,

$$P_y > \frac{K[\phi(0,0)-1]}{a[b+K(a+d_2)]}, \tag{5.3}$$

and invisible if,

$$P_y < \frac{K[\phi(0,0)-1]}{a[b+K(a+d_2)]}, \tag{5.4}$$

Equivalently, the tangent point T_2^2 is visible if $X_3^2 f_2(T_2^2) = \left\langle X_3^2(T_2^2), \mathrm{grad} X_3 f_2(T_2^2)\right\rangle > 0$, that is,

$$P_y < \frac{K[\phi(u_1,0)-1]}{(d_1+u_1)[b+K(a+d_2)]},$$

and invisible if,

$$P_y > \frac{K[\phi(u_1,0)-1]}{(d_1+u_1)[b+K(a+d_2)]}.$$

The vector field of the sliding segment $Z_2^s(p)$, with $p \in \Sigma_2^s$, is given by

$$Z_2^s(p) = \begin{pmatrix} 0 \\ aP_y - d_1 x \end{pmatrix},$$

with pseudo-equilibrium

$$PN_2 = \left(\frac{aP_y}{d_1}, P_y\right) \in \Sigma_2^s. \tag{5.5}$$

which corresponds locally to a stable pseudo-node.

On the other hand, if $\phi(0,0) < 1$ then the pseudo-equilibrium $PN_2 \notin \Sigma_2^s$. Consequently, and as stated in Theorem 1, the equilibrium P_0 is globally asymptotically stable in the Filippov system (5.1) for all $u_2, P_y > 0$.

Theorem 5. *If $\phi(0,0) < 1$, the pseudo-equilibrium PN_2 does not belong to the Filippov system* (5.1). *Therefore, the equilibrium P_0 is globally asymptotically stable.*

Proof. Suppose PN_2 exists when $PN_2 \in \overline{T_2^1 T_2^2}$. From (5.2) and (5.5) it follows that:

$$\frac{K[\phi(0,u_2)-1]}{d_1[b+K(a+d_2+u_2)]} < P_y < \frac{K[\phi(0,0)-1]}{d_1[b+K(a+d_2)]}. \tag{5.6}$$

Since $\phi(0,0) < 1$ it follows that $P_x < 0$ is contradictory. Analogous case for $PN_2 \in \overline{T_2^1 \frac{aK}{d_1}}$. Similar to what was shown in Theorem 4, can be used to prove the global stability of P_0. □

However, if $\phi(0, u_2) > 1$ then the equilibrium $P_{(0,u_2)}$, $P_{(0,0)}$ and the pseudo-equilibrium PN_2 do not coexist in the Filippov system (4.1), whose proof is similar to the one presented in Theorems 2 and 3, therefore the following result follows.

Theorem 6. *If $\phi(0, u_2) > 1$ then the equilibrium $P_{(0,u_2)}$, $P_{(0,0)}$ and the pseudo-equilibrium PN_2 do not coexist in the Filippov system* (4.1). *Therefore, PN_2, $P_{(0,u_2)}$ or $P_{(0,0)}$ are globally asymptotically stable nodes.*

5.2. Bifurcation diagram

The cases in which the parameters u_2 and P_y significantly alter the phase diagrams of the Filippov system (5.1) are analyzed using the results found by the qualitative analysis presented in Section 5.1.

For the case where $\phi(0, 0) < 1$, the Filippov system (4.1) possesses a globally asymptotically stable P_0 equilibrium for all $u_2 > 0$ and $0 < P_y \leq K$. However, the existence and shape of the sliding segment Σ_2^s depends on the existence of the tangent points T_2^1 and T_2^2. From (5.2), we have two bifurcation curves given by:

$$\begin{aligned} P_y &= \frac{bK}{b + K(a + d_2)}, \\ P_y &= \frac{bK}{b + K(a + d_2 + u_2)}, \end{aligned} \tag{5.7}$$

which generates three regions as observed in Figure 12(a). In Region 1, as shown in Figure 12(b), shows the presence of two tangent points and $\Sigma_2^s = \overline{T_2^1 T_2^2}$. In region two, as see in Figure 12(c), shows the existence of only one tangent point T_2^1, and in the case that $\frac{cKP_y(a+d_2)}{bK - P_y[b+K(a+d_2)]} < \frac{aK}{d_1}$, we have that $\Sigma_2^s = \overline{T_2^1 \frac{aK}{d_1}}$. In region 3, as shown in Figure 12(d), we have that $\Sigma_2^s = \emptyset$.

For the case where $\phi(0, u_2) > 1$, the collision of PN_2 with T_2^1 or with T_2^2 are represented, respectively, by the curves:

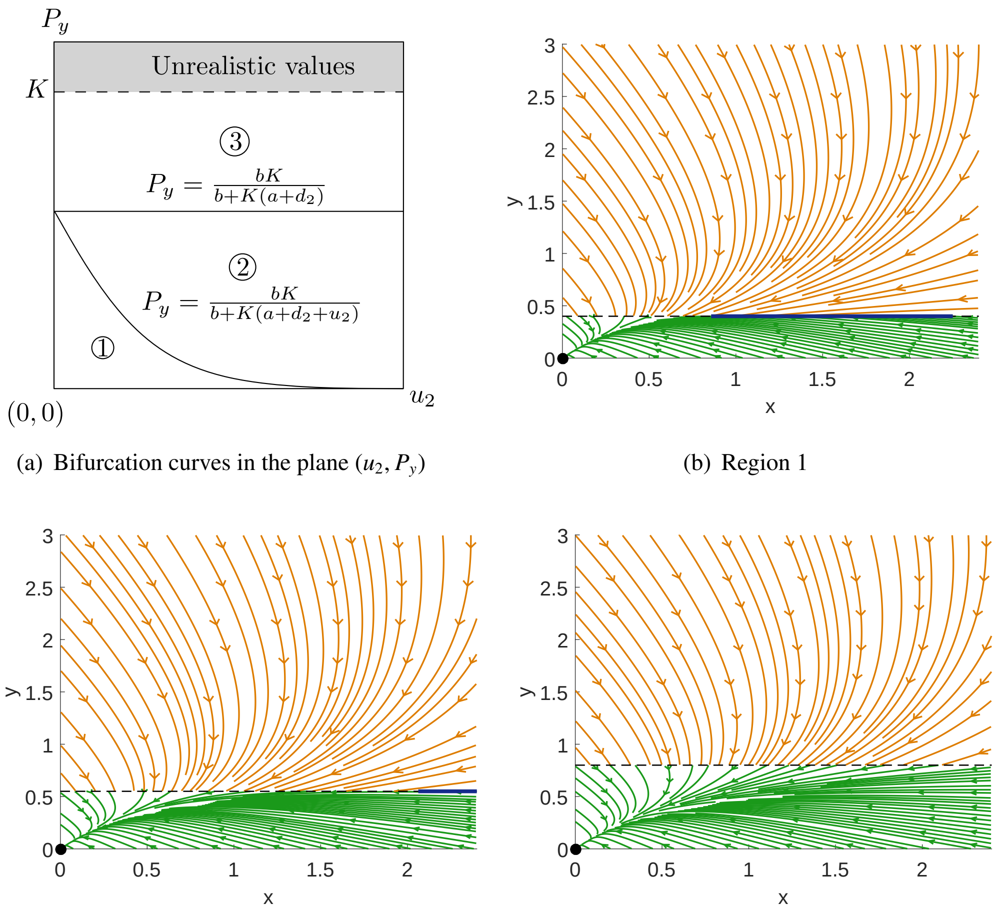
$$\begin{aligned} P_y &= \frac{K[\phi(0, 0) - 1]}{a[b + K(a + d_2)]}, \\ P_y &= \frac{K[\phi(0, u_2) - 1]}{a[b + K(a + d_2 + u_2)]}, \end{aligned} \tag{5.8}$$

which, together with the curves (5.7), generates six bifurcation regions as shown in Figure 13(a).

In this case, the equilibrium $P_{(0,u_2)}$ exists in region 1, as shown in Figure 13(b), with a sliding segment $\Sigma_2^s = \overline{T_2^1 T_2^2}$. The pseudo-equilibrium exists in regions 2 and 3, as shown in Figures 12(c,d), respectively, where the sliding segment $\Sigma_2^s = \overline{T_2^1 T_2^2}$ exists in region 2 and $\Sigma_2^s = \overline{T_2^1 \frac{aK}{d_1}}$ exists in region 3. Equilibrium $P_{(0,0)}$ exists in regions 4–6, as shown in Figures 12(e)–(g), respectively, where there is no presence of the sliding segment in region 6. Figure 12 shows the phase portraits for regions 1 to 6 of the Filippov system (4.1).

6. Conclusions

In this work, a global qualitative analysis of the preliminary model and the modified Filippov systems model was carried out to analyze the behavior of a species without interaction with other species and subject to control parameters with critical values. Unlike the continuous model that controls the stages of the species all times, the discontinuous models are used to control the dynamics of the species



(a) Bifurcation curves in the plane (u_2, P_y)

(b) Region 1

(c) Region 2

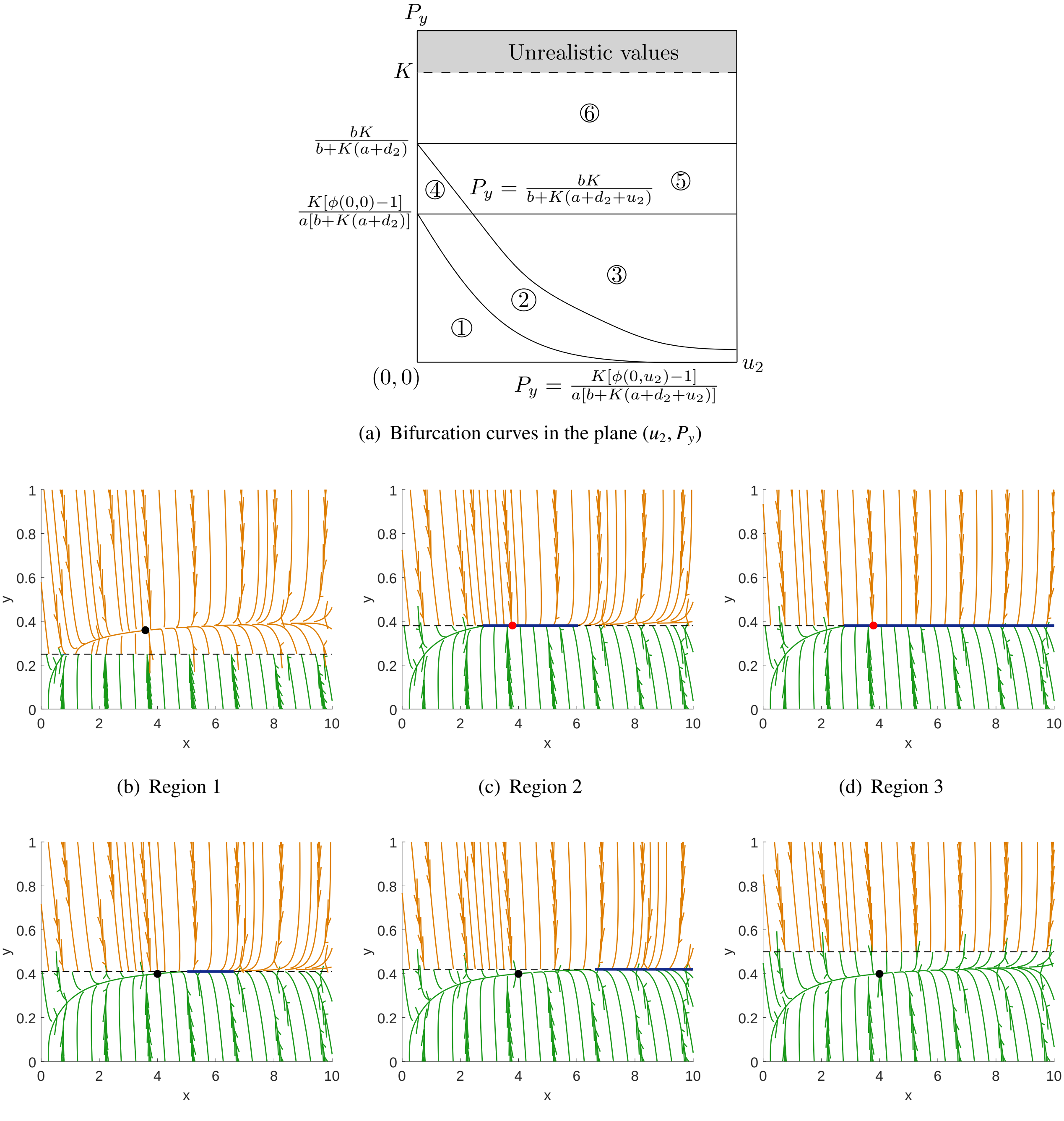
(d) Region 3

Figure 12. Bifurcation diagram of the Filippov system (5.1) in the plane (u_2, P_y), and phase portraits characterizing each region, with fixed parameters: $a = 0.8$, $b = c = d_1 = 1$, $K = 3$, $d_2 = 0.2$. Black point: $P_{(0,0)}$. Red point: PN_2. Blue line: sliding segment.

only when the quantity in one of its stages is above a critical value. This type of models are used, for example, when it is desired to apply a control strategy, chemical or biological, of a species until it converges to a desired value, as in the case of timber production in exploited forests, the growth of the Aedes aegypty mosquito, fishing activities or species to be used as anti-pests without undesired growth in the same species.

Regarding the global analysis of the models, whether continuous or not, we conclude that there are no limit cycles, so that their solutions do not oscillate over time but converge to an equilibrium. In the case of the continuous model, the dynamics of the species converges either to an internal equilibrium or disappears over time as long as the growth threshold is greater or less than one, respectively, so that the alteration of the control parameters influences the extinction or conservation of the species.

However, for discontinuous models, the species tends to disappear when the growth threshold, with zero control parameters, is less than one, otherwise the species converges to an equilibrium point. For the stage of which it has the control parameter, its dynamics converges to the value coinciding with the discontinuity of the model provided that the pseudo-equilibrium exists. Otherwise, the dynamics



(a) Bifurcation curves in the plane (u_2, P_y)

(b) Region 1

(c) Region 2

(d) Region 3

(e) Region 4

(f) Region 5

(g) Region 6

Figure 13. Bifurcation diagram of the Filippov system (5.1) in the plane (u_2, P_y), and phase portraits characterizing each region with fixed parameters: $a = b = c = 1$, $K = 1$, $d_1 = 0.1$ and $d_2 = 0.2$. Black point: $P_{(0,0)}$ or $P_{(0,u_2)}$. Red point: PN_2. Blue line: sliding segment.

converges to one of the equilibria over the vector fields of the Filippov system.

From the biological point of view, the bifurcation regions for each discontinuous model is a useful tool to show conditions in the control parameters which allow the inner equilibrium point not to be located above the threshold value. Thus, it is not necessary to consider an all-weather control strategy, chemical or biological, when one does not have sufficient resources, economic or social, to maintain such an activated control. On the contrary, a continuous model is useful to describe the dynamics of the species and to establish conditions in the control parameters to reach the desired objective for the species.

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Conflict of interest

The author declare there is no conflict of interest.

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