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Research article

Odd-order differential equations with deviating arguments: asymptomatic behavior and oscillation

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Abstract: Despite the growing interest in studying the oscillatory behavior of delay differential equations of even-order, odd-order equations have received less attention. In this work, we are interested in studying the oscillatory behavior of two classes of odd-order equations with deviating arguments. We get more than one criterion to check the oscillation in different methods. Our results are an extension and complement to some results published in the literature.

Keywords: differential equation with deviating argument; odd-order; neutral; oscillation; asymptotic behavior

1. Introduction

This study investigates the oscillatory and asymptotic behavior of delay differential equations (DDEs) of odd-order

$$\left(a(\eta)\left(\psi^{(n-1)}(\eta)\right)^{\kappa}\right)' + q(\eta) f(\psi(\phi(\eta))) = 0 \tag{1.1}$$

and

$$\left(a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}\right)' + q(\eta) f(\psi(\phi(\eta))) = 0, \tag{1.2}$$

where $n \geq 3$ is an odd integer and $\upsilon(\eta) = \psi(\eta) + p(\eta)\psi(\tau(\eta))$. Further, we assume that:

(i) κ is a ratio of odd natural numbers;
(ii) $q, p \in C([\eta_0, \infty), (0, \infty))$ and $0 \leq p(\eta) < 1$;
(iii) $a, \tau, \phi \in C^1([\eta_0, \infty))$, $a(\eta) > 0,\ a'(\eta) \geq 0,\ \phi(\eta) \geq \eta \geq \tau(\eta), \lim_{\eta\to\infty}\tau(\eta) = \infty$;
(iv) $f \in C(\mathbb{R}, \mathbb{R})$, $f(\psi) \geq k\psi^{\kappa}$ and

$$\pi(\eta) = \int_{\eta_0}^{\eta} \frac{1}{a^{1/\kappa}(s)} \mathrm{d}s \to \infty \text{ as } \eta \to \infty. \tag{1.3}$$

If there exists a $\eta_\psi \geq \eta_0$ with a continuous function ψ satisfies (1.1), $a(\eta)\left(\psi^{(n-1)}\right)^{\kappa}(\eta) \in C^1\left(\left[\eta_\psi, \infty\right), \mathbb{R}\right)$, and $\sup\{|\psi(\eta)| : \eta_1 \leq \eta\} > 0$ for every $\eta_1, \eta \in \left[\eta_\psi, \infty\right)$, then ψ is said to be a proper solution of (1.1). A solution ψ of (1.1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.

Lately, great attention has been devoted to the theory of oscillation in DDEs. The works [1–10] develop techniques and methods for studying the oscillations of second-order DDEs. This development was necessarily reflected in the study of the oscillation of even-order DDEs, and this can be seen through the works, for example [11–18]. On the other hand, odd-order DDEs have received less attention compared to even-order DDEs. The development of the study of such equations can be traced through papers, [19–29], and the references cited therein.

Baculikova and Dzurina [30] studied the asymptotic properties of neutral DDE

$$\left(a(\eta)\left((\psi(\eta) \pm p(\eta)\psi(\delta(\eta)))''\right)^{\kappa}\right)' + q(\eta)\psi^{\kappa}(\phi(\eta)) = 0.$$

Li and Rogovchenko [31] investigated the oscillation of neutral DDE

$$\left(a(\eta)\left(\upsilon''(\eta)\right)^{\kappa}\right)' + q(\eta)\psi^{\kappa}(\phi(\eta)) = 0,$$

where $\upsilon(\eta) = \psi(y) + p_0\psi(\eta - \delta_0)$ and $\delta_0 \geq 0$ (delayed argument) or $\delta_0 \leq 0$ (advanced argument). Lackova [32] deduced oscillatory and asymptotic behavior of neutral DDE

$$(\psi(\eta) + p(\eta)\psi(\tau(\eta)))^{(n)} + q(\eta) f(\psi(\phi(\eta))) = 0,$$

where $\phi(\eta)$ is a delayed argument and $n \geq 2$, $f(\varrho)\,\mathrm{sgn}\,\varrho \geq k|\varrho|^{\kappa},\ \kappa \geq 1,\ k > 0$.

The half-linear differential equations arise in the study of p Laplace equations, porous medium problems, chemotaxis models, and so forth; see, for instance, the papers [33–35] for more details. In this study, we investigate the oscillatory and asymptotic properties of solutions for half-linear differential equations (1.1) and (1.2) under the conditions mentioned above and present some new results which are complementary and extend to [30–32]. We will support the results obtained with two examples.

2. Auxiliary lemmas

We start with some lemmas that we will need to use later. The next result is a well-known result; see [36, Lemma 2], also see [37, Lemma 2.2.1].

Lemma 2.1. *If ψ is a solution of (1.1) and positive eventually, then $\psi^{(k)}(\eta)$, $1 \leq k \leq n-1$, are of constant signs, $a(\eta)\left(\psi^{(n-1)}(\eta)\right)^{\kappa}$ is decreasing. Moreover, ψ satisfies either*

$$\psi'(\eta) > 0,\ \psi''(\eta) > 0,\ \psi^{(n-1)}(\eta) > 0,\ \psi^{(n)}(\eta) < 0 \tag{2.1}$$

or

$$(-1)^m \psi^{(m)} > 0,\ m = 1, 2, ..., n. \tag{2.2}$$

Lemma 2.2. *[36] Let $\psi \in C^n([\eta_0,\infty),(0,\infty))$, $\psi^{(n-1)}(\eta)\psi^{(n)}(\eta) \leq 0$ for $\eta \geq \eta_\psi$ and assume that* $\lim_{\eta\to\infty}\psi(\eta) \neq 0$, *then there exists an $\eta_\theta \in \left[\eta_\psi,\infty\right)$ with*

$$\psi(\eta) \geq \frac{\theta}{(n-1)!}\eta^{n-1}\left|\psi^{(n-1)}(\eta)\right| \text{ for all } \eta \in [\eta_\theta,\infty) \text{ and } \theta \in (0,1).$$

Lemma 2.3. *Assume that $\psi^{(i)}(\eta) > 0$ for $i = 0, 1, 2$, eventually. Then, for all $\delta_0 \in (0,1)$,*

$$\psi'(\eta) \geq \frac{\delta_0}{\eta}\psi(\eta)$$

and

$$\psi(\phi(\eta)) \geq \left(\frac{\phi(\eta)}{\eta}\right)^{\delta_0}\psi(\eta). \tag{2.3}$$

Proof. Assume that $\psi^{(i)}(\eta) > 0$ for $i = 0, 1, 2$ and for all $\eta \geq \eta_1 \geq \eta_0$, η_1 large enough. Then, we get

$$\psi(\phi(\eta)) - \psi(\eta) = \int_\eta^{\phi(\eta)} \psi'(s)\,\mathrm{d}s \geq \psi'(\eta)(\phi(\eta)-\eta). \tag{2.4}$$

It is easy to notice that $\lim_{\eta\to\infty}\psi(\eta) = \infty$. Hence, there exists $\eta_2 \geq \eta_1$ large enough such that

$$\delta_0\psi(\eta) \leq \psi(\eta) - \psi(\eta_2) = \int_{\eta_2}^{\eta}\psi'(s)\,\mathrm{d}s \leq \psi'(\eta)(\eta-\eta_2) \leq \eta\psi'(\eta), \tag{2.5}$$

for all $\delta_0 \in (0,1)$. By integrating this inequality from η to $\phi(\eta)$, we find

$$\psi(\phi(\eta)) \geq \left(\frac{\phi(\eta)}{\eta}\right)^{\delta_0}\psi(\eta).$$

This completes the proof of this Lemma.

Lemma 2.4. *Suppose that ψ is a positive solution of (1.2). Then, $\left(a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}\right)' < 0$, $\upsilon^{(i)}(\eta)$, $1 \leq i \leq n-1$, are of constant signs, and $\upsilon(\eta)$ satisfies either*

$$\textit{Case } (1): \upsilon(\eta) > 0,\ \upsilon'(\eta) > 0,\ \upsilon''(\eta) > 0,\ \upsilon^{(n-1)}(\eta) > 0 \text{ and } \upsilon^{(n)}(\eta) \leq 0 \tag{2.6}$$

or

$$\textit{Case } (2): (-1)^k\upsilon^{(k)}(\eta) > 0, \text{ for } k = 1, 2, ..., n. \tag{2.7}$$

Proof. Suppose that ψ is a solution of (1.2) and positive eventually. Produces directly from (1.2) that

$$\left(a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}\right)' \leq -kq(\eta)\psi^{\kappa}(\phi(\eta)) < 0. \tag{2.8}$$

Now, from the above inequality we find either $\upsilon^{(n-1)}(\eta) > 0$ or $\upsilon^{(n-1)}(\eta) < 0$.
If $\upsilon^{(n-1)}(\eta) < 0$, then

$$a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa} < -c < 0,$$

integration from η_1 to η, we have

$$\upsilon^{(n-2)}(\eta) < \upsilon^{(n-2)}(\eta_1) - c^{1/\kappa}\int_{\eta_1}^{\eta}\frac{1}{a^{1/\kappa}(s)}\mathrm{d}s,$$

by using (1.3) we have $\upsilon^{(n-2)}(\eta) \to -\infty$ at $\eta \to \infty$, and by doing this process several times we get $\upsilon(\eta) \to -\infty$. This contradicts the positive $\upsilon(\eta)$, then $\upsilon^{(n-1)}(\eta) > 0$. Since $\upsilon^{(n-1)}(\eta) > 0$, we have that either $\upsilon^{(n-2)}(\eta) > 0$ or $\upsilon^{(n-2)}(\eta) < 0$. But, $\upsilon^{(n-2)}(\eta) > 0$ leads to $\upsilon^{(i)}(\eta) > 0$ for $0 \leq i \leq n-2$. Repeating these considerations, we verify that $\upsilon(\eta)$ satisfies either (2.6) or (2.7).
Now since $\upsilon^{(n-1)}(\eta) > 0$ and $a' \geq 0$. Then we have

$$0 > \left(a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}\right)' = a'(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa} + \kappa a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa-1}\upsilon^{(n)}(\eta),$$

which shows us that $\upsilon^{(n)}(\eta) < 0$. This completes the proof of this lemma.

Lemma 2.5. *Let Case* (1) *hold. Then*

$$\frac{\upsilon(\phi(\eta))}{\upsilon(\eta)} \geq \left(\frac{\phi(\eta)}{\eta}\right)^{\delta_0}, \tag{2.9}$$

for all $\delta_0 \in (0,1)$.

Proof. The proof of the above lemma is similar to that of Lemma 2.3 and so it is omitted.

Next, we will present the basic definitions and notations that we will use in our results.
$\{h_m(\eta)\}_{m=0}^{\infty}$ is a sequence of continuous functions defined as follows

$$h_0(\eta) = k\Psi(\eta),\ k \in (0,1) \text{ fixed},$$

$$h_{m+1}(\eta) = h_0(\eta) + \frac{\kappa k}{(n-2)!}\int_{\eta}^{\infty} h_m^{(\kappa+1)/\kappa}(s)\frac{s^{n-2}}{a^{1/\kappa}(s)}\mathrm{d}s,\ m = 0,1,..., \tag{2.10}$$

$$\Psi(\eta) = \int_{\eta}^{\infty} q(s)\left(\frac{\phi(s)}{s}\right)^{\kappa\delta_0}(1-p(\phi(s)))^{\kappa}\mathrm{d}s$$

and

$$\Theta(\eta) = \frac{k^{1/\kappa}}{a^{1/\kappa}(\eta)}\left(\int_{\eta}^{\infty} q(s)\mathrm{d}s\right)^{1/\kappa},$$

where $\delta_0 \in (0,1)$.

3. Main results

Now, we present our results for (1.1) and (1.2).

Theorem 3.1. *Assume that*

$$\liminf_{\eta\to\infty}\frac{1}{\Upsilon(\eta)}\int_{\eta}^{\infty}\frac{s^{n-2}\Upsilon^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s>\frac{(n-2)!}{(\kappa+1)^{(\kappa+1)/\kappa}} \tag{3.1}$$

and

$$\int_{\eta_0}^{\infty}s^{n-2}\frac{1}{a^{1/\kappa}(s)}\left(\int_{s}^{\infty}q(\varrho)\,\mathrm{d}\varrho\right)^{1/\kappa}\mathrm{d}s=\infty, \tag{3.2}$$

where

$$\Upsilon(\eta)=\int_{\eta}^{\infty}q(s)\left(\frac{\phi(s)}{s}\right)^{\kappa\delta_0}\mathrm{d}s,$$

then $\lim_{\eta\to\infty}\psi(\eta)=0$, *for every nonoscillatory solution* $\psi(\eta)$ *of (1.1).*

Proof. Assume that ψ is a solution of (1.1), positive eventually, and satisfies (2.1). By (3.1), we have

$$\liminf_{\eta\to\infty}\frac{k^{(\kappa+1)/\kappa}}{\Upsilon(\eta)}\int_{\eta}^{\infty}\frac{s^{n-2}\Upsilon^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s>\frac{(n-2)!}{(\kappa+1)^{1+1/\kappa}}, \tag{3.3}$$

for some $k\in(0,1)$. From Lemma 2.3 and (1.1), we have

$$\left(a(\eta)\left(\psi^{(n-1)}(\eta)\right)^{\kappa}\right)'+kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}\psi^{\kappa}(\eta)\leq 0. \tag{3.4}$$

Now we define $\varpi(\eta)$ as follows

$$\varpi(\eta)=\frac{a(\eta)\left(\psi^{(n-1)}(\eta)\right)^{\kappa}}{\psi^{\kappa}(\eta)}, \tag{3.5}$$

then

$$\varpi'(\eta)=\frac{\left(a(\eta)\left(\psi^{(n-1)}(\eta)\right)^{\kappa}\right)'}{\psi^{\kappa}(\eta)}-\kappa\frac{a(\eta)\left(\psi^{(n-1)}(\eta)\right)^{\kappa}\psi'(\eta)}{\psi^{\kappa+1}(\eta)}.$$

By using (3.4) and (3.5), we get

$$\varpi'(\eta)\leq -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}-\kappa\varpi(\eta)\frac{\psi'(\eta)}{\psi(\eta)}.$$

By using Lemma 2.2, we have

$$\varpi'(\eta)\leq -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}-\kappa\varpi(\eta)\frac{k\eta^{n-2}}{(n-2)!}\frac{\psi^{(n-1)}(\eta)}{\psi(\eta)},$$

from (3.5), we get

$$\varpi'(\eta)\leq -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}-\kappa\frac{k\eta^{n-2}}{a^{1/\kappa}(\eta)(n-2)!}\varpi^{1+1/\kappa}(\eta). \tag{3.6}$$

Integrating (3.6) from η to ∞, we have

$$\varpi(\eta) \geq k\Upsilon(\eta) + \frac{\kappa k}{(n-2)!}\int_{\eta}^{\infty} \frac{s^{n-2}}{a^{1/\kappa}(s)}\varpi^{1+1/\kappa}(s)\,\mathrm{d}s \tag{3.7}$$

or

$$\frac{\varpi(\eta)}{k\Upsilon(\eta)} \geq 1 + \frac{\kappa k^{(\kappa+1)/\kappa}}{(n-2)!\Upsilon(\eta)}\int_{\eta}^{\infty} \frac{s^{n-2}\Upsilon^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\left(\frac{\varpi(s)}{k\Upsilon(s)}\right)^{(\kappa+1)/\kappa}\mathrm{d}s,$$

eventually, let us say $\eta \geq \eta_1$. Since

$$\varpi(\eta) > k\Upsilon(\eta),$$

then

$$\inf_{\eta\geq\eta_1}\frac{\varpi(\eta)}{k\Upsilon(\eta)} = \zeta \geq 1,$$

thus

$$\frac{\varpi(\eta)}{k\Upsilon(\eta)} \geq 1 + \frac{\kappa(k\zeta)^{(\kappa+1)/\kappa}}{(n-2)!\Upsilon(\eta)}\int_{\eta}^{\infty} \frac{s^{n-2}\Upsilon^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s, \tag{3.8}$$

from (3.3), we have

$$\frac{k^{(\kappa+1)/\kappa}}{(n-2)!\Upsilon(\eta)}\int_{\eta}^{\infty} \frac{s^{n-2}\Upsilon^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s > \alpha > (\kappa+1)^{-(\kappa+1)/\kappa}, \tag{3.9}$$

for some positive α. From (3.8) and (3.9), we have

$$\frac{\varpi(\eta)}{k\Upsilon(\eta)} \geq 1 + \kappa\zeta^{(\kappa+1)/\kappa}\alpha, \tag{3.10}$$

therefore

$$\zeta \geq 1 + \kappa\zeta^{(\kappa+1)/\kappa}\alpha > 1 + \kappa\zeta^{(\kappa+1)/\kappa}(\kappa+1)^{-(\kappa+1)/\kappa},$$

that is,

$$0 \geq \frac{1}{\kappa+1} + \frac{\kappa}{\kappa+1}\left(\frac{\zeta}{\kappa+1}\right)^{(\kappa+1)/\kappa} - \frac{\zeta}{\kappa+1}.$$

But, we have

$$f(\vartheta) = \frac{1}{\kappa+1} + \frac{\kappa}{\kappa+1}\vartheta^{(\kappa+1)/\kappa} - \vartheta$$

is a non-negative function for every $\vartheta > 0$. Thus, we obtain that ψ cannot satisfy (2.1).
Next, we suppose that (2.2) holds. Then there exists a finite $\lim_{\eta\to\infty}\psi(\eta) = D$. Suppose the contrary that $D > 0$. By integrating (1.1) over $[\eta, \infty)$, we have

$$a(\eta)(\psi^{(n-1)}(\eta))^{\kappa} \geq k\int_{\eta}^{\infty} q(s)\psi^{\kappa}(\phi(s))\,\mathrm{d}s \geq kD^{\kappa}\int_{\eta}^{\infty} q(s)\,\mathrm{d}s,$$

that is,

$$\psi^{(n-1)}(\eta) \geq \frac{k^{1/\kappa}D}{a^{1/\kappa}(\eta)}\left(\int_{\eta}^{\infty} q(s)\,\mathrm{d}s\right)^{1/\kappa}. \tag{3.11}$$

Integrating (3.11) $n-2$ times, we obtain

$$-\psi'(\eta) \geq \frac{k^{1/\kappa}D}{(n-3)!}\int_{\eta}^{\infty}\left((s-\eta)^{n-3}\frac{1}{a^{1/\kappa}(s)}\left(\int_{s}^{\infty}q(\varrho)\,\mathrm{d}\varrho\right)^{1/\kappa}\right)\mathrm{d}s, \tag{3.12}$$

integrating (3.12) from η_1 to ∞, we get

$$\begin{aligned}\psi(\eta_1) &\geq \frac{k^{1/\kappa}D}{(n-2)!}\int_{\eta_1}^{\infty}\left((s-\eta_1)^{n-2}\frac{1}{a^{1/\kappa}(s)}\left(\int_{s}^{\infty}q(\varrho)\,\mathrm{d}\varrho\right)^{1/\kappa}\right)\mathrm{d}s\\ &\geq \frac{k^{1/\kappa}D}{2^{n-2}(n-2)!}\int_{2\eta_1}^{\infty}\left(s^{n-2}\frac{1}{a^{1/\kappa}(s)}\left(\int_{s}^{\infty}q(\varrho)\,\mathrm{d}\varrho\right)^{1/\kappa}\right)\mathrm{d}s,\end{aligned}$$

which contradicts (3.2). Then, $\lim_{\eta\to\infty}\psi(\eta)=0$. The proof is complete.

Theorem 3.2. *If*

$$\liminf_{\eta\to\infty}\frac{1}{\Psi(\eta)}\int_{\eta}^{\infty}\frac{s^{n-2}\Psi^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s > \frac{(n-2)!}{(\kappa+1)^{(\kappa+1)/\kappa}} \tag{3.13}$$

and

$$\int_{\eta_0}^{\infty}s^{n-2}\Theta(s)\,\mathrm{d}s=\infty, \tag{3.14}$$

then every nonoscillatory solution $\psi(\eta)$ *of (1.2) satisfies* $\lim_{\eta\to\infty}\psi(\eta)=0$.

Proof. Suppose that ψ is a solution of (1.2) and positive eventually. Assume that Case (1) holds. Since

$$\psi(\eta)=\upsilon(\eta)-p(\eta)\psi(\tau(\eta))\geq\upsilon(\eta)-p(\eta)\upsilon(\tau(\eta)),$$

From $\eta\geq\tau(\eta)$, we have

$$\psi(\eta)\geq\upsilon(\eta)-p(\eta)\upsilon(\tau(\eta))\geq\upsilon(\eta)-p(\eta)\upsilon(\eta)=\upsilon(\eta)(1-p(\eta))$$

and so

$$\psi(\phi(\eta))\geq\upsilon(\phi(\eta))(1-p(\phi(\eta))), \tag{3.15}$$

from Lemma 2.2, we have

$$\upsilon'(\eta)\geq\frac{k\eta^{n-2}}{(n-2)!}\upsilon^{(n-1)}(\eta). \tag{3.16}$$

From (2.9), (3.15) and (2.8), we have

$$\left(a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}\right)'\leq-kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}(1-p(\phi(\eta)))^{\kappa}\upsilon^{\kappa}(\eta). \tag{3.17}$$

Now, we define the function $\varphi(\eta)$ as follows

$$\varphi(\eta)=\frac{a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}}{\upsilon^{\kappa}(\eta)}. \tag{3.18}$$

Differentiating (3.18) and using (3.17), we get

$$\varphi'(\eta) \le -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}(1-p(\phi(\eta)))^{\kappa} - \frac{\kappa a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}\upsilon'(\eta)}{\upsilon^{\kappa+1}(\eta)}, \tag{3.19}$$

by using (3.16) and (3.18), we have

$$\varphi'(\eta) \le -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}(1-p(\phi(\eta)))^{\kappa} - \frac{\kappa k\eta^{n-2}}{(n-2)!a^{1/\kappa}(\eta)}\varphi^{(\kappa+1)/\kappa}(\eta). \tag{3.20}$$

Integrating (3.20) from η to ∞, we have

$$\varphi(\eta) \ge k\Psi(\eta) + \frac{\kappa k}{(n-2)!}\int_{\eta}^{\infty}\frac{s^{n-2}}{a^{1/\kappa}(s)}\varphi^{(\kappa+1)/\kappa}(s)\,\mathrm{d}s \tag{3.21}$$

or equivalently,

$$\frac{\varphi(\eta)}{k\Psi(\eta)} \ge 1 + \frac{\kappa k^{(1+\kappa)/\kappa}}{(n-2)!\Psi(\eta)}\int_{\eta}^{\infty}\frac{s^{n-2}\Psi^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\frac{\varphi^{(\kappa+1)/\kappa}(s)}{(k\Psi(s))^{(\kappa+1)/\kappa}}\mathrm{d}s, \tag{3.22}$$

eventually, let us say $\eta \ge \eta_1$. From (3.22), we have

$$\frac{\varphi(\eta)}{k\Psi(\eta)} \ge 1,$$

then

$$\inf_{\eta\ge\eta_1}\frac{\varphi(\eta)}{k\Psi(\eta)} = \varrho \ge 1, \tag{3.23}$$

by using (3.22) and (3.23), we have

$$\frac{\varphi(\eta)}{k\Psi(\eta)} \ge 1 + \frac{\kappa(\varrho k)^{(\kappa+1)/\kappa}}{(n-2)!\Psi(\eta)}\int_{\eta}^{\infty}\frac{s^{n-2}\Psi^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s, \tag{3.24}$$

by using (3.13), we have

$$\liminf_{\eta\to\infty}\frac{k^{(1+\kappa)/\kappa}}{\Psi(\eta)}\int_{\eta}^{\infty}\frac{s^{n-2}\Psi^{(1+\kappa)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s > \frac{(n-2)!}{(\kappa+1)^{(1+\kappa)/\kappa}}, \tag{3.25}$$

for some $k \in (0,1)$, from above inequality there exists some positive ξ such that

$$\frac{k^{(1+\kappa)/\kappa}}{(n-2)!\Psi(\eta)}\int_{\eta}^{\infty}\frac{s^{n-2}\Psi^{(1+\kappa)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s > \xi > \frac{1}{(\kappa+1)^{(1+\kappa)/\kappa}}. \tag{3.26}$$

From (3.26) and (3.24), we get

$$\frac{\varphi(\eta)}{k\Psi(\eta)} \ge 1 + \kappa\xi\varrho^{(1+\kappa)/\kappa}, \tag{3.27}$$

therefore, from (3.23), we have

$$\varrho \ge 1 + \kappa\xi\varrho^{(1+\kappa)/\kappa} > 1 + \kappa\left(\frac{\varrho}{\kappa+1}\right)^{(1+\kappa)/\kappa},$$

that is,

$$\frac{1}{\kappa+1}+\frac{\kappa}{\kappa+1}\left(\frac{\varrho}{\kappa+1}\right)^{(1+\kappa)/\kappa}-\frac{\varrho}{\kappa+1}<0.$$

This contradicts the fact that the function

$$f(\varsigma)=\frac{1}{\kappa+1}+\frac{\kappa}{\kappa+1}\varsigma^{(1+\kappa)/\kappa}-\varsigma\geq 0,$$

for all $\varsigma>0$. Thus $\upsilon(\eta)$ cannot satisfy Case (1).

Assume that Case (2) holds. Then there exists a constant $c\geq 0$ such that $\lim_{\eta\to\infty}\psi(\eta)=c$. Suppose that $c>0$. Integrating (1.2), we see that

$$a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}\geq k\int_{\eta}^{\infty}q(s)\psi^{\kappa}(\phi(s))\,\mathrm{d}s\geq kc^{\kappa}\int_{\eta}^{\infty}q(s)\,\mathrm{d}s, \tag{3.28}$$

that is,

$$\upsilon^{(n-1)}(\eta)\geq c\Theta(\eta). \tag{3.29}$$

Integrating (3.29) twice, we obtain

$$\upsilon^{(n-3)}(\eta)\geq c\int_{\eta}^{\infty}\left(\int_{\varrho}^{\infty}\Theta(s)\,\mathrm{d}s\right)\mathrm{d}\varrho=c\int_{\eta}^{\infty}\Theta(s)(s-\eta)\,\mathrm{d}s, \tag{3.30}$$

integrating (3.30) $n-4$ times, we get

$$-\upsilon'(\eta)\geq\frac{c}{(n-3)!}\int_{\eta}^{\infty}\Theta(s)(s-\eta)^{n-3}\,\mathrm{d}s, \tag{3.31}$$

integrating (3.31) from η_1 to ∞, we get

$$\upsilon(\eta_1)\geq\frac{c}{(n-2)!}\int_{\eta_1}^{\infty}\Theta(s)(s-\eta_1)^{n-2}\,\mathrm{d}s\geq\frac{c}{2^{n-2}(n-2)!}\int_{2\eta_1}^{\infty}s^{n-2}\Theta(s)\,\mathrm{d}s,$$

which contradicts (3.14), and so we have verified that $\lim_{\eta\to\infty}\psi(\eta)=0$. The proof is complete.

Theorem 3.3. *Let ψ be a nonoscillatory solution of (1.2), (3.14) hold, and*

$$\int_{\eta_0}^{\infty}q(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}(1-p(\phi(\eta)))^{\kappa}\exp\left(\frac{\kappa k}{(n-2)!}\int_{\eta_0}^{\eta}h_m^{1/\kappa}(s)\frac{s^{n-2}}{a^{1/\kappa}(s)}\mathrm{d}s\right)\mathrm{d}\eta=\infty, \tag{3.32}$$

for some $k\in(0,1)$ and some $m=0,1,\ldots$. Then $\lim_{\eta\to\infty}\psi(\eta)=0$.

Proof. Suppose that ψ is a solution of (1.2) and positive eventually, we conclude from Lemma 2.4 that $\upsilon(\eta)$ satisfies Case (1) or Case (2). If Case (1) holds, then from proof of Theorem 3.2, we find that (3.21) holds. By induction, using (3.21), we have that the sequence $\{h_m(\eta)\}_{m=0}^{\infty}$ is nondecreasing and $\varphi(\eta)\geq h_m(\eta)$. Thus the sequence $\{h_m(\eta)\}_{m=0}^{\infty}$ converges to $h(\eta)$. By the Lebesgue monotone convergence theorem and letting $m\to\infty$ in (2.10), we have

$$h(\eta)=h_0(\eta)+\frac{\kappa k}{(n-2)!}\int_{\eta}^{\infty}h^{(\kappa+1)/\kappa}(s)\frac{s^{n-2}}{a^{1/\kappa}(s)}\mathrm{d}s,$$

since $\varphi(\eta) \geq h_m(\eta)$, gives

$$\begin{aligned} h'(\eta) &= -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}(1-p(\phi(\eta)))^{\kappa} - \frac{\kappa k}{(n-2)!}h^{(\kappa+1)/\kappa}(\eta)\frac{\eta^{n-2}}{a^{1/\kappa}(\eta)} \\ &\leq -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}(1-p(\phi(\eta)))^{\kappa} - \frac{\kappa k}{(n-2)!}h(\eta)h_m^{1/\kappa}(\eta)\frac{\eta^{n-2}}{a^{1/\kappa}(\eta)}. \end{aligned}$$

Therefore,

$$h'(\eta) + \frac{\kappa k}{(n-2)!}h(\eta)h_m^{1/\kappa}(\eta)\frac{\eta^{n-2}}{a^{1/\kappa}(\eta)} \leq -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}(1-p(\phi(\eta)))^{\kappa},$$

that is,

$$\begin{aligned} &\left(h(\eta)\exp\left(\frac{\kappa k}{(n-2)!}\int_{\eta_1}^{\eta} h_m^{1/\kappa}(s)\frac{s^{n-2}}{a^{1/\kappa}(s)}\mathrm{d}s\right)\right)' \\ \leq\ & -kq(\eta)\left(\frac{\phi(\eta)}{\eta}\right)^{\kappa\delta_0}(1-p(\phi(\eta)))^{\kappa}\exp\left(\frac{\kappa k}{(n-2)!}\int_{\eta_1}^{\eta} h_m^{1/\kappa}(s)\frac{s^{n-2}}{a^{1/\kappa}(s)}\mathrm{d}s\right). \end{aligned}$$

Integrating the above inequality from η_1 to η we obtain

$$\begin{aligned} &h(\eta)\exp\left(\frac{\kappa k}{(n-2)!}\int_{\eta_1}^{\eta} h_m^{1/\kappa}(s)\frac{s^{n-2}}{a^{1/\kappa}(s)}\mathrm{d}s\right) \\ \leq\ & h(\eta_1) - k\int_{\eta_1}^{\eta} q(\varrho)\left(\frac{\phi(\varrho)}{\varrho}\right)^{\kappa\delta_0}(1-p(\phi(\varrho)))^{\kappa}\exp\left(\frac{\kappa k}{(n-2)!}\int_{\eta_1}^{\varrho} h_m^{1/\kappa}(s)\frac{s^{n-2}}{a^{1/\kappa}(s)}\mathrm{d}s\right)\mathrm{d}\varrho. \end{aligned} \tag{3.33}$$

Since,

$$h(\eta)\exp\left(\frac{\kappa k}{(n-2)!}\int_{\eta_1}^{\eta} h_m^{1/\kappa}(s)\frac{s^{n-2}}{a^{1/\kappa}(s)}\mathrm{d}s\right) \geq 0, \tag{3.34}$$

letting $\eta \to \infty$ in (3.33) and using (3.32), we obtain a contradiction with (3.34).
If Case (2) holds, then from proof of Theorem 3.2, condition (3.14) insures that $\psi(\eta)$ tends to zero at $\eta \to \infty$. This completes the proof of the theorem.

Theorem 3.4. *Let ψ be a nonoscillatory solution of (1.2), (3.14) hold, and*

$$\limsup_{\eta\to\infty} k\frac{\eta^{\kappa(n-1)}}{a(\eta)}h_m(\eta) > ((n-1)!)^{\kappa}, \tag{3.35}$$

for some $h_m(\eta)$ and for some $k \in (0,1)$. Then $\lim_{\eta\to\infty}\psi(\eta) = 0$.

Proof. Suppose that ψ is an eventually positive solution of (1.2), we conclude from Lemma 2.4 that $\upsilon(\eta)$ satisfies Case (1) or Case (2). Assume that Case (1) holds. From Lemma 2.2, we have

$$\upsilon(\eta) \geq \frac{k^{1/\kappa}}{(n-1)!}\eta^{n-1}\upsilon^{(n-1)}(\eta),$$

where k is the same as in $h_m(\eta)$. Then

$$\frac{1}{\varphi(\eta)} = \frac{1}{a(\eta)} \frac{\upsilon^{\kappa}(\eta)}{\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}} \geq \frac{k}{a(\eta)\left((n-1)!\right)^{\kappa}} \eta^{\kappa(n-1)},$$

that is,

$$((n-1)!)^{\kappa} \geq \frac{k}{a(\eta)} \eta^{\kappa(n-1)} \varphi(\eta).$$

Since $\varphi(\eta) \geq h_m(\eta)$, we have

$$((n-1)!)^{\kappa} \geq \frac{k}{a(\eta)} \eta^{\kappa(n-1)} h_m(\eta), \tag{3.36}$$

from (3.35) and (3.36) we get a contradiction. This completes the proof of the theorem.

Corollary 1. *Let (3.14) be satisfied, and*

$$\limsup_{\eta\to\infty} \frac{\eta^{\kappa(n-1)}}{a(\eta)} \int_{\eta}^{\infty} q(s)\left(\frac{\phi(s)}{s}\right)^{\kappa\delta_0} (1-p(\phi(s)))^{\kappa}\, \mathrm{d}s > ((n-1)!)^{\kappa}. \tag{3.37}$$

Then every nonoscillatory solution $\psi(\eta)$ *of (1.2) satisfies* $\lim_{\eta\to\infty} \psi(\eta) = 0$.

Proof. From (3.37) there exists some $k \in (0,1)$ such that

$$\limsup_{\eta\to\infty} k^2 \frac{\eta^{\kappa(n-1)}}{a(\eta)} \int_{\eta}^{\infty} q(s)\left(\frac{\phi(s)}{s}\right)^{\kappa\delta_0} (1-p(\phi(s)))^{\kappa}\, \mathrm{d}s > ((n-1)!)^{\kappa},$$

that is,

$$\limsup_{\eta\to\infty} k \frac{\eta^{\kappa(n-1)}}{a(\eta)} h_0(\eta) > ((n-1)!)^{\kappa}.$$

The assertion now follows from Theorem 3.4.

Example 3.5. Consider the differential equation of third order

$$\left(\eta\left(\psi''(\eta)\right)^3\right)' + \frac{q_0}{\eta^6}\psi^3(2\eta) = 0. \tag{3.38}$$

From (3.38), we find that $n = 3$, $\kappa = 3$, $a(\eta) = \eta$, $q(\eta) = q_0/\eta^6$, $q_0 > 0$ and $\phi(\eta) = 2\eta$.
Now, from Theorem 3.1 we notice that condition (3.2) is satisfied and condition (3.1) is satisfied if $q_0 > \frac{625}{256}$. Thus, we obtain that all nonoscillatory solutions of (3.38) tend to zero at infinity when $q_0 > \frac{625}{256}$.

Example 3.6. Consider the third-order neutral differential equation

$$\left(\eta^{1/3}\left((\psi(\eta) + p_0\psi(\lambda\eta))''\right)^{1/3}\right)' + \frac{q_0}{\eta^{4/3}}\psi^{1/3}(\varpi\eta) = 0, \tag{3.39}$$

where p_0 and q_0 are constants, $\kappa = 1/3$, $n = 3$, $a(\eta) = \eta^{1/3}$, $\tau(\eta) = \lambda\eta$, $\lambda \in (0,1)$, $\varpi \geq 1$, $\phi(\eta) = \varpi\eta$, $p(\eta) = p_0$, $q(\eta) = q_0/\eta^{4/3}$ and $q_0 > 0$.
It is easy to get

$$\Psi(\eta) = \int_{\eta}^{\infty} q(s)\left(\frac{\phi(s)}{s}\right)^{\kappa\delta_0} (1-p(\phi(s)))^{\kappa}\, \mathrm{d}s = q_0\varpi^{\delta_0/3}(1-p_0)^{1/3} \frac{3}{\eta^{1/3}}$$

and

$$\Theta(\eta) = \frac{k^{1/\kappa}}{a^{1/\kappa}(\eta)}\left(\int_{\eta}^{\infty} q(s)\,\mathrm{d}s\right)^{1/\kappa} = k^3 q_0^3 \frac{3^3}{\eta^2}$$

and condition (3.14) holds

$$\int_{\eta_0}^{\infty} s^{n-2}\Theta(s)\,\mathrm{d}s = \int_{\eta_0}^{\infty} sk^3 q_0^3 \frac{3^3}{s^2}\mathrm{d}s = 3^3 k^3 q_0^3 \int_{\eta_0}^{\infty} \frac{1}{s}\mathrm{d}s = \infty$$

and

$$\liminf_{\eta\to\infty} \frac{1}{\Psi(\eta)} \int_{\eta}^{\infty} \frac{s^{n-2}\Psi^{(\kappa+1)/\kappa}(s)}{a^{1/\kappa}(s)}\mathrm{d}s = 3^4 q_0^3 \varpi^{\delta_0}(1-p_0),$$

then condition (3.13) reduces to

$$3^4 q_0^3 \varpi^{\delta_0}(1-p_0) > \frac{1}{(4/3)^4},$$

which, by Theorem 3.2, guarantees that all nonoscillatory solutions of (3.39) tend to zero at infinity.

4. Conclusions

In this paper, several new results for (1.1) and (1.2) have been presented which complement and expand some results introduced in the cited papers in the introduction. We obtain the conditions by using Riccati transformation and some analytical skill. In fact, our results are applicable in the case where κ is a ratio of odd positive integers. We supported the results obtained in this paper with two examples. An interesting problem for further research is to study the problem of nonoscillation for $\left(a(\eta)\left(\upsilon^{(n-1)}(\eta)\right)^{\kappa}\right)' + q(\eta) f(\psi(\phi(\eta))) = 0$ where $f(\psi) \geq k\psi^{\gamma}$ and $\gamma \neq \kappa$.

Conflict of interest

There are no competing interests.

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