Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 19(2): 1239–1250.
DOI: 10.3934/mbe.2022057
Received: 13 October 2021
Accepted: 26 November 2021
Published: 1 December 2021

Research article

Some new results on bathtub-shaped hazard rate models

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Abstract: The most common non-monotonic hazard rate situations in life sciences and engineering involves bathtub shapes. This paper focuses on the quantile residual life function in the class of lifetime distributions that have bathtub-shaped hazard rate functions. For this class of distributions, the shape of the α-quantile residual lifetime function was studied. Then, the change points of the α-quantile residual life function of a general weighted hazard rate model were compared with the corresponding change points of the basic model in terms of their location. As a special weighted model, the order statistics were considered and the change points related to the order statistics were compared with the change points of the baseline distribution. Moreover, some comparisons of the change points of two different order statistics were presented.

Keywords: bathtub-shaped hazard rate function; burn-in; α-quantile residual life; weighted hazard rate models; order statistics

1. Introduction

In many real-life situations, we are dealing with an object (a manufactured device or a born living thing) that goes through three distinct phases of life: the early risky life phase, the use phase, and the wear and tear phase. In the early life phase, the object is naturally exposed to a higher than average hazard rate (HR), which decreases over time, and at the end of this phase the object reaches a more reliable state. After the use phase, in which the object is exposed to a rather low risk, the wear phase begins. In the wear phase, the HR due to erosion or aging increases over time. In medical and industrial applications, such aging behavior suggests a life model with a bathtub-shaped hazard rate (BSHR).

There is a vast literature dealing with BSHR lifetime distribution. Rajarshi and Rajarshi [1] and Lai et al. [2] have presented two good surveys of BSHR models. Many authors have introduced models with such behavior, e.g., Glaser [3], Mudholkar and Sirvastava [4], Navarro and Hernandez [5], Xie et al. [6], Wang [7] and many others. In addition, Lai and Xie [8] and Nadarajah [9] have presented lists of such models. Mi [10] and Gupta and Akman [11] characterized the mean residual life (MRL) function and showed that when the HR function is bathtub-shaped, the MRL function is decreasing or upside-down bathtub-shaped (UBS). Mi [10] applied this result to determine the optimal time for burn-in. In addition, Block et al. [12] characterized the average HR function, the MRL function, the average mean residual life function, the harmonic average mean residual life function, the variance residual life function, and the residual entropy function when the HR function is bathtub-shaped. They investigated the coefficient of variation of the residual life as a criterion for burn-in.

The α-quantile residual life function (α-QRL), which describes the α-quantile of the remaining life span survival to a certain time, was introduced by Haines and Singpurwalla [13]. After that, Arnold and Brockett [14], Gupta and Longford [15], Joe and Proschan [16], Joe [17], Launer [18], Song and Cho [19], Lin [20], Shafaei Noughabi and Kayid [21], Kayid et al. [22], Shafaei Noghabi et al. [23], Shafaei Noughabi and Franco-Pereira [24, 25] and many others have conducted researches on this measure.

Franco-Pereira et al. [26] pointed out that the α-QRL function is preferable to the MRL function in some situations, e.g., when the concerned distribution is skewed or heavily tailed. They studied some features of this function, in particular for the case when the HR is bathtub-shaped, and presented a characterization of BSHR distribution via the percentile residual life function.

Shen et al. [27] have shown that there are relationships between the change points of the MRL functions (the point that maximizes the MRL function) in series or parallel systems and the change points of the MRL function of the components. Shafaei Noughabi et al. [28] considered a more general weighted HR model and proved similar results for series and parallel systems as special cases of the study. The aim of this paper is to study the relationship between points maximizing the α-QRL of the basic model and such points for a general weighted HR model and order statistics as a special case. The rest of the paper is organized as follows. In Section 2, we present a new characterization of the α-QRL function BSHR distributions. In Section 3, the maximization points of the α-QRL function of a general weighted HR model are compared with the change points corresponding to the baseline distribution with respect to its location. Then, the order statistic was considered as a special case.

2. Characterization of QRL functions

Let lifetime of an object be represented by the random variable T following the distribution function $F(t)$ and the reliability function $\bar{F}(t)$. When F is absolutely continuous with the density function $f(t)$, the HR function is defined by $\lambda(t) = \frac{f(t)}{\bar{F}(t)}$, $t \in (0, u)$, $u = \sup\{t : \bar{F}(t) > 0\} \in (0, \infty]$. Here, we consider the following definition for the class of BSHR distributions.

Definition 1. The distribution function F is said to be BSHR with the change points t_0 and t_1, $0 < t_0 \leq t_1 < u$, if its corresponding HR function is strictly decreasing in $(0, t_0)$, constant in (t_0, t_1), and strictly increasing in (t_1, u).

Definition 2. A function $q(t)$ has an UBS form with change points a and b when it is strictly increasing on $(0, a)$, constant on (a, b) and strictly decreasing for $t \geq b$.

The conditional residual life of an item with variable lifetime T given survival up to time t is

$T_t = (T - t | T \geq t)$. Then, the conditional reliability survival up to time t can be written as

$$\bar{F}_t(x) = P(T_t > x) = \frac{\bar{F}(t+x)}{\bar{F}(t)}. \tag{2.1}$$

Based on this measure, the α-QRL function is defined to be

$$\begin{aligned} q_\alpha(t) &= \inf\{x : \quad \bar{F}_t(x) \leq \bar{\alpha}\} \\ &= \inf\{x : \quad \bar{F}(t+x) \leq \bar{\alpha}\bar{F}(t)\} \\ &= \inf\{y : \quad \bar{F}(y) \leq \bar{\alpha}\bar{F}(t)\} - t \\ &= \bar{F}^{-1}(\bar{\alpha}\bar{F}(t)) - t, \end{aligned} \tag{2.2}$$

where $\bar{\alpha} = 1 - \alpha$. The following basic theorem from Joe and Proschan [16] will be applied repeatedly in this paper.

Theorem 1. Assume that $F(t)$ is absolutely continuous with the density function $f(t)$ and the HR function $\lambda(t)$. Then, $q_\alpha(t)$ satisfies the condition

$$\int_t^{t+q_\alpha(t)} \lambda(u)\, du = -\ln \bar{\alpha}, \quad 0 \leq t + q_\alpha(t) < u.$$

Therefore, for continuous $\lambda(t)$ and differentiable $q_\alpha(t)$ we have

$$q'_\alpha(t) = \frac{\lambda(t)}{\lambda(t + q_\alpha(t))} - 1. \tag{2.3}$$

As an immediate result of Theorem 1, assume that there exists one t_* such that $\lambda(t) \leq \lambda(t_*)$ for all $0 \leq t \leq t_*$, and $\lambda(t)$ is increasing for $t \geq t_*$. Then, the α-QRL function is decreasing for all $\alpha_0 \leq \alpha < 1$ where $\alpha_0 = F(t_*)$. To make this statement clear, note that for $t \geq t_*$, we have $\lambda(t) \leq \lambda(t + q_\alpha(t))$ so $q'_\alpha(t) \leq 0$. Also, if $0 \leq t < t_*$, then

$$t_* = \bar{F}^{-1}(1 - \alpha_0) \leq \bar{F}^{-1}((1 - \alpha)\bar{F}(t)) = t + q_\alpha(t).$$

Thus, $\lambda(t) \leq \lambda(t + q_\alpha(t))$ and in turn $q'_\alpha(t) \leq 0$.

For example, if F is a BSHR distribution and there exists a t_* such that $\lambda(t_*) \geq \lambda(0)$, then the corresponding α-QRL is decreasing for all $F(t_*) \leq \alpha < 1$, refer to Joe and Proschan [16] for more details.

Launer [18] assumed a critical point for the α-QRL function at which this function is maximized or minimized, and showed that this critical point precedes the critical point of the HR function. The critical point of the hazard rate was defined to be the point that the hazard rate is minimized (in the case of BSHR) or maximized (in the case of upside down bathtub shaped hazard rate). Then, he assumed that $\lambda(x)$ has a bathtub shape and satisfies $\lambda(0) = \lambda(t_0)$ for a $t_0 > 0$, and showed that the corresponding α-QRL function is decreasing for all $\alpha \geq \alpha_0$ for one $0 < \alpha_0 < 1$ while for $\alpha < \alpha_0$, it has a maximum preceding the first change point of $\lambda(t)$. Moreover, Franco-Pereira et al. [26] presented some results related to the form of the α-QRL function when the HR is bathtub shaped.

Here we provide a new characterization of the α-QRL function in terms of the bathtub-shaped HR functions, which provides for a more detailed behavior of the quantile residual lifetime function and is useful for the next part of our study.

Definition 3. A function g is said to be *wide UBS* in an interval $(0, b)$ when it satisfies the following conditions.
(a) $g'(x) = 0$ implies $g'(s) \leq 0$ for $x < s \leq b$, and $g'(r) \geq 0$ for $0 < r < x$,
(b) $g'(x) < 0$ implies $g'(s) < 0$ for $x < s \leq b$,
and
(c) $g'(x) > 0$ implies $g'(r) > 0$ for $0 < r < x$.

The following lemma will be applied in the next results.

Lemma 1. Let F be a BSHR distribution with the change points t_1 and t_2. Then $q_\alpha(t)$ is a wide UBS function in the interval $(0, t_2]$.

Proof: From Eq (2.3) the sign of the continuous function

$$d(t) = \lambda(t) - \lambda(t + q_\alpha(t)) = \lambda(t + q_\alpha(t))q'_\alpha(t), \tag{2.4}$$

equals the sign of $q'_\alpha(t)$, because $\lambda(t + q_\alpha(t)) > 0$.

Now, we show that $d(x) = 0$ implies that $d(s) \leq 0$ for $x < s < t_2$ and $d(r) \geq 0$ for $0 < r < x < t_2$.

Let $d(x) = 0$, i.e., $\lambda(x) = \lambda(x + q_\alpha(x))$, and $x < s < t_2$. Clearly, $\lambda(x) \geq \lambda(s)$ since the HR is decreasing in the interval $[0, t_2]$. On the other hand, we must have

$$t_1 < x + q_\alpha(x) < s + q_\alpha(s). \tag{2.5}$$

The first inequality of Eq (2.5) is true since $x + q_\alpha(x) \leq t_1$ contradicts with the assumption $\lambda(x) = \lambda(x + q_\alpha(x))$ and the second inequality results from the fact that $\bar{F}^{-1}(\bar{\alpha}\bar{F}(t)) = t + q_\alpha(t)$ is increasing in t. Consequently $\lambda(s + q_\alpha(s)) \geq \lambda(x + q_\alpha(x))$. Thus,

$$d(s) = \lambda(s) - \lambda(s + q_\alpha(s)) \leq \lambda(x) - \lambda(x + q_\alpha(x)) = 0.$$

For the other argument, let $d(x) = 0$, and $0 < r < x < t_2$. If $r + q_\alpha(r) \leq t_1$, then clearly, $d(r) \geq 0$ since the HR function is strictly decreasing in $[0, t_1]$. Assume that $r + q_\alpha(r) > t_1$. Then, we have $\lambda(x) \leq \lambda(r)$ since $r < x < t_2$ and the HR is decreasing in $[0, t_2]$. Also, $\lambda(r + q_\alpha(r)) \leq \lambda(x + q_\alpha(x))$ since $t_1 < r + q_\alpha(r) < x + q_\alpha(x)$. Thus, we can write

$$d(r) = \lambda(r) - \lambda(r + q_\alpha(r)) \geq \lambda(x) - \lambda(x + q_\alpha(x)) = 0.$$

The statements (b) and (c) of Definition 3 can be checked in a similar way. □

Proposition 1. Let F be a BSHR distribution with the change points t_1 and t_2. Then, for every $\alpha \in (0, 1)$, $q_\alpha(t)$ is decreasing or has an UBS form.

Proof: The HR function is strictly increasing in the interval $[t_2, u)$, so $\lambda(t + q_\alpha(t)) > \lambda(t)$ in this interval. This proves that $q_\alpha(t)$ is strictly decreasing in the interval $[t_2, u)$.

To check the behavior of $q_\alpha(t)$ for $t \leq t_2$, we note that by Lemma 1, $q_\alpha(t)$ is a wide UBS function in the interval $(0, t_2]$. Then, by the fact that $q_\alpha(t)$ is continuous and $q'_\alpha(t_2) < 0$ (since the HR function is strictly increasing in $[t_2, u)$ it results that $\lambda(t_2) < \lambda(t_2 + q_\alpha(t_2))$ or equivalently $q'_\alpha(t_2) < 0$), all possible forms of it consists of strictly increasing then constant then strictly decreasing, strictly increasing then strictly decreasing, strictly decreasing or constant then strictly decreasing. □

Note that the conditions of Proposition 1 do not guarantee a strict UBS form for the α-QRL function. The following theorem provides a condition at which the α-QRL function exhibits an early strictly increasing interval.

Proposition 2. Let F be a BSHR distribution with the change points t_1 and t_2, $t_1 \leq t_2$, with the continuous HR function λ and $\lambda(0) > \lim_{t\to\infty} \lambda(t)$. Then, for every $\alpha \in (0, 1)$, $q_\alpha(t)$ has an UBS form.

Proof: Consider a point $t^* < t_1$ that satisfies the condition

$$\lambda(t^*) = \lim_{t\to\infty} \lambda(t). \tag{2.6}$$

Thus, $\lambda(t) > \lambda(s)$ for all $t < s$, $t \leq t^*$. So, for every $\alpha \in (0, 1)$, we have

$$q'_\alpha(t) = \frac{\lambda(t)}{\lambda(t + q_\alpha(t))} - 1 > 0, \quad 0 < t < t^*.$$

Proposition 1 states that the α-QRL function of a BSHR distribution is decreasing or UBS. Therefore, it results that if

$$q'_\alpha(0) \leq 0, \qquad \text{or equivalently} \qquad \lambda(0) \leq \lambda(q_\alpha(0)), \tag{2.7}$$

then the α-QRL will be a decreasing function and if

$$q'_\alpha(0) > 0, \qquad \text{or equivalently} \qquad \lambda(0) > \lambda(q_\alpha(0)), \tag{2.8}$$

it will show an UBS form. □

Suppose that $m(t)$ is the mean residual life function of a lifetime model. Gupta and Akman [11] proved similar results for $m(t)$ which state that for a BSHR distribution, when $\lambda(0) < (\geq)\frac{1}{m(0)}$, the mean residual life function is decreasing (UBS).

Mi [10] considered the expected lifetime as an important measure of the quality of a product and proposed the point at which the MRL function is maximized as the optimal burn-in time. Here, we propose the α-QRL function as another criterion for burn-in, especially when $\alpha = 0.5$ (which is the median residual life). Based on this criterion, the optimal burn-in time is the smallest time that maximizes the α-QRL function. This burn-in time, which is a function of α, is denoted by $bq(\alpha)$. For a BSHR distribution, by Eqs (2.6) and (2.7), it results that $bq(\alpha) = 0$ if $\lambda(0) \leq \lambda(q_\alpha(0))$.

3. The α-QRL of weighted HR models

The weighted HR model defined in the following was considered by Shafaei Noughabi et al. [28]. Moreover, many special cases of such models appeared in the literature, e.g., Takahasi [29], Nagaraja [30], Misra et al. [31], Navarro et al. [32] and Samaniego [33].

Definition 4. For a baseline HR function λ, the weighted HR model is characterized by the HR function

$$\lambda_w(t) = w(t)\lambda(t), \tag{3.1}$$

where $w(t) \geq 0$ and $\int_0^t \lambda_w(x)dx < +\infty$ for all $t \in (0, u)$ and $\int_0^u \lambda_w(x)dx = +\infty$. Moreover, F and F_w are taken to be the corresponding distribution functions to λ and λ_w respectively.

Two special cases of this general weighted model are the proportional HR model and the coherent system with possibly dependent and identically distributed components, see relation Eq (2.8) of Navarro et al. [32] and Eq (5) of Samaniego [33]. The order statistics are special cases of such coherent systems.

Definition 5. Let F_1 and F_2 be two distribution functions with continuous HR functions and UBS or decreasing α-QRL function $q_{1,\alpha}$ and UBS α-QRL function $q_{2,\alpha}$ respectively with change points a_1 and b_1 (in the case that $q_{1,\alpha}$ is UBS), and change points a_2 and b_2. Then, we say that the burn-in period of F_1 is less than F_2 when $a_1 \leq a_2$ and $b_1 \leq b_2$ or $q_{1,\alpha}$ is decreasing.

In this paper, we assume that the baseline HR function λ does not vanish at the range. This mild condition is necessary for proving the following Lemma and subsequently next results.

Lemma 2. Assume that $0 < w(t) < 1$ and F and F_w are as described in Eq (3.1). Then, F_w is strictly greater than F in the α-QRL, i.e.,

$$q_{w,\alpha}(t) > q_\alpha(t), \quad 0 < t < u, \tag{3.2}$$

for all $0 < \alpha < 1$.

Proof. Since $\lambda_w(t) \leq \lambda(t)$, i.e., $F_w \geq F$ in the HR order, it results that $q_{w,\alpha}(t) \geq q_\alpha(t)$ for all $0 \leq t \leq u$ and $0 < \alpha < 1$, refer Joe and Proschan [16]. We show that under the conditions of this lemma, the inequality is strict, i.e., $q_{w,\alpha}(t) > q_\alpha(t)$. For every $t > 0$ and $0 < \alpha < 1$ we have

$$\int_t^{t+q_{w,\alpha}(t)} \lambda(x)dx > \int_t^{t+q_{w,\alpha}(t)} w(t)\lambda(x)dx = \int_t^{t+q_\alpha(t)} \lambda(x)dx = -\ln\bar{\alpha}. \tag{3.3}$$

The strict inequality results from the fact that $0 < w(t) < 1$ and the equality can be concluded by Theorem 1. Then, Eq (3.3) implies that

$$\int_{t+q_\alpha(t)}^{t+q_{w,\alpha}(t)} \lambda(x)dx > 0, \tag{3.4}$$

and since $\lambda(x) > 0$, we must have $q_{w,\alpha}(t) > q_\alpha(t)$ for all t in the support and all $0 < \alpha < 1$. □

The following result can be used to compare the burn-in time of a weighted HR model with the baseline model.

Proposition 3. Consider the baseline BSHR distribution F with the HR function $\lambda(t)$ and the α-QRL function $q_\alpha(t)$ and the weighted HR distribution F_w with the HR function λ_w and decreasing or UBS α-QRL function $q_{w,\alpha}(t)$. If $q_\alpha(t)$ is an UBS function, $w(t)$ is an increasing function, and $0 < w(t) < 1$, then the burn-in period of F_w is less than F. If $q_\alpha(t)$ is decreasing, then $q_{w,\alpha}(t)$ is decreasing too.

Proof: By Proposition 1, $q_\alpha(t)$ is decreasing or UBS. Let $q_\alpha(t)$ be an UBS function, a and b show change points of $q_\alpha(t)$ and a_w and b_w be change points of $q_{w,\alpha}(t)$ (in the case that $q_{w,\alpha}(t)$ be an UBS function). Note that $q_{w,\alpha}(t)$ may be decreasing and in this case a_w and b_w do not exist. So, we are comparing two functions which one of them is UBS and the other is decreasing or UBS. By this fact, in order to show that $a_w \leq a$ and $b_w \leq b$ (in the case that $q_{w,\alpha}(t)$ is UBS) or $q_{w,\alpha}(t)$ is decreasing, it is sufficient to show that

(i) $q'_{w,\alpha}(t) > 0$ implies $q'_\alpha(t) > 0$,

and

(ii) $q'_{w,\alpha}(t) = 0$ implies $q'_\alpha(t) \geq 0$,

respectively.

Assume that $q'_{w,\alpha}(t) > 0$. Then, since $w(t)$ is increasing, we have

$$1 < q'_{w,\alpha}(t) + 1 = \frac{w(t)\lambda(t)}{w(t + q_{w,\alpha}(t))\lambda(t + q_{w,\alpha}(t))} \leq \frac{\lambda(t)}{\lambda(t + q_{w,\alpha}(t))}, \tag{3.5}$$

so $\lambda(t) > \lambda(t + q_{w,\alpha}(t))$. In the sequel, we use one simple property of bathtub shaped functions. For a bathtub shaped HR function $\lambda(t)$, if $x < y < z$ be three positive values, then $\lambda(x) > \lambda(z)$ implies $\lambda(x) > \lambda(y)$. By Lemma 2, we have three ordered points $t < t + q_\alpha(t) < t + q_{w,\alpha}(t)$ and by Eq (3.5), it results that $\lambda(t) > \lambda(t + q_{w,\alpha}(t))$. Thus, we can conclude that $\lambda(t) > \lambda(t + q_\alpha(t))$, which by Theorem 1, gives $q'_\alpha(t) > 0$ and proves (i).

Assume that $q'_{w,\alpha}(t) = 0$. Again, applying Theorem 1 and properties of the weight function $w(t)$, we have

$$q'_{w,\alpha}(t) + 1 = \frac{w(t)\lambda(t)}{w(t + q_{w,\alpha}(t))\lambda(t + q_{w,\alpha}(t))} \leq \frac{\lambda(t)}{\lambda(t + q_{w,\alpha}(t))}, \tag{3.6}$$

so $\lambda(t) \geq \lambda(t + q_{w,\alpha}(t))$. Similar to part (i), we apply one simple property of bathtub shaped functions. It states that if $x \leq y \leq z$ are positive ordered values, then $\lambda(x) \geq \lambda(z)$ implies $\lambda(x) \geq \lambda(y)$.

Since $t < t + q_\alpha(t) < t + q_{w,\alpha}(t)$, and $\lambda(t) \geq \lambda(t + q_{w,\alpha}(t))$, we can conclude that $\lambda(t) \geq \lambda(t + q_\alpha(t))$, which by Theorem 1 implies that $q'_\alpha(t) \geq 0$ and completes the proof of (ii).

Similarly, it can be shown that $q'_\alpha(t) \leq 0$ implies $q'_{w,\alpha}(t) < 0$ which proves the last statement of the proposition. □

3.1. Order statistics

Let $T_1, T_2, ..., T_n$ represent n independent and identically distributed (iid) lifetimes following the distribution function F and the reliability function $\bar{F}$, and $T_{k:n},\ 1 \leq k \leq n$ is the k^{th} order statistic such that $T_{i:n} \leq T_{j:n},\ 1 \leq i \leq j \leq n$. The k^{th} order statistic $T_{k:n}$ describes the lifetime of a $n-k+1$-out-of-n system in the reliability literature. Two special cases $T_{1:n}$ and $T_{n:n}$ are considered as the lifetimes of the series and parallel system, respectively. The HR function of $T_{k:n}$ is given by

$$\lambda_{k:n}(t) = k\binom{n}{k}\frac{(\frac{F(t)}{\bar{F}(t)})^{k-1}}{\sum_{i=0}^{k-1}\binom{n}{i}(\frac{F(t)}{\bar{F}(t)})^{i}}\lambda(t), \tag{3.7}$$

where λ is the baseline HR corresponding to the distribution function F. It has been proven that the coefficient of λ is increasing in t, see Samaniego [33]. Furthermore, we can express the HR of some coherent systems based on the HR function $\lambda_{k:n}(t)$ as in the following.

$$\lambda_{k+1:n}(t) = \frac{(k+1)\binom{n}{k+1}}{k\binom{n}{k}}\lambda_{k:n}(t)g_1^{k:n}(\xi(t)), \tag{3.8}$$

$$\lambda_{k:n-1}(t) = \frac{\binom{n-1}{k}}{\binom{n}{k}}\lambda_{k:n}(t)g_2^{k:n}(\xi(t)), \tag{3.9}$$

and

$$\lambda_{k+1:n+1}(t) = \frac{(k+1)\binom{n+1}{k+1}}{k\binom{n}{k}}\lambda_{k:n}(t)g_3^{k:n}(\xi(t)), \tag{3.10}$$

where $\xi(t) = \frac{F(t)}{\bar{F}(t)}$ is an increasing function of t, and $g_1^{k:n}(x)$, $g_2^{k:n}(x)$ and $g_3^{k:n}(x)$ are respectively defined by

$$g_1^{k:n}(x) = \frac{\sum_{j=0}^{k-1}\binom{n}{j}x^{j+1}}{\sum_{j=0}^{k}\binom{n}{j}x^{j}}, \tag{3.11}$$

$$g_2^{k:n}(x) = \frac{\sum_{j=0}^{k-1}\binom{n}{j}x^{j}}{\sum_{j=0}^{k-1}\binom{n-1}{j}x^{j}}, \tag{3.12}$$

and

$$g_3^{k:n}(x) = \frac{\sum_{j=0}^{k-1}\binom{n}{j}x^{j+1}}{\sum_{j=0}^{k}\binom{n+1}{j}x^{j}}. \tag{3.13}$$

Shafaei Noughabi et al. [28] proved that these functions are increasing.

We apply Proposition 3 to state some results about the burn-in time of a system. Let $F_{k:n}$ denote the distribution function of $T_{k:n}$.

Proposition 4. Suppose that F be a BSHR distribution and $F_{n:n}$, the distribution function of the parallel system has a decreasing or UBS α-QRL function. Then, the burn-in period of the parallel system is less than the baseline component.

Proof: Applying Eq (3.7), the HR function of the parallel system can be written as a weighted HR model $\lambda_{n:n}(t) = w_n(t)\lambda(t)$ where

$$w_n(t) = n\frac{\xi(t)^{n-1}}{\sum_{i=0}^{n-1}\binom{n}{i}\xi(t)^i},$$

is an increasing function and $0 < w_n(t) < 1$. Therefore, Proposition 3 gives the result. □

Proposition 5. Suppose that F is a BSHR distribution and $F_{1:n}$ is the distribution function of the series system. Then, the burn-in period of the baseline model is less than that of the series system.

Proof: Applying Eq (3.7), we can write $\lambda(t) = \frac{1}{n}\lambda_{1:n}(t)$. Thus, Proposition 3 proves the result immediately. □

Proposition 6. Suppose that $F_{k:n}$ is a BSHR distribution and $F_{k+1:n}$, $F_{k:n-1}$ and $F_{k+1:n+1}$ have decreasing or UBS α-QRL functions. Then, we have
(i) The burn-in period of $F_{k+1:n}$ is less than $F_{k:n}$,
(ii) The burn-in period of $F_{k:n-1}$ is less than $F_{k:n}$,
(iii) The burn-in period of $F_{k+1:n+1}$ is less than $F_{k:n}$.

Proof: Applying Eq (3.8), we can write the HR function of $F_{k+1:n}$ as a weighted HR form of $F_{k:n}$,

$$\lambda_{k+1:n}(t) = w_{k,n}(t)\lambda_{k:n}(t), \tag{3.14}$$

where

$$w_{k,n}(t) = \frac{(k+1)\binom{n}{k+1}}{k\binom{n}{k}} g_1^{k:n}(\xi(t)). \tag{3.15}$$

As stated above, this function is increasing. Applying the HR ordering results of order statistics it results that $0 < w_{k,n}(t) < 1$, refer to Shaked and Shanthikumar [34]. Then, by Proposition 3, (i) is proved. The proof for (ii) and (iii) is completely similar. □

Example 1. To confirm the results, we consider the well-known Weibull additive HR model with the HR function

$$\lambda(t) = ab(at)^{b-1} + ac(at)^{c-1}, \quad a > 0,\ b > 0,\ c > 0,\ t > 0. \tag{3.16}$$

When $b < 1$ and $c > 1$, the HR will show a bathtub shape. We assume $a = 0.1$, $b = 0.4$ and $c = 1.8$. Figure 1 draws the HR function for this model along with the HR of some order statistics (systems). Moreover, this figure draws the α-QRL functions of this model and the selected order statistics. All α-QRL functions have one maximum point which can be regarded as burn-in point. For F, $F_{2:5}$, $F_{3:5}$, $F_{2:4}$, $F_{1:3}$ and $F_{2:2}$, the maximum points are 1.3091, 1.4369, 0, 1.1334, 1.9992 and 0.1457, respectively and confirms the results.

4. Conclusions

The α-QRL function is a reliability measure that can be a good candidate for describing the aging behavior of an object. A special case of it is the median residual life function, which may be more acceptable than the MRL function for reliability studies when the distribution is skewed or the data consists of outliers. In this work, we have focused on BSHR distributions. For them, we proved that the α-QRL function is decreasing or UBS. To improve the reliability of a product (with BSHR lifetime distribution), a common approach is burn-in. In this way, it could be more natural to consider the point that maximizes the α-QRL function than the point that minimizes the hazard rate function. For example, the median residual life function at time t indicates when 50 percent of the surviving products have failed at time t. A UBS α-QRL function reaches its maximum at its change point.

The change point of the α-QRL function of the distribution is compared to the corresponding change point of a general weighted HR model with respect to its location. As special cases, the change points corresponding to the order statistics are then compared with those of the baseline distribution. The change points of a parallel system lie before the change points of the components, while the change points of a series system lie after the change points of the components. We have also shown that there are relationships between the change points of some order statistics. The results studied here may be

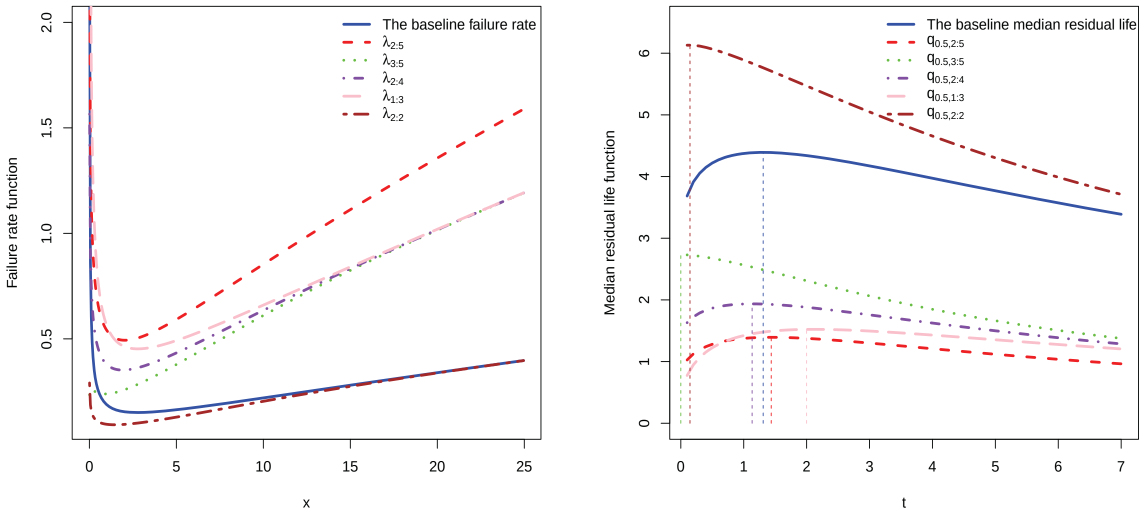

Figure 1. The HR function (left side) and the median residual life function (right side) of the baseline and some systems described in Example 1.

useful if we are interested in the optimal burn-in time of a weighted HR lifetime model. For example, the results are useful in studying the burn-in times of coherent systems.

Note that for some models it may happen that the weighted HR model has BSHR or UBS α-QRL functions, but that this property does not hold for the baseline distributions (e.g., exponential, Weibull, Pareto,...). The results obtained in this paper are not applicable in such cases.

Acknowledgments

The author would like to thank the two anonymous reviewers for their suggestions and very constructive comments, which improved the presentation and readability of the paper. This work is supported by Researchers Supporting Project number (RSP-2021/392), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare there is no conflict of interest.

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