



Research article

Analysis and control of *Aedes Aegypti* mosquitoes using sterile-insect techniques with Wolbachia

Rajivganthi Chinnathambi^{1,*} and Fathalla A. Rihan^{2,*}

¹ Division of Mathematics, School of Advances Sciences, Vellore Institute of Technology, Chennai, Tamilnadu-600 127, India

² Department of Mathematical Sciences, College of Science, United Arab Emirates University, Al-Ain 15551, UAE

* **Correspondence:** Email: rajivganthi.c@vit.ac.in, frihan@uaeu.ac.ae.

Abstract: Combining Sterile and Incompatible Insect techniques can significantly reduce mosquito populations and prevent the transmission of diseases between insects and humans. This paper describes impulsive differential equations for the control of a mosquito with Wolbachia. Several interesting conditions are created when sterile male mosquitoes are released impulsively, ensuring both open- and closed-loop control. To determine the wild mosquito population size in real-time, we propose an open-loop control system, which uses impulsive and constant releases of sterile male mosquitoes. A closed-loop control scheme is also being investigated, which specifies the release of sterile mosquitoes according to the size of the wild mosquito population. To eliminate or reduce a mosquito population below a certain threshold, the Sterile insect technique involves mass releases of sterile insects. Numerical simulations verify the theoretical results.

Keywords: *Aedes Aegypti* mosquitoes; feedback control; stability; sterile insect technique; Wolbachia

1. Introduction

Dengue hemorrhagic fever is emerging as one of the most serious vector-borne diseases worldwide. The *Aedes Aegypti* mosquito is a major vector for the transmission of dengue and other arboviral infections from an infected individual to a susceptible individual. In recent decades, people have been paying more attention to the prevalence, mortality, and massive economic impact of diseases like malaria, chikungunya, dengue, etc, due to their morbidity burdens; see [1–4]. For a long time, there is no available vaccines for the mosquito-borne diseases, even though the researchers is trying to find an effective vaccine. Multerer et al. [5] discussed the dynamical behaviors of *Aedes Aegypti* mosquito

in terms of partial differential equations and also described an Allee effect to capture extinction events and optimal control analysis to identify the release strategy that eliminates the mosquitoes. Zheng et al. [6] investigated the existence and stability results for the mosquito and human populations model in which contains dengue serotype circulates, the delay terms capture the respective intrinsic and extrinsic incubation periods, as well as the maturation delay between mating and emergence of adult mosquitoes [7, 8].

There are mainly two biological control techniques for mosquito populations: Incompatible Insect Techniques (IIT) and Sterile Insect Techniques (SIT). The IIT contains an intracellular bacterium, *Wolbachia*, commonly found in insects including mosquitoes. The technique of releasing mosquitoes carrying *Wolbachia* to replace the wild mosquito population is a type of population replacement strategy. If the females mosquitoes infected *Wolbachia*, then its produced offsprings will also be infected. If the females are uninfected, mate with male with infected *Wolbachia*, then the offsprings will be dead [9]. Ormaetxe et al. [10] reported the availability of different *Wolbachia* strains, which are successfully implanted to *Aedes Aegypti* mosquitoes. The important thing of such techniques is stopping the spread of mosquitoes and reducing the mosquitoes life span. The authors in [11], discussed the augmentation control strategies of *Aedes Aegypti* mosquitoes with infected *Wolbachia*. Li et al. [12] investigated the spread of *Wolbachia* in sex-structured mosquitoes model with birth pulse and studied the sufficient condition for stability of total replacement periodic solution. Zheng et al. [13] studied the qualitative properties of *Aedes Aegypti* mosquitoes with infected *Wolbachia* and maturation delay terms. Li et al. [14] introduced the stage-structured mosquitoes model with *Wolbachia* infection and key factors like, including male killing effect, *cytoplasmic incompatibility* (CI), fecundity cost due to fitness effect, different mortality rates for infected individuals and maternal transmission are discussed. The authors in [15] discussed the dynamical behaviours the *Wolbachia* infected mosquito model with delays and effective control techniques.

In the SIT system, natural mosquito reproduction is disrupted and modified by physical, chemical, and radical methods into male mosquitoes that are sterile. These sterile mosquitoes are then released into the environment to mate with wild mosquitoes that are present in the environment. Repeating the process of sterile mosquitoes releases may control or wipe out the wild mosquitoes population. The authors [16, 17] reported the mathematical modelling of SIT to *Aedes albopictus*, which transmits the Chikungunya diseases and discussed pulsed periodic releases, which is useful to prevent, eliminate, reduce the diseases. Almeida et al. [18] discussed the control techniques, SIT and IIT to the mosquitoes model and optimizing the dissemination protocol for each of these strategies, in order to get as close as possible to these objectives. Nowadays, some researchers focused on the dynamical results of SIT into mosquitoes models; see [19–22].

In the most recent literature, continuous dynamical models and continuous mosquito releases were discussed. Instantaneous releases are a key feature of sterile mosquito releases, and they should be repeated numerous times to bring the mosquito population under control. Generally, continuous dynamical models have disadvantages in defining such control analysis with the impulsiveness nature of releases. Impulsive sterile releases can be applied to make up for a lack of such capacity. Huang et al. [23] reported the dynamical behavior of the interaction of sterile and wild mosquitoes and also discussed the impact of periodic impulsive sterile mosquitoes releases. Zhang et al. [24] described the stroboscopic maps which used to define the numbers of mosquitoes with uninfected and infected *Wolbachia* immediately after each birth pulse at the discrete times. Li et al. [25] studied the dynamical

ical behaviors of mosquitoes and *Wolbachia* model with impulsive general birth and death rate functions and the sufficient conditions of mosquito extinction are discussed. Bliman et al. [26] reported the open and closed-loop control strategies for releasing impulsive sterile male insect techniques in the wild mosquito population, and they studied the open-loop control technique. In cases where the size of the wild mosquito population cannot be determined in real-time, cyclical impulsive releases of sterile males with constant release sizes are used. A closed-loop control strategy is proposed if periodic assessments of wild population size (synchronized with releases or sparser) are available in real-time [27]. The authors in [28, 29] described the modelling of sterile and incompatible insect techniques for the mosquitoes model and suppression of the population. Mosquitoes epidemic models described through the impulsive differential equations have received much attention from the researchers (see [30–32]).

Herein, we provide new models of *Aedes Aegypti* mosquitos with infected *Wolbachia* and impulsive discharges of sterile mosquitos, inspired by earlier studies. We investigate periodic impulsive sterile emissions and formulate parameters for effective eradication based on the magnitude and frequency of the releases. We use a combination of SIT and IIT approaches to produce high sterile male mosquitoes in these models. Periodic releases are calculated so as to maintain the mosquito-free equilibrium. In this environment, open-loop techniques can be designed that ensure mosquito eradication in a finite amount of time without estimating the size of the wild mosquito population. By using the open-loop control technique, sterile mosquitoes are released every τ days in order to eliminate wild mosquitoes. We can also create closed-loop control qualities to determine the size of the wild population.

In this paper, we develop a *Wolbachia Aedes Aegypti* Mosquito model and examine the findings of local stability in Section 2. The open and closed loop control features of the underlying model are studied in Section 3. In Section 4, the numerical results are discussed in order to confirm the theoretical results. Section 5 ends with a conclusion.

2. Modeling of *Aedes Aegypti* mosquitoes with *Wolbachia*

Assume that $M_u(t)$, $M_w(t)$ and $F_u(t)$, $F_w(t)$ denote the population density of male and female *Aedes Aegypti* mosquitoes with uninfected, infected *Wolbachia* respectively. Consider a *Aedes Aegypti* Mosquitoes with *Wolbachia* model of the form

$$\begin{aligned}\dot{M}_u &= r\rho_u F_u \frac{M_u}{M_u + \gamma_w M_w} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_u} M_u, \\ \dot{F}_u &= (1 - r)\rho_u F_u \frac{M_u}{M_u + \gamma_w M_w} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_u} F_u, \\ \dot{M}_w &= r\rho_w F_w e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_w} M_w, \\ \dot{F}_w &= (1 - r)\rho_w F_w e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_w} F_w.\end{aligned}\tag{2.1}$$

This model assumes that all females mate equally. The parameter β includes direct or indirect effects of competition at different stages (larvae, pupae, adults). Let $r \in (0, 1)$ define the primary sex ratio in offspring, and ρ_u, ρ_w represent the mean number of eggs produced by a single female (uninfected and infected *Wolbachia*) who can deposit on average per day. γ_w measures the competition between male mosquitoes with uninfected and infected *Wolbachia* for the female mates. $\mu_{M_u}, \mu_{F_u}, \mu_{M_w}, \mu_{F_w}$ are mean death rates of adult mosquitoes with uninfected, infected *Wolbachia* respectively. The basic offspring

numbers are described as $N_{M_u} = \frac{r\rho_u}{\mu_{M_u}}$, $N_{F_u} = \frac{(1-r)\rho_u}{\mu_{F_u}}$, $N_{M_w} = \frac{r\rho_w}{\mu_{M_w}}$, and $N_{F_w} = \frac{(1-r)\rho_w}{\mu_{F_w}}$. Generally, the male mortality is larger than female, so let us take $\mu_{M_u} \geq \mu_{F_u}$, $\mu_{M_w} \geq \mu_{F_w}$.

Table 1. The maternal transmission and CI reproductive phenotype induced by *Wolbachia*.

Adults	Uninfected Male	<i>Wolbachia</i> infected Male
Uninfected Female	Uninfected	Inviatile eggs
<i>Wolbachia</i> infected Female	<i>Wolbachia</i> Infected	<i>Wolbachia</i> Infected

Now, we discuss the local asymptotic stability of (2.1). The model (2.1) has four steady states:

- (i) Mosquito free equilibrium point $\mathcal{E}_0 = (0, 0, 0, 0)$.
(ii) If $N_{F_u} > 1$, *Wolbachia* free equilibrium point $\mathcal{E}_1 = (M_u^*, F_u^*, 0, 0)$,
where $M_u^* = \frac{N_{M_u}}{N_{M_u} + N_{F_u}} \frac{1}{\beta} \ln N_{F_u}$, $F_u^* = \frac{N_{F_u}}{N_{M_u} + N_{F_u}} \frac{1}{\beta} \ln N_{F_u}$ and $M_u^* + F_u^* = \frac{1}{\beta} \ln N_{F_u}$.
(iii) If $N_{F_w} > 1$, all *Wolbachia* infected equilibrium point $\mathcal{E}_2 = (0, 0, M_w^*, F_w^*)$ where $M_w^* = \frac{N_{M_w}}{N_{M_w} + N_{F_w}} \frac{1}{\beta} \ln N_{F_w}$, $F_w^* = \frac{N_{F_w}}{N_{M_w} + N_{F_w}} \frac{1}{\beta} \ln N_{F_w}$ and $M_w^* + F_w^* = \frac{1}{\beta} \ln N_{F_w}$.
(iv) If $N_{F_u}, N_{F_w} > 1$, the interior equilibrium point $\mathcal{E}_3 = (M_u^*, F_u^*, M_w^*, F_w^*)$, where

$$M_u^* = \frac{\gamma_w}{\beta Q_c} N_{M_u} N_{M_w} N_{F_w} \ln N_{F_w},$$

$$F_u^* = \frac{\gamma_w}{\beta Q_c} N_{F_u} N_{M_w} N_{F_w} \ln N_{F_w},$$

$$M_w^* = \frac{1}{\beta Q_c} N_{M_u} N_{M_w} (N_{F_u} - N_{F_w}) \ln N_{F_w},$$

$$F_w^* = \frac{1}{\beta Q_c} N_{M_u} N_{F_w} (N_{F_u} - N_{F_w}) \ln N_{F_w},$$

$$\text{and } Q_c = N_{M_u} (N_{F_u} - N_{F_w}) (N_{M_w} + N_{F_w}) + \gamma_w N_{M_w} N_{F_w} (N_{M_u} + N_{F_u}).$$

The Jacobian matrix of the model (2.1) is described by

$$J = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_9 & a_{10} & a_{11} \\ a_{12} & a_{12} & a_{12} & a_{13} \end{pmatrix},$$

where

$$a_1 = r\rho_u b_2 c \{ -\beta M_u^* + b_6 \} - \mu_{M_u}, a_2 = r\rho_u b_1 (1 - \beta F_u^*) c,$$

$$a_3 = -r\rho_u F_u^* b_1 c \frac{\gamma_w + \beta(M_u^* + \gamma_w M_w^*)}{M_u^* + \gamma_w M_w^*},$$

$$a_4 = -r\beta\rho_u F_u^* b_1 c, a_5 = (1 - r)\rho_u b_2 c (b_6 - \beta M_u^*),$$

$$a_6 = (1 - r)\rho_u b_1 c (1 - \beta F_u^*) - \mu_{F_u}, a_7 = b_3 (\gamma_w + \beta(M_u^* + \gamma_w M_w^*)) \frac{1}{M_u^* + \gamma_w M_w^*},$$

$$a_8 = b_3 \beta, a_9 = -r\beta\rho_w F_w^* c, a_{10} = a_9 - \mu_{M_w},$$

$$\begin{aligned}
a_{11} &= r\rho_w c(1 - \beta F_w^*), a_{12} = -(1 - r)\beta\rho_w F_w^* c, \\
a_{13} &= (1 - r)\rho_w c(1 - \beta F_w^*) - \mu_{F_w}, b_1 = \frac{M_u^*}{M_u^* + \gamma_w M_w^*}, b_2 = \frac{F_u^*}{M_u^* + \gamma_w M_w^*}, \\
b_3 &= -(1 - r)\rho_u M_u^* b_2 c, b_4 = -r\beta\rho_w F_w^* c, b_5 = -(1 - r)\beta\rho_w F_w^* c, \\
b_6 &= \frac{\gamma_w M_w^*}{M_u^* + \gamma_w M_w^*}, c = \frac{1}{N_{F_w}}.
\end{aligned}$$

The Jacobian matrix at \mathcal{E}_0 is

$$J(\mathcal{E}_0) = \begin{pmatrix} -\mu_{M_u} & 0 & 0 & 0 \\ 0 & -\mu_{F_u} & 0 & 0 \\ 0 & 0 & -\mu_{M_w} & r\rho_w \\ 0 & 0 & 0 & (1-r)\rho_w - \mu_{F_w} \end{pmatrix}.$$

Then, the trivial equilibrium point \mathcal{E}_0 is locally stable if $N_{F_w} < 1$.

The characteristic equation at *Wolbachia* free equilibrium point \mathcal{E}_1 is defined as

$$\begin{aligned}
(\lambda + \mu_{M_w})(\lambda + \mu_{F_w} + (1 - r)\rho_w e^{-\beta(M_u^* + F_u^*)})(\lambda^2 + p_1\lambda + p_2) &= 0, \\
\lambda_1 = -\mu_{M_w}, \lambda_2 = -(\mu_{F_w} + (1 - r)\rho_w e^{-\beta(M_u^* + F_u^*)}) \quad \text{and} \quad (\lambda^2 + p_1\lambda + p_2) &= 0,
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= \mu_{F_u} + \mu_{M_u} + \rho_u e^{-\beta(M_u^* + F_u^*)}(r\beta F_u^* - (1 - r)(1 - \beta F_u^*)), \\
p_2 &= \mu_{M_u}\mu_{F_u} + \rho_u e^{-\beta(M_u^* + F_u^*)}(r\beta F_u^*\mu_{F_u} - \mu_{M_u}(1 - r)(1 - \beta F_u^*)).
\end{aligned}$$

$p_1 > 0, p_2 > 0$ are necessary conditions for all roots in the characteristic equation $\lambda^2 + p_1\lambda + p_2 = 0$ to have negative real parts.

Lemma 1. *Suppose that $N_{F_u} > 1$. Then *Wolbachia* free equilibrium point \mathcal{E}_1 is locally asymptotically stable if $p_1 > 0, p_2 > 0$.*

The characteristic equation at all *Wolbachia* infected equilibrium point \mathcal{E}_2 is defined as

$$\begin{aligned}
(\lambda + \mu_{M_u})(\lambda + \mu_{F_u})(\lambda^2 + r_1\lambda + r_2) &= 0, \\
\lambda_1 = -\mu_{M_u}, \lambda_2 = -\mu_{F_u} \quad \text{and} \quad (\lambda^2 + r_1\lambda + r_2) &= 0,
\end{aligned}$$

where

$$\begin{aligned}
r_1 &= \mu_{F_w} + \mu_{M_w} + \rho_w e^{-\beta(M_w^* + F_w^*)}(r\beta F_w^* - (1 - r)(1 - \beta F_w^*)), \\
r_2 &= \mu_{M_w}\mu_{F_w} + \rho_w e^{-\beta(M_w^* + F_w^*)}(r\beta F_w^*\mu_{F_w} - \mu_{M_w}(1 - r)(1 - \beta F_w^*)).
\end{aligned}$$

$r_1 > 0, r_2 > 0$ are necessary conditions for all roots in the characteristic equation $\lambda^2 + r_1\lambda + r_2 = 0$ to have negative real parts.

Lemma 2. *Suppose that $N_{F_w} > 1$, then all *Wolbachia* infected equilibrium point \mathcal{E}_2 is locally asymptotically stable if $r_1 > 0, r_2 > 0$.*

The characteristic equation at interior equilibrium point \mathcal{E}_3 is defined as

$$\lambda^4 + s_1\lambda^3 + s_2\lambda^2 + s_3\lambda + s_4 = 0,$$

where

$$\begin{aligned} s_1 &= -(1 + a_{10}a_{13}), \\ s_2 &= a_2a_5 + (a_1 + a_6)(a_{10} + a_{13}) + a_{12}(a_4 - a_8 - a_{11}) - a_9(a_3 + a_7) + a_1a_6 + a_{10}a_{13}, \\ s_3 &= -a_2a_5(a_{10} + a_{13}) + a_2(a_7a_9 + a_8a_{12}) + a_3(a_5(a_{12} - a_9) - a_{11}a_{12}) + a_9(a_4a_{12} \\ &\quad + a_7(a_1 + a_{13})) - a_1a_6(a_{10} + a_{13}) - a_{10}a_{13}(a_1 + a_6) + a_{12}(a_{11}(a_1 + a_6 - a_7)) \\ &\quad + a_8(a_1 - a_9 + a_{10}) + a_3a_9(a_6 + a_{13}) - a_4a_{12}(a_6 + a_{10}), \\ s_4 &= a_6a_{12}(a_3a_{11} - a_4a_9) + (a_1a_6 + a_2a_5)(a_{10}a_{13} - a_{11}a_{12}) + (a_4a_5 + a_1a_8)(a_{12}(a_9 - a_{10})) \\ &\quad + (a_9a_{13} - a_{11}a_{12})(a_3a_5 - a_7(a_1 + a_2)) + a_2a_8a_{12}(a_9 - a_{10}) + a_6(a_4a_{10}a_{12} - a_3a_9a_{13}). \end{aligned}$$

According to the Routh-Hurwitz criterion, all roots $\lambda_{1,2,3,4}$ of the characteristic equation $\lambda^4 + s_1\lambda^3 + s_2\lambda^2 + s_3\lambda + s_4 = 0$ must be negative real parts. The conditions are

$$s_1 > 0, s_3 > 0, s_4 > 0, s_1s_2s_3 > s_3^2 + s_1^2s_4.$$

We arrive at the following Lemma.

Lemma 3. *The interior steady state \mathcal{E}_3 is locally asymptotically stable if $s_1 > 0, s_3 > 0, s_4 > 0, s_1s_2s_3 > s_3^2 + s_1^2s_4$.*

2.1. Continuous releases of sterile male mosquitoes

Here, we incorporate continuous releases sterile male mosquitoes into model (2.1) and assume $\gamma_w = 1$, the revised model takes the form

$$\begin{aligned} \dot{M}_u &= r\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_u} M_u, \\ \dot{F}_u &= (1 - r)\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_u} F_u, \\ \dot{M}_w &= r\rho_w F_w \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_w} M_w, \\ \dot{F}_w &= (1 - r)\rho_w F_w \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_w} F_w, \\ \dot{M}_s &= \Lambda - \mu_{M_s} M_s. \end{aligned} \tag{2.2}$$

The sterile male population density at time t is $M_s(t)$. At the beginning of each release period, Λ is the number of sterile male mosquitoes released. The mean death rate of sterile mosquitoes is μ_{M_s} . γ_s be a relative reproductive efficiency and the value is smaller than one. The recruitment terms in (2.2) only include the successful mating of uninfected females F_u and *Wolbachia* infected females F_w , i.e., those leading to viable offspring, that are detected with probabilities $\frac{M_u}{M_u + M_w + \gamma_s M_s}$ and $\frac{M_u + M_w}{M_u + M_w + \gamma_s M_s}$, respectively. From (2.2), the equilibrium point $M_s^* = \frac{\Lambda}{\mu_{M_s}}$. Moreover, $\mu_{M_s} \geq \mu_{M_u}$ and $\mu_{M_s} \geq \mu_{M_w}$.

Theorem 1. Assume $N_{F_w} > 1$, then the Λ^{crit} value described as

$$\Lambda^{crit} = \frac{\mu_{M_s}(N_{F_w} e^{-\beta\left(\frac{N_{M_u}}{N_{M_u}+N_{F_u}} + \frac{N_{F_u}}{N_{M_u}+N_{F_u}} + \frac{N_{M_w}}{N_{M_w}+N_{F_w}} + \frac{N_{F_w}}{N_{M_w}+N_{F_w}}\right)} - 1)}{\gamma} \left(\frac{N_{M_u}}{N_{M_u} + N_{F_u}} + \frac{N_{M_w}}{N_{M_w} + N_{F_w}} \right).$$

Proof: From the model (2.2)

$$\begin{aligned} \frac{M_u^*}{M_u^* + M_w^* + \gamma_s M_s^*} e^{-\beta(M_u^* + F_u^* + M_w^* + F_w^*)} &= \frac{1}{N_{F_u}}, \\ \frac{M_u^* + M_w^*}{M_u^* + M_w^* + \gamma_s M_s^*} e^{-\beta(M_u^* + F_u^* + M_w^* + F_w^*)} &= \frac{1}{N_{F_w}}, \\ M_u^* &= \frac{N_{M_u}}{N_{M_u} + N_{F_u}}, \quad F_u^* = \frac{N_{F_u}}{N_{M_u} + N_{F_u}}, \\ M_w^* &= \frac{N_{M_w}}{N_{M_w} + N_{F_w}}, \quad F_w^* = \frac{N_{F_w}}{N_{M_w} + N_{F_w}}, \end{aligned}$$

and

$$\begin{aligned} (1-r)\rho_w \frac{M_u^* + M_w^*}{M_u^* + M_w^* + \gamma_s M_s^*} e^{-\beta(M_u^* + F_u^* + M_w^* + F_w^*)} &= \mu_{F_w}, \\ N_{F_w} e^{-\beta(M_u^* + F_u^* + M_w^* + F_w^*)} &= 1 + \frac{\gamma_s M_s^*}{M_u^* + M_w^*}, \\ \Lambda^{crit} &= \frac{\mu_{M_s} \left(N_{F_w} e^{-\beta(M_u^* + F_u^* + M_w^* + F_w^*)} - 1 \right)}{\gamma_s} (M_u^* + M_w^*). \end{aligned}$$

If $N_{F_w} > 1$, there exists $\Lambda^{crit} > 0$ such that (2.2) have one non negative steady state at $\Lambda = \Lambda^{crit}$ and no non negative steady state at $\Lambda > \Lambda^{crit}$. The asymptotic stability results of the model (2.2) are similar to those in the previous subsection, so it is omitted.

2.2. Impulsive releases of sterile male mosquitoes

Now, we incorporate periodic impulsive sterile male mosquitoes into model (2.2), the model becomes

$$\begin{aligned} \dot{M}_u &= r\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_u} M_u, \\ \dot{F}_u &= (1-r)\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_u} F_u, \\ \dot{M}_w &= r\rho_w F_w \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_w} M_w, \\ \dot{F}_w &= (1-r)\rho_w F_w \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_w} F_w, \\ \dot{M}_s &= -\mu_{M_s} M_s, \quad \text{for any } t \in \bigcup_{n \in \mathbb{N}} (n\tau, (n+1)\tau), \\ M_s(n\tau^+) &= \tau\Lambda_n + M_s(n\tau^-), \quad n = 1, 2, 3, \dots \end{aligned} \tag{2.3}$$

Make Λ_n a constant and drop consequently the subindex n . $M_s(n\tau^\pm)$ represents the right and left limits of $M_s(t)$ at $t = n\tau$. In other terms, model (2.3) evolves according first four equations of (2.3) on the union of open intervals $(n\tau, (n+1)\tau)$. While M_s undergoes jumps at each $n\tau$, accounting for the released sterile male mosquitoes. For such release schedule, when $t \rightarrow \infty$, the function M_s converges towards the following periodic solution

$$M_s^{per}(t) = \frac{\tau\Lambda}{1 - e^{-\mu_{M_s}\tau}} e^{-\mu_{M_s}(t-n\tau^+)}.$$

We define the periodic system

$$\begin{aligned} \dot{M}_u &= r\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_u} M_u, \\ \dot{F}_u &= (1-r)\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_u} F_u, \\ \dot{M}_w &= r\rho_w F_w \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_w} M_w, \\ \dot{F}_w &= (1-r)\rho_w F_w \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_w} F_w. \end{aligned} \quad (2.4)$$

The model (2.3) has the same mosquito free equilibrium point \mathcal{E}_0 . Now, we are going to study the conditions under which mosquitoes free steady state is asymptotically stable. For the such study, we find the mean value of $\frac{1}{M_s^{per}}$,

$$\left\langle \frac{1}{M_s^{per}} \right\rangle := \frac{1}{\tau} \int_0^\tau \frac{1}{M_s^{per}(t)} dt = \frac{(1 - e^{-\mu_{M_s}\tau})}{\tau^2 \Lambda} \int_0^\tau e^{\mu_{M_s}t} dt = \frac{2(\cosh(\mu_{M_s}\tau) - 1)}{\mu_{M_s} \tau^2 \Lambda}. \quad (2.5)$$

3. Stability analysis and main results

Theorem 2. Assume that

$$\Lambda \geq \Lambda_{per}^{crit} = \frac{2(\cosh(\mu_{M_s}\tau) - 1)}{e\beta\gamma_s\mu_{M_s}\tau^2} \min\{N_{M_u}, N_{F_u}, N_{M_w}, N_{F_w}\}, \quad \tau > 0, \quad (3.1)$$

Therefore, the solution of (2.4) converges globally exponentially to the steady-state \mathcal{E}_0 .

Proof: From (2.4),

$$\dot{F}_u = \left((1-r)\rho_u \frac{M_u}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_u} \right) F_u.$$

For any $F_u \geq 0$ and $t \geq 0$ and use $\alpha = \{xe^{-\beta x}; \quad x \geq 0\} = \frac{1}{e\beta}$

$$\frac{M_u}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + F_u + M_w + F_w)} \leq \frac{M_u}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u)} \leq \frac{\alpha_1}{\gamma M_s^{per}}.$$

Integrate from $n\tau$ to t with $n\tau < t$, we get

$$F_u(t) \leq F_u(n\tau) e^{\int_{n\tau}^t \left((1-r)\rho_u \frac{\alpha_1}{\gamma_s M_s^{per}} - \mu_{F_u} \right) ds}.$$

Taking $t = (n + 1)\tau$, $n \in \mathbb{N}$,

$$F_u((n + 1)\tau) \leq e^{\left((1-r)\rho_u \frac{\alpha_1}{\gamma_s} \left\langle \frac{1}{M_s^{per}} \right\rangle - \mu_{F_u}\right)\tau} F_u(n\tau).$$

Thus, the sequence $\{F_u(n\tau)\}_{n \in \mathbb{N}}$ approaches to 0,

$$\begin{aligned} (1-r)\rho_u \frac{\alpha_1}{\gamma_s} \left\langle \frac{1}{M_s^{per}} \right\rangle &< \mu_{F_u}, \\ \left\langle \frac{1}{M_s^{per}} \right\rangle &< e\beta\gamma_s \frac{1}{N_{F_u}}. \end{aligned}$$

From the first equation of (2.4), similarly, we can prove $\left\langle \frac{1}{M_s^{per}} \right\rangle < e\beta\gamma_s \frac{1}{N_{M_u}}$. Let us consider

$$\begin{aligned} \dot{F}_w &= ((1-r)\rho_w \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_w}) F_w, \\ \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + F_u + M_w + F_w)} &\leq \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s^{per}} e^{-\beta(M_u + M_w)} \leq \frac{\alpha_2}{\gamma_s M_s^{per}}. \end{aligned}$$

Integrate from $n\tau$ to t with $n\tau < t$,

$$F_w(t) \leq F_w(n\tau) e^{\int_{n\tau}^t \left((1-r)\rho_w \frac{\alpha_2}{\gamma_s M_s^{per}} - \mu_{F_w} \right) ds}.$$

Taking $t = (n + 1)\tau$, $n \in \mathbb{N}$,

$$F_w((n + 1)\tau) \leq e^{\left((1-r)\rho_w \frac{\alpha_2}{\gamma_s} \left\langle \frac{1}{M_s^{per}} \right\rangle - \mu_{F_w}\right)\tau} F_w(n\tau).$$

Thus, the sequence $\{F_w(n\tau)\}_{n \in \mathbb{N}}$ decreases towards 0, provided that

$$\left\langle \frac{1}{M_s^{per}} \right\rangle < e\beta\gamma_s \frac{1}{N_{F_w}}.$$

Similarly, from (2.4), we have $\left\langle \frac{1}{M_s^{per}} \right\rangle < e\beta\gamma_s \frac{1}{N_{M_w}}$.

Providing the necessary conditions $\left\langle \frac{1}{M_s^{per}} \right\rangle$ leads to sufficient conditions for asymptotic stability at \mathcal{E}_0 ;

$$\begin{aligned} \left\langle \frac{1}{M_s^{per}} \right\rangle &= \frac{2(\cosh(\mu_{M_s} \tau) - 1)}{\mu_{M_s} \tau^2 \Lambda} < e\beta\gamma_s \max\left\{ \frac{1}{N_{M_u}}, \frac{1}{N_{F_u}}, \frac{1}{N_{M_w}}, \frac{1}{N_{F_w}} \right\}, \\ \Lambda &\geq \frac{2(\cosh(\mu_{M_s} \tau) - 1)}{e\beta\gamma_s \mu_{M_s} \tau^2} \min\{N_{M_u}, N_{F_u}, N_{M_w}, N_{F_w}\}. \end{aligned}$$

Lemma 4. Let k_1, k_2 be a real number such that $0 < k_1 < \frac{1}{N_{F_u}}, 0 < k_2 < \frac{1}{N_{F_w}}$. Then, every solution of (2.2) such that $\frac{M_u}{M_u + M_w + \gamma M_s} \leq k_1$ and $\frac{M_u + M_w}{M_u + M_w + \gamma M_s} \leq k_2$, $t \geq 0$, converges exponentially to \mathcal{E}_0 .

Proof: By using the assumptions $\frac{M_u}{M_u + M_w + \gamma M_s} \leq k_1$ and $\frac{M_u + M_w}{M_u + M_w + \gamma M_s} \leq k_2$, the model (2.2) becomes

$$\dot{M}_u = r\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{M_u} M_u,$$

$$\begin{aligned}
&\leq r\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s} - \mu_{M_u} M_u, \\
&\leq -\mu_{M_u} M_u + r\rho_u k_1 F_u, \\
\dot{F}_u &\leq (1-r)\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s} - \mu_{F_u} F_u, \\
&\leq (-\mu_{F_u} + (1-r)\rho_u k_1) F_u, \\
\dot{M}_w &\leq -\mu_{M_w} M_w + r\rho_w k_2 F_w, \\
\dot{F}_w &\leq (-\mu_{F_w} + (1-r)\rho_w k_2) F_w.
\end{aligned}$$

The autonomous linear system

$$\begin{pmatrix} \dot{M}'_u \\ \dot{F}'_u \\ \dot{M}'_w \\ \dot{F}'_w \end{pmatrix} = \begin{pmatrix} -\mu_{M_u} & r\rho_u k_1 & 0 & 0 \\ 0 & -\mu_{F_u} + (1-r)\rho_u k_1 & 0 & 0 \\ 0 & 0 & -\mu_{M_w} & r\rho_w k_2 \\ 0 & 0 & 0 & -\mu_{F_w} + (1-r)\rho_w k_2 \end{pmatrix} \begin{pmatrix} M'_u \\ F'_u \\ M'_w \\ F'_w \end{pmatrix} \quad (3.2)$$

is monotone (Metzler matrix involved) (see [33]), and it can be used as a comparison system for the evolution of (2.2). Thus, it is deduced that

$$0 \leq M_u(t) \leq M'_u(t), \quad 0 \leq F_u(t) \leq F'_u(t), \quad 0 \leq M_w(t) \leq M'_w(t), \quad 0 \leq F_w(t) \leq F'_w(t), \quad t \geq 0.$$

Here, (M'_u, F'_u, M'_w, F'_w) be the solution of the linear system (3.2) obtained by same initial condition as the solution (M_u, F_u, M_w, F_w) of (2.2). Furthermore, the linear system (3.2) is asymptotically stable if $0 < k_1 < \frac{1}{N_{F_u}}, 0 < k_2 < \frac{1}{N_{F_w}}$, i.e., (M'_u, F'_u, M'_w, F'_w) asymptotically converges to \mathcal{E}_0 . Based on this, (M_u, F_u, M_w, F_w) also asymptotically converges to \mathcal{E}_0 .

Remark 1. The upper bound of k_1 and k_2 are fixed on the ratio $\frac{M_u}{M_u + M_w + \gamma M_s}$ and $\frac{M_u + M_w}{M_u + M_w + \gamma M_s}$ respectively, in order to make the apparent basic offspring number $k_1 N_{F_u}$ and $k_2 N_{F_w}$ is smaller than 1.

Here, we want to verify the condition $\frac{M_u + M_w}{M_u + M_w + \gamma M_s} \leq k_2$, based on the sufficient impulse sterile releases Λ_n . Before, the value of M_s on $(n\tau, (n+1)\tau]$ is described as

$$M_s(t) = M_s(n\tau^+) e^{-\mu_{M_s}(t-n\tau)} = (\tau\Lambda_n + M_s(n\tau)) e^{-\mu_{M_s}(t-n\tau)}. \quad (3.3)$$

We impose the stronger condition instead of $\frac{M_u + M_w}{M_u + M_w + \gamma M_s} \leq k_2$, on $(n\tau, (n+1)\tau]$

$$\gamma M_s(t) \geq \left(\frac{1}{k_2} - 1\right)(M'_u(t) + M'_w(t)), \quad t \geq 0, \quad (3.4)$$

where $M'_u(t), M'_w(t)$ refers to super solution of $M_u(t), M_w(t)$ (Lemma 4).

Lemma 5. The solution of (3.2) on $(n\tau, (n+1)\tau]$ with initial conditions $(M'_u(n\tau), F'_u(n\tau), M'_w(n\tau), F'_w(n\tau)) = (M_u(n\tau), F_u(n\tau), M_w(n\tau), F_w(n\tau))$ is defined by

$$\begin{pmatrix} M'_u(n\tau) \\ F'_u(n\tau) \\ M'_w(n\tau) \\ F'_w(n\tau) \end{pmatrix} = \begin{pmatrix} p_1 & d_1 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & d_2 \\ 0 & 0 & 0 & p_4 \end{pmatrix} \begin{pmatrix} M_u(n\tau) \\ F_u(n\tau) \\ M_w(n\tau) \\ F_w(n\tau) \end{pmatrix}, \quad (3.5)$$

where $p_1 = e^{-\mu_{M_u}(t-n\tau)}$, $p_2 = e^{-(\mu_{F_u}-(1-r)\rho_u k_1)(t-n\tau)}$, $p_3 = e^{-\mu_{M_w}(t-n\tau)}$, $p_4 = e^{-(\mu_{F_w}-(1-r)\rho_w k_2)(t-n\tau)}$, $d_1 = \frac{r\rho_u k_1}{\mu_{M_u} - \mu_{F_u} + (1-r)\rho_u k_1} (e^{-(\mu_{F_u}-(1-r)\rho_u k_1)(t-n\tau)} - e^{-\mu_{M_u}(t-n\tau)})$, and $d_2 = \frac{r\rho_w k_2}{\mu_{M_w} - \mu_{F_w} + (1-r)\rho_w k_2} (e^{-(\mu_{F_w}-(1-r)\rho_w k_2)(t-n\tau)} - e^{-\mu_{M_w}(t-n\tau)})$.

Now, we define the feedback control analysis, on any $(n\tau, (n+1)\tau]$, substitute the values of (3.3) and (3.5) into (3.4), we get

$$\begin{aligned} \gamma M_s(t) &\geq \left(\frac{1}{k_2} - 1\right)(M'_u(t) + M'_w(t)), \\ &\quad \gamma(\Lambda_n \tau + M_s(n\tau))e^{-\mu_{M_s}(t-n\tau)} \\ &\geq \frac{1-k_2}{k_2} \left(e^{-\mu_{M_u}(t-n\tau)} M_u(n\tau) + \frac{r\rho_u k_1}{\mu_{M_u} - \mu_{F_u} + (1-r)\rho_u k_1} (e^{-(\mu_{F_u}-(1-r)\rho_u k_1)(t-n\tau)} \right. \\ &\quad \left. - e^{-\mu_{M_u}(t-n\tau)}) F_u(n\tau) + e^{-\mu_{M_w}(t-n\tau)} M_w(n\tau) + \frac{r\rho_w k_2}{\mu_{M_w} - \mu_{F_w} + (1-r)\rho_w k_2} \right. \\ &\quad \left. \times (e^{-(\mu_{F_w}-(1-r)\rho_w k_2)(t-n\tau)} - e^{-\mu_{M_w}(t-n\tau)}) F_w(n\tau) \right), \\ \Lambda_n \tau &\geq -M_s(n\tau) + \frac{1-k_2}{\gamma k_2} e^{(\mu_{M_s} - \mu_{M_u})s} \left(M_u(n\tau) + \frac{r\rho_u k_1}{\mu_{M_u} - \mu_{F_u} + (1-r)\rho_u k_1} \times \right. \\ &\quad \left. (e^{(\mu_{M_u} - (\mu_{F_u} - (1-r)\rho_u k_1))s} - 1) F_u(n\tau) + M_w(n\tau) + \frac{r\rho_w k_2}{\mu_{M_w} - \mu_{F_w} + (1-r)\rho_w k_2} \right. \\ &\quad \left. \times (e^{(\mu_{M_w} - (\mu_{F_w} - (1-r)\rho_w k_2))s} - 1) F_w(n\tau) \right), \quad s \in [0, \tau]. \end{aligned}$$

We arrive the following theorem:

Theorem 3. For a given $k_1 \in (0, \frac{1}{N_{F_u}})$, $k_2 \in (0, \frac{1}{N_{F_w}})$, assuming, for $n \in \mathbb{N}$,

$$\begin{aligned} \tau \Lambda_n &\geq \left| \Gamma \begin{pmatrix} M_u(n\tau) \\ F_u(n\tau) \\ M_w(n\tau) \\ F_w(n\tau) \end{pmatrix} - M_s(n\tau) \right|_+, \quad (3.6) \\ \Gamma &= \begin{pmatrix} \frac{1-k_2}{\gamma k_2} e^{(\mu_{M_s} - \mu_{M_u})\tau} \\ \frac{(1-k_2)}{k_2} \frac{r\rho_u k_1}{\gamma(\mu_{M_u} - \mu_{F_u} + (1-r)\rho_u k_1)} (e^{(\mu_{M_s} - (\mu_{F_u} - (1-r)\rho_u k_1))\tau} - e^{(\mu_{M_s} - \mu_{M_u})\tau}) \\ \frac{1-k_2}{\gamma k_2} e^{(\mu_{M_s} - \mu_{M_w})\tau} \\ \frac{r\rho_w(1-k_2)}{\gamma(\mu_{M_w} - \mu_{F_w} + (1-r)\rho_w k_2)} (e^{(\mu_{M_s} - (\mu_{F_w} - (1-r)\rho_w k_2))\tau} - e^{(\mu_{M_s} - \mu_{M_w})\tau}) \end{pmatrix}^T \end{aligned}$$

Then, every solution of (2.3) exponentially converges to \mathcal{E}_0 with a rate of convergence restricted from below by a value unrelated to the initial value. Moreover,

$$\tau \Lambda_n \leq \Gamma \begin{pmatrix} M_u(n\tau) \\ F_u(n\tau) \\ M_w(n\tau) \\ F_w(n\tau) \end{pmatrix}, \quad (3.7)$$

then the series $\sum_{n=0}^{+\infty} \Lambda_n$ also converges.

Proof: Suppose $(M_u(n\tau), F_u(n\tau), M_w(n\tau), F_w(n\tau)) = (0, 0, 0, 0)$, there is no impulsion effect Λ_n on the evolution of (M_u, F_u, M_w, F_w) . Let us consider $(M_u(n\tau), F_u(n\tau), M_w(n\tau), F_w(n\tau)) \neq (0, 0, 0, 0)$.

Case(i) Assume the strict inequality of (3.6), one can easily get

$$\gamma M_s(t) > \left(\frac{1-k_2}{k_2}\right)(M'_u(t) + M'_w(t)), \quad \text{for every } t \in (n\tau, (n+1)\tau], \quad (3.8)$$

where the solution (M'_u, F'_u, M'_w, F'_w) of (3.2) starting from $(M_u(n\tau), F_u(n\tau), M_w(n\tau), F_w(n\tau))$ at $t = n\tau$. Initially, we establishes like that

$$M_u(t) < M'_u(t), F_u(t) < F'_u(t), M_w(t) < M'_w(t), F_w(t) < F'_w(t), \text{ for every } t \in [n\tau, (n+1)\tau]. \quad (3.9)$$

Let $t_0 \in [n\tau, (n+1)\tau)$ such that $M_u(t_0) \leq M'_u(t_0), F_u(t_0) \leq F'_u(t_0), M_w(t_0) \leq M'_w(t_0), F_w(t_0) \leq F'_w(t_0)$ with at least one equality. Let us show the existence of t_1 such that $t_0 < t_1 < (n+1)\tau$,

$$M_u(t) < M'_u(t), F_u(t) < F'_u(t), M_w(t) < M'_w(t), F_w(t) < F'_w(t), \text{ for every } t \in (t_0, t_1). \quad (3.10)$$

Based on (3.8) and the definition of t_0 ,

$$\gamma M_s(t_0) > \left(\frac{1-k_2}{k_2}\right)(M'_u(t_0) + M'_w(t_0)) \geq \left(\frac{1-k_2}{k_2}\right)(M_u(t_0) + M_w(t_0)),$$

when $t_0 = n\tau$, and $M_s(t_0) = M_s(n\tau^+)$. The functions $M_u(t), F_u(t), M_w(t), F_w(t)$ and $M_s(t)$ are continuous on $(n\tau, (n+1)\tau)$, there exists t_1 such that $t_0 < t_1 < (n+1)\tau$

$$\gamma M_s(t) > \left(\frac{1-k_2}{k_2}\right)(M_u(t) + M_w(t)), \quad \text{for every } t \in (t_0, t_1).$$

It can be shown as in Lemma 4 that $(M'_u(t), F'_u(t), M'_w(t), F'_w(t)) \geq (M_u(t), F_u(t), M_w(t), F_w(t))$ for any $t \in (t_0, t_1)$, Also $(M'_u(t), F'_u(t), M'_w(t), F'_w(t)) > (M_u(t), F_u(t), M_w(t), F_w(t))$ because the functions defining the right hand sides of (2.3) take on strictly smaller values than those defining the r.h.s of (3.2). Therefore, for any $t_0 \in (n\tau^+) \cup (n\tau, (n+1)\tau)$, there exists $t_1 > t_0$ such that (3.10) holds. From (3.10) and the fact that $(M_u(n\tau), F_u(n\tau), M_w(n\tau), F_w(n\tau)) = (M'_u(n\tau), F'_u(n\tau), M'_w(n\tau), F'_w(n\tau))$, one deduces that (3.10) is true with $t_1 = (n+1)\tau$, and (3.9) is true. Finally, putting together (3.8) and (3.9) yields

$$\gamma M_s(t) > \left(\frac{1-k_2}{k_2}\right)(M_u(t) + M_w(t)), \quad \text{for every } t \in (n\tau, (n+1)\tau]. \quad (3.11)$$

Case(ii) Assume (3.6) holds, and instead of (3.11), considering the Λ_n values convergent from above to the quantity in the r.h.s of this inequality, and trust the flow's consistency with respect to Λ_n .

$$\gamma M_s(t) \geq \left(\frac{1-k_2}{k_2}\right)(M_u(t) + M_w(t)), \quad \text{for every } t \in (n\tau, (n+1)\tau]. \quad (3.12)$$

From (3.12), for any $t \in (n\tau, (n+1)\tau]$, $\frac{M_u}{M_u+M_w+M_s} \leq k_1$ and $\frac{M_u+M_w}{M_u+M_w+M_s} \leq k_2$.

$$\begin{aligned} \dot{F}_u &= (1-r)\rho_u F_u \frac{M_u}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u+F_u+M_w+F_w)} - \mu_{F_u} F_u, \\ &\leq ((1-r)\rho_u k_1 - \mu_{F_u}) F_u, \end{aligned}$$

$$\begin{aligned} \dot{F}_w &= (1-r)\rho_w F_w \frac{M_u + M_w}{M_u + M_w + \gamma_s M_s} e^{-\beta(M_u + F_u + M_w + F_w)} - \mu_{F_w} F_w, \\ &\leq ((1-r)\rho_w k_2 - \mu_{F_w}) F_w. \end{aligned}$$

From the values of Γ , there exists $\epsilon_1, \epsilon_2 > 0$ such that $\mu_{F_u} - (1-r)\rho_u k_1 > \epsilon_1, \mu_{F_w} - (1-r)\rho_w k_2 > \epsilon_2$ and then $\dot{F}_u \leq -\epsilon_1 F_u, \dot{F}_w \leq -\epsilon_2 F_w$. Therefore, $F_u(t), F_w(t)$ exponentially converges to 0. It is then deduced from first and third equation of (2.3) that $M_u(t), M_w(t)$ also exponentially converges to 0. $(M_u(t), F_u(t), M_w(t), F_w(t))$ converges to \mathcal{E}_0 .

Finally, Λ_n satisfies (3.6) and (3.7), based on exponentially stability concepts, there exist $C, \epsilon > 0$ such that $M_u(t) < C e^{-\epsilon t}, F_u(t) < C e^{-\epsilon t}, M_w(t) < C e^{-\epsilon t}, F_w(t) < C e^{-\epsilon t}, t \geq 0$. We can easily obtain that

$$\begin{aligned} \Lambda_n &\leq \frac{(1-k_2)}{\tau\gamma k_2} e^{(\mu_{M_s} - \mu_{M_u})\tau} M_u(n\tau) + \frac{r\rho_u k_1(1-k_2)}{\tau\gamma k_2(\mu_{M_u} - \mu_{F_u} + (1-r)\rho_u k_1)} (e^{(\mu_{M_s} - (\mu_{F_u} - (1-r)\rho_u k_1))\tau} \\ &\quad - e^{(\mu_{M_s} - \mu_{M_u})\tau}) F_u(n\tau) + \frac{(1-k_2)}{\tau\gamma k_2} e^{(\mu_{M_s} - \mu_{M_w})\tau} M_w(n\tau) + \frac{r\rho_w(1-k_2)}{\tau\gamma(\mu_{M_w} - \mu_{F_w} + (1-r)\rho_w k_2)} \\ &\quad \times (e^{(\mu_{M_s} - (\mu_{F_w} - (1-r)\rho_w k_2))\tau} - e^{(\mu_{M_s} - \mu_{M_w})\tau}) F_w(n\tau), \\ &\leq \frac{C}{\gamma\tau} \left(\frac{(1-k_2)}{k_2} e^{(\mu_{M_s} - \mu_{M_u})\tau} + \frac{r\rho_u(1-k_2)k_1}{k_2(\mu_{M_u} - \mu_{F_u} + (1-r)\rho_u k_1)} (e^{(\mu_{M_s} - (\mu_{F_u} - (1-r)\rho_u k_1))\tau} - e^{(\mu_{M_s} - \mu_{M_u})\tau}) \right. \\ &\quad \left. + \frac{(1-k_2)}{k_2} e^{(\mu_{M_s} - \mu_{M_w})\tau} + \frac{r\rho_w(1-k_2)}{(\mu_{M_w} - \mu_{F_w} + (1-r)\rho_w k_2)} (e^{(\mu_{M_s} - (\mu_{F_w} - (1-r)\rho_w k_2))\tau} - e^{(\mu_{M_s} - \mu_{M_w})\tau}) \right) e^{-n\epsilon\tau}, \\ \sum_{n=0}^{+\infty} \Lambda_n &\leq \frac{C}{\gamma\tau} \left(\frac{(1-k_2)}{k_2} e^{(\mu_{M_s} - \mu_{M_u})\tau} + \frac{r\rho_u(1-k_2)k_1}{k_2(\mu_{M_u} - \mu_{F_u} + (1-r)\rho_u k_1)} (e^{(\mu_{M_s} - (\mu_{F_u} - (1-r)\rho_u k_1))\tau} \right. \\ &\quad \left. - e^{(\mu_{M_s} - \mu_{M_u})\tau}) + \frac{(1-k_2)}{k_2} e^{(\mu_{M_s} - \mu_{M_w})\tau} + \frac{r\rho_w(1-k_2)}{(\mu_{M_w} - \mu_{F_w} + (1-r)\rho_w k_2)} (e^{(\mu_{M_s} - (\mu_{F_w} - (1-r)\rho_w k_2))\tau} \right. \\ &\quad \left. - e^{(\mu_{M_s} - \mu_{M_w})\tau}) \right) \frac{1}{1 - e^{-\epsilon\tau}}. \end{aligned}$$

The above series is convergent.

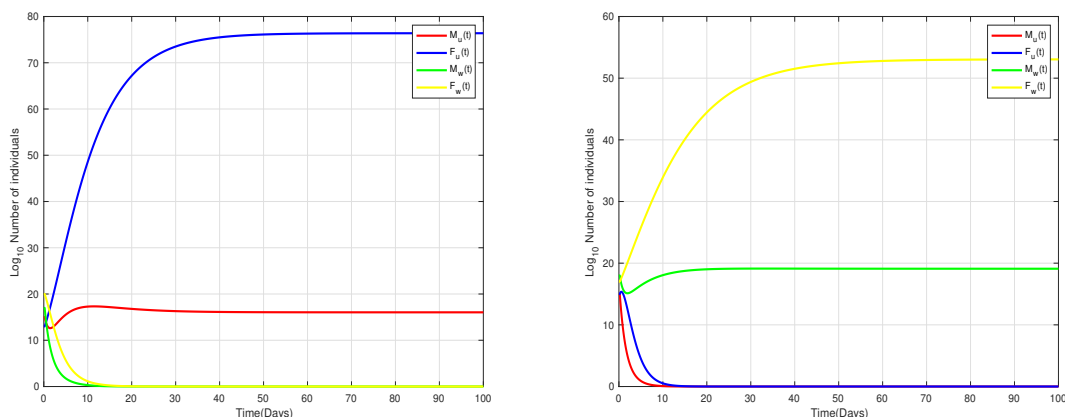


Figure 1. The *Wolbachia* free equilibrium point \mathcal{E}_1 , which is asymptotically stable when $N_{F_u} > 1$ (left) and *Wolbachia* infected equilibrium point \mathcal{E}_2 , which is asymptotically stable when $N_{F_w} > 1$ (right) for model (2.1).

4. Numerical simulations

We consider the parameter values of the model: $r = 0.5$, $\rho_u = 6.4$, $\rho_w = 3.5$, $\gamma_s = 0.9$, $\mu_{M_u} = 0.07$, $\mu_{F_u} = 0.06$, $\mu_{M_w} = 0.04$, $\mu_{F_w} = 0.03$, $\mu_{M_s} = 0.11$, $\sigma = 0.05$, $K = 155$, $\beta = \frac{\sigma}{K} = 3.22 \times 10^{-4}$. The basic offspring numbers: $N_{M_u} \approx 45.71$, $N_{F_u} \approx 53.33$, $N_{M_w} \approx 43.75$, $N_{F_w} \approx 58.33$. The basic offspring figures show the average number of children produced over a person's lifespan. Figures 1 and 2 display the stable behaviour of the steady states $\mathcal{E}_1, \mathcal{E}_2$ and \mathcal{E}_3 , respectively. Let's starting with the regular rash discharges of sterile male mosquitoes. The releasing method derived in Theorem 2 is demonstrated. For open-loop periodic impulsive releases carried out every 7 and 14 days. Consider the smallest value in (3.1) to estimate the number of sterile male mosquitoes to release, i.e., $\Lambda_{per}^{crit} \times 7 = 1525 \times 7 = 10,675$ and $\Lambda_{per}^{crit} \times 14 = 1595 \times 14 = 22,330$, sterile male mosquitoes per hectare and per two weeks, respectively. Figure 3 depicts the corresponding simulations.

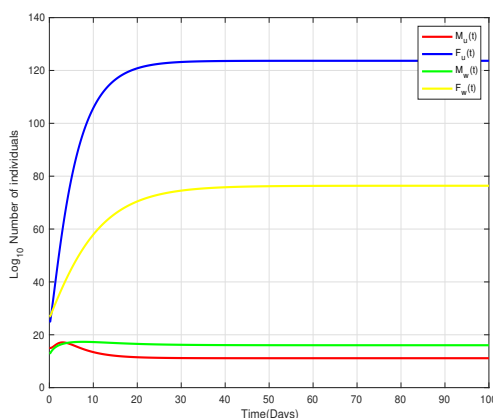


Figure 2. The interior equilibrium point \mathcal{E}_3 , which is asymptotically stable when $N_{F_u} > 1$ and $N_{F_w} > 1$ for model (2.1).

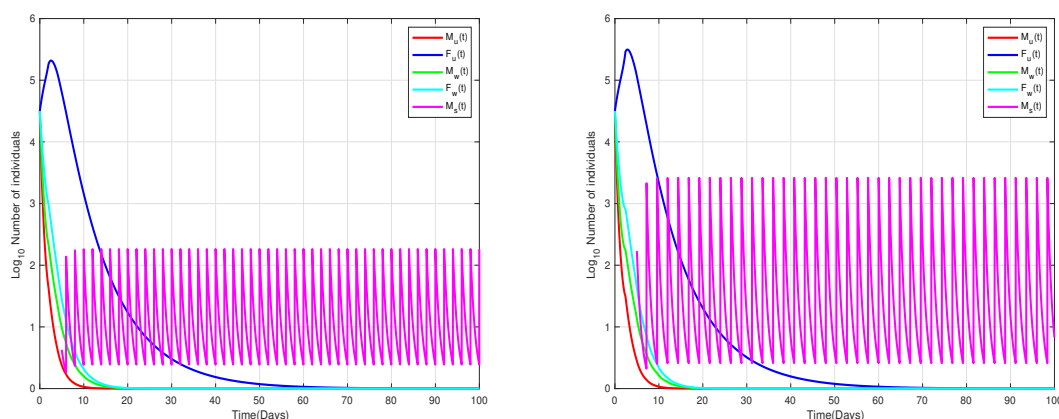


Figure 3. Open loop periodic impulsive control of (2.3) with $\tau = 7$ days (left) and $\tau = 14$ days (right).

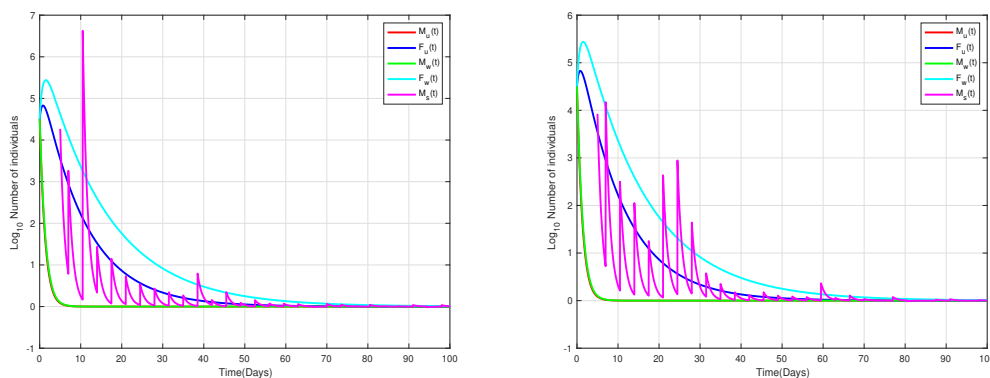


Figure 4. Closed loop periodic impulsive control of (2.3) with $k_1 N_{F_u} = 0.2$, $k_2 N_{F_w} = 0.25$, $\tau = 7$ days (left) and $k_1 N_{F_u} = 0.2$, $k_2 N_{F_w} = 0.25$, $\tau = 14$ days (right).

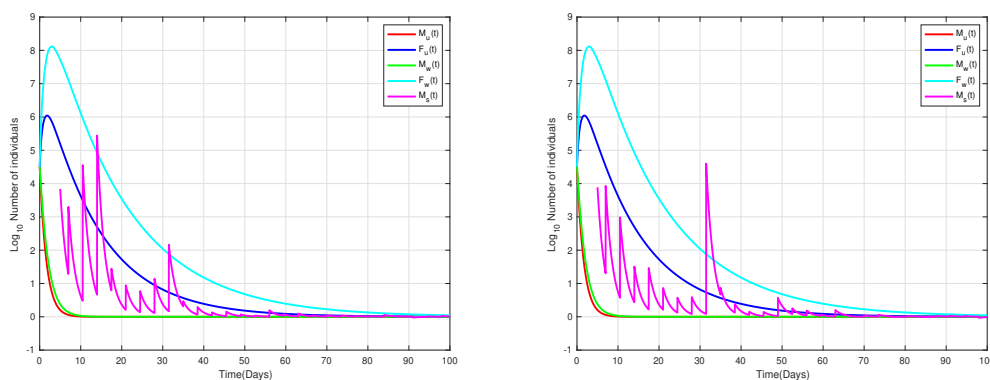


Figure 5. Closed loop periodic impulsive control of (2.3) with $k_1 N_{F_u} = 0.7$, $k_2 N_{F_w} = 0.8$, $\tau = 7$ days (left) and $k_1 N_{F_u} = 0.7$, $k_2 N_{F_w} = 0.8$, $\tau = 14$ days (right).

The closed-loop method can be utilized to reduce the total number of sterile insects released. Theorem 3 shows how the method can be used to lower the total number of sterile insects released. We consider the wild population every τ days both here and in the examination of the feedback technique. We also look at the values of k_1 and k_2 to show the tradeoff between treatment duration and control effort. A small k_1 and k_2 results in a larger control effort and a faster convergence to \mathcal{E}_0 . For $k_1 N_{F_u} = 0.2$, $k_2 N_{F_w} = 0.25$, $\tau = 7$ and $k_1 N_{F_u} = 0.2$, $k_2 N_{F_w} = 0.25$, $\tau = 14$, the diagrams displayed in Figure 4 show that the wild population is close to extinction with the help of SIT treatment. The control effort is smaller and convergence should be delayed for bigger values of k_1 and k_2 . For $k_1 N_{F_u} = 0.7$, $k_2 N_{F_w} = 0.8$, $\tau = 7$ and $k_1 N_{F_u} = 0.7$, $k_2 N_{F_w} = 0.8$, $\tau = 14$, the diagrams are displayed in Figure 5. The size Λ_n of the n^{th} release is taken equal to the right-hand side of (3.6). As it can be clearly seen that k_1 , k_2 and τ have a significant impact on the mosquito population convergence to \mathcal{E}_0 .

5. Concluding remarks

Our study utilized a *Wolbachia*-infected *Aedes Aegypti* mosquitoes model, followed by continuous and impulsive releases of sterile male mosquitoes. Mosquitoes can be controlled by releasing ster-

ile mosquitoes or by replacing the wild population with one that carries *Wolbachia* bacteria, which prevents the transmission of viruses from mosquitoes to humans. Our analysis suggests that despite the lower fitness of the *Wolbachia*-carrying population in comparison to the wild population, the CI-reproductive phenotype gives the system an advantage when shifting from a wild to a *Wolbachia*-carrying population. A number of interesting sufficient conditions have been derived for the model's local asymptotic stability. The conditions required for open-loop or closed-loop control systems were also assessed using sterile male mosquitoes released impulsively. Based on wild population estimates, release sizes are determined in closed-loop control. As shown by (3.6), the released volume is essentially proportional to the measured population. Using sterile insect techniques may lead to periodic outbreaks of mosquitoes due to a periodic oscillation in the system. The theoretical results confirm the numerical solution to the proposed model.

The use of control variables (open, closed-loop, and mixed control techniques) will be considered in future research to determine the best SIT technique, mixed control, and mosquito population elimination strategies.

Acknowledgments

This research was funded by the UAEU Research, fund # 12S005-2021.

Conflict of interest

The authors declare no conflicts of interest.

References

1. H. Hughes, N. F. Britton, Modelling the use of *Wolbachia* to control dengue fever transmission, *Bull. Math. Biol.*, **75** (2013), 796–818. <https://doi.org/10.1007/s11538-013-9835-4>
2. Y. Hui, J. Yu, Global asymptotic stability in a non-autonomous delay mosquito population suppression model, *Appl. Math. Lett.*, **124** (2022), 107599. <https://doi.org/10.1016/j.aml.2021.107599>
3. M. Z. Ndi, R. I. Hickson, D. Allingham, G. N. Mercer, Modelling the transmission dynamics of dengue in the presence of *Wolbachia*, *Math. Biosci.*, **262** (2015), 157–166. <https://doi.org/10.1016/j.mbs.2014.12.011>
4. P. A. Bliman, M. S. Aronna, F. C. Coelho, M. A. H. B. Silva, Ensuring successful introduction of *Wolbachia* in natural populations of *Aedes Aegypti* by means of feedback control, *J. Math. Biol.*, **76** (2018), 1269–1300. <https://doi.org/10.1007/s00285-017-1174-x>
5. L. Multerer, T. Smith, N. Chitnis, Modeling the impact of sterile males on an *Aedes Aegypti* population with optimal control, *Math. Biosci.*, **311** (2019), 91–102. <https://doi.org/10.1016/j.mbs.2019.03.003>
6. B. Zheng, L. Chen, Q. Sun, Analyzing the control of dengue by releasing *Wolbachia* infected male mosquitoes through a delay differential equation model, *Math. Biosci. Eng.*, **16** (2019), 5531–5550. <http://dx.doi.org/10.3934/mbe.2019275>

7. F. A. Rihan, *Delay Differential Equations and Applications to Biology*, Springer, Singapore, 2021. <https://doi.org/10.1007/978-981-16-0626-7>
8. A. Aghriche, R. Yafia, M. A. A. Alaoui, A. Tridane, F. A. Rihan, Oscillations induced by quiescent adult female in a reaction-diffusion model of wild *Aedes Aegypti* mosquitoes, *Int. J. Bifurcation Chaos*, **29** (2019), 1950189. <https://doi.org/10.1142/S021812741950189X>
9. S. P. Sinkins, Wolbachia and cytoplasmic incompatibility in mosquitoes, *Insect Biochem. Mol. Biol.*, **34** (2004), 723–729. <https://doi.org/10.1016/j.ibmb.2004.03.025>
10. I. Iturbe-Ormaetxe, T. Walker, S. L. O’Neill, Wolbachia and the biological control of mosquito-borne disease, *EMBO Rep.*, **12** (2011), 508–518. <https://doi.org/10.1038/embor.2011.84>
11. X. Zhang, S. Tang, R. A. Cheke, H. Zhu, Modeling the effects of augmentation strategies on the control of dengue fever With an impulsive differential equation, *Bull. Math. Biol.*, **78** (2016), 1968–2010. <https://doi.org/10.1007/s11538-016-0208-7>
12. Y. Li, X. Liu, A sex-structured model with birth pulse and release strategy for the spread of Wolbachia in mosquito population, *J. Theor. Biol.*, **448** (2018), 53–65. <https://doi.org/10.1016/j.jtbi.2018.04.001>
13. B. Zheng, M. Tang, J. Yu, Modeling Wolbachia spread in mosquitoes through delay differential equations, *SIAM J. Appl. Math.*, **74** (2014), 743–770. <https://doi.org/10.1137/13093354X>
14. D. Li, H. Wan, The threshold infection level for Wolbachia invasion in a two sex mosquito population model, *Bull. Math. Biol.*, **81** (2019), 2596–2624. <https://doi.org/10.1007/s11538-019-00620-1>
15. Z. Zhang, B. Zheng, Dynamics of a mosquito population suppression model with a saturated Wolbachia release rate, *Appl. Math. Lett.*, **129** (2022), 107933. <https://doi.org/10.1016/j.aml.2022.107933>
16. Y. Dumont, J. M. Tchuente, Mathematical studies on the sterile insect technique for the Chikungunya disease and *Aedes albopictus*, *J. Math. Biol.*, **65** (2012), 809–854. <https://doi.org/10.1007/s00285-011-0477-6>
17. Y. Dumont, I. V. Yatat-Djeument, Sterile insect technique with accidental releases of sterile females. Impact on mosquito-borne diseases control when viruses are circulating, *Math. Biosci.*, **343** (2022), 108724. <https://doi.org/10.1016/j.mbs.2021.108724>
18. L. Almeida, M. Duprez, Y. Privat, N. Vauchelet, Mosquito population control strategies for fighting against arboviruses, *Math. Biosci. Eng.*, **16** (2019), 6274–6297. <https://doi.org/10.3934/mbe.2019313>
19. S. Ai, M. Fox, Four positive equilibria in a model for sterile and wild mosquito populations, *Appl. Math. Lett.*, **121** (2021), 107409. <https://doi.org/10.1016/j.aml.2021.107409>
20. S. Xue, M. Li, J. Ma, J. Li, Sex-structured wild and sterile mosquito population models with different release strategies, *Math. Biosci. Eng.*, **16** (2019), 1313–1333. <https://doi.org/10.3934/mbe.2019064>
21. S. S. Lee, R. E. Baker, E. A. Gaffney, S. M. White, Modelling *Aedes Aegypti* mosquito control via transgenic and sterile insect techniques Endemics and emerging outbreaks, *J. Theor. Biol.*, **331** (2013), 78–90. <https://doi.org/10.1016/j.jtbi.2013.04.014>

22. R. Anguelov, Y. Dumont, J. Lubuma, Mathematical modeling of sterile insect technology for control of anopheles mosquito, *Comput. Math. Appl.*, **64** (2012), 374–389. <https://doi.org/10.1016/j.camwa.2012.02.068>
23. M. Huang, X. Song, J. Li, Modelling and analysis of impulsive releases of sterile mosquitoes, *J. Biol. Dyn.*, **11** (2017), 147–171. <https://doi.org/10.1080/17513758.2016.1254286>
24. X. Zhang, S. Tang, R. A. Cheke, Birth-pulse models of Wolbachia-induced cytoplasmic incompatibility in mosquitoes for dengue virus control, *Nonlinear Anal. Real World Appl.*, **22** (2015), 236–258. <https://doi.org/10.1016/j.nonrwa.2014.09.004>
25. Y. Li, X. Liu, An impulsive model for Wolbachia infection control of mosquito-borne diseases with general birth and death rate functions, *Nonlinear Anal. Real World Appl.*, **37** (2017), 412–432. <https://doi.org/10.1016/j.nonrwa.2017.03.003>
26. P. A. Bliman, D. C. Salgadob, Y. Dumont, O. Vasilieva, Implementation of control strategies for sterile insect techniques, *Math. Biosci.*, **314** (2019), 43–60. <https://doi.org/10.1016/j.mbs.2019.06.002>
27. V. A. Dyck, J. Hendrichs, A. S. Robinson, *The Sterile Insect Technique, Principles and Practice in Area-wide Integrated Pest Management*, Springer, Dordrecht, 2006. <https://doi.org/10.1201/9781003035572>
28. B. Zheng, J. Yu, J. Li, Modeling and analysis of the implementation of the Wolbachia incompatible and sterile insect technique for mosquito population suppression, *SIAM J. Appl. Math.*, **81** (2021), 718–740. <https://doi.org/10.1137/20M1368367>
29. X. Zheng, D. Zhang, Y. Li, S. M. White, Incompatible and sterile insect techniques combined eliminate mosquitoes, *Nature*, **572** (2019), 56–61. <https://doi.org/10.1038/s41586-019-1407-9>
30. D. O. Carvalho, J. A. Torres-Monzon, P. Koskinioti, N. D. A. D. Wijegunawardana, X. Liang, G. Pillwax, et al., *Aedes Aegypti* lines for combined sterile insect technique and incompatible insect technique applications: the importance of host genomic background, *Entomol. Exp. Appl.*, **168** (2020), 560–572. <https://doi.org/10.1111/eea.12892>
31. X. Xu, Y. Xiao, R. A. Cheke, Models of impulsive culling of mosquitoes to interrupt transmission of West Nile virus to birds, *Appl. Math. Modell.*, **39** (2015), 3549–3568. <https://doi.org/10.1016/j.apm.2014.10.072>
32. Y. Li, X. Liu, A sex-structured model with birth pulse and release strategy for the spread of Wolbachia in mosquito population, *J. Theor. Biol.*, **448** (2018), 53–65. <https://doi.org/10.1016/j.jtbi.2018.04.001>
33. H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems* (Mathematical Surveys And Monographs), American Mathematical Society, 1995. <https://doi.org/10.1090/surv/041>