Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 19(10): 10143–10159.
DOI: 10.3934/mbe.2022475
Received: 17 May 2022
Revised: 18 June 2022
Accepted: 07 July 2022
Published: 19 July 2022

Research article

Stability of a class of nonlinear hierarchical size-structured population model

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Abstract: This paper investigates the existence of positive equilibrium as well as the stability of positive equilibrium and zero equilibrium in a nonlinear size-structured hierarchical population model. Under the condition that larger individuals are more competitive advantages than smaller ones, a non-zero fixed point theorem is used to show that there is at lest one positive equilibrium in the system. Moreover, we obtain the stability results of positive equilibrium and zero equilibrium by deriving characteristic equations and establishing Liapunov function. Finally, some numerical experiments are presented.

Keywords: stability; hierarchical population; non-zero fixed point theorem; positive equilibrium; numerical experiments

1. Introduction

Many experiments have shown that the hierarchical structure is prevalent in biological populations in nature. In 1982, Dewsbury [1] listed hundreds of species with hierarchical structures including mammals, reptiles, and many plant species. Subsequently, in the past few decades, many scholars have carried out more research on hierarchical model. In a general hierarchical model, the competition for resources that determines individual vital rates are depend on some hierarchy which related to age or individual size. For example, in 1994, Cushing [2] proposed a hierarchical population model, in which the vital rates of an individual at the age of a depend on the number of individuals who are older than a and smaller than a. After that, Blayneh [3] established a hierarchical size-structured population model, where the individual fertility and mortality rates of individuals with a scale of s are related to the number of individuals with a scale larger than s and smaller than s.

In addition, Calsina and Saldaña [4] studied a hierarchical population model and demonstrated the existence and uniqueness of the model solution and the gradual behavior of the solution. Jang and Cushing [5] proposed a discrete with hierarchical age and size structure model to analyze intra-specific competition. Ackleh et al. [6] established a comparison principle and built monotone sequences, they proved the existence and uniqueness of solution for a size structure population system which environment contain hierarchy. On this basis, Liu and He [7] proposed a nonlinear hierarchical population model and applied the comparative principle to prove the existence uniqueness theorem of solution of model. Ackleh et al. [8] studied a class of finite difference approximations with hierarchical size-structured model, demonstrated the existence uniqueness of weak solution and the convergence of finite difference approximations. Kraev [9] used coordinate transformation to demonstrate the global existence of continuous solution in height structured hierarchical population system. For biological populations, the hierarchical population model was more favorable to the individuals with hierarchical advantages, such as Henson et al. [10], who studied the dynamic consequences of intra-specific scramble competition and contest competition, and found that the ability resource absorption rate of individual is an important determinant. Cushing [11] believed that in some species, the larger individuals have the advantages of resource absorption and competition. Gurney and Nisbet [12] argued that predator populations with hierarchical model were more effective in biological control than simple predators without such social structures.

On the other hand, the population stability plays an important role in the survival of organism populations and has received widespread attention from scholars. Farkas and Hagen [13] studied a class of nonlinear size-structured population dynamics model, applied semi-population and spectral methods to analyze the stability results of the stationary solutions of the model. Li [14] discussed a class of single population equation with random periods, analyzed the stability conditions of the positive equilibrium solution when the control function was taken as E. Farkas and Hinow [15] analyzed the stability of population model distribution. In recent years, He et al. [16] proposed a class of hierarchical age-structured single-population, and discussed the stability conditions of the zero equilibrium solution of the model through linearization. In the same year, He and Zhou [17] proposed a class of competing population model with hierarchical age-structured, used the semi-group theory to obtain the stability criterion of equilibrium solution.

However, compare with the results of age-structured models and size-structured models, the research on hierarchical models are inadequate. In addition, the study on the stability of hierarchical size-structured population models are also insufficient. Therefore, we propose a hierarchical size-structured population model, which the individual's life rate functions depend on the internal environment $E(p)$. The non-zero fixed-point theorem is applied to prove the existence of positive equilibrium in the model. Moreover, we prove the stability of positive equilibrium and zero equilibrium by deriving the characteristic equation and present several numerical experiments for zero equilibrium state.

2. The existence of positive equilibrium

In this paper, we study the following population model

$$
\begin{cases}
\frac{\partial p}{\partial t} + \frac{\partial gp}{\partial s} = -\mu(s, E(p)(s,t))p(s,t) - u(s, E(p)(s,t))p(s,t), & (s,t) \in Q, \\
g(0)p(0,t) = \int_0^L \beta(s, E(p)(s,t))p(s,t)\mathrm{d}s, & t \in (0,T), \\
P(t) = \int_0^L p(s,t)\mathrm{d}s, & t \in (0,T), \\
p(s,0) = p_0(s), & s \in [0,L], \\
E(p)(s,t) = \alpha \int_0^s p(r,t)dr + \int_s^L p(r,t)\mathrm{d}r, & (s,t) \in Q,
\end{cases}
\tag{2.1}
$$

where $Q = (0,L) \times (0,+\infty)$, suppose a finite maximal size denoted by L and the size of the newborn is considered to be 0. The population is described by means of a density function $p(s,t)$, $P(t)$ is the total population size at time t. $g(s)$ represents the growth rate of size. The functions μ,β and u are respectively the mortality, fertility and harvesting effort which based on size s and on the internal environment $E(p)$, α is called small individuals discount factor with $0 \le \alpha < 1$.

This paper makes the following assumptions on the model ingredients

(H_1) $\mu(s,E) > 0, 0 \le \beta(s,E) \le M_1$, and $u(s,E) > 0, \forall (s,E) \in [0,L] \times [0,+\infty)$, M_1 is constant;

(H_2) $\mu(s,E)$, $\beta(s,E)$ and $u(s,E)$ are continuous functions, $\mu'_E = \frac{\partial \mu}{\partial E}$ and $\beta'_E = \frac{\partial \beta}{\partial E}$ exist with $0 \le \mu'_E < \infty$ and $-\infty < \beta'_E < 0$, respectively;

(H_3) μ,β and u are locally Lipschitz functions, there exists Lipschitz constants $L_i > 0, i = 1,2,3$, such that

$$
|\mu(s,E_1) - \mu(s,E_2)| \le L_1|E_1 - E_2|;
$$
$$
|\beta(s,E_1) - \beta(s,E_2)| \le L_2|E_1 - E_2|;
$$
$$
|u(s,E_1) - u(s,E_2)| \le L_3|E_1 - E_2|;
$$

a.e. $s \in [0,L)$ and $E_1, E_2 \in [0,\infty)$;

(H_4) $0 < M_2 \le g(s) \le M_3$, $\forall (s,E) \in [0,L] \times [0,+\infty)$, M_2, M_3 are constants. Furthermore, $g(L) = 0$.

Moreover, assume the functions β, μ and $u \in C^1$.

In [7], the authors have proved that system (2.1) has a unique non-negative solution on $t \in [0,T]$, according to the extension theorem of the solution, it can be obtained that system (2.1) has a unique non-negative solution for $t \in [0,\infty)$.

If system (2.1) has equilibria solutions $p_1(s)$ then it has to satisfy the following equations

$$
\begin{cases}
\frac{\mathrm{d}gp_1}{\mathrm{d}s} = -\mu(s, E(p_1)(s))p_1(s) - u(s, E(p_1)(s))p_1(s), & s \in (0,L), \\
g(0)p_1(0) = \int_0^L \beta(s, E(p_1)(s))p_1(s)\mathrm{d}s, & s \in (0,L), \\
P_1 = \int_0^L p_1(s)\mathrm{d}s, & s \in (0,L), \\
E(p_1)(s) = \alpha \int_0^s p_1(r)\mathrm{d}r + \int_s^L p_1(r)\mathrm{d}r, & s \in (0,L).
\end{cases}
\tag{2.2}
$$

From the first equation of system (2.2)

$$
p_1(s) = p_1(0)\exp\Big\{ - \int_0^s \frac{\mu(r, E(p_1)(r)) + u(r, E(p_1)(r)) + g'(r)}{g(r)}\mathrm{d}r\Big\},
\tag{2.3}
$$

$p_1(0) = 0$ and $p_1(0) > 0$ are respectively called the zero equilibrium and positive equilibrium of system (2.2). Substituting (2.3) into the second equation of system (2.2), then

$$p_1(0) = \frac{p_1(0)}{g(0)} \int_0^L \exp\Big\{ - \int_0^s \frac{\mu(r, E(p_1)(r)) + u(r, E(p_1)(r)) + g'(r)}{g(r)} \mathrm{d}r \Big\} \beta(s, E(p_1)(s)) \mathrm{d}s, \tag{2.4}$$

when $p_1(0) > 0$, it can be obtained that $S(p_1) = 1$ from (2.4), where

$$S(p_1) = \frac{1}{g(0)} \int_0^L \exp\Big\{ - \int_0^s \frac{\mu(r, E(p_1)(r) + u(r, E(p_1)(r)) + g'(r)}{g(r)} \mathrm{d}r \Big\} \beta(s, E(p_1)(s)) \mathrm{d}s,$$

which is called the net reproduction number.

Now introducing non-zero fixed point theorem ([18], Theorem A) to prove system (2.1) existence positive equilibria.

Lemma 2.1. *Let Z be a Banach space, $K \subset Z$ a closed convex cone, $K_r = K \bigcap B_r(0)$, $F : K_r \to K$ continuous such that $F(K_r)$ is relatively compact. Assume*

(i) $Fz \neq \lambda z$ for all $\|z\| = r$;

(ii) there are $\rho \in (0, r), e \in K \backslash \{0\}$ such that $z - Fz \neq \lambda e$ for all $\|z\| = \rho, \lambda > 0$.

Then F has at least on fixed point $z_0 \in \{z \in K : \rho \leq \|z\| \leq r\}$.

Proof. Take Banach space $Z = L^1(0, L) \times R$, define norm $\|(v, c)\| = \|v\| + |c|$ on the space Z, where $\|v\| = \int_0^L |v(s)| \mathrm{d}s$. Consider closed convex cone $K = \{(v, c) \in Z : v(s) \geq 0, c \geq 0\}, K_r = K \cap B_r(0)$.

Define mapping $F : K_r \to K$, which

$$F(v, c) = \begin{pmatrix} c \exp\{- \int_0^s \frac{\mu(r,E(v)(r))+u(r,E(v)(r))+g'(r)}{g(r)} \mathrm{d}r\} \\ \frac{c}{g(0)} \int_0^L \beta(s, E(v)(s)) \exp\{- \int_0^s \frac{\mu(r,E(v)(r))+u(r,E(v)(r))+g'(r)}{g(r)} \mathrm{d}r\} \mathrm{d}s \end{pmatrix}.$$

Firstly, prove the mapping F is continuous.

Let $(v, c) \to (v_0, c_0)$, that is $\|v - v_0\| \to 0$, $|c - c_0| \to 0$. From the fourth equation in system (2.2), then

$$\begin{aligned} \|E(v) - E(v_0)\| &= \int_0^L \Big| \alpha \int_0^s [v(r) - v(r_0)] \mathrm{d}r + \int_s^L [v(r) - v(r_0)] \mathrm{d}r \Big| \mathrm{d}s \\ &\leq \int_0^L \Big| \int_0^L [v(r) - v(r_0)] \mathrm{d}r \Big| \mathrm{d}s \\ &\leq \int_0^L \|v(r) - v(r_0)\| \mathrm{d}s \to 0, \end{aligned}$$

thus $E(v)$ is continuous at v_0. Then

$$\begin{aligned} &\frac{c}{g(0)} \int_0^L \beta(s, E(v)) \exp\Big\{ - \int_0^s \frac{\mu(r, E(v)) + u(r, E(v)) + g'(r)}{g(r)} \mathrm{d}r \Big\} \mathrm{d}s \\ &\to \frac{c_0}{g(0)} \int_0^L \beta(s, E(v_0)) \exp\Big\{ - \int_0^s \frac{\mu(r, E(v_0)) + u(r, E(v_0)) + g'(r)}{g(r)} \mathrm{d}r \Big\} \mathrm{d}s. \end{aligned}$$

On the other hand

$$\Big\|\exp\Big\{-\int_0^s \frac{\mu(r,E(v))+u(r,E(v))+g'(r)}{g(r)}\mathrm{d}r\Big\}-\exp\Big\{-\int_0^s \frac{\mu(r,E(v_0))+u(r,E(v_0))+g'(r)}{g(r)}\mathrm{d}r\Big\}\Big\|$$

$$\le \int_0^L \int_0^s |\frac{1}{g(r)}|\|[\mu(r,E(v))+u(r,E(v))]-[\mu(r,E(v_0))+u(r,E(v_0))]|\mathrm{d}r\mathrm{d}s$$

$$\le (L_1+L_3)\int_0^L \int_0^s |\frac{1}{g(r)}|\|E(v)-E(v_0)|\mathrm{d}r\mathrm{d}s$$

$$\le (L_1+L_3)\int_0^L \|\frac{1}{g(s)}[E(v)-E(v_0)]\|\mathrm{d}s \to 0,$$

above results mean that F is a continuous mapping.

Secondly, apply the Fréchet-Kolmogorov theorem [21] to prove the relative compactness of mapping $F(K_r)$.

Extending the domain of the function $v(s)$ to $(-\infty,\infty)$. Let $v(s)=0$ when $s\notin[0,L]$, for any $(v,c)\in K_r$, it can be obtained that

$$F^1(v,c)(s)=\begin{cases} c\exp\{-\int_0^s \frac{\mu(\theta,E(v))+u(\theta,E(v))+g'(\theta)}{g(\theta)}\mathrm{d}\theta\}, & s\in[0,L],\\ 0, & s\in\mathbb{R}\setminus[0,L],\end{cases}$$

$$F^2(v,c)(s)=\begin{cases} \frac{c}{g(0)}\int_0^L \beta(s,E(v))\exp\{-\int_0^s \frac{\mu(\theta,E(v))+u(\theta,E(v))+g'(\theta)}{g(\theta)}\mathrm{d}\theta\}\mathrm{ds}, & s\in[0,L],\\ 0, & s\in\mathbb{R}\setminus[0,L].\end{cases}$$

Then

$$\begin{aligned}\sup_{(v,c)\in K_r}\|F^1(v,c)\| &= \sup\Big(\int_0^L \Big|c\exp\Big\{-\int_0^s \frac{\mu(\theta,E(v))+u(\theta,E(v))+g'(\theta)}{g(\theta)}\mathrm{d}\theta\Big\}\Big|\mathrm{d}s\Big)\\ &= \int_0^L \Big|c\exp\Big\{-\int_0^s \frac{\mu(\theta,E(v))+u(\theta,E(v))}{g(\theta)}\mathrm{d}\theta\Big\}\exp\Big\{\int_0^s \frac{g'(r)}{g(r)}\mathrm{d}r\Big\}\Big|\mathrm{d}s\\ &\le \int_0^L |\frac{cg(0)}{M_2}|\mathrm{d}s\\ &\le |\frac{cg(0)}{M_2}|L\\ &< \infty.\end{aligned}$$

On the other hand

$$\begin{aligned}\sup_{(v,c)\in K_r}\|F^2(v,c)\| &= \sup\Big(\Big|\int_0^L \frac{c\beta(s,E(v))}{g(s)}\exp\Big\{-\int_0^s \frac{\mu(\theta,E(v))+u(\theta,E(v))}{g(\theta)}\mathrm{d}\theta\Big\}\mathrm{d}s\Big|\Big)\\ &\le |\frac{c}{M_2}\int_0^L \beta(s,E(v))\mathrm{d}s|\\ &\le \frac{cM_1L}{M_2}\\ &< \infty.\end{aligned}$$

By the the continuity of $\mu(s,E)$ and $u(s,E)$, when $s\in[0,L-\varepsilon]$, $\varepsilon>0$ and sufficiently small, then

$$\begin{aligned}
&\int_S |F^1(v,c)(s+t)-F^1(v,c)(s)|\mathrm{d}s\\
\le&\int_0^{L-\varepsilon}\frac{cg(0)}{g(s)}\Big|\exp\Big\{-\int_0^{s+t}\frac{\mu(\theta,E(v))+u(\theta,E(v))}{g(\theta)}\mathrm{d}\theta\Big\}-\exp\Big\{-\int_0^{s}\frac{\mu(\theta,E(v))+u(\theta,E(v))}{g(\theta)}\mathrm{d}\theta\Big\}\Big|\\
\le&\frac{cg(0)}{M_2}\int_0^{L-\varepsilon}\Big|\int_s^{s+t}\frac{\mu(\theta,E(v))+u(\theta,E(v))}{g(\theta)}\mathrm{d}\theta\Big|\mathrm{d}s\\
\le&\frac{cg(0)}{M_2^2}(L-\varepsilon)(\mu_\varepsilon+u_\varepsilon)|t|\to 0\ \ (t\to 0),
\end{aligned}$$

where μ_ε, u_ε are upper bounds for $\mu(s,E)$, $u(s,E)$ at $[0,L-\varepsilon]$ respectively.

When $s\in(L-\varepsilon,L)$, it can be obtained that

$$\begin{aligned}
&\int_{L-\varepsilon}^{L}|F^1(v,c)(s+t)-F^1(v,c)(s)|\mathrm{d}s\\
\le&\int_{L-\varepsilon}^{L}\frac{cg(0)}{g(s)}\Big|\exp\Big\{-\int_0^{s+t}\frac{\mu(\theta,E(v))+u(\theta,E(v))}{g(\theta)}\mathrm{d}\theta\Big\}+\exp\Big\{-\int_0^{s}\frac{\mu(\theta,E(v))+u(\theta,E(v))}{g(\theta)}\mathrm{d}\theta\Big\}\Big|\mathrm{d}s\\
\le&\frac{2cg(0)\varepsilon}{M_2}\to 0\ \ (\varepsilon\to 0).
\end{aligned}$$

From the extended definition of function in $F(K_r)$, if $\gamma\ge L$, then $|s|>\gamma\ge L$, it can be obtained that

$$\|F(v,c)\|=\|F^1(v,c)\|+F^2(v,c)=0.$$

According to Fréchet-Kolmogorov theorem, it can be obtained that $C(M_r)$ is relatively compact.

Thirdly, prove $F(v,c)\neq\lambda(v,c)$ for $\|(v,c)\|=r>0$ and $\lambda>1$.

If $F(v,c)=\lambda(v,c)$, from the definition of the mapping F, then

$$\begin{cases}\lambda v(s)=c\exp\Big\{-\int_0^s\frac{\mu(r,E(v))+u(r,E(v))+g'(r)}{g(r)}\mathrm{d}r\Big\},\\ \lambda c=\frac{c}{g(0)}\int_0^L\beta(s,E(v))\exp\Big\{-\int_0^s\frac{\mu(r,E(v))+u(r,E(v))+g'(r)}{g(r)}\mathrm{d}r\Big\}\mathrm{d}s,\end{cases}\tag{2.5}$$

from the first equation in (2.5), when $c=0$, then $v(s)=0$ which is absurd due to $\|(v,c)\|=r>0$.

In the following, consider the cases of $c\neq 0$.

(i) If $F(v,c)=\lambda(v,c)$ for $\forall\|(v,c)\|=r,\exists\lambda>1$. By the first equation in (2.5), it can be obtained that

$$\|v\|\le\lambda\|v\|=c\Big\|\exp\Big\{-\int_0^s\frac{\mu(r,E(v))+u(r,E(v))+g'(r)}{g(r)}\mathrm{d}r\Big\}\Big\|\le\frac{cLg(0)}{M_2}:=K,$$

thus $\|(v,c)\|=\|v\|+c\le(1+K)c$, it implies $\|(v,c)\|\to 0$ when $c\to 0$, contradictory with the condition $\|(v,c)\|=r>0$;

(ii) If $F(v,c)=\lambda(v,c)$ for $\forall\lambda>1,\exists\|(v,c)\|=r$. From Eq (2.5), it can be obtained that

$$c=\frac{1}{g(0)}\int_0^L\beta(s,E(v))v(s)\mathrm{d}s\le\frac{M_1}{g(0)}\|v\|,$$

and

$$v(s) = \frac{c}{\lambda} \exp\Big\{ - \int_0^s \frac{\mu(r, E(v)) + u(r, E(v)(r)) + g'(r)}{g(r)} \mathrm{d}r \Big\},$$

when $\lambda \to +\infty$, then $\|v\| \to 0$, it implies $r = \|(v, c)\| = \|v\| + c \to 0$, contradictory with the conditions.

Finally, prove the following.

There are exist $\rho \in (0, r), e \in K\backslash\{0\}$ such that $z - Fz \neq \lambda e$ for all $\|z\| = \rho$.

Suppose the opposite of the conclusion is true and treat the peoblem in the four cases:

(i)$\forall \rho \in (0, r),\ \exists\ (\overline{v}, \overline{c}) \in K\backslash\{0\}$ such that $(v, c) - F(v, c) = \lambda(\overline{v}, \overline{c})$ for all $\lambda > 0$ and some (v, c) satisfying $\|(v, c)\| = \rho$. From the definition of mapping F, then

$$\begin{cases} v(s) - c \exp\{- \int_0^s \frac{\mu(r,E(v))+u(r,E(v))+g'(r)}{g(r)} \mathrm{d}r\} = \lambda \overline{v}(s), \\ c - \frac{c}{g(0)} \int_0^L \beta(s, E(v)) \exp\{- \int_0^s \frac{\mu(r,E(v))+u(r,E(v))+g'(r)}{g(r)} \mathrm{d}r\} \mathrm{d}s = \lambda \overline{c}, \end{cases}$$

$\|(v.c)\| \to 0$ when $\rho \to 0$, which implies $\|(\overline{v}.\overline{c})\| \to 0$, it contradicts $\|(v, c)\| = r > 0$;

(ii)$\forall \rho \in (0, r),\ \exists\ (\overline{v}, \overline{c}) \in K\backslash\{0\}$ such that $(v, c) - F(v, c) = \lambda(\overline{v}, \overline{c})$ for some $\lambda_0 > 0$ and all (v, c) satisfying $\|(v, c)\| = \rho$. Then

$$(v, c) - F(v, c) = \lambda_0(\overline{v}, \overline{c}),$$

let $\rho \to 0$, then $\lambda_0(\overline{v}, \overline{c}) = 0$, contradictory with $\lambda_0(\overline{v}, \overline{c}) \neq 0$;

(iii)$\exists\ \rho \in (0, r),\ \forall (\overline{v}, \overline{c}) \in K\backslash\{0\}$ such that $(v, c) - F(v, c) = \lambda(\overline{v}, \overline{c})$ for some $\lambda_0 > 0$ and all (v, c) satisfying $\|(v, c)\| = \rho$. Let $(\overline{v}, \overline{c}) = \frac{2}{\lambda_0}(v, c)$, then $F(v, c) = -(v, c)$, which is absurd due to $F(v, c)$ is non-negative;

(iv)$\exists\ \rho \in (0, r),\ \forall (\overline{v}, \overline{c}) \in K\backslash\{0\}$ such that $(v, c) - F(v, c) = \lambda(\overline{v}, \overline{c})$ for all $\lambda > 0$ and some (v, c) satisfying $\|(v, c)\| = \rho$. Let $\lambda \to 0$, then $(v, c) = F(v, c)$, it implies (v, c) is the positive equilibrium of F.

In summary, F satisfies all the conditions in Lemma 2.1, then F has at least one non-zero fixed point p_1, which is the positive equilibrium of the system (2.1). □

3. The stability of positive equilibrium

In this section, we demonstrate the stability conditions of positive equilibrium of system (2.1). Now introducing the variation for positive equilibrium $p_1(s)$

$$w(s, t) = p(s, t) - p_1(s),$$

which satisfies the following differential equation

$$\frac{\partial w(s, t)}{\partial t} + \frac{\partial g(s) w(s, t)}{\partial s} = \frac{\partial p(s, t)}{\partial t} + \frac{\partial g(s) p(s, t)}{\partial s} - \frac{\partial g(s) p_1(s)}{\partial s},$$

where

$$\frac{\partial p(s, t)}{\partial t} + \frac{\partial g(s) p(s, t)}{\partial s} = -\mu(s, E(p)) p(s, t) - u(s, E(p)) p(s, t),$$

$$\frac{\partial g(s) p_1(s)}{\partial s} = -\mu(s, E(p_1)) p_1(s) - u(s, E(p_1)) p_1(s),$$

then

$$\frac{\partial w(s,t)}{\partial t}+\frac{\partial g(s)w(s,t)}{\partial s}=-\mu(s,E(p))p(s,t)-u(s,E(p))p(s,t)$$
$$+\mu(s,E(p_1))p_1(s)+u(s,E(p_1))p_1(s),$$

after linearizing in $E(p_1)$, it can be obtained that

$$\frac{\partial w(s,t)}{\partial t}+\frac{\partial g(s)w(s,t)}{\partial s}=-[\mu(s,E(p_1))+u(s,E(p_1))]w(s,t)$$
$$-[\mu'_E(s,E(p_1))+u'_E(s,E(p_1))]p_1(s)E(w).$$

When $g(0)=1$, then

$$\begin{aligned}w(0,t)&=p(0,t)-p_1(0)\\&=\int_0^L\beta(s,E(p_1))w(s,t)\mathrm{d}s+\int_0^L\beta'_E(s,E(p_1))p_1(s)\mathrm{d}sE(w).\end{aligned}$$

Assume above linear problem has solutions of the form $w(s,t)=e^{\lambda t}W(s)$, and applying the notation $\overline{W}=E(W)(s)$, then

$$\begin{aligned}W'(s)=&-W(s)\frac{\mu(s,E(p_1))+u(s,E(p_1))+g'(s)+\lambda}{g(s)}\\&-\overline{W}\frac{[\mu'_E(s,E(p_1)(s))+u'_E(s,E(p_1))]p_1(s)}{g(s)},\end{aligned}\tag{3.1}$$

$$W(0)=\int_0^L\beta(s,E(p_1))W(s)\mathrm{d}s+\overline{W}\int_0^L\beta'_E(s,E(p_1))p_1(s)\mathrm{d}s.\tag{3.2}$$

The solution of (3.1) and (3.2) is

$$\begin{aligned}W(s)=&\Big(W(0)-\int_0^s\frac{\overline{W}[\beta'_E(a,E(p_1))+u'_E(a,E(p_1))]p_1(a)}{g(a)}\\&\times\exp\Big\{\int_0^a\frac{\mu(r,E(p_1)+u(r,E(p_1))+g'(r)+\lambda}{g(r)}\mathrm{d}r\Big\}\mathrm{d}a\Big)\\&\times\exp\Big\{-\int_0^s\frac{\mu(a,E(p_1))+u(a,E(p_1))+g'(a)+\lambda}{g(a)}\mathrm{d}a\Big\}.\end{aligned}\tag{3.3}$$

Substituting (2.3) into (3.3) and integrating from 0 to L, then

$$\overline{W}=A_{11}(\lambda)W(0)+A_{12}(\lambda)\overline{W},$$

where

$$A_{11}(\lambda)=\int_0^L\exp\Big\{-\int_0^s\frac{\mu(a,E(p_1))+u(a,E(p_1))+g'(a)+\lambda}{g(a)}\mathrm{d}a\Big\}\mathrm{d}s,$$

$$A_{12}(\lambda) = -p_1(0)\int_0^L \exp\Big\{-\int_0^s \frac{\mu(a,E(p_1))+u(a,E(p_1))+g'(a)+\lambda}{g(a)}\mathrm{d}a\Big\}$$
$$\times \int_0^s \frac{\mu'_E(a,E(p_1))+u'_E(a,E(p_1))}{g(a)}\exp\Big\{\int_0^a \frac{\lambda}{g(r)}\mathrm{d}r\Big\}\mathrm{d}a\mathrm{d}s.$$

Substituting $W(s)$ into (3.2), it can be obtained that

$$W(0) = W(0)A_{21}(\lambda) + \overline{W}A_{22}(\lambda),$$

where

$$A_{21}(\lambda) = \int_0^L \beta(s,E(p_1))\exp\Big\{-\int_0^s \frac{\mu(a,E(p_1))+u(a,E(p_1))+g'(a)+\lambda}{g(a)}\mathrm{d}a\Big\}\mathrm{d}s,$$

$$A_{22}(\lambda) = p_1(0)\int_0^L \Big(\exp\Big\{-\int_0^s \frac{\mu(a,E(p_1))+u(a,E(p_1))+g'(a)}{g(a)}\mathrm{d}a\Big\}\beta'_E(s,E(p_1))$$
$$-\exp\Big\{-\int_0^s \frac{\mu(a,E(p_1))+u(a,E(p_1))+g'(a)+\lambda}{g(a)}\mathrm{d}a\Big\}\beta(s,E(p_1))$$
$$\times \int_0^s \frac{\mu'_E(a,E(p_1))+u'_E(a,E(p_1))}{g(a)}\exp\Big\{\int_0^a \frac{\lambda}{g(r)}\mathrm{d}r\Big\}\mathrm{d}a\Big)\mathrm{d}s.$$

Then obtain the following linear system and the Lemma 3.1

$$W(0)A_{11}(\lambda) + (A_{12}(\lambda)-1)\overline{W} = 0,$$

$$W(0)(A_{21}(\lambda)-1) + A_{22}(\lambda)\overline{W} = 0.$$

Lemma 3.1. *in [19]: The positive equilibrium $p_1(s)$ is asymptotically stable (resp. unstable) if all the roots of the following equation have a negative real part (resp. it has a root with a positive real part)*

$$K(\lambda) = A_{11}(\lambda)A_{22}(\lambda) - A_{12}(\lambda)A_{21}(\lambda) + A_{12}(\lambda) + A_{21}(\lambda) = 1.$$

Next prove the stability result for the positive equilibrium.

Theorem 3.2. *In the case of g(0)=1, $\cos(y\Gamma(s)) > 0$, the positive equilibrium $p_1(s)$ is asymptotically stable if $\mu'_E + u'_E = 0$.*

Proof. Introduce the following notations

$$T(s,E(p_1),\lambda) = \exp\Big\{-\int_0^s \frac{\mu(a,E(p_1))+u(a,E(p_1))+g'(a)+\lambda}{g(a)}\mathrm{d}a\Big\},$$

$$T(s,E(p_1)) = \exp\Big\{-\int_0^s \frac{\mu(a,E(p_1))+u(a,E(p_1))+g'(a)}{g(a)}\mathrm{d}a\Big\}, \qquad \Gamma(s) = \int_0^s \frac{1}{g(a)}\mathrm{d}a.$$

Then

$$A_{11}(\lambda) = \int_0^L T(s,E(p_1),\lambda)\mathrm{d}s, \qquad A_{21}(\lambda) = \int_0^L \beta(s,E(p_1))T(s,E(p_1),\lambda)\mathrm{d}s,$$

$$A_{12}(\lambda) = -\, p_1(0)\int_0^L T(s,E(p_1),\lambda)\int_0^s \frac{[\mu'_E(a,E(p_1)) + u'_E(a,E(p_1))]\exp\{\lambda\Gamma(a)\}}{g(a)}\mathrm{d}a\mathrm{d}s,$$

$$\begin{aligned} A_{22}(\lambda) =& p_1(0)\int_0^L T(s,E(p_1))\beta'_E(s,E(p_1)) - T(s,E(p_1),\lambda)\beta(s,E(p_1)) \\ &\times \int_0^s \frac{[\mu'_E(a,E(p_1)) + u'_E(a,E(p_1))]}{g(a)}\exp\{\lambda\Gamma(a)\}\mathrm{d}a\mathrm{d}s. \end{aligned}$$

Thus, the following characteristic equation can be derived

$$\begin{aligned} K(\lambda) =& p_1(0)\int_0^L T(s,E(p_1))\exp\{-\lambda\Gamma(s)\}\mathrm{d}s \int_0^L T(s,E(p_1))\beta'_E(s,E(p_1))\mathrm{d}s \\ &+ \int_0^L T(s,E(p_1),\lambda)\beta(s,E(p_1))\mathrm{d}s. \end{aligned}$$

The real part of all roots of the characteristic equation $K(\lambda)$ are negative, assume that there exists a root $\lambda = x + iy$, if $x \geq 0$, it can be obtained that

$$\begin{aligned} \mathrm{Re}(K(\lambda)) =& p_1(0)\int_0^L T(s,E(p_1))\exp\{-x\Gamma(s)\}\cos(y\Gamma(s))\mathrm{d}s \int_0^L T(s,E(p_1))\beta'_E(s,E(p_1))\mathrm{d}s \\ &+ \int_0^L T(s,E(p_1)\beta(s,E(p_1))\exp\{-x\Gamma(s)\}\cos(y\Gamma(s))\mathrm{d}s \\ =& 1. \end{aligned}$$

Let

$$Q(p_1) = p_1(0)\int_0^L T(s,E(p_1))\exp\{-x\Gamma(s)\}\cos(y\Gamma(s))\mathrm{d}s,$$

for $x \geq 0$, and $\exp\{-x\Gamma(s)\} \leq 1, 0 < \cos(y\Gamma(s)) \leq 1$, then

$$\begin{aligned} \mathrm{Re}(K(\lambda)) \leq& Q(p_1)\int_0^L T(s,E(p_1))\beta'_E(s,E(p_1))\mathrm{d}s + \int_0^L T(s,E(p_1)\beta(s,E(p_1))\mathrm{d}s \\ =& Q(p_1)\int_0^L T(s,E(p_1))\beta'_E(s,E(p_1))\mathrm{d}s + g(0) \\ <& 1. \end{aligned}$$

Contradictory with $K(\lambda) = 1$, it means that the positive equilibrium $p_1(s)$ is asymptotically stable if $\mu'_E + u'_E = 0$. □

4. The stability of zero equilibrium

The linearization of the system (2.1) in the zero equilibrium is as follows

$$\begin{cases} \frac{\partial p}{\partial t} + \frac{\partial gp}{\partial s} = -\mu(s,0)p(s,t) - u(s,0)p(s,t), & (s,t) \in Q, \\ g(0)p(0,t) = \int_0^L \beta(s,0)p(s,t)ds, & t \in (0,\infty), \\ p(s,0) = p_0(s), & s \in [0,L]. \end{cases} \tag{4.1}$$

Consider the system (4.1) has solutions of the form $p(s,t) = e^{\lambda t}P(s)$, by the first equation in the system (4.1), it can be obtained that

$$\lambda P(s) + g'(s)P(s) + g(s)P'(s) = -[\mu(s,0) + u(s,0)]P(s),$$

and

$$P(s) = P(0)\exp\Big\{-\int_0^s \frac{\mu(a,0) + u(a,0) + g'(a) + \lambda}{g(a)}\mathrm{d}a\Big\}.$$

Let $\exp\{\lambda\Gamma(s)\} = \exp\{-\int_0^s \frac{\lambda}{g(r)}\mathrm{d}r\}$, it can be obtained that the following characteristic equation

$$\begin{aligned}1 = K(\lambda) =& \frac{1}{g(0)}\int_0^L \beta(s,0)\exp\Big\{-\int_0^s \frac{\mu(a,0) + u(a,0) + g'(a) + \lambda}{g(a)}\mathrm{d}a\Big\}\mathrm{d}s\\ =& \frac{1}{g(0)}\int_0^L \beta(s,0)\exp\Big\{-\int_0^s \frac{\mu(a,0) + u(a,0) + g'(a)}{g(a)}\mathrm{d}a\Big\}\exp\{-\lambda\Gamma(s)\}\mathrm{d}s.\end{aligned}$$

Theorem 4.1. *(i) The zero equilibrium of system (2.1) is unstable if* $K(0) > 1$*; (ii) the zero equilibrium of system (2.1) is asymptotically stable if* $K(0) < 1$*, moveover, it is globally asymptotically stable when* $\mu(s,0) + u(s,E) \geq \beta(s,0)$*, a.e.* $s \in (0,L)$*.*

Proof. From the characteristic equation, it is clear that $K(\lambda)$ is a strictly monotone decreasing function with respect to λ, it can be obtained that $\lim_{\lambda\to\infty} K(\lambda) = 0$ when $K(0) > 1$. Therefore, when $K(0) > 1$, characteristic equation has a unique positive characteristic root λ_0, which means that the zero equilibrium of system (2.1) is unstable.

On the other hand, the characteristic equation has only negative real roots when $K(0) < 1$, denoted as λ_0, assume that there are exist another root $\lambda = x + iy$, then $x \leq \lambda_0$, if $x > \lambda_0$, it can be obtained that

$$\begin{aligned}1 =& \Big|\mathrm{Re}\frac{1}{g(0)}\int_0^L \beta(s,0)\exp\Big\{-\int_0^s \frac{\mu(a,0) + u(a,0) + g'(a)}{g(a)}\mathrm{d}a\Big\}\exp\{-(x+iy)\Gamma(s)\}\mathrm{d}s\Big|\\ \leq& \frac{1}{g(0)}\int_0^L \Big|\exp\Big\{-\int_0^s \frac{\mu(a,0) + u(a,0) + g'(a)}{g(a)}\mathrm{d}a\Big\}\exp\{-x\Gamma(s)\}\cos(y\Gamma(s))\Big|\mathrm{d}s\\ \leq& \frac{1}{g(0)}\int_0^L \exp\Big\{-\int_0^s \frac{\mu(a,0) + u(a,0) + g'(a)}{g(a)}\mathrm{d}a\Big\}\exp\{-x\Gamma(s)\}\mathrm{d}s\\ <& \frac{1}{g(0)}\int_0^L \exp\Big\{-\int_0^s \frac{\mu(a,0) + u(a,0) + g'(a)}{g(a)}\mathrm{d}a\Big\}\exp\{-\lambda_0\Gamma(s)\}\mathrm{d}s\\ =&1.\end{aligned}$$

Contradictory with $K(\lambda) = 1$.

Establish the Liapunov function $V(p(t)) = \int_0^L p(s,t)\mathrm{d}s$, when $\mu(s,0) + u(s,E) \geq \beta(s,0)$, a.e. $s \in$

[0.L], then

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \int_0^L \frac{\partial P}{\partial t}\mathrm{d}s = -\int_0^L [\frac{\partial gp}{\partial s} + \mu(s, E(p))p(s,t) + u(s, E(p))p(s,t)]\mathrm{d}s \\
&= \int_0^L [\beta(s, E(p)) - \mu(s, E(p)) - u(s, E(p)(s,t))]p(s,t)\mathrm{d}s \\
&< \int_0^L [\beta(s,0) - \mu(s,0) - u(s, E(p)]p(s,t)\mathrm{d}s \\
&<0,
\end{aligned}
$$

it is clear that the zero equilibrium is globally asymptotically stable. □

5. Numerical examples

Selected parameters as $L = T = 10, \alpha = 0.5$, growth rate $g(s) = 1 - 0.025s$, mortality as follows

$$
\mu(s, E) = \begin{cases} 2\sin^2(3s) + 0.009E, & 0 \le s \le 8; \\ +\infty, & \text{otherwise.} \end{cases}
$$

Fertility function

$$
\beta(s, E) = \begin{cases} 0.8[\cos^2(2s) + 0.8], & 1 \le s \le 8; \\ 0, & \text{otherwise.} \end{cases}
$$

Harvesting effort function

$$
u(s, E) = \begin{cases} 0.12(1 - s) - 0.004E, & 1 \le s \le 8; \\ 0, & \text{otherwise.} \end{cases}
$$

Initial distribution of population

$$
p_0^1(s) = \begin{cases} 2(9 - s)^2[sin^2(2s + \frac{\pi}{3}) + 1], & 0 \le s \le 8; \\ 0, & \text{otherwise.} \end{cases}
$$

after calculation, obtain $K_1(0) = 0.1717$. If other parameters are the same, select initial distribution $p_0^2(s) = 2(9 - s)^2 sin^2(s + \frac{\pi}{4})$, then $K_2(0) = 0.6697$.

Choose $L = T = 10, \alpha = 0.5$, $g(s) = 1 - 0.025s$, initial distribution of population $p_0^3(s) = 3(9 - s)^2 cos^2(s + \frac{\pi}{4})$ and the mortality function as follows

$$
\mu(s, E) = \begin{cases} 3\sin^2(s + \frac{\pi}{3}) + 0.004E, & 0 \le s \le 8; \\ +\infty, & \text{otherwise.} \end{cases}
$$

Fertility function

$$
\beta(s, E) = \begin{cases} 0.5[\cos^2(4s) + 0.7], & 1 \le s \le 8; \\ 0, & \text{otherwise.} \end{cases}
$$

Harvesting effort function

$$u(s,E)=\begin{cases} u(s,E)=0.12(1-s)-0.002E, & 1\le s\le 8;\\ 0, & \text{otherwise.}\end{cases}$$

It can be obtained that $K_3(0) = 0.8906$, select initial distribution of population $p_0^4(s) = (8-s)^2[sin^2(s+\frac{\pi}{3})+0.8]$, then $K_4(0) = 0.9309$.

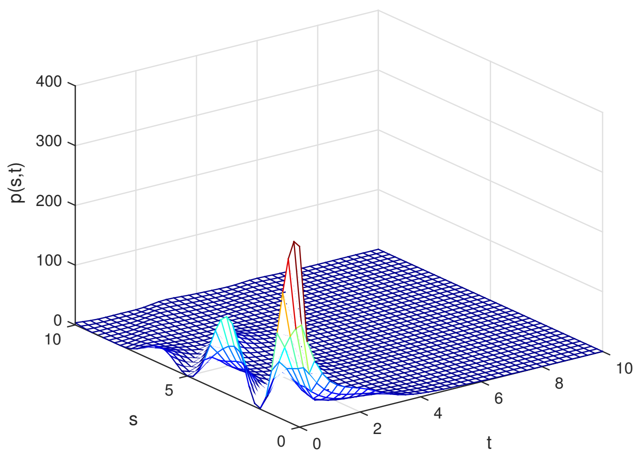

Figure 1. $K_1(0) = 0.1717$.

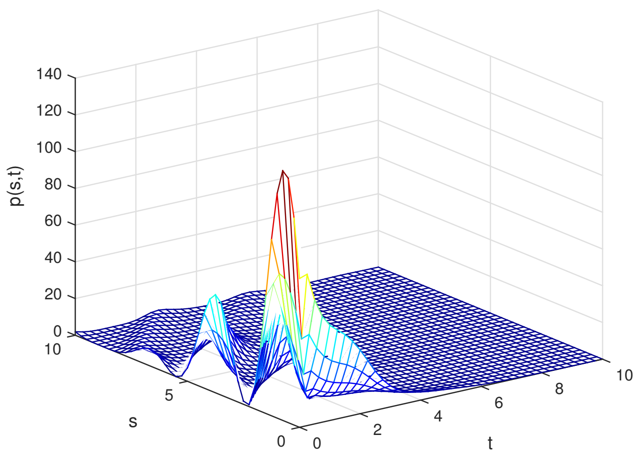

Figure 2. $K_2(0) = 0.6697$.

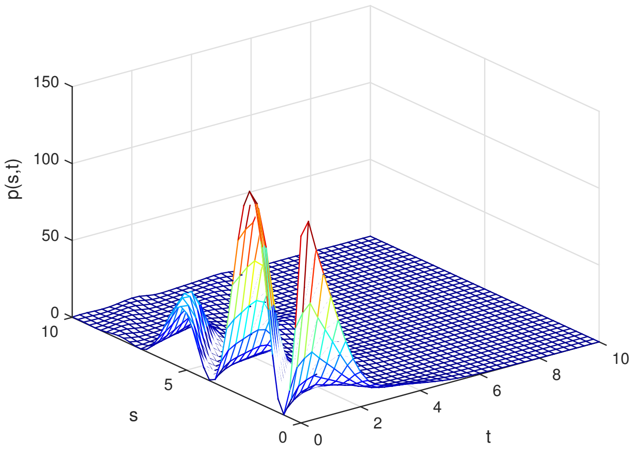

Figure 3. $K_3(0) = 0.8906$.

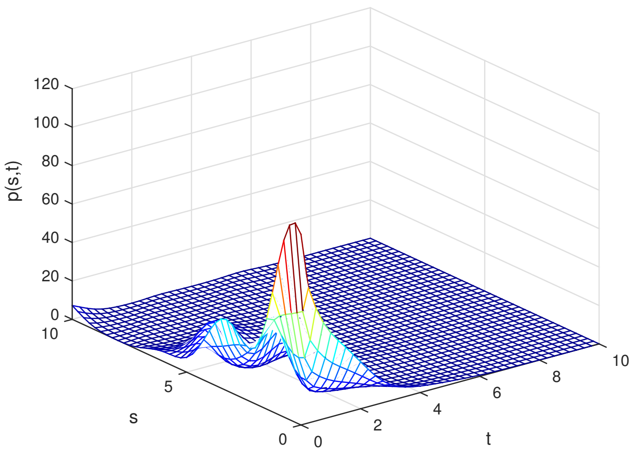

Figure 4. $K_4(0) = 0.9309$.

In these cases, $K_1(0)$ is close to 0, $K_2(0)$ belong to the middle of [0,1], $K_3(0)$ and $K_4(0)$ is close to 1. From Figures 1–4, it can be seen that the initial distribution of population $p_0(s)$ has multiple peaks. However, all the initial distribution of populations gradually approach zero as time goes by, moreover, $p_0(s)$ no longer fluctuates with time after tend to zero, it means that the zero equilibrium is stable.

Selected parameters as $L = T = 10, \alpha = 0.7$, growth rate $g(s) = 1 - 0.001s$, harvesting effort function $u(s, E) = 0.1(1 - s) - 0.35E$, initial distribution of population $p_0^5(s) = 0.5(10 - s)^2 \sin^2(s + \frac{\pi}{3})$, mortality function as follows

$$\mu(s, E) = \begin{cases} 0.35[\cos^2(s + \frac{\pi}{4}) + 1.003E + 1.1], & 0 \le s \le 8; \\ +\infty, & \text{otherwise.} \end{cases}$$

Fertility function

$$\beta(s,E)=\begin{cases}0.5[\cos^2(s+\frac{\pi}{4})+1.6], & 1\le s\le 8;\\ 0, & \text{otherwise.}\end{cases}$$

after calculation, obtain $K_5(0) = 1.1347$.

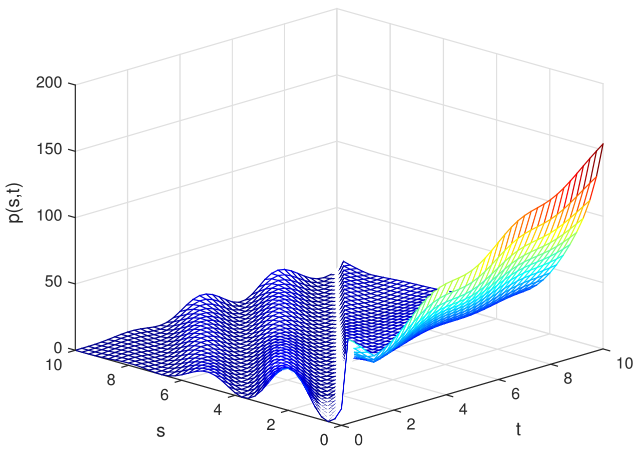

Figure 5. $K_5(0) = 1.1347$.

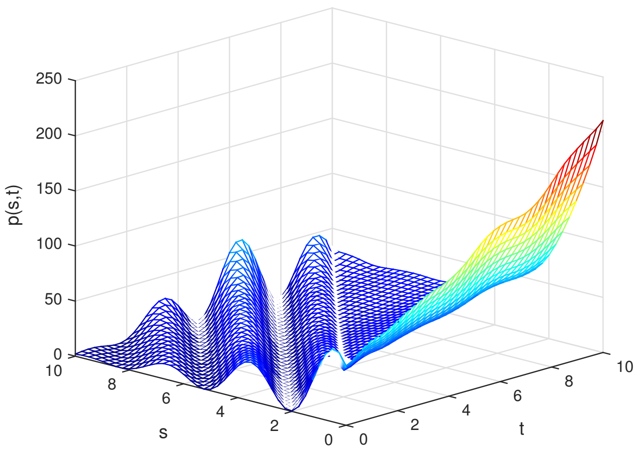

Figure 6. $K_6(0) = 4.4981$.

Choose $L = T = 10, \alpha = 0.3$, $g(s) = 1 - 0.001s$, initial distribution of population $p_0^6(s) = 0.5(12 - s)^2 \sin^2(s + \frac{\pi}{3})$, and the parameters $\beta(s, E)$, $\mu(s, E)$, $u(s, E)$ as follows

$$\beta(s,E) = 0.48[\sin^2(3s + \frac{\pi}{4}) + 0.9];$$

$$\mu(s, E) = 0.2[\sin^2(s) + 1.003E + 1.7];$$

$$u(s, E) = 0.1(1 - s) - 0.2E.$$

Then $K_6(0) = 4.4981$. The stability results of the zero equilibrium are shown by the figures.

In these cases, $K_5(0)$ is close to 1 and $K_6(0)$ is much larger than 1. From Figures 5–6, it can be seen that as time goes by the initial distribution of population $p_0(s)$ gradually deviate from the zero equilibrium surface, it means that $p_0(s)$ is in runaway state. Thus the zero equilibrium is unstable. The results of above figures are consistent with the conclusions in Theorem 4.1.

Acknowledgments

The research has obtained the support from the Priority Research and Projects for Ningxia in China (2020BEG03021), Science and Technique Research Foundation of Ningxia Institutions of Higher Education (NGY2020010), the Natural Science Foundation of Ningxia Province (2020AAC03058), the First-class Major Foundation of Ningxia Institutions of High Education in China (NXYLXK2021A03).

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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