



Research article

Analysis of some Katugampola fractional differential equations with fractional boundary conditions

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Abstract: In this work, some class of the fractional differential equations under fractional boundary conditions with the Katugampola derivative is considered. By proving the Lyapunov-type inequality, there are deduced the conditions for existence, and non-existence of the solutions to the considered boundary problem. Moreover, we present some examples to demonstrate the effectiveness and applications of the new results.

Keywords: fractional calculus; Lyapunov inequality; Green function; fractional differential equations; Katugampola derivative; eigenvalue of fractional differential equation

1. Introduction

The Lyapunov inequality, proved in 1907 by Russian mathematician Aleksandr Mikhailovich Lyapunov [1], is very useful in various problems related with oscillation theory, differential and difference equations and eigenvalue problems (see [2–7] and the references therein). The Lyapunov result states that, if a nontrivial solution to the following boundary value problem

$$\begin{cases} u''(t) + g(t)u(t) = 0, & a < t < b, \\ u(a) = u(b) = 0 \end{cases}$$

exists, where g is a continuous function, then the following inequality

$$\int_a^b |g(s)| ds > \frac{4}{b-a} \quad (1.1)$$

holds. This theorem formulates a necessary condition for the existence of solutions and allows to deduce sufficient conditions for non-existence of solutions to the considered boundary problem.

Recently, the research of Lyapunov-type inequality has gained more and more popularity, because by replacing the classical derivative with some fractional operators allows to model processes exhibiting memory effect. The first work in this direction is due to Ferreira [8] in 2013 by using the Riemann-Liouville derivative of order $\alpha \in (1, 2]$. In [9] was investigated the Lyapunov-type inequality for the boundary value problem with the Katugampola fractional derivative. In [10] was used the Caputo fractional derivative and in [11] the Hadamard fractional derivative.

Moreover, together with raising popularity of fractional operators, many modifications of the Lyapunov inequality appeared by changing the boundary conditions, for example $u(a) = u'(b) = 0$ or $u'(a) = u(b) = 0$ (see [12–14]).

Motivated by the above works, we consider in this paper the Katugampola fractional differential equation under boundary condition involving the Katugampola fractional derivative. We choose this special fractional derivative because it generalizes two other fractional operators, that is the Riemann-Liouville and the Hadamard fractional derivatives and classical derivative of integer order. More precisely, we consider the boundary value problem

$$\begin{cases} D_{a+}^{\alpha, \rho} u(t) + g(t)u(t) = 0 \\ u(a) = D_{a+}^{\beta, \rho} u(b) = 0 \end{cases} \quad (1.2)$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$ and $g : [a, b] \rightarrow \mathbb{R}$ is a continuous function. Thanks to the detailed analysis of the integral equation equivalent to (1.2) we are able to obtain a corresponding Lyapunov-type inequality. After that, we show some applications to present the effectiveness of the new Lyapunov-type inequality. We deduce some existence and non-existence results for the considered problem (1.2) which are very helpful for other researchers in this field. Furthermore, at the end of the article there will be graphs illustrating the applications of the proven theorems.

2. Preliminaries

In this section, we introduce the definitions and properties of the Katugampola fractional operators which are needed to prove the main results. For more details, we refer to [15–17].

Definition 2.1. Let $\alpha > 0$, $\rho > 0$, $-\infty < a < b < \infty$. The operators

$$I_{a+}^{\alpha, \rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau,$$

$$I_{b-}^{\alpha, \rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\alpha}} f(\tau) d\tau,$$

for $t \in (a, b)$ are called the left-sided and right-sided Katugampola integrals of fractional order α , respectively. The operators $I_{a+}^{\alpha, \rho} f$ and $I_{b-}^{\alpha, \rho} f$ are defined for $f \in L^p(a, b)$, $p \geq 1$.

Definition 2.2. Let $\alpha > 0$, $\rho > 0$, $n = [\alpha] + 1$, $0 < a < t < b \leq \infty$. The operators

$$D_{a+}^{\alpha, \rho} f(t) = \left(t^{1-\rho} \frac{d}{dt} \right)^n I_{a+}^{n-\alpha, \rho} f(t)$$

$$D_{b+}^{\alpha, \rho} f(t) = \left(-t^{1-\rho} \frac{d}{dt} \right)^n I_{b-}^{n-\alpha, \rho} f(t)$$

for $t \in (a, b)$ are called the left-sided and right-sided Katugampola derivatives of fractional order α , respectively.

The Katugampola derivative generalizes two other fractional operators, by introducing a new parameter $\rho > 0$ in the definition. Indeed, if we take $\rho \rightarrow 1$, we have the Riemann-Liouville fractional derivative, i.e.,

$$\lim_{\rho \rightarrow 1} D_{a+}^{\alpha, \rho} f(t) = \left(\frac{d}{dt} \right)^n \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau.$$

Moreover, if we take $\rho \rightarrow 0^+$, we get the Hadamard fractional derivative, i.e.,

$$\lim_{\rho \rightarrow 0^+} D_{a+}^{\alpha, \rho} f(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt} \right)^n \int_a^t \left(\log \frac{t}{\tau} \right)^{n - \alpha - 1} f(\tau) \frac{d\tau}{\tau}.$$

Example 2.3. ([9]) For $\rho > 0$, $\alpha > 0$, $\lambda > \alpha - 1$, we have

$$D_{a+}^{\alpha, \rho} \left(\frac{t^\rho - a^\rho}{\rho} \right)^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - \alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\lambda - \alpha},$$

The higher order Katugampola fractional operators satisfy the following properties, which were precisely discussed and proven in [9, 16, 17].

Lemma 2.4. ([16]) Let $\alpha > 0$, $\rho > 0$ and $f \in L[a, b]$ then

$$D_{a+}^{\alpha, \rho} I_{a+}^{\alpha, \rho} f(t) = f(t).$$

Lemma 2.5. ([16]) Let $\alpha > \beta > 0$, $\rho > 0$ and $f \in L[a, b]$ then

$$D_{a+}^{\beta, \rho} I_{a+}^{\alpha, \rho} f(t) = I_{a+}^{\alpha - \beta, \rho} f(t), \quad a < t < b.$$

Lemma 2.6. ([17]) Let $n - 1 < \alpha < n$, $n \in \mathbb{N}$, $\rho > 0$ and $f \in L[a, b]$ then

$$I_{a+}^{\alpha, \rho} D_{a+}^{\alpha, \rho} f(t) = f(t) + \sum_{i=0}^{n-1} \tilde{c}_i \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i - n + \alpha}$$

where \tilde{c}_i are real constants.

It is worth to mention that the complex formula for the Katugampola operator is established in [18].

3. Main result

We start with writing problem (1.2) in its equivalent integral form.

Theorem 3.1. Function $u \in C[a, b]$ is a solution to the boundary value problem (1.2) if and only if u is a solution to the integral equation

$$u(t) = \int_a^b G(t, s) g(s) u(s) ds, \quad (3.1)$$

where the Green function G is given by

$$G(t, s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \begin{cases} (t^\rho - a^\rho)^{\alpha-1} \left(\frac{b^\rho - a^\rho}{b^\rho - s^\rho} \right)^{\beta-\alpha+1}, & a \leq t \leq s \leq b \\ (t^\rho - a^\rho)^{\alpha-1} \left(\frac{b^\rho - a^\rho}{b^\rho - s^\rho} \right)^{\beta-\alpha+1} - (t^\rho - s^\rho)^{\alpha-1}, & a \leq s < t \leq b. \end{cases} \quad (3.2)$$

Proof. Integrating equation from (1.2) and using Lemma 2.6 we obtain that general solution is of the form

$$u(t) = c_1 \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + c_2 \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-2} - I_{a+}^{\alpha,\rho} [g(t)u(t)], \quad (3.3)$$

where c_1 and c_2 are some real constants. Since $u(a) = 0$, we get $c_2 = 0$. Moreover, differentiating (3.3) in Katugampola sense with $c_2 = 0$, we have

$$D_{a+}^{\beta,\rho} u(t) = c_1 D_{a+}^{\beta,\rho} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} - D_{a+}^{\beta,\rho} I_{a+}^{\alpha,\rho} [g(t)u(t)].$$

By Example 2.3 and Lemma 2.5 we obtain

$$D_{a+}^{\beta,\rho} u(t) = \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha - \beta)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-\beta-1} - \frac{\rho^{\beta-\alpha+1}}{\Gamma(\alpha - \beta)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{\beta-\alpha+1}} g(\tau)u(\tau) d\tau.$$

Since $D_{a+}^{\beta,\rho} u(b) = 0$, we get

$$c_1 = \frac{(b^\rho - a^\rho)^{\beta-\alpha+1}}{\Gamma(\alpha)} \int_a^b \frac{\tau^{\rho-1}}{(b^\rho - \tau^\rho)^{\beta-\alpha+1}} g(\tau)u(\tau) d\tau.$$

Therefore,

$$\begin{aligned} u(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left[\int_a^t \left((t^\rho - a^\rho)^{\alpha-1} \left(\frac{b^\rho - a^\rho}{b^\rho - \tau^\rho} \right)^{\beta-\alpha+1} - (t^\rho - \tau^\rho)^{\alpha-1} \right) \tau^{\rho-1} g(\tau)u(\tau) d\tau \right. \\ &\quad \left. + \int_t^b (t^\rho - a^\rho)^{\alpha-1} \left(\frac{b^\rho - a^\rho}{b^\rho - \tau^\rho} \right)^{\beta-\alpha+1} \tau^{\rho-1} g(\tau)u(\tau) d\tau \right] \end{aligned}$$

which ends the proof. \square

The below theorem present the properties of the Green function G obtained in (3.2)

Theorem 3.2. Let $0 < a < b < \infty$, $\alpha \in (1, 2]$, $\beta \in (0, 1]$, $\alpha > \beta + 1$ and $\rho > 0$. The function G given by (3.2) satisfies the following estimates

- (i) $G(t, s) \geq 0$, $t, s \in [a, b]$,
- (ii) $\max_{t \in [a, b]} G(t, s) = G(s, s) \leq \frac{4^\beta \max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)} \left(\frac{b^\rho - a^\rho}{4\rho} \right)^{\alpha-1}$, $s \in [a, b]$.

Proof. First we prove the positivity of function G . For $t \leq s$ it is obvious, but for $s < t$ we can rewrite

function G in the form

$$\begin{aligned}
 G(t, s) &= \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \frac{(t^\rho - a^\rho)^{\alpha-1}}{(b^\rho - a^\rho)^{\alpha-\beta-1} (b^\rho - s^\rho)^\beta} \\
 &\quad \times \left[(b^\rho - s^\rho)^{\alpha-1} - \left(\frac{b^\rho - s^\rho}{b^\rho - a^\rho} \right)^\beta \left(\frac{(b^\rho - a^\rho)(t^\rho - s^\rho)}{t^\rho - a^\rho} \right)^{\alpha-1} \right] \\
 &\geq \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \frac{(t^\rho - a^\rho)^{\alpha-1}}{(b^\rho - a^\rho)^{\alpha-\beta-1} (b^\rho - s^\rho)^\beta} \\
 &\quad \times \left[(b^\rho - s^\rho)^{\alpha-1} - \left(\frac{(b^\rho - a^\rho)(t^\rho - s^\rho)}{t^\rho - a^\rho} \right)^{\alpha-1} \right] \\
 &= \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \frac{(t^\rho - a^\rho)^{\alpha-1}}{(b^\rho - a^\rho)^{\alpha-\beta-1} (b^\rho - s^\rho)^\beta} \\
 &\quad \times \left[(b^\rho - s^\rho)^{\alpha-1} - \left(b^\rho - \left(a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \right) \right)^{\alpha-1} \right].
 \end{aligned}$$

Let us see that there is the following estimation

$$s^\rho \leq a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \leq b^\rho,$$

because

$$\frac{(s^\rho - a^\rho)(b^\rho - t^\rho)}{t^\rho - a^\rho} \geq 0 \quad \text{and} \quad \frac{(b^\rho - a^\rho)(s^\rho - t^\rho)}{t^\rho - a^\rho} \leq 0.$$

Thus the function G is positive also for $s < t$.

(ii) Now, we prove that $G(t, s) \leq G(s, s)$. Firstly, we consider the interval $a \leq t \leq s \leq b$. Differentiating G with respect to t we have

$$\frac{\partial G}{\partial t} = \frac{\rho^{2-\alpha}(\alpha-1)}{\Gamma(\alpha)} s^{\rho-1} t^{\rho-1} \left(\frac{b^\rho - a^\rho}{b^\rho - s^\rho} \right)^{\beta-\alpha+1} (t^\rho - a^\rho)^{\alpha-2} \geq 0.$$

Therefore,

$$G(t, s) \leq G(s, s), \quad \text{for } t \leq s, \quad s \in [a, b], \quad (3.4)$$

because the function G with respect to t is increasing on the considered interval.

Now, let we take the interval $a \leq s < t \leq b$. Taking the derivative of function G with respect to t ,

we obtain

$$\begin{aligned}
 \frac{\partial G}{\partial t} &= \\
 &= \frac{\rho^{2-\alpha}(\alpha-1)s^{\rho-1}t^{\rho-1}}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-2} \left[\frac{(b^\rho - s^\rho)^{\alpha-\beta-1}}{(b^\rho - a^\rho)^{1-\beta}} - \left(\frac{b^\rho - a^\rho}{t^\rho - a^\rho}\right)^{\alpha-2} (t^\rho - s^\rho)^{\alpha-2} \right] \\
 &\leq \frac{\rho^{2-\alpha}(\alpha-1)}{\Gamma(\alpha)} s^{\rho-1} t^{\rho-1} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-2} \\
 &\quad \times \left[(b^\rho - a^\rho)^{\alpha-2} - \left(\frac{(b^\rho - a^\rho)(t^\rho - s^\rho)}{t^\rho - a^\rho}\right)^{\alpha-2} \right] \\
 &= \frac{\rho^{2-\alpha}(\alpha-1)}{\Gamma(\alpha)} s^{\rho-1} t^{\rho-1} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-2} \\
 &\quad \times \left[(b^\rho - a^\rho)^{\alpha-2} - \left(b^\rho - \left(a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \right) \right)^{\alpha-2} \right].
 \end{aligned}$$

Note that $\frac{\partial G}{\partial t} \leq 0$ because

$$a^\rho \leq a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \leq b^\rho.$$

Therefore,

$$G(t, s) \leq G(s, s), \quad \text{for } t > s, s \in [a, b], \quad (3.5)$$

because the function G is decreasing with respect to t on the considered interval. From (3.4) and (3.5) we get

$$\begin{aligned}
 G(t, s) &\leq G(s, s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)(b^\rho - a^\rho)^{\alpha-\beta-1}} \frac{(s^\rho - a^\rho)^{\alpha-1}}{(b^\rho - s^\rho)^{\beta-\alpha+1}} \\
 &\leq \frac{\rho^{1-\alpha} \max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)(b^\rho - a^\rho)^{\alpha-\beta-1}} [(s^\rho - a^\rho)(b^\rho - s^\rho)]^{\alpha-\beta-1} (s^\rho - a^\rho)^\beta \\
 &\leq \frac{\rho^{1-\alpha} \max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)(b^\rho - a^\rho)^{\alpha-2\beta-1}} [(s^\rho - a^\rho)(b^\rho - s^\rho)]^{\alpha-\beta-1}
 \end{aligned}$$

for $s \in [a, b], t \in [a, b]$. Now let us define a function f

$$f(s) = [(s^\rho - a^\rho)(b^\rho - s^\rho)]^{\alpha-\beta-1}, \quad s \in [a, b].$$

In order to find the maximum value of this function, we check the sign of the derivative on the interior (a, b) . We have

$$f'(s) = \rho(\alpha - \beta - 1)s^{\rho-1} [(s^\rho - a^\rho)(b^\rho - s^\rho)]^{\alpha-\beta-2} [b^\rho - 2s^\rho + a^\rho].$$

It follows that $f'(\hat{s}) = 0$ if and only if

$$\hat{s} = \left(\frac{a^\rho + b^\rho}{2} \right)^{\frac{1}{\rho}}.$$

It is easily seen that $f'(s) < 0$ for $\hat{s} < s$ and $f'(s) > 0$ for $\hat{s} > s$.

Therefore

$$\max_{s \in [a, b]} f(s) = f(\hat{s}) = \left[\left(\frac{b^\rho - a^\rho}{2} \right)^2 \right]^{\alpha - \beta - 1}.$$

It ends the proof. \square

We are ready to state and prove our main results in the Banach space $C[a, b]$ with the maximum norm $\|u\| = \max_{t \in [a, b]} |u(t)|$.

Theorem 3.3. *If a nontrivial continuous solution of the fractional boundary value problem (1.2) exists, where g is a real and continuous function and $\alpha > \beta + 1$, then*

$$\int_a^b |g(s)| ds \geq \frac{\Gamma(\alpha)}{4^\beta \max\{a^{\rho-1}, b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho} \right)^{\alpha-1}.$$

Proof. It follows from Theorem 3.1 that solution of the fractional boundary value problem (1.2) satisfies the integral equation (3.1). Thus

$$|u(t)| \leq \int_a^b |G(t, s)| |g(s)| |u(s)| ds, \quad t \in [a, b].$$

Using the estimation of the function G which was obtained in Theorem 3.2 we get

$$\|u\| \leq \frac{4^\beta \max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)} \left(\frac{b^\rho - a^\rho}{4\rho} \right)^{\alpha-1} \|u\| \int_a^b |g(s)| ds.$$

Thus, we have

$$\int_a^b |g(s)| ds \geq \frac{\Gamma(\alpha)}{4^\beta \max\{a^{\rho-1}, b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho} \right)^{\alpha-1}.$$

The proof is completed. \square

In particular, if we chose $\beta = 0$ in Theorem 3.3, we obtain the main theorem of the work [9]. Moreover, taking $\alpha = 2$, $\beta = 0$, and $\rho = 1$ we recover the classical Lyapunov's inequality (1.1).

Due to the fact, that the Katugampola derivative has an additional parameter ρ (which by taking $\rho \rightarrow 0^+$ reduces to the Hadamard fractional derivative and for parameter $\rho = 1$ become the Riemann-Liouville fractional derivative) we get the Lyapunov-type inequality for both the Riemann-Liouville derivative D_{a+}^α and the Hadamard derivative ${}^H D_{a+}^\alpha$.

Corollary 3.4. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned} D_{a+}^\alpha u(t) + g(t)u(t) &= 0, \quad a < t < b, \quad 1 < \alpha \leq 2 \\ u(a) = D_{a+}^\beta u(b) &= 0, \quad \beta < \alpha - 1 \end{aligned}$$

exists, where g is a real and continuous function, then

$$\int_a^b |g(s)| ds \geq \frac{\Gamma(\alpha)}{4^\beta} \left(\frac{4}{b-a} \right)^{\alpha-1}.$$

In particular, if we take, in Corollary 3.4, $\beta = 0$ we obtain the main result of the work [10] proved by Ferreiro.

Corollary 3.5. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned} {}^H D_{a+}^\alpha u(t) + g(t)u(t) &= 0, \quad a < t < b, \quad 1 < \alpha \leq 2 \\ u(a) &= {}^H D_{a+}^\beta u(b) = 0, \quad \beta < \alpha - 1 \end{aligned}$$

exists, where g is a real and continuous function, then

$$\int_a^b |g(s)| ds \geq \frac{\Gamma(\alpha)}{4^\beta \max\{a, b\}} \left(\frac{\ln \frac{b}{a}}{4} \right)^{1-\alpha}.$$

4. Applications and examples

In this section, we apply the results on the Lyapunov-type inequalities obtained previously to study the nonexistence of solutions for certain fractional boundary value problems.

Theorem 4.1. *If*

$$\int_a^b |g(s)| ds < \frac{\Gamma(\alpha)}{4^\beta \max\{a^{\rho-1}, b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho} \right)^{\alpha-1},$$

then (1.2) has no nontrivial solution.

The proof of this theorem is trivial and it is left for the reader.

Example 4.2. Let us consider the following boundary value problem

$$\begin{cases} D_{1+}^{\frac{3}{2}\rho} u(t) + \lambda t^{\rho-1} u(t) = 0, & 1 < t < 2 \\ u(1) = D_{1+}^{\frac{1}{4}\rho} u(2) = 0. \end{cases} \quad (4.1)$$

By Theorem 3.3, if the continuous solution to the problem (4.1) exists, then necessarily

$$|\lambda| \frac{2^\rho - 1}{\rho} > \min\{1, 2^{1-\rho}\} \sqrt{\frac{\pi\rho}{2^{\rho+1} - 2}}.$$

Note, that inequality depends on two parameters λ and ρ . Taking $\rho = 1$ we obtain $\lambda > 1.2534$, but for $\rho \rightarrow 0^+$ we have $\lambda > 2.1719$. In particular, for $\lambda = 1$, $\rho = 1$ and $\rho \rightarrow 0^+$ the solution to problem (4.1) does not exist. For $\lambda = 2$, the solution does not exist for $\rho \rightarrow 0^+$, but for $\rho = 1$ the solution to (4.1) may exist. Moreover, for $\lambda = 3$ the solution may exist for $\rho = 1$ and $\rho \rightarrow 0^+$.

Let us consider problem (1.2) with $g(t) = \lambda$

$$\begin{cases} D_{a^+}^{\alpha, \rho} u(t) + \lambda u(t) = 0, & a < t < b, & 1 < \alpha \leq 2 \\ u(a) = D_{a^+}^{\beta, \rho} u(b) = 0. \end{cases} \quad (4.2)$$

If problem (4.2) admits a nontrivial solution u_λ , we say that λ is an eigenvalue of problem (4.2). We have the following result which provides a lower bound of the eigenvalues of problem (4.2).

Corollary 4.3. *If λ is an eigenvalue of problem (4.2), then*

$$|\lambda| \geq \frac{\Gamma(\alpha)}{4^\beta(b-a) \max\{a^{\rho-1}, b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho} \right)^{\alpha-1}.$$

The proof follows immediately from Theorem 3.3.

Furthermore, if

$$|\lambda| < \frac{\Gamma(\alpha)}{4^\beta(b-a) \max\{a^{\rho-1}, b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho} \right)^{\alpha-1},$$

then λ is not an eigenvalue of problem (4.2).

Example 4.4. Let us consider problem (4.2) with $\beta = \frac{1}{4}$, $\rho = 1$. If

$$|\lambda| < C_\alpha(b-a) := \frac{\Gamma(\alpha)4^{\alpha-2}}{\sqrt{2}(b-a)^\alpha},$$

then the solution of the eigenvalue of problem (4.2) does not exist. The plot below shows the upper estimation of λ depending on order of considered equation, for which (4.2) does not have a solution. The red line is for cases $b-a=0.5$, and the blue line is for $b-a=1$.

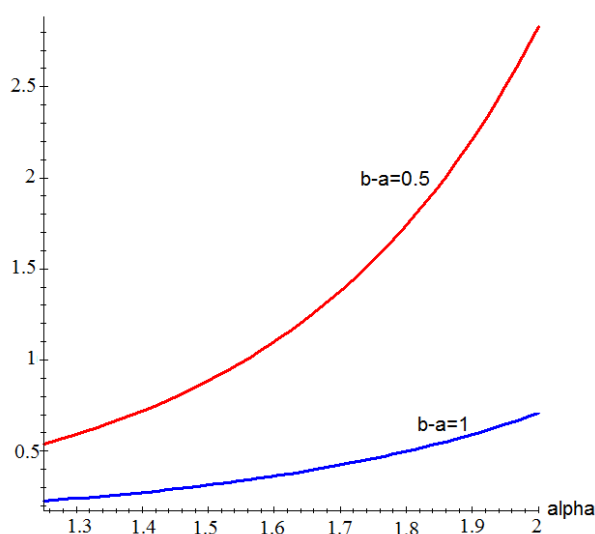


Figure 1. The function $C_\alpha(1)$ and $C_\alpha(0.5)$ for $\alpha \in (\frac{5}{4}, 2]$.

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Conflict of interest

The authors declare that they have no conflict of interest.

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