Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 18(4): 4390–4401.
DOI: 10.3934/mbe.2021221
Received: 20 February 2021
Accepted: 29 April 2021
Published: 20 May 2021

Research article

Oscillation behavior for neutral delay differential equations of second-order

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Abstract: In this paper, new criteria for oscillation of neutral delay differential equations of second-order are presented. One objective of this study is to complement and extend some well-known related results in the literature. To support our main results, we give illustrating examples.

Keywords: delay differential equations; oqscillation; neutral

1. Introduction

In this paper, we study the oscillatory behavior for neutral delay differential equations of second-order

$$\left(a(\xi)\vartheta'(\xi)\right)' + q(\xi) f(\varkappa(\sigma(\xi))) = 0, \tag{1.1}$$

where $\xi \geq \xi_0$ and

$$\vartheta(\xi) := \varkappa^{\gamma}(\xi) + p(\xi)\varkappa(\tau(\xi)).$$

Throughout this article, we assume:

(M1) γ is a quotient of odd positive integers and $\gamma \geq 1$;

(M2) $a \in C^1([\xi_0, \infty), (0, \infty))$, $p, q \in C([\xi_0, \infty), [0, \infty))$, $0 \le p(\xi) < 1$ and

$$\int_{\xi_0}^{\infty} \frac{1}{a(\rho)} d\rho = \infty;$$

(M3) $\tau, \sigma \in C([\xi_0, \infty), \mathbb{R})$, $\tau(\xi) \le \xi$, $\sigma(\xi) < \xi$ and $\lim_{\xi \to \infty} \tau(\xi) = \lim_{\xi \to \infty} \sigma(\xi) = \infty$;

(M4) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(\varkappa)/\varkappa^{\gamma} \ge k$ for $\varkappa \ne 0$ and a constant $k > 0$.

By a solution of Eq (1.1), we mean a nontrivial real-valued function $\varkappa \in C([\xi_{\varkappa}, \infty), \mathbb{R})$ with $\xi_{\varkappa} := \min\{\tau(\xi_{\varkappa}), \sigma(\xi_{\varkappa})\}$ for some $\xi_{\varkappa} \ge \xi_0$, which has the property $a\vartheta' \in C^1([\xi_0, \infty), \mathbb{R})$ and satisfies Eq (1.1) on $[\xi_0, \infty)$. We will consider only those solutions of Eq (1.1) which exist on some half-line $[\xi_{\varkappa}, \infty)$ and satisfy the condition

$$\sup\{|\varkappa(\xi)| : \xi_c \le \xi < \infty\} > 0 \text{ for any } \xi_c \ge \xi_{\varkappa}.$$

If y is either positive or negative, eventually, then y is called nonoscillatory; otherwise it is called oscillatory.

Due to the many applications for differential equations of the second-order in various problems of economics, biology, and physics, there is constant interest in obtaining new sufficient conditions for the oscillation or nonoscillation of the solutions of varietal types for differential equations. The development in the study of second-order delay differential equations with a canonical operator can be followed through the works [1–6], while the equations with a noncanonical operator [7–11]. The works [12–16] extended the results from the second-order to the higher-order delay differential equations .

For some related works, Baculikova and Dzurina [5] considered the oscillation of the neutral differential equation

$$(a(\xi)(\varkappa(\xi) + p(\xi)\varkappa(\tau(\xi)))')' + q(\xi)\varkappa(\sigma(\xi)) = 0,$$

under the condition

$$0 \le p(\xi) \le p_0 < \infty \text{ and } \tau \circ \sigma = \sigma \circ \tau. \tag{1.2}$$

Grace and Lalli [3] studied the oscillation of the equation

$$(a(\xi)(\varkappa(\xi) + p(\xi)\varkappa(\xi - \tau))')' + q(\xi)f(\varkappa(\xi - \tau)) = 0.$$

Dong [2], Liu and Bai [17] and Xu and Meng [18, 19] investigated the oscillation of equation

$$(a(\xi)(\varkappa(\xi) + p(\xi)\varkappa(\tau(\xi)))^{\gamma})' + q(\xi)\varkappa^{\beta}(\sigma(\xi)) = 0,$$

where $0 \le p(\xi) < 1$. Li and Han [20–22] considered the oscillation of the second-order neutral differential equation

$$(\varkappa(\xi) + p(\xi)\varkappa(\tau(\xi)))'' + q(\xi)\varkappa(\sigma(\xi)) = 0$$

for the case where Eq (1.2) holds. Recently, Moaaz [6] obtained sufficient conditions for the oscillation of neutral differential equations second order

$$(a(\xi)((\varkappa(\xi) + p(\xi)\varkappa(\tau(\xi))))^{\gamma})' + f(\xi, \varkappa(\sigma(\xi))) = 0,$$

where

$$\int_{\xi_0}^{\infty} \left(\frac{1}{a(\rho)}\right)^{1/\gamma} d\rho = \infty.$$

The objective of this paper is to establish new oscillation results for Eq (1.1). This paper is structured as follows: Firstly, by applying the theorems of comparison that compare the second-order equations with first-order delay equations, we establish a new criterion for oscillation of Eq (1.1). Secondly, we present new results for oscillation of Eq (1.1) by using the Riccati technique. Finally, some examples are considered to illustrate the main results.

2. Main results I

Throughout this paper, we will be employing the next notations:

$$\begin{aligned} U(\xi) &:= kq(\xi)(1-p(\sigma(\xi))), \\ \eta(\xi) &:= \int_{\xi_1}^{\xi} a^{-1}(\rho)\,\mathrm{d}\rho, \\ \widetilde{\eta}(\xi) &:= \eta(\xi) + \int_{\xi_1}^{\xi} \eta(\rho)\,\eta(\sigma(\rho))\,U(\rho)\,\mathrm{d}\rho \end{aligned}$$

and

$$\widehat{\eta}(\xi) := \exp\left(-\int_{\sigma(\xi)}^{\xi} \frac{\mathrm{d}u}{\widetilde{\eta}(u)\,a(u)}\right).$$

To prove the oscillation criteria, we need the next lemmas.

Lemma 2.1. *[1, Lemma 3] Let $\varkappa$ be a positive solution of Eq (1.1) on $[\xi_0, \infty)$, then there exists a $\xi_1 \geq \xi_0$ such that*

$$\vartheta(\xi) > 0,\ \vartheta'(\xi) > 0 \text{ and } \left(a(\xi)(\vartheta'(\xi))^{\gamma}\right)' \leq 0. \tag{2.1}$$

Proof. Assume that $\varkappa(\xi) > 0$ is a solution of Eq (1.1). From Eq (1.1), we get

$$(a(\xi)\,\vartheta'(\xi))' \leq -kq(\xi)\,\varkappa^{\gamma}(\sigma(\xi)) < 0$$

Therefore, $(a(\xi)\,\vartheta'(\xi))'$ is decreasing. Thus $\vartheta'(\xi) > 0$ or $\vartheta'(\xi) < 0$ for $\xi \geq \xi_1$. If $\vartheta'(\xi) < 0$, then there exists a constant c such that

$$\vartheta'(\xi) \leq -\frac{c}{a(\xi)} < 0$$

Integrating from ξ_1 to ξ, we have

$$\vartheta(\xi) \leq \vartheta(\xi_1) - c\int_{\xi_1}^{\xi} \frac{1}{a(s)}\mathrm{d}s \to -\infty \text{ as } \xi \to \infty$$

This is a contradiction and we conclude that $\vartheta'(\xi) > 0$.

Theorem 2.2. *If the first order delay differential equation*

$$w'(\xi) + U(\xi)\,\widetilde{\eta}(\sigma(\xi))\,w(\sigma(\xi)) = 0 \tag{2.2}$$

is oscillatory, then all solutions of Eq (1.1) are oscillatory.

Proof. Assume that Eq (1.1) has a non-oscillatory solution $\varkappa$ on $[\xi_0, \infty)$. Without loss of generality, we assume that there exists a $\xi_1 \geq \xi_0$ such that $\varkappa(\xi) > 0$, $\varkappa(\tau(\xi)) > 0$ and $\varkappa(\sigma(\xi)) > 0$ for $\xi \geq \xi_1$. By the definition of ϑ, using $\tau(\xi) \leq \xi$ and $\vartheta'(\xi) > 0$, we obtain, for $\xi \geq \xi_1$,

$$\begin{aligned} \varkappa^{\gamma}(\xi) &= \vartheta(\xi) - p(\xi)\varkappa(\tau(\xi)) \geq \vartheta(\xi) - p(\xi)\vartheta(\tau(\xi)) \\ &\geq (1 - p(\xi))\vartheta(\xi), \end{aligned}$$

which together with Eq (1.1) implies that

$$\begin{aligned} (a(\xi)\vartheta'(\xi))' &\leq -kq(\xi)(1 - p(\sigma(\xi)))\vartheta(\sigma(\xi)) \\ &\leq -U(\xi)\vartheta(\sigma(\xi)). \end{aligned} \tag{2.3}$$

From Lemma 2.1, we see that

$$\vartheta(\xi) = \vartheta(\xi_1) + \int_{\xi_1}^{\xi} \frac{1}{a(\rho)} a(\rho)\vartheta'(\rho)\,\mathrm{d}\rho \geq \eta(\xi)a(\xi)\vartheta'(\xi).$$

By simple computations, we see that

$$(\vartheta(\xi) - \eta(\xi)a(\xi)\vartheta'(\xi))' = -\eta(\xi)(a(\xi)\vartheta'(\xi))' \geq \eta(\xi)U(\xi)\vartheta(\sigma(\xi)). \tag{2.4}$$

Integrating Eq (2.4) from ξ_1 to ξ, we get

$$\vartheta(\xi) \geq \eta(\xi)a(\xi)\vartheta'(\xi) + \int_{\xi_1}^{\xi} \eta(\rho)U(\rho)\vartheta(\sigma(\rho))\,\mathrm{d}\rho.$$

Thus, from the fact that $(a(\xi)(\vartheta'(\xi))^{\gamma})' \leq 0$, we arrive at

$$\begin{aligned} \vartheta(\xi) &\geq \eta(\xi)a(\xi)\vartheta'(\xi) + \int_{\xi_1}^{\xi} \eta(\rho)U(\rho)\eta(\sigma(\rho))a(\sigma(\rho))\vartheta'(\sigma(\rho))\,\mathrm{d}\rho \\ &\geq \eta(\xi)a(\xi)\vartheta'(\xi) + \int_{\xi_1}^{\xi} \eta(\rho)U(\rho)\eta(\sigma(\rho))a(\rho)\vartheta'(\rho)\,\mathrm{d}\rho \\ &\geq a(\xi)\vartheta'(\xi)\left(\eta(\xi) + \int_{\xi_1}^{\xi} \eta(\rho)U(\rho)\eta(\sigma(\rho))\,\mathrm{d}\rho\right) \\ &\geq a(\xi)\vartheta'(\xi)\widetilde{\eta}(\xi). \end{aligned} \tag{2.5}$$

Next, we set $w(\xi) = a(\xi)\vartheta'(\xi)$. Using Eqs (2.3) and (2.5), we note that w be a positive solution of

$$w'(\xi) + U(\xi)\widetilde{\eta}(\sigma(\xi))w(\sigma(\xi)) \leq 0.$$

Using [25, Theorem1], we have that Eq (2.2) also has a positive solution, and so, we arrive at a contradiction. This ends the proof.

Corollary 1. *If*

$$\limsup_{\xi\to\infty} \int_{\sigma(\xi)}^{\xi} U(\rho)\widetilde{\eta}(\sigma(\rho))\,\mathrm{d}\rho > 1,\ \sigma \text{ is non-decreasing} \tag{2.6}$$

or

$$\liminf_{\xi\to\infty} \int_{\sigma(\xi)}^{\xi} U(\rho)\widetilde{\eta}(\sigma(\rho))\,\mathrm{d}\rho > \frac{1}{e}, \tag{2.7}$$

then all solutions of Eq (1.1) are oscillatory.

Proof. Using [23, Theorem 2.1.1], we note that the conditions Eqs (2.6) or (2.7) ensure oscillation of Eq (2.2). Thus, from Theorem 2.2, all solutions of Eq (1.1) are oscillatory.

Lemma 2.3. *[4, Lemma 4] Let Eq (1.1) has an eventually positive solution $\varkappa$. Suppose that σ is strictly increasing. Assume for some $\delta > 0$ that*

$$\liminf_{\xi\to\infty} \int_{\sigma(\xi)}^{\xi} U(\rho)\widetilde{\eta}(\sigma(u))\,\mathrm{d}u \geq \delta. \tag{2.8a}$$

Then

$$\frac{\omega(\sigma(\xi))}{\omega(\xi)} \geq \theta_n(\delta) \tag{2.9}$$

for every $n \geq 0$ and ξ large enough, where $w(\xi) := a(\xi)\vartheta'(\xi)$,

$$\theta_0(u) := 1 \text{ and } \theta_{n+1}(u) := \exp(u\theta_n(u)), \quad n = 0, 1, \ldots. \tag{2.10}$$

Theorem 2.4. *Assume that σ is strictly increasing and Eq (2.8a) holds for some $\delta > 0$. If there exists a function $\varphi \in C^1([\xi_0, \infty), (0, \infty))$ such that*

$$\limsup_{\xi\to\infty} \int_{\xi_1}^{\xi} \left(U(\rho)\varphi(\rho) - \frac{\left(\left(\varphi'(\rho)^2\right) a(\sigma(\rho))\right)}{4\varphi(\rho)\sigma'(\rho)\theta_n(\delta)} \right) \mathrm{d}\rho = \infty, \tag{2.11}$$

for sufficiently large $\xi \geq \xi_1$ and for some $n \geq 0$, where $\theta_n(\delta)$ is defined as in Eq (2.10) and $\varphi'_+(\xi) = \max\{0, \varphi'(\xi)\}$, then all solutions of Eq (1.1) are oscillatory.

Proof. Assume that there is a positive solution $\varkappa$ of Eq (1.1) on $[\xi_0, \infty)$. Thus, there is a $\xi_1 \geq \xi_0$ such that $\varkappa(\xi) > 0$, $\varkappa(\tau(\xi)) > 0$ and $\varkappa(\sigma(\xi)) > 0$ for $\xi \geq \xi_1$. It follows from Lemma 2.3 that

$$\frac{\vartheta(\sigma(\xi))}{\vartheta(\xi)} \geq \left(\frac{\theta_n(\delta) a(\xi)}{a(\sigma(\xi))} \right). \tag{2.12}$$

We define the function $\Phi(\xi)$ by

$$\Phi(\xi) := \varphi(\xi) a(\xi) \left(\frac{\vartheta'(\xi)}{\vartheta(\sigma(\xi))} \right). \tag{2.13}$$

Then, $\Phi(\xi) > 0$ for $\xi \geq \xi_1$. Differentiating Eq (2.13), we get

$$\Phi'(\xi) = \frac{\varphi'(\xi)}{\varphi(\xi)}\Phi(\xi) + \varphi(\xi)\frac{(a(\xi)\vartheta'(\xi))'}{\vartheta(\sigma(\xi))} - \varphi(\xi)\sigma'(\xi)a(\xi)\left(\frac{\vartheta'(\xi)}{\vartheta(\sigma(\xi))}\right)\left(\frac{\vartheta'(\sigma(\xi))}{\vartheta(\sigma(\xi))}\right).$$

From Eqs (2.3), (2.12) and (2.11), we obtain

$$\begin{aligned}
\Phi'(\xi) &\leq -\varphi(\xi)U(\xi) + \frac{\varphi'_+(\xi)}{\varphi(\xi)}\Phi(\xi) - \left(\frac{\sigma'(\xi)\theta_n(\delta)}{\varphi(\xi)a(\sigma(\xi))}\right)\Phi^2(\xi) \\
&\leq -\varphi(\xi)U(\xi) + \frac{(\varphi'(\rho))^2 a(\sigma(\rho))}{4\varphi(\rho)\sigma'(\rho)\theta_n(\delta)}.
\end{aligned}$$

Integrating this inequality from ξ_1 to ξ, we conclude

$$\limsup_{\xi\to\infty} \int_{\xi_1}^{\xi} \left(U(\rho)\varphi(\rho) - \frac{(\varphi'(\rho))^2 a(\sigma(\rho))}{4\varphi(\rho)\sigma'(\rho)\theta_n(\delta)} \right) \mathrm{d}\rho \leq \Phi(\xi_1),$$

which contradicts with Eq (2.11). This ends the proof.

Theorem 2.5. *Assume that there exists a function* $\phi \in C^1([\xi_0, \infty), (0, \infty))$ *such that*

$$\limsup_{\xi \to \infty} \int_{\xi_1}^{\xi} \left(\phi(\rho) U(\rho) \widehat{\eta}(\xi) - \frac{(\varphi'(\rho))^2 a(\rho)}{4\phi(\rho)} \right) d\rho = \infty, \tag{2.14}$$

for some sufficiently large $\xi \geq \xi_1$, *where* $\phi'_+(\xi) = \max\{0, \phi'(\xi)\}$. *Then all solutions of Eq (1.1) are oscillatory.*

Proof. Assume that there is a positive solution $\varkappa$ of Eq (1.1) on $[\xi_0, \infty)$. Thus, there is a $\xi_1 \geq \xi_0$ such that $\varkappa(\xi) > 0$, $\varkappa(\tau(\xi)) > 0$ and $\varkappa(\sigma(\xi)) > 0$ for $\xi \geq \xi_1$. From Lemma 2.1, we have that Eq (2.1) holds. As in the proof of Theorem 2.2, we arrive at Eq (2.5). From Eq (2.5), we have

$$\frac{\vartheta'(\xi)}{\vartheta(\xi)} \leq \frac{1}{\widetilde{\eta}(\xi) a(\xi)}.$$

Integrating this inequality from $\sigma(\xi)$ to ξ, we get

$$\frac{\vartheta(\sigma(\xi))}{\vartheta(\xi)} \geq \exp\left(-\int_{\sigma(\xi)}^{\xi} \frac{du}{\widetilde{\eta}(u) a(u)} \right). \tag{2.15}$$

Combining Eqs(2.3) and (2.15), we have

$$\frac{(a(\xi)\vartheta'(\xi))'}{\vartheta(\xi)} \leq -U(\xi) \left(\frac{\vartheta(\sigma(\xi))}{\vartheta(\xi)} \right) \leq -U(\xi) \widehat{\eta}(\xi). \tag{2.16}$$

Define the function

$$\Psi(\xi) = \phi(\xi) a(\xi) \left(\frac{\vartheta'(\xi)}{\vartheta(\xi)} \right). \tag{2.17}$$

Then $\Psi(\xi) > 0$ for $\xi > \xi_{1.}$ Differentiating Eq (2.17), we arrive at

$$\Psi'(\xi) \leq \frac{(a(\xi)\vartheta'(\xi))'}{\vartheta(\xi)} \phi(\xi) - \frac{1}{\phi(\xi) a(\xi)} \Psi^2(\xi) + \frac{\phi'(\xi)}{\phi(\xi)} \Psi(\xi). \tag{2.18}$$

From Eqs (2.16), (2.17) and (2.18), we deduce that

$$\begin{aligned} \Psi'(\xi) &\leq -\phi(\xi) U(\xi) \widehat{\eta}(\xi) - \frac{1}{\phi(\xi) a(\xi)} \Psi^2(\xi) + \frac{\phi'_+(\xi)}{\phi(\xi)} \Psi(\xi) \\ &\leq -\phi(\xi) U(\xi) \widehat{\eta}(\xi) + \frac{(\phi'_+(\xi))^2 a(\xi)}{4\phi(\xi)}. \end{aligned}$$

Integrating this inequality from ξ_1 to ξ, we find

$$\limsup_{\xi \to \infty} \int_{\xi_1}^{\xi} \left(\phi(\rho) U(\rho) \widehat{\eta}(\xi) - \frac{\left(\varphi'(\rho)\right)^2 a(\rho)}{4\phi(\rho)} \right) d\rho \leq \Psi(\xi_1),$$

which contradicts with Eq (2.14). This ends the proof.

Theorem 2.6. *If*

$$\liminf_{\xi\to\infty}\frac{1}{\psi(\xi)}\int_{\xi}^{\infty}a^{-1}(u)\psi^{2}(u)\,\mathrm{d}u>\frac{1}{4}, \tag{2.19}$$

where

$$\psi(\xi):=\int_{\xi}^{\infty}U(u)\widehat{\eta}(u)\,\mathrm{d}u$$

then all solutions of Eq (1.1) are oscillatory.

Proof. Proceeding as in the proof of Theorem Eq (2.5), we arrive at Eq (2.18). Using Eq (2.18) with $\phi(\xi)=1$, we obtain

$$\Psi'(\xi)\le\frac{(a(\xi)\vartheta'(\xi))'}{\vartheta(\xi)}-\frac{1}{a(\xi)}\Psi^{2}(\xi)$$

Thus, we obtain

$$\Psi'(\xi)\le -U(\xi)\widehat{\eta}(\xi)-\frac{1}{a(\xi)}\Psi^{2}(\xi)<0. \tag{2.20}$$

By integrating Eq (2.20) from ξ to ρ, we get

$$\int_{\xi}^{\rho}U(u)\widehat{\eta}(u)\,\mathrm{d}u+\int_{\xi}^{\rho}a^{-1}(u)\Psi^{2}(u)\,\mathrm{d}u\le\Psi(\xi)-\Psi(\rho),$$

Since $\Psi>0$ and $\Psi'<0$, we see that $\lim_{\rho\to\infty}\Psi(\rho)=c\ge 0$. Thus the previous inequality becomes

$$\psi(\xi)+\int_{\xi}^{\infty}a^{-1}(u)\Psi^{2}(u)\,\mathrm{d}u\le\Psi(\xi),$$

Hence

$$1+\frac{1}{\psi(\xi)}\int_{\xi}^{\infty}a^{-1}(u)\psi^{2}(u)\left(\frac{\Psi(u)}{\psi(u)}\right)^{2}\mathrm{d}u\le\frac{\Psi(\xi)}{\psi(\xi)}, \tag{2.21}$$

Set

$$\delta:=\inf_{\xi\ge\xi_1}\frac{\Psi(\xi)}{\psi(\xi)}.$$

From Eq (2.21), $\delta\ge 1$. Taking Eqs (2.19) and (2.21) into account, we find $1+\frac{1}{4}\delta^{2}\le\delta$, which not possible with the permissible value $\delta\ge 1$. Thus, the proof is complete.

Example 2.7. Consider the differential equation

$$\left(\varkappa(\xi)+\frac{1}{2}\varkappa\left(\frac{\xi}{\mathrm{e}}\right)\right)''+\frac{q_0}{\xi^{2}}\varkappa\left(\frac{\xi}{\mathrm{e}}\right)=0, \tag{2.22}$$

where $\xi>0$. By apply Theorem 2.1 in [24] or Theorem 1 in [26], Eq (2.22) is oscillatory if $q_0>1.3591$. From Theorem 2.5, Eq (2.22) is oscillatory if $q_0>1.1425$. Thus, our results improves results in [24, 26].

3. Mian results II: improved criteria

$$Q(\xi) = \min\{q(\xi), q(\tau(\xi))\}.$$

$$0 \le p(\xi) \le p_0 < \infty.$$

Lemma 3.1. *[27] Let α be a ratios of two odd positive integers. Then*

$$Kv - Lv^{(\alpha+1)/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{K^{\alpha+1}}{L^\alpha},\ L > 0.$$

Theorem 3.2. *Assume that $\gamma = 1$, $a'(\xi) \ge 0$, $\sigma'(\xi) > 0$, $\sigma(\xi) \le \tau(\xi)$, $\tau' \ge \tau_0 > 0$ and $\sigma \circ \tau = \tau \circ \sigma$. Furthermore, Assume that there exists a function $\rho(\xi) \in C^1([\xi_0, \infty), (0, \infty))$, for all sufficiently large $\xi_1 \ge \xi_0$, there is a $\xi_2 > \xi_1$ such that*

$$\limsup_{\xi\to\infty} \int_{\xi_2}^{\xi} \left(k\rho(s) Q(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{1}{4} \frac{a(s)(\rho'_+(s))^2}{\rho(s)\sigma'(s)} \right) \mathrm{d}s = \infty, \tag{3.1}$$

where $\rho'_+(\xi) = \max\{0, \rho'(\xi)\}$. Then Eq (1.1) is oscillatory.

Proof. Assume that there is a positive solution $\varkappa$ of Eq (1.1) on $[\xi_0, \infty)$. Thus, there is a $\xi_1 \ge \xi_0$ such that $\varkappa(\xi) > 0$, $\varkappa(\tau(\xi)) > 0$ and $\varkappa(\sigma(\xi)) > 0$ for $\xi \ge \xi_1$.Now, from Eq (1.1), we obtain

$$\begin{aligned} 0 \ \ge \ & (a(\xi)\vartheta'(\xi))' + \frac{p_0}{\tau_0}(a(\tau(\xi))\vartheta'(\tau(\xi)))' + kq(\xi)\varkappa(\sigma(\xi)) \\ & + kp_0 q(\tau(\xi))\varkappa(\sigma(\tau(\xi))), \end{aligned}$$

which follows from $\sigma \circ \tau = \tau \circ \sigma$ that

$$(a(\xi)\vartheta'(\xi))' + \frac{p_0}{\tau_0}(a(\tau(\xi))\vartheta'(\tau(\xi)))' + kQ(\xi)\vartheta(\sigma(\xi)) \le 0. \tag{3.2}$$

Next, we define a function $\omega(\xi)$ by

$$\omega(\xi) = \rho(\xi) \frac{a(\xi)(\vartheta'(\xi))}{\vartheta(\sigma(\xi))}, \tag{3.3}$$

then $\omega(\xi) > 0$. Differentiating Eq (3.3) with respect to ξ, we have

$$\omega'(\xi) = \frac{\rho'(\xi)}{\rho(\xi)}\omega(\xi) + \rho(\xi)\frac{(a(\xi)(\vartheta'(\xi)))'}{\vartheta(\sigma(\xi))} - \rho(\xi)\frac{a(\xi)(\vartheta'(\xi))\vartheta'(\sigma(\xi))\sigma'(\xi)}{\vartheta^2(\sigma(\xi))}, \tag{3.4}$$

since $\vartheta''(\xi) \le 0$ and $\sigma(\xi) < \xi$, we get

$$\omega'(\xi) \le \frac{\rho'(\xi)}{\rho(\xi)}\omega(\xi) + \rho(\xi)\frac{(a(\xi)(\vartheta'(\xi)))'}{\vartheta(\sigma(\xi))} - \rho(\xi)\frac{a(\xi)(\vartheta'(\xi))^2\sigma'(\xi)}{\vartheta^2(\sigma(\xi))}. \tag{3.5}$$

It follows from Eqs (3.3) and (3.5) that

$$\omega'(\xi) \le \frac{\rho'(\xi)}{\rho(\xi)}\omega(\xi) + \rho(\xi)\frac{(a(\xi)(\vartheta'(\xi)))'}{\vartheta(\sigma(\xi))} - \frac{\sigma'(\xi)}{a(\xi)\rho(\xi)}\omega^2(\xi). \tag{3.6}$$

Similarly, define another function ψ by

$$\psi(\xi) = \rho(\xi)\frac{a(\tau(\xi))(\vartheta'(\tau(\xi)))}{\vartheta(\sigma(\xi))}, \tag{3.7}$$

then $\psi(\xi) > 0$. Differentiating Eq (3.7) with respect to ξ, we have

$$\psi'(\xi) = \frac{\rho'(\xi)}{\rho(\xi)}\psi(\xi) + \rho(\xi)\frac{(a(\tau(\xi))(\vartheta'(\tau(\xi))))'}{\vartheta(\sigma(\xi))} - \rho(\xi)\frac{a(\tau(\xi))(\vartheta'(\tau(\xi)))\vartheta'(\sigma(\xi))\sigma'(\xi)}{\vartheta^2(\sigma(\xi))}, \tag{3.8}$$

since $\vartheta''(\xi) \leq 0$ and $\sigma(\xi) < \tau(\xi)$, we get

$$\psi'(\xi) \leq \frac{\rho'(\xi)}{\rho(\xi)}\psi(\xi) + \rho(\xi)\frac{(a(\tau(\xi))(\vartheta'(\tau(\xi))))'}{\vartheta(\sigma(\xi))} - \rho(\xi)\frac{a(\tau(\xi))(\vartheta'(\tau(\xi)))^2\sigma'(\xi)}{\vartheta^2(\sigma(\xi))}. \tag{3.9}$$

It follows from Eqs (3.7) and (3.9) that

$$\psi'(\xi) \leq \frac{\rho'(\xi)}{\rho(\xi)}\psi(\xi) + \rho(\xi)\frac{(a(\tau(\xi))(\vartheta'(\tau(\xi))))'}{\vartheta(\sigma(\xi))} - \frac{\sigma'(\xi)}{\rho(\xi)a(\xi)}\psi^2(\xi). \tag{3.10}$$

Multiplying Eq (3.10) by p_0/τ_0 and combining it with Eq (3.6), we get

$$\begin{aligned} \omega'(\xi) + \frac{p_0}{\tau_0}\psi'(\xi) \quad &\leq \quad \rho(\xi)\left(\frac{(a(\xi)(\vartheta'(\xi)))'}{\vartheta(\sigma(\xi))} + \frac{p_0}{\tau_0}\frac{(a(\tau(\xi))(\vartheta'(\tau(\xi))))'}{\vartheta(\sigma(\xi))}\right) \\ &\quad + \frac{\rho'_+(\xi)}{\rho(\xi)}\omega(\xi) - \frac{\sigma'(\xi)}{a(\xi)\rho(\xi)}\omega^2(\xi) \\ &\quad + \frac{p_0}{\tau_0}\left(\frac{\rho'_+(\xi)}{\rho(\xi)}\psi(\xi) - \frac{\sigma'(\xi)}{\rho(\xi)a(\xi)}\psi^2(\xi)\right). \end{aligned}$$

From Eq (3.2), we obtain

$$\begin{aligned} \omega'(\xi) + \frac{p_0}{\tau_0}\psi'(\xi) \quad &\leq \quad -k\rho(\xi)Q(\xi) + \frac{\rho'_+(\xi)}{\rho(\xi)}\omega(\xi) - \frac{\sigma'(\xi)}{a(\xi)\rho(\xi)}\omega^2(\xi) \\ &\quad + \frac{p_0}{\tau_0}\left(\frac{\rho'_+(\xi)}{\rho(\xi)}\psi(\xi) - \frac{\sigma'(\xi)}{\rho(\xi)a(\xi)}\psi^2(\xi)\right). \end{aligned} \tag{3.11}$$

From Lemma 3.1, Eq (3.11), becomes

$$\omega'(\xi) + \frac{p_0}{\tau_0}\psi'(\xi) \leq -k\rho(\xi)Q(\xi) + \frac{1}{4}\frac{(\rho'_+(\xi))^2 a(\xi)}{\rho(\xi)\sigma'(\xi)} + \frac{p_0}{\tau_0}\frac{1}{4}\frac{(\rho'_+(\xi))^2 a(\xi)}{\rho(\xi)\sigma'(\xi)} \tag{3.12}$$

integrating Eq (3.12) from ξ_2 $(\xi_2 \geq \xi_1)$ to ξ, we get

$$\int_{\xi_2}^{\xi}\left(k\rho(s)Q(s) - \left(1 + \frac{p_0}{\tau_0}\right)\frac{1}{4}\frac{a(s)(\rho'_+(s))^2}{\rho(s)\sigma'(s)}\right)\mathrm{d}s \leq \omega'(\xi_2) + \frac{p_0}{\tau_0}\psi'(\xi_2),$$

which contradicts Eq (3.1). This ends the proof.

Example 3.3. Consider the differential equation

$$\left(\varkappa(\xi)+2\varkappa\left(\frac{\xi}{\mathrm{e}}\right)\right)''+\frac{q_0}{\xi^2}\varkappa\left(\frac{\xi}{\mathrm{e}}\right)=0, \tag{3.13}$$

where $\gamma = 1$ and $q_0 > 0$. We note that $a'(\xi) \geq 0$, $p(\xi) = 2$, $\sigma'(\xi) = 1/\mathrm{e} > 0$, $\sigma(\xi) = \tau(\xi) = \xi/\mathrm{e}$, $q(\xi) = q_0/\xi^2$, $\tau_0 = 1/\mathrm{e} > 0$ and $\sigma \circ \tau = \tau \circ \sigma = \xi/\mathrm{e}^2$. It's easy to verify that

$$Q(\xi) = q_0/\xi^2.$$

By choosing $\rho(\xi) = \xi^2$, the condition Eq (3.1) is satisfied if $q_0 > 17.496$.
Thus, from Theorem 3.2, we see that Eq (3.13) is oscillatory if $q_0 > 17.496$.

4. Conclusions

In this paper, by different techniques and criteria, the oscillatory behavior of a class of second-order neutral delay differential equations has been studied. The results obtained are an extension and supplement to the relevant results in the literature. It is interesting to extend the results in this paper to Emden-Fowler delay differential equations with a sublinear neutral term.

Acknowledgments

The authors present their sincere thanks to the editors and two anonymous referees.

Funding

National Natural Science Foundation of China (No. 71601072), Key Scientific Research Project of Higher Education Institutions in Henan Province of China (No. 20B110006) and the Fundamental Research Funds for the Universities of Henan Province (No. NSFRF210314).

Conflict of interest

There are no competing interests.

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