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Research article

Flip bifurcation of a discrete predator-prey model with modified Leslie-Gower and Holling-type III schemes

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Abstract: The continuous predator-prey model is one of the main models studied in recent years. The dynamical properties of these models are so complex that it is an urgent topic to be studied. In this paper, we transformed a continuous predator-prey model with modified Leslie-Gower and Holling-type III schemes into a discrete mode by using Euler approximation method. The existence and stability of fixed points for this discrete model were investigated. Flip bifurcation analyses of this discrete model was carried out and corresponding bifurcation conditions were obtained. Provided with these bifurcation conditions, an example was given to carry out numerical simulations, which shows that the discrete model undergoes flip bifurcation around the stable fixed point. In addition, compared with previous studies on the continuous predator-prey model, our discrete model shows more irregular and complex dynamic characteristics. The present research can be regarded as the continuation and development of the former studies.

Keywords: flip bifurcation; Euler approximation method; discrete predator-prey model; stability; center manifold theorem

1. Introduction

In biological systems, the continuous predator-prey model has been successfully investigated and many interesting results have been obtained (cf. [1–9] and the references therein). Moreover, based on the continuous predator-prey model, many human factors, such as time delay [10–12], impulsive effect [13–20], Markov Switching [21], are considered. The existing researches mainly focus on stability, periodic solution, persistence, extinction and boundedness [22–28].

In 2011, the authors [28] considered the system incorporating a modified version of Leslie-Gower functional response as well as that of the Holling-type III:

$$\begin{cases} \dot{x}(t) = x(a_1 - bx - \frac{c_1 y^2}{x^2+k_1}), \\ \dot{y}(t) = y(a_2 - \frac{c_2 y}{x+k_2}). \end{cases} \tag{1}$$

With the diffusion of the species being also taken into account, the authors [28] studied a reaction-diffusion predator-prey model, and gave the stability of this model.

In model (1) x represents a prey population, y represents a predator with population, a_1 and a_2 represent the growth rate of prey x and predator y respectively, constant b represents the strength of competition among individuals of prey x, c_1 measures the maximum value of the per capita reduction rate of prey x due to predator y, k_1 and k_2 represent the extent to which environment provides protection to x and to y respectively, c_2 admits a same meaning as c_1. All the constants $a_1, a_2, b, c_1, c_2, k_1, k_2$ are positive parameters.

However, provided with experimental and numerical researches, it has been obtained that bifurcation is a widespread phenomenon in biological systems, from simple enzyme reactions to complex ecosystems. In general, the bifurcation may put a population at a risk of extinction and thus hinder reproduction, so the bifurcation has always been regarded as a unfavorable phenomenon in biology [29]. This bifurcation phenomenon has attracted the attention of many mathematicians, so the research on bifurcation problem is more and more abundant [30–40].

Although the continuous predator-prey model has been successfully applied in many ways, its disadvantages are also obvious. It requires that the species studied should have continuous and overlapping generations. In fact, we have noticed that many species do not have these characteristics, such as salmon, which have an annual spawning season and are born at the same time each year. For the population with non-overlapping generation characteristics, the discrete time model is more practical than the continuous model [38], and discrete models can generate richer and more complex dynamic properties than continuous time models [39]. In addition, since many continuous models cannot be solved by symbolic calculation, people usually use difference equations for approximation and then use numerical methods to solve the continuous model.

In view of the above discussion, the study of discrete system is paid more and more attention by mathematicians. Many latest research works have focused on flip bifurcation for different models, such as, discrete predator-prey model [41,42]; discrete reduced Lorenz system [43]; coupled thermoacoustic systems [44]; mathematical cardiac system [45]; chemostat model [46], etc.

For the above reasons, we will study from different perspectives in this paper, focusing on the discrete scheme of Eq (1).

In order to get a discrete form of Eq (1), we first let

$$u = \frac{b}{a_1}x, v = \frac{c_1}{a_1}y, \tau = a_1 t,$$

and rewrite u, v, τ as x, y, t, then (1) changes into:

$$\begin{cases} \dot{x}(t) = x(1 - x - \frac{\beta_1 y^2}{x^2+h_1}), \\ \dot{y}(t) = \alpha y(1 - \frac{\beta_2 y}{x+h_2}), \end{cases} \tag{2}$$

where $\beta_1 = \frac{b^2}{c_1 a_1}, h_1 = \frac{b^2 k_1}{a_1^2}, \alpha = \frac{a_2}{c_1}, \beta_2 = \frac{c_2 b}{c_1 a_2}, h_2 = \frac{b k_2}{a_1}$.

Next, we use Euler approximation method, i.e., let

$$\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{\triangle t}, \qquad \frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\triangle t},$$

where $\triangle t$ denotes a time step, x_n, y_n and x_{n+1}, y_{n+1} represent consecutive points. Provided with Euler approximation method with the time step $\triangle t = 1$, (2) changes into a two-dimensional discrete dynamical system:

$$\begin{cases} x_{n+1} = x_n + x_n(1 - x_n - \frac{\beta_1 y_n^2}{x_n^2 + h_1}), \\ y_{n+1} = y_n + \alpha y_n(1 - \frac{\beta_2 y_n}{x_n + h_2}). \end{cases} \tag{3}$$

For the sake of analysis, we rewrite (3) in the following map form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + x(1 - x - \dfrac{\beta_1 y^2}{x^2 + h_1}) \\ y + \alpha y(1 - \dfrac{\beta_2 y}{x + h_2}) \end{pmatrix}. \tag{4}$$

In this paper, we will consider the effect of the coefficients of map (4) on the dynamic behavior of the map (4). Our goal is to show how a flipped bifurcation of map (4) can appear under some certain conditions.

The remainder of the present paper is organized as follows. In section 2, we discuss the fixed points of map (4) including existence and stability. In section 3, we investigate the flip bifurcation at equilibria E_2 and E^*. It has been proved that map (4) can undergo the flip bifurcation provided with that some values of parameters be given certain. In section 4, we give an example to support the theoretical results of the present paper. As the conclusion, we make a brief discussion in section 5.

2. Existence and stability of fixed points

Obviously, $E_1(1, 0)$ and $E_2(0, \frac{h_2}{\beta_2})$ are fixed points of map (4). Given the biological significance of the system, we focus on the existence of an interior fixed point $E^*(x^*, y^*)$, where $x^* > 0, y^* > 0$ and satisfy

$$1 - x^* = \frac{\beta_1 (y^*)^2}{(x^*)^2 + h_1}, x^* + h_2 = \beta_2 y^*,$$

i.e., x^* is the positive root of the following cubic equation:

$$\beta_2^2 x^3 + (\beta_1 - \beta_2^2) x^2 + (\beta_2^2 h_1 + 2\beta_1 h_2) x + \beta_1 h_2^2 - \beta_2^2 h_1 = 0. \tag{5}$$

Based on the relationship between the roots and the coefficients of Eq (5), we have

Lemma 2.1 Assume that $\beta_1 h_2^2 - \beta_2^2 h_1 < 0$, then Eq (5) has least one positive root, and in particular

(i) a unique positive root, if $\beta_1 \geq \beta_2^2$;

(ii) three positive roots, if $\beta_1 < \beta_2^2$.

The proof of Lemma 2.1 is easy, and so it is omitted.

In order to study the stability of equilibria, we first give the Jacobian matrix $J(E)$ of map (4) at any a fixed point $E(x, y)$, which can be written as

$$J(E) = \begin{pmatrix} 2 - 2x - \frac{\beta_1 y^2(h_1 - x^2)}{(x^2+h_1)^2} & -\frac{2\beta_1 xy}{x^2+h_1} \\ \frac{\alpha\beta_2 y^2}{(x+h_2)^2} & 1 + \alpha - \frac{2\alpha\beta_2 y}{x+h_2} \end{pmatrix}.$$

For equilibria E_1, we have

$$J(E_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 + \alpha \end{pmatrix}.$$

The eigenvalues of $J(E_1)$ are $\lambda_1 = 0, \lambda_2 = 1 + \alpha$ with $\lambda_2 > 1$ due to the constant $\alpha > 0$, so $E_1(1, 0)$ is a saddle.

For equilibria E_2, note that

$$J(E_2) = \begin{pmatrix} 2 - \frac{\beta_1 h_2^2}{\beta_2^2 h_1} & 0 \\ \frac{\alpha}{\beta_2} & 1 - \alpha \end{pmatrix},$$

then the eigenvalues of $J(E_2)$ are $\lambda_1 = 2 - \frac{\beta_1 h_2^2}{\beta_2^2 h_1}, \lambda_2 = 1 - \alpha$, and so we get

Lemma 2.2 The fixed point $E_2(0, \frac{h_2}{\beta_2})$ is

(i) a sink if $1 < \frac{\beta_1 h_2^2}{\beta_2^2 h_1} < 3$ and $0 < \alpha < 2$;

(ii) a source if $\frac{\beta_1 h_2^2}{\beta_2^2 h_1} < 1$ or $\frac{\beta_1 h_2^2}{\beta_2^2 h_1} > 3$ and $\alpha > 2$;

(iii) a a saddle if $1 < \frac{\beta_1 h_2^2}{\beta_2^2 h_1} < 3$ and $\alpha > 2$, or, $\frac{\beta_1 h_2^2}{\beta_2^2 h_1} < 1$ or $\frac{\beta_1 h_2^2}{\beta_2^2 h_1} > 3$ and $0 < \alpha < 2$;

(iv) non-hyperbolic if $\frac{\beta_1 h_2^2}{\beta_2^2 h_1} = 1$ or $\frac{\beta_1 h_2^2}{\beta_2^2 h_1} = 3$ or $\alpha = 2$.

3. Flip bifurcation

In this section, we will use the relevant results of literature [38–40] to study the flip bifurcation at equilibria E_2 and E^*.

3.1. Flip bifurcation at equilibria E_2

Based on (iii) in Lemma 2.2, it is known that if $\alpha = 2$, the eigenvalues of $J(E_2)$ are: $\lambda_1 = 2 - \frac{\beta_1 h_2^2}{\beta_2^2 h_1}, \lambda_2 = -1$. Define

$$Fl = \{(\beta_1, \beta_2, h_1, h_2, \alpha) : \alpha = 2, \beta_1, \beta_2, h_1, h_2 > 0\}.$$

We conclude that a flip bifurcation at $E_2(0, \frac{h_2}{\beta_2})$ of map (4) can appear if the parameters vary in a small neighborhood of the set Fl.

To study the flip bifurcation, we take constant α as the bifurcation parameter, and transform $E_2(0, \frac{h_2}{\beta_2})$ into the origin. Let $e = 2 - \frac{\beta_1 h_2^2}{\beta_2^2 h_1}, \alpha_1 = \alpha - 2$, and

$$u(n) = x(n), v(n) = y(n) - \frac{h_2}{\beta_2},$$

then map (4) can be turned into

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} eu - u^2 - \frac{2\beta_1 h_2}{\beta_2 h_1} uv + O((|u| + |v| + |\alpha_1|)^3) \\ \frac{2}{\beta_2} u - v - \frac{2}{\beta_2 h_2} u^2 - \frac{2\beta_2}{h_2} v^2 + \frac{4}{h_2} uv + \frac{\alpha_1}{\beta_2} u - \alpha_1 v - \frac{\alpha_1}{\beta_2 h_2} u^2 \\ -\frac{\alpha_1 \beta_2}{h_2} v^2 + \frac{2\alpha_1}{h_2} uv + O((|u| + |v| + |\alpha_1|)^3) \end{pmatrix}. \tag{6}$$

Let

$$T_1 = \begin{pmatrix} 1+e & 0 \\ \frac{2}{\beta_2} & 1 \end{pmatrix},$$

then by the following invertible transformation:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T_1 \begin{pmatrix} s \\ w \end{pmatrix},$$

map (6) turns into

$$\begin{pmatrix} s \\ w \end{pmatrix} \mapsto \begin{pmatrix} es - (1+e)s^2 - \frac{2\beta_1 h_2}{\beta_2 h_1} s(\frac{2s}{\beta_2} + w) + O(|s| + |w| + |\alpha_1|)^3 \\ -w + F_2(s, w, \alpha_1) \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} F_2 = {} & \frac{2}{\beta_2}[(1+e)s^2 + \frac{2\beta_1 h_2}{\beta_2 h_1} s(\frac{2s}{\beta_2} + w)] - \frac{2}{\beta_2 h_2}(1+e)^2 s^2 - \frac{2\beta_2}{h_2}(\frac{2s}{\beta_2} + w)^2 + \frac{4(1+e)}{h_2} s(\frac{2s}{\beta_2} + w) \\ & + \frac{(1+e)\alpha_1}{\beta_2} s - \alpha_1(\frac{2s}{\beta_2} + w) - \frac{(1+e)^2 \alpha_1}{\beta_2 h_2} s^2 - \frac{\alpha_1 \beta_2}{h_2}(\frac{2s}{\beta_2} + w)^2 \\ & + \frac{2(1+e)\alpha_1}{h_2} s(\frac{2s}{\beta_2} + w) + O(|s| + |w| + |\alpha_1|)^3. \end{aligned}$$

Provided with the center manifold theorem (Theorem 7 in [40]), it can be obtained that there will exist a center manifold $W^c(0,0)$ for map (7), and the center manifold $W^c(0,0)$ can be approximated as:

$$W^c(0,0) = \{(w, s, \alpha_1) \in R^3 : s = aw^2 + bw\alpha_1 + c(\alpha_1)^2 + O(|w| + |\alpha_1|)^3\}.$$

As the center manifold satisfies:

$$\begin{aligned} s &= a(-w + F_2)^2 + b(-w + F_2)\alpha_1 + c(\alpha_1)^2 \\ &= e(aw^2 + bw\alpha_1 + c(\alpha_1)^2) - (1+e)(aw^2 + bw\alpha_1 + c(\alpha_1)^2)^2 \\ &\quad - \tfrac{2\beta_1 h_2}{\beta_2 h_1}(aw^2 + bw\alpha_1 + c(\alpha_1)^2)(\tfrac{2}{\beta_2}(aw^2 + bw\alpha_1 + c(\alpha_1)^2) + w) \\ &\quad + O(|s| + |w| + |\alpha_1|)^3, \end{aligned}$$

it can be obtained by comparing the coefficients of the above equality that $a = 0, b = 0, c = 0$, so the center manifold of map (7) at $E_2(0, \frac{h_2}{\beta_2})$ is $s = 0$. Then map (7) restricted to the center manifold turns into

$$w(n+1) = -w(n) - \alpha_1 w(n) - \frac{2\beta_2}{h_2} w^2(n) - \frac{\alpha_1 \beta_2}{h_2} w^2(n) + O(|w(n)| + |\alpha_1|)^3$$

$$\triangleq f(w, \alpha_1).$$

Obviously,

$$f_w(0,0) = -1, \quad f_{ww}(0,0) = -\frac{4\beta_2}{h_2},$$

so

$$\frac{(f_{ww}(0,0))^2}{2} + \frac{f_{www}(0,0)}{3} \neq 0, \quad f_{w\alpha_1}(0,0) = -1 \neq 0.$$

Therefore, Theorem 4.3 in [38] guarantees that map (3) undergoes a flip bifurcation at $E_2(0, \frac{h_2}{\beta_2})$. □

3.2. Flip bifurcation at equilibria E^*

Note that

$$J(E^*) = \begin{pmatrix} 2 - 2x^* - \frac{\beta_1 (y^*)^2 (h_1 - (x^*)^2)}{((x^*)^2 + h_1)^2} & -\frac{2\beta_1 x^* y^*}{(x^*)^2 + h_1} \\ \frac{\alpha}{\beta_2} & 1 - \alpha \end{pmatrix},$$

then the characteristic equation of Jacobian matrix $J(E^*)$ of map (3) at $E^*(x^*, y^*)$ is:

$$\lambda^2 - (1 + \alpha_0 - \alpha)\lambda + (1 - \alpha)\alpha_0 - \eta\alpha = 0, \tag{8}$$

where

$$\alpha_0 = 2 - 2x^* - \frac{\beta_1 (y^*)^2 (h_1 - (x^*)^2)}{((x^*)^2 + h_1)^2}, \eta = -\frac{2\beta_1 x^* y^*}{\beta_2((x^*)^2 + h_1)}.$$

Firstly, we discuss the stability of the fixed point $E^*(x^*, y^*)$. The stability results can be described as the the following Lemma, which can be easily proved by the relations between roots and coefficients of the characteristic Eq (8), so the proof has been omitted.

Lemma 3.1 The fixed point $E^*(x^*, y^*)$ is

(i) a sink if one of the following conditions holds.

(i.1) $0 < \alpha_0 + \eta < 1$, and $\frac{\alpha_0 - 1}{\alpha_0 + \eta} < \alpha < \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}$;

(i.2) $-1 < \alpha_0 + \eta < 0$, and $\alpha < \min\{\frac{\alpha_0 - 1}{\alpha_0 + \eta}, \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}\}$;

(i.3) $\alpha_0 + \eta < -1$, and $\frac{\alpha_0 - 1}{\alpha_0 + \eta} > \alpha > \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}$;

(ii) a source if one of the following conditions holds.

(ii.1) $0 < \alpha_0 + \eta < 1$, and $\alpha < \min\{\frac{\alpha_0 - 1}{\alpha_0 + \eta}, \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}\}$;

(ii.2) $-1 < \alpha_0 + \eta < 0$, and $\frac{\alpha_0 - 1}{\alpha_0 + \eta} < \alpha < \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}$;

(ii.3) $\alpha_0 + \eta < -1$, and $\alpha > \max\{\frac{\alpha_0 - 1}{\alpha_0 + \eta}, \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}\}$;

(iii) a saddle if one of the following conditions holds.

(iii.1) $-1 < \alpha_0 + \eta < 1$, and $\alpha > \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}$;

(iii.2) $\alpha_0 + \eta < -1$, and $\alpha < \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}$;

(iv) non-hyperbolic if one of the following conditions holds.

(iv.1) $\alpha_0 + \eta = 1$;

(iv.2) $\alpha_0 + \eta \neq -1$; and $\alpha = \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}$;

(iv.3) $\alpha_0 + \eta \neq 0, \alpha = \frac{\alpha_0 - 1}{\alpha_0 + \eta}$ and $(1 + \alpha_0 - \alpha)^2 < 4((1 - \alpha)\alpha_0 - \eta\alpha)$.

Then based on (iv.2) of Lemma 3.1 and $\alpha \neq 1+\alpha_0, 3+\alpha_0$, we get that one of the eigenvalues at $E^*(x^*, y^*)$ is -1 and the other satisfies $|\lambda| \neq 1$. For $\alpha, \beta_1, \beta_2, h_1, h_2 > 0$, let us define a set:

$$Fl = \{(\beta_1, \beta_2, h_1, h_2, \alpha) : \alpha = \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}, \alpha_0+\eta \neq -1, \alpha \neq 1+\alpha_0, 3+\alpha_0\}.$$

We assert that a flip bifurcation at $E^*(x^*, y^*)$ of map (3) can appear if the parameters vary in a small neighborhood of the set Fl.

To discuss flip bifurcation at $E^*(x^*, y^*)$ of map (3), we choose constant α as the bifurcation parameter and adopt the central manifold and bifurcation theory [38–40].

Let parameters $(\alpha_1, \beta_1, \beta_2, h_1, h_2) \in Fl$, and consider map (3) with $(\alpha_1, \beta_1, \beta_2, h_1, h_2)$, then map (3) can be described as

$$\begin{cases} x_{n+1} = x_n + x_n(1 - x_n - \frac{\beta_1 y_n^2}{x_n^2+h_1}), \\ y_{n+1} = y_n + \alpha_1 y_n(1 - \frac{\beta_2 y_n}{x_n+h_2}). \end{cases} \tag{9}$$

Obviously, map (9) has only a unique positive fixed point $E^*(x^*, y^*)$, and the eigenvalues are $\lambda_1 = -1, \lambda_2 = 2+\alpha_0-\alpha$, where $|\lambda_2| \neq 1$.

Note that $(\alpha_1, \beta_1, \beta_2, h_1, h_2) \in Fl$, then $\alpha_1 = \frac{2(1+\alpha_0)}{1+\alpha_0+\eta}$. Let $|\alpha^*|$ small enough, and consider the following perturbation of map (9) described by

$$\begin{cases} x_{n+1} = x_n + x_n(1 - x_n - \frac{\beta_1 y_n^2}{x_n^2+h_1}), \\ y_{n+1} = y_n + (\alpha_1+\alpha^*) y_n(1 - \frac{\beta_2 y_n}{x_n+h_2}), \end{cases} \tag{10}$$

with α^* be a perturbation parameter.

To transform $E^*(x^*, y^*)$ into the origin, we let $u = x - x^*, v = y - y^*$, then map (10) changes into

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} a_1u + a_2v + a_3u^2 + a_4uv + a_5v^2 + a_6u^3 + a_7u^2v \\ +a_8uv^2 + a_9v^3 + O((|u|+|v|)^4) \\ b_1u + b_2v + b_3u^2 + b_4uv + b_5v^2 + c_1u\alpha^* + c_2v\alpha^* + c_3u^2\alpha^* \\ +c_4uv\alpha^* + c_5v^2\alpha^* + b_6u^3 + b_7u^2v + b_8uv^2 + b_9v^3 \\ +O((|u|+|v|+|\alpha^*|)^4) \end{pmatrix}, \tag{11}$$

where

$a_1 = 2 - 2x^* - \beta_1(y^*)^2 f(0) - \beta_1 x^*(y^*)^2 f'(0)$; $a_2 = -2\beta_1 x^* y^* f(0)$;
$a_3 = -1 - \beta_1(y^*)^2 f'(0) - \frac{1}{2}\beta_1 x^*(y^*)^2 f''(0)$; $a_4 = -2\beta_1 y^* f(0) - 2\beta_1 x^* y^* f'(0)$;
$a_5 = -\beta_1 x^* f(0)$; $a_6 = -\frac{1}{2}\beta_1(y^*)^2 f''(0) - \frac{1}{6}\beta_1 x^*(y^*)^2 f'''(0)$;
$a_7 = -\beta_1 x^* y^* f''(0) - 2\beta_1 y^* f'(0)$; $a_8 = -\beta_1 f(0) - \beta_1 x^* f'(0)$, $a_9 = 0$;
$f(0) = \frac{1}{(x^*)^2+h_1}, f'(0) = \frac{-2x^*}{[(x^*)^2+h_1]^2}, f''(0) = \frac{6(x^*)^2-2h_1}{[(x^*)^2+h_1]^3}, f'''(0) = \frac{24x^*(h_1-(x^*)^2)}{[(x^*)^2+h_1]^4}$.
$b_1 = \frac{\alpha_1\beta_2(y^*)^2}{(x^*+h_2)^2}$; $b_2 = 1 + \alpha_1 - \frac{\alpha_1\beta_2 y^*}{x^*+h_2}$; $b_3 = -\frac{\alpha_1\beta_2(y^*)^2}{(x^*+h_2)^3}$; $b_4 = \frac{2\alpha_1\beta_2 y^*}{(x^*+h_2)^2}$;
$b_5 = -\frac{\alpha_1\beta_2}{x^*+h_2}$; $c_1 = \frac{\beta_2(y^*)^2}{(x^*+h_2)^2}$; $c_2 = 1 - \frac{\beta_2 y^*}{x^*+h_2}$; $c_3 = -\frac{\beta_2(y^*)^2}{(x^*+h_2)^3}$;
$c_4 = \frac{2\beta_2 y^*}{(x^*+h_2)^2}$; $c_5 = -\frac{\beta_2}{x^*+h_2}$; $b_6 = \frac{\alpha_1\beta_2(y^*)^2}{(x^*+h_2)^4}$; $b_7 = -\frac{2\alpha_1\beta_2 y^*}{(x^*+h_2)^3}$;

$$b_8 = \frac{\alpha_1\beta_2}{(x^*+h_2)^2}; \quad b_9 = 0.$$

Now let's construct an matrix

$$T_2 = \begin{pmatrix} a_2 & a_2 \\ -1-a_1 & \lambda_2 - a_1 \end{pmatrix}.$$

It's obvious that the matrix T_2 is invertible due to $\lambda_2 \neq -1$, and then we use the following invertible translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T_2 \begin{pmatrix} s \\ w \end{pmatrix},$$

map (11) can be described by

$$\begin{pmatrix} s \\ w \end{pmatrix} \mapsto \begin{pmatrix} -s + f_1(s,w,\alpha^*) \\ \lambda_2 w + f_2(s,w,\alpha^*) \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} f_1(s,w,\alpha^*) = & \tfrac{(\lambda_2-a_1)a_3-a_2b_3}{a_2(\lambda_2+1)}u^2 + \tfrac{(\lambda_2-a_1)a_4-a_2b_4}{a_2(\lambda_2+1)}uv + \tfrac{(\lambda_2-a_1)a_5-a_2b_5}{a_2(\lambda_2+1)}v^2 + \tfrac{(\lambda_2-a_1)a_6-a_2b_6}{a_2(\lambda_2+1)}u^3 \\ & + \tfrac{(\lambda_2-a_1)a_7-a_2b_7}{a_2(\lambda_2+1)}u^2v + \tfrac{(\lambda_2-1)a_8-a_2b_8}{a_2(\lambda_2+1)}uv^2 + \tfrac{(\lambda_2-a_1)a_9-a_2b_9}{a_2(\lambda_2+1)}v^3 - \tfrac{a_2c_1}{a_2(\lambda_2+1)}u\alpha^* \\ & - \tfrac{a_2c_2}{a_2(\lambda_2+1)}v\alpha^* - \tfrac{a_2c_3}{a_2(\lambda_2+1)}u^2\alpha^* - \tfrac{a_2c_4}{a_2(\lambda_2+1)}uv\alpha^* - \tfrac{a_2c_5}{a_2(\lambda_2+1)}v^2\alpha^* \\ & + O((|s|+|w|+|\alpha^*|)^4), \\ f_2(s,w,\alpha^*) = & \tfrac{(a_1+1)a_3+a_2b_3}{a_2(\lambda_2+1)}u^2 + \tfrac{(a_1+1)a_4+a_2b_4}{a_2(\lambda_2+1)}uv + \tfrac{(a_1+1)a_5+a_2b_5}{a_2(\lambda_2+1)}v^2 + \tfrac{(a_1+1)a_6+a_2b_6}{a_2(\lambda_2+1)}u^3 \\ & + \tfrac{(a_1+1)a_7+a_2b_7}{a_2(\lambda_2+1)}u^2v + \tfrac{(a_1+1)a_8+a_2b_8}{a_2(\lambda_2+1)}uv^2 + \tfrac{(a_1+1)a_9+a_2b_9}{a_2(\lambda_2+1)}v^3 + \tfrac{a_2c_1}{a_2(\lambda_2+1)}u\alpha^* \\ & + \tfrac{a_2c_2}{a_2(\lambda_2+1)}v\alpha^* + \tfrac{a_2c_3}{a_2(\lambda_2+1)}u^2\alpha^* + \tfrac{a_2c_4}{a_2(\lambda_2+1)}uv\alpha^* + \tfrac{a_2c_5}{a_2(\lambda_2+1)}v^2\alpha^* \\ & + O((|s|+|w|+|\alpha^*|)^4), \end{aligned}$$

with

$$\begin{aligned} & u = a_2(s+w), v = (\lambda_2 - a_1)w - (a_1+1)s; \\ & u^2 = (a_2(s+w))^2; \\ & uv = (a_2(s+w))((\lambda_2-a_1)w-(a_1+1)s); \\ & v^2 = ((\lambda_2-a_1)w-(a_1+1)s)^2; \\ & u^3 = (a_2(s+w))^3; \\ & u^2v = (a_2(s+w))^2((\lambda_2-a_1)w-(a_1+1)s); \\ & uv^2 = (a_2(s+w))((\lambda_2-a_1)w-(a_1+1)s)^2; \\ & v^3 = ((\lambda_2-a_1)w-(a_1+1)s)^3. \end{aligned}$$

In the following, we will study the center manifold of map (12) at fixed point (0,0) in a small neighborhood of $\alpha^* = 0$. The well-known center manifold theorem guarantee that a center manifold $W^c(0,0)$ can exist, and it can be approximated as follows

$$W^c(0,0) = \{(s,w,\alpha^*) \in R^3 : w = d_1s^2 + d_2s\alpha^* + d_3(\alpha^*)^2 + O((|s|+|\alpha^*|)^3)\},$$

which satisfies

$$\begin{aligned} w &= d_1(-s+f_1(s,w,\alpha^*))^2 + d_2(-s+f_1(s,w,\alpha^*))\alpha^* + d_3(\alpha^*)^2 \\ &= \lambda_2(d_1s^2 + d_2s\alpha^* + d_3(\alpha^*)^2) + f_2(s,w,\alpha^*). \end{aligned}$$

By comparing the coefficients of the above equation, we have

$$d_1 = \frac{a_2((a_1+1)a_3+a_2b_3)}{1-\lambda_2^2} - \frac{(a_1+1)((a_1+1)a_4+a_2b_4)}{1-\lambda_2^2} + \frac{(a_1+1)^2((a_1+1)a_5+a_2b_5)}{1-\lambda_2^2},$$

$$d_2 = \frac{c_2(a_1+1) - a_2 c_1}{(1+\lambda_2)^2}, \qquad d_3 = 0.$$

So, restricted to the center manifold $W^c(0,0)$, map (12) turns into

$$\begin{aligned} s \mapsto & -s + e_1 s^2 + e_2 s\alpha^* + e_3 s^2 \alpha^* + e_4 s(\alpha^*)^2 + e_5 s^3 + O((|s| + |\alpha^*|)^4) \\ & \triangleq F_2(s, \alpha^*), \end{aligned} \tag{13}$$

where

$e_1 = A_1 a_2^2 - A_2 a_2(a_1+1) + A_3(a_1+1)^2$;
$e_2 = -A_8 a_2 + A_9(a_1+1)$;
$e_3 = 2A_1 d_2 a_2^2 + A_2 a_2 d_2(\lambda_2 - 2a_1 - 1) - 2A_3 d_2(\lambda_2 - a_1)(a_1+1) - A_8 a_2 d_1$
$-A_9(\lambda_2 - a_1)d_1 - A_{10} a_2^2 + A_{11} a_2(a_1+1) - A_{12}(a_1+1)^2$;
$e_4 = -A_8 a_2 d_2 - A_9(\lambda_2 - a_1)d_2$;
$e_5 = 2A_1 a_2^2 d_1 + A_2 a_2 d_1(\lambda_2 - 2a_1 - 1) - 2A_3 d_1(\lambda_2 - a_1)(a_1+1) + A_4 a_2^3$
$-A_5 a_2^2(a_1+1) + A_6 a_2(a_1+1)^2 - A_7(a_1+1)^3$;

with

$A_1 = \frac{(\lambda_2-a_1)a_3-a_2b_3}{a_2(\lambda_2+1)}$; $A_2 = \frac{(\lambda_2-a_1)a_4-a_2b_4}{a_2(\lambda_2+1)}$; $A_3 = \frac{(\lambda_2-a_1)a_5-a_2b_5}{a_2(\lambda_2+1)}$; $A_4 = \frac{(\lambda_2-a_1)a_6-a_2b_6}{a_2(\lambda_2+1)}$;
$A_5 = \frac{(\lambda_2-a_1)a_7-a_2b_7}{a_2(\lambda_2+1)}$; $A_6 = \frac{(\lambda_2-1)a_8-a_2b_8}{a_2(\lambda_2+1)}$; $A_7 = \frac{(\lambda_2-a_1)a_9-a_2b_9}{a_2(\lambda_2+1)}$; $A_8 = \frac{a_2c_1}{a_2(\lambda_2+1)}$;
$A_9 = \frac{a_2c_2}{a_2(\lambda_2+1)}$; $A_{10} = \frac{a_2c_3}{a_2(\lambda_2+1)}$; $A_{11} = \frac{a_2c_4}{a_2(\lambda_2+1)}$; $A_{12} = \frac{a_2c_5}{a_2(\lambda_2+1)}$.

To study the flip bifurcation of map (13), we define the following two discriminatory quantities

$$\mu_1 = \left(\frac{\partial^2 F_2}{\partial s \partial \alpha^*} + \frac{1}{2}\frac{\partial F_2}{\partial \alpha^*}\frac{\partial^2 F_2}{\partial s^2}\right)|_{(0,0)},$$

and

$$\mu_2 = \left(\frac{1}{6}\frac{\partial^3 F_2}{\partial s^3} + \left(\frac{1}{2}\frac{\partial^2 F_2}{\partial s^2}\right)^2\right)|_{(0,0)}$$

which can be showed in [38]. Then provided with Theorem 3.1 in [38], the following result can be given as

Theorem 3.1. Assume that μ_1 and μ_2 are not zero, then a flip bifurcation can occur at $E^*(x^*, y^*)$ of map (3) if the parameter α^* varies in a small neighborhood of origin. And that when $\mu_2 > 0(< 0)$, the period-2 orbit bifurcated from $E^*(x^*, y^*)$ of map (3) is stable (unstable).

4. Example

As application, we now give an example to support the theoretical results of this paper by using MATLAB. Let $\beta_1 = 1, \beta_2 = 0.5, h_1 = 0.05, h_2 = 0.1$, then we get from (5) that map (3) has only one positive point $E^*(0.0113, 0.2226)$. And we further have $\mu_1 = e_2 = 0.1134 \neq 0, \mu_2 = e_5 + e_1^2 = -4.4869 \neq 0$, which implies that all conditions of Theorem 3.1 hold, a flip bifurcation comes from E^* at the bifurcation parameter $\alpha = 2.2238$, so the flip bifurcation is supercritical, i.e., the period-2 orbit is unstable.

According to Figures 1 and 2, the positive point $E^*(0.0113, 0.2226)$ is stable for $2 \leq \alpha \leq 2.4$ and loses its stability at the bifurcation parameter value $\alpha = 2.2238$. Which implies that map (3) has complex dynamical properties.

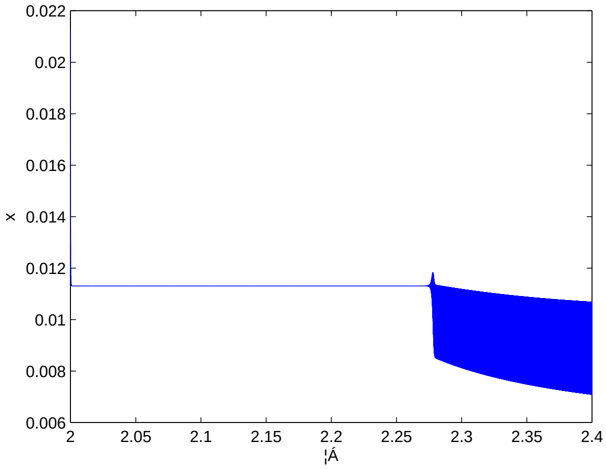

Figure 1. Flip bifurcation diagram of map (3) in the (α, x) plane for $\beta_1 = 1, \beta_2 = 0.5, h_1 = 0.05, h_2 = 0.1$. The initial value is (0.0213 , 0.2326).

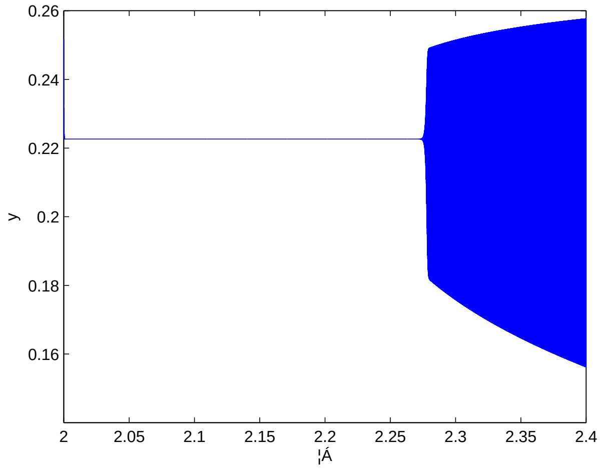

Figure 2. Flip bifurcation diagram of map (3) in the (α, y) plane for $\beta_1 = 1, \beta_2 = 0.5, h_1 = 0.05, h_2 = 0.1$. The initial value is (0.0213 , 0.2326).

5. Conclusions

In this paper, a predator-prey model with modified Leslie-Gower and Holling-type III schemes is considered from another aspect. The complex behavior of the corresponding discrete time dynamic system is investigated. we have obtained that the fixed point E_1 of map (4) is a saddle, and the fixed points E_2 and E^* of map (4) can undergo flip bifurcation. Moreover, Theorem 3.1 tell us that the period-2 orbit bifurcated from $E^*(x^*, y^*)$ of map (3) is stable under some sufficient conditions, which means that the predator and prey can coexist on the stable period-2 orbit. So, compared with previous studies [28] on the continuous predator-prey model, our discrete model shows more irregular and complex dynamic characteristics. The present research can be regarded as the continuation and development of the former studies in [28].

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Conflict of interest

The authors declare that they have no competing interests.

YYL carried out the proofs of main results in the manuscript. FXZ and XLZ participated in the design of the study and drafted the manuscripts. All the authors read and approved the final manuscripts.

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