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Research article

Dynamics of a reaction-diffusion SIRI model with relapse and free boundary

Qian Ding[1], **Yunfeng Liu**[1], **Yuming Chen**[2] **and Zhiming Guo**[1,*]

[1] School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, 510006, China
[2] Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5, Canada

* **Correspondence:** Email: guozm@gzhu.edu.cn.

Abstract: This paper is concerned with the free boundary problem for a reaction-diffusion SIRI model with relapse and bilinear incidence rate. After studying the (global) existence and uniqueness of solutions, we provide some sufficient conditions on the disease spreading-vanishing dichotomies for both cases with and without relapse.

Keywords: relapse; reaction-diffusion equation; free boundary; spreading-vanishing dichotomy

1. Introduction

Though medicine and living conditions have been constantly improving, infectious diseases are still a global concern. Mathematical modeling can not only enhance our understanding of the transmission mechanisms underlying them but also help us assess the efficacy of control strategies. Among the deterministic models described by ordinary differential equations are compartmental models. One of the basic models is the Kermack-McKendric model,

$$\begin{cases} \dfrac{dS}{dt} = -\beta S I, \\ \dfrac{dI}{dt} = \beta S I - \gamma I, \\ \dfrac{dR}{dt} = \gamma I, \end{cases}$$

where S, I, and R are the densities (or numbers) of susceptible, infectious, and recovered individuals, respectively; β is the transmission rate while γ is the recovery rate. The incidence rate is the bilinear one, $\beta S I$. To better reflect the actual biology of a given disease, the above model has been significantly modified.

In this paper, we consider the factor of relapse. For certain diseases such as herpes, tuberculosis, simplex virus type 2 (a human disease transmitted by close physical or sexual contacts), recovered individuals may experience relapse, which means that they can revert to the infectious class with the reactivation of a latent infection. For example, this feature of recurrence for tuberculosis is often due to incomplete treatment. Tudor [1] was the first to study relapse, who built the so-called SIRI model. In this model, the bilinear incidence rate is used. Tudor investigated the existence and local stability of equilibria. Later on, Moreira and Wang [2] modified this model with an incidence rate depending on the size of the susceptible population. By means of an elementary analysis of Liénard's equation and Lyapunov's direct method, they established sufficient conditions on the global asymptotic stability of the disease-free and endemic equilibria.

In the above mentioned studies on relapse, the population size is constant. In particular, there are no disease-induced deaths. Thus, in 2013, Vargas-De-León [3] proposed two epidemiological models with relapse and disease-induced deaths. One of them is the following one with the bilinear incidence rate,

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \beta SI - \mu S, \\ \dfrac{dI}{dt} = \beta SI - (\alpha + \gamma + \mu)I + \eta R, \\ \dfrac{dR}{dt} = \gamma I - (\mu + \eta)R, \end{cases} \tag{1.1}$$

where Λ represents the recruitment rate, β is the transmission rate, μ is the natural death rate, α is the disease-induced death rate, γ is the recovery rate, and η is the relapse rate. All the parameters are positive. They constructed suitable Lyapunov functions to obtain threshold dynamics determined by the basic reproduction number R_0. If $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable and hence the disease dies out. On the other hand, if $R_0 > 1$, the endemic equilibrium is globally asymptotically stable and hence the disease remains endemic. For more works on SIRI models described by ordinary differential equations, we refer to [4–6] and references therein.

Note that, due to mobility, the distribution of individuals in an area is not even. Modeling this phenomenon often results in reaction-diffusion equations. Consequently, inspired by [3], we have formulated a diffusive epidemic model with relapse and bilinear incidence as follows,

$$\begin{cases} S_t(x,t) = d\Delta S(x,t) + \Lambda - \beta S(x,t)I(x,t) - \mu S(x,t), & t > 0, x \in \Omega, \\ I_t(x,t) = d\Delta I(x,t) + \beta S(x,t)I(x,t) - (\alpha + \gamma + \mu)I(x,t) + \eta R(x,t), & t > 0, x \in \Omega, \\ R_t(x,t) = d\Delta R(x,t) + \gamma I(x,t) - (\mu + \eta)R(x,t), & t > 0, x \in \Omega, \\ S(x,0) = S_0(x) \geq 0, I(x,0) = I_0(x) \geq, \not\equiv 0, R(x,0) = R_0(x) \geq 0, & x \in \overline{\Omega}; \\ \dfrac{\partial S}{\partial n}(x,t) = \dfrac{\partial I}{\partial n}(x,t) = \dfrac{\partial R}{\partial n}(x,t) = 0, & t > 0, x \in \partial\Omega. \end{cases} \tag{1.2}$$

Here $S(x,t)$, $I(x,t)$, and $R(x,t)$ are the densities of susceptible, infective, and recovered individuals at time t and position $x \in \Omega$, respectively; Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$; Δ is the usual Laplacian operator; $\frac{\partial}{\partial n}$ is the outward normal derivative to $\partial\Omega$; d is the diffusion rate which represents the ability of random mobility of individuals; and the meanings of the other parameters are the same as those in (1.1). Note that the Neumann boundary conditions imply that individuals cannot move across the boundary $\partial\Omega$.

It should be pointed out that any solution of (1.2) is always positive for any time $t > 0$ no matter what the nonnegative nontrivial initial condition is. Thus the disease spreads to the whole area immediately, even though the infectious are confined to a quite small part of the habitat at the beginning. This does not agree with the observed fact that diseases always spread gradually. To compensate for the gradual disease spreading progress, a better modeling technique is to introduce free boundary.

The equation governing the free boundary, $h'(t) = -\mu I_x(h(t), t)$, is a special case of the well-known Stefan condition, which has been established in [7] for diffusive populations and used in the modeling of a number of applied problems. For example, it was used to describe the melting of ice in contact with water [8] and to model oxygen in muscles [9] as well as wound healing [10]. There is a vast literature on Stefan problems. Some important recent theoretical advances can be found in [11]. As a typical case, in 2013, Kim et al. [12] studied a diffusive SIR epidemic model in a radially symmetric domain with free boundary. They provided sufficient conditions on disease vanishing and spreading.

Motivated by the above discussion, in this paper, we investigate the behavior of nonnegative solutions $(S(x,t), I(x,t), R(x,t); h(t))$ of the following reaction-diffusion SIRI epidemic with free boundary,

$$\begin{cases} S_t(x,t) = dS_{xx}(x,t) + \Lambda - \beta S(x,t)I(x,t) - \delta S(x,t), & x > 0, t > 0, \\ I_t(x,t) = dI_{xx}(x,t) + \beta S(x,t)I(x,t) - (\alpha + \gamma + \delta)I(x,t) + \eta R(x,t), & 0 < x < h(t), t > 0, \\ R_t(x,t) = dR_{xx}(x,t) + \gamma I(x,t) - (\delta + \eta)R(x,t), & 0 < x < h(t), t > 0, \\ S_x(0,t) = I_x(0,t) = R_x(0,t) = 0, & t > 0, \\ I(x,t) = R(x,t) = 0, & x \geq h(t), t > 0, \\ h'(t) = -\mu I_x(h(t), t), & t > 0, \\ h(0) = h_0, & \\ S(x,0) = S_0(x) \geq 0, I(x,0) = I_0(x) \geq 0, R(x,0) = R_0(x) \geq 0, & x \geq 0, \end{cases} \tag{1.3}$$

where $x = h(t)$ is the moving boundary to be determined, μ represents the moving rate of the free boundary, δ is the natural death rate, and the meanings of the rest parameters are the same as those in model (1.2). All parameters are assumed to be positive. The nonnegative initial functions S_0, I_0 and R_0 satisfy

$$\begin{cases} S_0 \in C^2([0, +\infty)), \qquad I_0, R_0 \in C^2([0, h_0]), \\ I_0(x) = R_0(x) = 0 \text{ for } x \in [h_0, +\infty) \text{ and } I_0(x) > 0 \text{ for } x \in [0, h_0). \end{cases} \tag{1.4}$$

In reality, $I_0(x) = 0$ for $x \in [h_0, +\infty)$ and $I_0 \not\equiv 0$ on $[0, h_0)$. Since for $t > 0$, the solution though the initial condition $(S_0, I_0, R_0; h_0)$ with such an I_0 satisfy $I(x,t) > 0$ on $[0, h(t))$ and $I(x,t) = 0$ for $x \in [h(t), +\infty)$. Thus, without loss of generality, we make the assumption (1.4). Biologically, model (1.3) means that beyond the free boundary $x = h(t)$, there are only susceptible individuals. We will also consider the case without relapse, that is, $\eta = 0$. In this case, (1.3) reduces to

$$\begin{cases} S_t(x,t) = dS_{xx}(x,t) + \Lambda - \beta S(x,t)I(x,t) - \delta S(x,t), & x > 0, t > 0, \\ I_t(x,t) = dI_{xx}(x,t) + \beta S(x,t)I(x,t) - (\alpha + \gamma + \delta)I(x,t), & 0 < x < h(t), t > 0, \\ R_t(x,t) = dR_{xx}(x,t) + \gamma I(x,t) - \delta R(x,t), & 0 < x < h(t), t > 0, \\ S_x(0,t) = I_x(0,t) = R_x(0,t) = 0, & t > 0, \\ I(x,t) = R(x,t) = 0, & x \geq h(t), t > 0, \\ h'(t) = -\mu I_x(t, h(t)), & t > 0, \\ h(0) = h_0, & \\ S(x,0) = S_0(x) \geq 0, I(x,0) = I_0(x) \geq 0, R(x,0) = R_0(x) \geq 0, & x \geq 0. \end{cases} \tag{1.5}$$

The remainder of this paper is organized as follows. In Section 2, we prove some general results on the existence and uniqueness of solutions to (1.3)–(1.4). In particular, solutions are global. Then, in Section 3, we provide some sufficient conditions on disease spreading and vanishing. For (1.5), the disease will die out either if the basic reproduction number $\mathcal{R}_0 < 1$ or if $\mathcal{R}_0 > 1$ and the initial infected area, boundary moving rate, and initial value of infected individuals are sufficiently small; while the disease will spread to the whole area if $\mathcal{R}_0 > 1$ and either the initial infected area is suitably large or the diffusion rate is suitably small. For (1.3), when the basic reproduction number $\tilde{\mathcal{R}}_0 \leq 1$, the disease will disappear, whereas when $\tilde{\mathcal{R}}_0 > \mathcal{R}_0 > 1$ and the initial infected area is suitably large, the disease will successfully spread. The paper ends with a brief conclusion and discussion.

2. Existence and uniqueness of solutions

First, we state the result on the local existence of solutions to (1.3)–(1.4), which can be proved with some modifications of the arguments in [10] and [13]. Hence we omit the proof to avoid repetition.

Theorem 2.1. *For any given* (S_0, I_0, R_0) *satisfying* (1.4) *and any* $r \in (0,1)$, *there is a* $T > 0$ *such that problem* (1.3) *admits a unique bounded solution*

$$(S, I, R; h) \in C^{1+r, \frac{(1+r)}{2}}(D_T^\infty) \times [C^{1+r, \frac{(1+r)}{2}}(D_T)]^2 \times C^{1+\frac{r}{2}}([0,T]);$$

moreover,

$$\|S\|_{C^{1+r,\frac{(1+r)}{2}}(D_T^\infty)} + \|I\|_{C^{1+r,\frac{(1+r)}{2}}(D_T)} + \|R\|_{C^{1+r,\frac{(1+r)}{2}}(D_T)} + \|h\|_{C^{1+\frac{r}{2}}([0,T])} \leq C,$$

where $D_T^\infty = \{(x,t) \in \mathbb{R}^2 : x \in [0, +\infty), t \in [0,T]\}$ *and* $D_T = \{(x,t) \in \mathbb{R}^2 : x \in [0, h(t)], t \in [0,T]\}$. *Here* C *and* T *only depend on* h_0, r, $\|S_0\|_{C^2([0,+\infty))}$, $\|I_0\|_{C^2([0,h_0])}$, *and* $\|R_0\|_{C^2([0,h_0])}$.

Next we make some preparations to show the global existence of solutions.

Lemma 2.1. *Problem* (1.3)–(1.4) *admits a unique and uniformly bounded solution* $(S, I, R; h)$ *on* $(0, T_0)$ *for some* $T_0 \in (0, +\infty]$, *that is, there exists a constant* M *independent of* T_0 *such that*

$$0 < S(x,t) \leq M \quad \text{for } 0 \leq x < +\infty, t \in (0, T_0).$$
$$0 < I(x,t), R(x,t) \leq M \quad \text{for } 0 \leq x < h(t), t \in (0, T_0).$$

Proof. As long as the solution exists, it is easy to see that $S \geq 0$, $I \geq 0$, and $R \geq 0$ on $[0, +\infty) \times [0, T_0]$. By applying the strong maximum principle to the equations on $\{(x, t) : x \in [0, h(t)], t \in [0, T_0]\}$, we immediately obtain

$$\begin{aligned} &S(x,t) > 0 \quad \text{for } 0 \leq x < +\infty, 0 < t < T_0, \\ &I(x,t), R(x,t) > 0 \quad \text{for } 0 \leq x < h(t), 0 < t < T_0. \end{aligned}$$

It remains to prove the uniform boundedness of the solution $(S(x,t), I(x,t), R(x,t); h(t))$. For this purpose, define

$$U(x,t) = S(x,t) + I(x,t) + R(x,t), \qquad 0 \leq x < +\infty, t \in (0, T_0).$$

A direct calculation gives

$$\begin{aligned} \frac{dU}{dt} &= dS_{xx} + dI_{xx} + dR_{xx} + \Lambda - \delta S - (\delta + \alpha)I - \delta R \\ &= dU_{xx} + \Lambda - \delta(S + I + R) - \alpha I \\ &\leq dU_{xx} + \Lambda - \delta U, \end{aligned}$$

which gives $U(x,t) \leq \max\left\{\|U_0\|_\infty, \frac{\Lambda}{\delta}\right\} \triangleq M$, where

$$\|U_0\|_\infty = \|S(x,0) + I(x,0) + R(x,0)\|_\infty .$$

Now the required result follows immediately. □

Finally, we show that the free boundary of (1.3)–(1.4) is strictly monotonically increasing.

Lemma 2.2. *Let $(S, I, R; h)$ be a solution to problem* (1.3)–(1.4) *defined for $t \in (0, T_0)$ for some $T_0 \in (0, +\infty]$. Then there exists a constant C_1 independent of T_0 such that*

$$0 < h'(t) \leq C_1 \quad \text{for} \quad t \in (0, T_0).$$

Proof. Using the strong maximum principle and Hopf boundary lemma to the equation of I, we can obtain $I_x(h(t), t) < 0$ for $t \in (0, T_0)$. This, combined with the Stefan condition $h'(t) = -\mu I_x(h(t), t)$, gives $h'(t) > 0$ for $t \in (0, T_0)$.

In order to get a bound for $h'(t)$, we denote

$$\Omega_N := \{(x,t) : h(t) - N^{-1} < x < h(t), 0 < t < T_0\},$$

and construct an auxiliary function

$$\omega_N(x,t) := M[2N(h(t) - x) - N^2(h(t) - x)^2].$$

We will choose N so that $\omega_N(x,t) \geq I(x,t)$ holds over Ω_N.

Clearly, for $(x,t) \in \Omega_N$,

$$\begin{aligned} (\omega_N)_t &= 2MNh'(t)[1 - N(h(t) - x)] \geq 0, \\ -(\omega_N)_{xx} &= 2MN^2, \end{aligned}$$

$$\beta SI - (\alpha + \gamma + \delta)I + \eta R \quad \leq \quad \beta M^2 + \eta M.$$

Therefore, if $N^2 \geq \frac{\beta M + \eta}{2d}$ then

$$(\omega_N)_t - d(\omega_N)_{xx} \geq 2dMN^2 \geq \beta M^2 + \eta M.$$

On the other hand, we have the boundary condition

$$\begin{aligned} \omega_N(h(t) - N^{-1}, t) &= M \geq I(h(t) - N^{-1}, t), \\ \omega_N(h(t), t) &= 0 = I(h(t), t). \end{aligned}$$

To employ the maximum principle to $(\omega_N - I)$ over Ω_N to deduce that $I(x, t) \leq \omega_N(x, t)$, we only have to find some N independent of T_0 such that $I_0(x) \leq \omega_N(x, 0)$ for $x \in [h_0 - N^{-1}, h_0]$. It would then follow that

$$\begin{aligned} I_x(h(t), t) &\geq (\omega_N)_x(h(t), t) = -2NM, \\ h'(t) &= -\mu I_x(h(t), t) \leq 2\mu NM. \end{aligned}$$

Note that

$$\begin{aligned} I_0(x) &= I_0(x) - I_0(h_0) \\ &= -\int_x^{h(t)} I_0'(s)ds \\ &\leq (h_0 - x)\left\|I_0'\right\|_{C[0,h_0]} \end{aligned}$$

and

$$\omega_N(x, 0) := M[2N(h_0 - x) - N^2(h_0 - x)^2] \geq MN(h_0 - x), \quad x \in [h_0 - N^{-1}, h_0].$$

It suffices to have

$$(h_0 - x)\left\|I_0'\right\|_{C[0,h_0]} \leq MN(h_0 - x).$$

Thus choosing

$$N := \max\left\{\sqrt{\frac{\beta M + \eta}{2d}}, \frac{\left\|I_0'\right\|_{C([0,h_0])}}{M}\right\}$$

completes the proof. □

By a similar argument as the one in [12, 13], we can have the following result.

Theorem 2.2. *The solution of problem* (1.3)–(1.4) *exists and is unique for all* $t \in (0, +\infty)$.

3. The spreading-vanishing dichotomy

This section is devoted to the spreading-vanishing dichotomy. We distinguish two cases, $\eta = 0$ and $\eta > 0$. We start with a sufficient condition on disease vanishing, which will be used in the coming discussion.

3.1. Disease vanishing

It follows from Lemma 2.2 that if $x = h(t)$ is monotonically increasing, then $h_\infty := \lim_{t\to\infty} h(t) \in (h_0, +\infty]$ is well defined.

Lemma 3.1. *If $h_\infty < +\infty$, then $\lim_{t\to+\infty} \|I(\cdot,t)\|_{C([0,h(t)])} = 0$. Moreover, $\lim_{t\to+\infty} \|R(\cdot,t)\|_{C([0,h(t)])} = 0$ and $\lim_{t\to+\infty} S(x,t) = \frac{\Lambda}{\delta}$ uniformly in any bounded subset of* $[0, +\infty)$.

Proof. Define

$$s = \frac{h_0 x}{h(t)}, \quad u(s,t) = S(x,t), \quad v(s,t) = I(x,t), \quad w(s,t) = R(x,t).$$

Then it is easy to see that

$$I_t = v_t - \frac{h'(t)}{h(t)} s v_s, \quad I_x = \frac{h_0}{h(t)} v_s, \quad I_{xx} = \frac{h_0^2}{h^2(t)} v_{ss}.$$

It follows that $v(s,t)$ satisfies

$$\begin{cases} v_t - \dfrac{h'(t)}{h(t)} s v_s - d\dfrac{h_0^2}{h^2(t)} v_{ss} = v[\beta u - (\alpha + \delta + \gamma)] + \eta w, & 0 < s < h_0, t > 0, \\ v_s(0,t) = v(h_0,t) = 0, & t > 0, \\ v(s,0) = I_0(s) \geq 0, & 0 \leq s \leq h_0. \end{cases}$$

This means that the transformation changes the free boundary $x = h(t)$ into the fixed line $s = h_0$ and hence we have an initial boundary value problem over a fixed area $s < h_0$.

Since $h_0 \leq h(t) < h_\infty < +\infty$, the differential operator is uniformly parabolic. With the bounds in Lemma 2.1 and Lemma 2.2, there exist positive constants M_1 and M_2 such that

$$\|v(\beta u - (\alpha + \mu + \gamma)) + \eta w\|_{L^\infty} \leq M_1 \quad \text{and} \quad \left\| \frac{h'(t)}{h(t)} s \right\|_{L^\infty} \leq M_2.$$

Applying the standard L^p theory and the Sobolev embedding theorem [14], we obtain that

$$\|v\|_{C^{1+\alpha, \frac{1+\alpha}{2}}([0,h_0]\times[0,+\infty))} \leq M_3$$

for some constant M_3 depending on α, h_0, M_1, M_2, and $\|I_0\|_{C^2[0,h_0]}$. It follows that there exists a constant $\tilde{C}$ depending on α, h_0, (S_0, I_0, R_0), and h_∞ such that

$$\|h\|_{C^{1+\frac{\alpha}{2}}([0,+\infty))} \leq \tilde{C}. \tag{3.1}$$

Assume $\limsup_{t\to+\infty} \|I(\cdot,t)\|_{C([0,h(t)])} = \varpi > 0$ by contradiction. Then there exists a sequence $\{(x_k, t_k)\}$ in $[0, h_\infty) \times (0, +\infty)$ such that $I(x_k, t_k) \geq \frac{\varpi}{2}$ for all $k \in \mathbb{N}$ and $t_k \to +\infty$ as $k \to +\infty$. Since $I(h(t), t) = 0$ and since (3.1) indicates that $|\, I_x(h(t), t)\,|$ is uniformly bounded for $t \in [0, +\infty)$, there exists $\sigma > 0$ such that $x_k \leq h(t_k) - \sigma$ for all $k \geq 1$. Then there is a subsequence of $\{x_k\}$ which converges to $x_0 \in [0, h_\infty - \sigma]$. Without loss of generality, we assume $x_k \to x_0$ as $k \to +\infty$. Correspondingly,

$$s_k := \frac{h_0 x_k}{h(t_k)} \to s_0 := \frac{h_0 x_0}{h_\infty} < h_0.$$

Define $S_k(x,t) = S(x, t_k + t)$, $I_k(x,t) = I(x, t_k + t)$, and $R_k(x,t) = R(x, t_k + t)$ for $(x,t) \in (0, h(t_k + t)) \times (-t_k, +\infty)$. It follows from the parabolic regularity that $\{(S_k, I_k, R_k)\}$ has a subsequence $\{(S_{k_i}, I_{k_i}, R_{k_i})\}$ such that $(S_{k_i}, I_{k_i}, R_{k_i}) \to (\tilde{S}, \tilde{I}, \tilde{R})$ as $i \to +\infty$. Since $\|h\|_{C^{1+\frac{\alpha}{2}}([0,+\infty))} \leq \tilde{C}$, $h'(t) > 0$, and $h(t) \leq h_\infty < +\infty$, it is necessary that $h'(t) \to 0$ as $t \to +\infty$. Hence $(\tilde{S}, \tilde{I}, \tilde{R})$ satisfies

$$\begin{cases} \tilde{S}_t - d_1 \tilde{S}_{xx} = \Lambda - \beta \tilde{S} \tilde{I} - \delta \tilde{S}, & 0 < x < h_\infty, t \in (-\infty, +\infty), \\ \tilde{I}_t - d_2 \tilde{I}_{xx} = \beta \tilde{S} \tilde{I} - (\alpha + \gamma + \delta) \tilde{I} + \eta \tilde{R}, & 0 < x < h_\infty, t \in (-\infty, +\infty), \\ \tilde{R}_t - d_3 \tilde{R}_{xx} = \gamma \tilde{I} - (\delta + \eta) \tilde{R}, & 0 < x < h_\infty,\ t \in (-\infty, +\infty). \end{cases}$$

Since $\tilde{I}(x_0, 0) \geq \frac{\varpi}{2}$, the maximum principle implies that $\tilde{I} > 0$ on $[0, h_\infty) \times (-\infty, +\infty)$. Thus we can apply the Hopf lemma to conclude that $\sigma_0 := \frac{\partial \tilde{I}}{\partial s}(h_0, 0) < 0$. It follows that

$$v_x(h(t_{k_i}), t_{k_i}) = \frac{\partial I_{k_i}(h_0, 0)}{\partial s} \frac{h_0}{h(t_{k_i})} \leq \frac{\sigma_0}{2} \frac{h_0}{h_\infty} < 0$$

for all large i. Hence $h'(t_{k_i}) \geq -\mu \frac{\sigma_0}{2} \frac{h_0}{h_\infty} > 0$ for all large i, which contradicts with $h'(t) \to 0$ as $t \to +\infty$. This proves $\lim\limits_{t \to +\infty} \|I(\cdot, t)\|_{C([0,h(t)])} = 0$.

Using a simple comparison argument, we can deduce that $\lim\limits_{t \to +\infty} \|R(\cdot, t)\|_{C([0,h(t)])} = 0$ and $\lim\limits_{t \to +\infty} S(x,t) = \frac{\Lambda}{\delta}$ uniformly in any bounded subset of $[0, +\infty)$. In fact, for any $\varepsilon > 0$, there exists a $T_0 \geq 0$ such that $I(x,t) \leq \varepsilon$ for $t \geq T_0$. Then, for $t \geq T_0$, we have

$$S_t \geq d S_{xx} + \Lambda - (\beta \varepsilon + \delta) S(x,t)$$

and

$$R_t \leq d R_{xx} + \gamma \varepsilon - (\delta + \eta) R(x,t).$$

It follows that

$$\liminf_{t \to +\infty} S(x,t) \geq \frac{\Lambda}{\beta \varepsilon + \delta} \quad \text{uniformly in any bounded subset of } [0, +\infty)$$

and

$$\limsup_{t \to +\infty} \|R(, \cdot, t)\|_{C([0,h(t)])} \leq \frac{\gamma \varepsilon}{\delta + \eta}.$$

As ε is arbitrarily, letting $\varepsilon \to 0^+$ gives us

$$\liminf_{t \to +\infty} S(x,t) \geq \frac{\Lambda}{\delta} \quad \text{uniformly in any bounded subset of } [0, +\infty)$$

and

$$\limsup_{t \to +\infty} \|R(, \cdot, t)\|_{C([0,h(t)])} \leq 0.$$

This immediately gives $\lim\limits_{t \to +\infty} \|R(, \cdot, t)\|_{C([0,h(t)])} = 0$. Moreover, for $t \geq 0$, we have

$$S_t \leq d S_{xx} + \Lambda - \delta S(x,t).$$

Then $S(x,t) \le \bar{S}(t)$ for $x \in (0,+\infty)$ and $t \in (0,+\infty)$, where

$$\bar{S}(t) := \frac{\Lambda}{\delta} + \left(\bar{S}(0) - \frac{\Lambda}{\delta}\right) e^{-\delta t}$$

is the solution of the problem

$$\frac{d\bar{S}(t)}{dt} = \Lambda - \delta \bar{S}(t), \qquad t > 0; \;\; \bar{S}(0) = \max\left\{\frac{\Lambda}{\delta}, \|S_0\|_\infty\right\}.$$

Since $\lim_{t\to+\infty} \bar{S}(t) = \frac{\Lambda}{\delta}$, we deduce that

$$\limsup_{t\to+\infty} S(x,t) \le \lim_{t\to+\infty} \bar{S}(t) = \frac{\Lambda}{\delta} \text{ uniformly for } x \in [0,+\infty).$$

Therefore, we have $\lim_{t\to+\infty} S(x,t) = \frac{\Lambda}{\delta}$ uniformly in any bounded subset of $[0,+\infty)$. □

3.2. The case where $\eta = 0$

Consider the following eigenvalue problem,

$$\begin{cases} d\phi_{xx} + \dfrac{\beta\Lambda}{\delta}\phi - (\alpha+\gamma+\delta)\phi + \lambda\phi = 0, & x \in (0,h_0), \\ \phi_x(0) = 0, \phi(h_0) = 0. \end{cases} \tag{3.2}$$

It admits a principal eigenvalue λ_1, where

$$\lambda_1 = \frac{d\pi^2}{4h_0^2} - \frac{\beta\Lambda}{\delta} + (\alpha+\gamma+\delta).$$

The basic reproduction number of (1.5) denoted by $\mathcal{R}_0$ is given by

$$\mathcal{R}_0 = \frac{\beta\Lambda}{\delta(\gamma+\alpha+\delta)}.$$

With the assistance of the expression of $\mathcal{R}_0$, we can rewrite the expression of λ_1 as

$$\lambda_1 = \frac{d\pi^2}{4h_0^2} - \frac{\beta\Lambda}{\delta} + (\alpha+\gamma+\delta) = \frac{d\pi^2}{4h_0^2} - \left(1 - \frac{1}{\mathcal{R}_0}\right)\frac{\beta\Lambda}{\delta}.$$

It follows that $\lambda_1 > 0$ either if $\mathcal{R}_0 \le 1$ or if $\mathcal{R}_0 > 1$ and $h_0 < \sqrt{\frac{d\delta\pi^2}{4\beta\Lambda(1-\frac{1}{\mathcal{R}_0})}}$.

We first give some sufficient conditions on disease vanishing.

Theorem 3.1. *If $\mathcal{R}_0 < 1$, then $\lim_{t\to+\infty} \|I(\cdot,t)\|_{C([0,h(t)])} = 0$ and $\lim_{t\to+\infty} \|R(\cdot,t)\|_{C([0,h(t)])} = 0$. Moreover, $\lim_{t\to+\infty} S(x,t) = \frac{\Lambda}{\delta}$ uniformly in any bounded subset of* $[0,+\infty)$.

Proof. From the proof of Lemma 3.1, we have obtained that

$$\limsup_{t\to+\infty} S(x,t) \le \frac{\Lambda}{\delta} \qquad \text{uniformly for } x \in [0,+\infty).$$

Since $\mathcal{R}_0 < 1$, there exists T_0 such that $S(x,t) \le \frac{\Lambda}{\delta}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0}$ on $[0,+\infty)\times(T_0,+\infty)$. Then $I(x,t)$ satisfies

$$\begin{cases} I_t(x,t) \le dI_{xx} + \left[\frac{\beta\Lambda}{\delta}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0} - (\alpha+\gamma+\delta)\right] I(x,t), & 0 < x < h(t),\ t > T_0, \\ I_x(0,t) = 0,\ I(h(t),t) = 0, & t > T_0, \\ I(x,T_0) > 0, & 0 \le x \le h(T_0). \end{cases}$$

We know that $I(x,t) \le \bar{I}(x,t)$ for $(x,t) \in \{(x,t) : x \in [0,h(t)], t \in (T_0,+\infty)\}$, where $\bar{I}(x,t)$ satisfies

$$\begin{cases} \bar{I}_t(x,t) = d\bar{I}_{xx} + \left[\frac{\beta\Lambda}{\delta}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0} - (\alpha+\gamma+\delta)\right] \bar{I}(x,t), & 0 < x < h(t),\ t > T_0, \\ \bar{I}_x(0,t) = \bar{I}(h(t),t) = 0, & t > T_0, \\ \bar{I}(x,T_0) \ge \|I(\cdot,T_0)\|_\infty > 0, & 0 \le x \le h(T_0). \end{cases}$$

Since $\frac{\beta\Lambda}{\delta}\frac{1+\mathcal{R}_0}{2\mathcal{R}_0} - (\alpha+\gamma+\delta) = \frac{(\alpha+\gamma+\delta)(\mathcal{R}_0-1)}{2} < 0$, we have $\lim_{t\to+\infty} \|\bar{I}(\cdot,t)\|_{C[0,h(t)]} = 0$. Then it follows from $I(x,t) \le \bar{I}(x,t)$ that $\|I(\cdot,t)\|_{C[0,h(t)]} \to 0$ as $t \to +\infty$. The remaining part follows from Lemma 3.1. □

Theorem 3.2. *Suppose $\mathcal{R}_0 > 1$. Then $h_\infty < +\infty$ for given initial condition $(S_0, I_0, R_0; h_0)$ satisfying $h_0 \le \min\left\{\sqrt{\frac{d}{16k_0}}, \sqrt{\frac{d}{16\gamma}}\right\}$ and $\mu \le \frac{d}{8K}$, where $k_0 = \beta M - \alpha - \gamma - \delta > 0$, $M = \max\left\{\|S_0\|_\infty, \frac{\Lambda}{\delta}\right\}$, and $K = \frac{4}{3}\max\{\|I_0\|_\infty, \|R_0\|_\infty\}$.*

Proof. Since $\mathcal{R}_0 > 1$, one can easily see that $k_0 > 0$. Inspired by [13], we define

$$\begin{aligned} \bar{S}(x,t) &= M, \\ \bar{I}(x,t) &= \begin{cases} Ke^{-\theta t}V(\frac{x}{\bar{h}(t)}), & 0 \le x \le \bar{h}(t), \\ 0, & x > \bar{h}(t), \end{cases} \\ \bar{R}(x,t) &= \begin{cases} Ke^{-\theta t}V(\frac{x}{\bar{h}(t)}), & 0 \le x \le \bar{h}(t), \\ 0, & x > \bar{h}(t), \end{cases} \\ V(y) &= 1 - y^2, \quad 0 \le y \le 1, \\ \bar{h}(t) &= 2h_0(2 - e^{-\theta t}), t \ge 0, \end{aligned}$$

where θ is a constant to be determined later. In the following, we show that $(\bar{S}, \bar{I}, \bar{R}; \bar{h})$ is an upper solution to (1.5).

For $0 < x < \bar{h}(t)$ and $t > 0$, direct computations yield

$$\begin{aligned} \bar{S}_t - d\bar{S}_{xx} = 0 &\ge \Lambda - \delta\bar{S}, \\ \bar{I}_t - d\bar{I}_{xx} - (\beta\bar{S} - \alpha - \gamma - \delta)\bar{I} &= \bar{I}_t - d\bar{I}_{xx} - k_0\bar{I} \\ &= Ke^{-\theta t}[-\theta V - x\bar{h}'\bar{h}^{-2}V' - d\bar{h}^{-2}V'' - k_0 V] \end{aligned}$$

$$
\begin{aligned}
&\geq \quad Ke^{-\theta t}\left[\frac{d}{8h_0^2}-\theta-k_0\right],\\
\bar{R}_t - d\bar{R}_{xx} - (\gamma\bar{I}-\delta\bar{R}) &\geq \quad Ke^{-\theta t}\left[\frac{d}{8h_0^2}-\theta-\gamma\right],\\
\bar{h}'(t) &= \quad 2h_0\theta e^{-\theta t},\\
-\mu\bar{I}_x(\bar{h}(t),t) &= \quad 2K\mu\bar{h}^{-1}(t)e^{-\theta t}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\bar{S}(x,0) &\geq S_0(x),\\
\bar{I}(x,0) &= K\left(1-\frac{x^2}{4h_0^2}\right)\geq\frac{3}{4}K \quad \text{for } x\in[0,h_0],\\
\bar{R}(x,0) &= K\left(1-\frac{x^2}{4h_0^2}\right)\geq\frac{3}{4}K \quad \text{for } x\in[0,h_0].
\end{aligned}
$$

Choose $\theta=\frac{d}{16h_0^2}$. Noting $\bar{h}(t)\leq 4h_0$, we have

$$
\begin{cases}
\bar{S}_t - d\bar{S}_{xx}\geq\Lambda-\delta\bar{S}, & x>0, t>0,\\
\bar{I}_t - d\bar{I}_{xx}\geq\beta\bar{S}\bar{I}-(\alpha+\gamma+\delta)\bar{I}, & 0<x<\bar{h}(t), t>0,\\
\bar{R}_t - d\bar{R}_{xx}\geq\alpha\bar{I}-\delta\bar{R}, & 0<x<\bar{h}(t), t>0,\\
\bar{S}_x(0,t)\geq 0,\ \bar{I}_x(0,t)\geq 0,\ \bar{R}_x(0,t)\geq 0, & t>0,\\
\bar{I}(x,t)=\bar{R}(x,t)=0, & x\geq\bar{h}(t), 0<t\leq T,\\
\bar{h}'(t)\geq-\mu\bar{I}_x(\bar{h}(t),t),\ \bar{h}(0)=2h_0\geq h_0, & t>0,\\
\bar{S}(x,0)\geq S_0(x),\ \bar{I}(x,0)\geq I_0(x),\ \bar{R}(x,0)\geq R_0(x), & 0\leq x\leq h_0.
\end{cases}
$$

This verifies that $(\bar{S},\bar{I},\bar{R};\bar{h})$ is an upper solution to (1.5). Then we can apply a result similar as [12, Lemma 4.1] (which can be proved in the same manner as [13, Lemma 5.6]) to conclude that $h(t)\leq\bar{h}(t)$ for $t>0$. This implies that $h_\infty\leq\lim_{t\to+\infty}\bar{h}(t)=4h_0<+\infty$. □

Theorem 3.3. *Assume that $\mathcal{R}_0>1$. For given initial condition $(S_0,I_0,R_0;h_0)$, we have $h_\infty<+\infty$ provided that $h_0<h_*:=\min\left\{\sqrt{\frac{d\pi^2}{4[\beta N-(\alpha+\gamma+\delta)]}},\frac{\sqrt{d\gamma}}{4\gamma}\right\}$ and both $\|I_0\|_\infty$ and $\|R_0\|_\infty$ are sufficiently small (which is specified in the proof), where $N=\max\{\frac{\Lambda}{\delta},\|S_0\|_\infty\}$.*

Proof. Note that h_* is well defined since $\mathcal{R}_0>1$. As in the proof of Theorem 3.2, we will construct a suitable upper solution to (1.5). Since $h_0<h_*$, there exists $\varepsilon_1>0$ such that $h_0<\sqrt{\frac{d\pi^2}{4[\beta(N+\varepsilon_1)-(\alpha+\gamma+\delta)]}}$. Then the principal eigenvalue of the eigenvalue problem

$$
\begin{cases}
d\phi_{xx}+\beta(N+\varepsilon_1)\phi-(\alpha+\gamma+\delta)\phi+\lambda\phi=0, & 0<x<h_0\\
\phi_x(0)=\phi(h_0)=0.
\end{cases}
$$

is

$$
\widetilde{\lambda}_1=\frac{d\pi^2}{4h_0^2}-\beta(N+\varepsilon_1)+\alpha+\delta+\gamma>0
$$

and it is has a normalized positive eigenfunction $\tilde{\phi}$ on $(0, h_0)$. Moreover, $\tilde{\phi}_x < 0$ on $(0, h_0]$. Choose $\varepsilon_2 \in (0, \gamma)$ such that

$$\widetilde{\lambda}_1 > [\beta(N + \varepsilon_1) + \varepsilon_2](1 + \varepsilon_2)^2 - \beta(N + \varepsilon_1) > 0.$$

Recall that $\limsup_{t\to+\infty} S(t, x) \le \frac{\Lambda}{\delta}$ uniformly for $x \in [0, +\infty)$. Thus there exists a $T_0 > 0$ such that $0 < S(x, t) \le (N + \varepsilon_1)$ in $[0, +\infty) \times [T_0, +\infty)$. As in [15], we define

$$\begin{aligned}
\vartheta(t) &= h_0\left(1 + \varepsilon_2 - \frac{\varepsilon_2}{2} e^{-\varepsilon_2 t}\right),\\
\bar{S}(x, t) &= (N + \varepsilon_1), \qquad t \ge T_0,\\
\bar{I}(x, t) &= \begin{cases} \iota e^{-\varepsilon_2 t}\tilde{\phi}(\frac{x h_0}{\vartheta(t)}), & 0 \le x \le \vartheta(t), t \ge T_0,\\ 0, & x > \vartheta(t), t \ge T_0,\end{cases}\\
\bar{R}(x, t) &= \begin{cases} \iota e^{-\varepsilon_2 t} V(\frac{x}{\vartheta(t)}), & 0 \le x \le \vartheta(t), t \ge T_0,\\ 0, & x > \vartheta(t), t \ge T_0.\end{cases}\\
V(y) &= 1 - y^2, \qquad 0 \le y \le 1,
\end{aligned}$$

where ι is a positive number to be determined later. As $\tilde{\phi}(h_0) = 0$, it follows that $\bar{I}(\vartheta(t), t) = 0$ for $t \ge T_0$, which implies that the function $\bar{I}(x, t)$ is continuous on $[0, +\infty) \times [0, +\infty)$. Similarly, as $V(1) = 0$, we know that $\bar{R}$ is also continuous on $[0, +\infty)\times[0, +\infty)$. Detailed calculations yield $\bar{S}_t - d\bar{S}_{xx} = 0 \ge \Lambda - \delta\bar{S}$ and, for $0 \le x \le \vartheta(t)$,

$$\begin{aligned}
&\bar{I}_t - d\bar{I}_{xx} - \beta\bar{S}\bar{I} + (\alpha + \gamma + \delta)\bar{I}\\
= {}& \iota e^{-\varepsilon_2 t}\left[-\varepsilon_2\tilde{\phi} - \frac{x h_0 \vartheta'(t)}{\vartheta^2(t)}\tilde{\phi}_x - \frac{d h_0^2}{\vartheta^2(t)}\tilde{\phi}_{xx} - \beta(N + \varepsilon_1)\tilde{\phi} + (\alpha + \gamma + \delta)\tilde{\phi}\right]\\
= {}& \iota e^{-\varepsilon_2 t}\left\{-\varepsilon_2\tilde{\phi} - \frac{x h_0 \vartheta'(t)}{\vartheta^2(t)}\tilde{\phi}_x - \frac{h_0^2}{\vartheta^2(t)}\left[-\beta(N + \varepsilon_1)\tilde{\phi} + (\alpha + \gamma + \delta)\tilde{\phi} - \widetilde{\lambda}_1\tilde{\phi}\right]\right.\\
&\left. -\beta(N + \varepsilon_1)\tilde{\phi} + (\alpha + \gamma + \delta)\tilde{\phi}\right\}\\
= {}& \iota e^{-\varepsilon_2 t}\left[-\varepsilon_2\tilde{\phi} - \frac{x h_0 \vartheta'(t)}{\vartheta^2(t)}\tilde{\phi}_x + \left(\frac{h_0^2}{\vartheta^2(t)} - 1\right)\beta(N + \varepsilon_1)\tilde{\phi} + \left(1 - \frac{h_0^2}{\vartheta^2(t)}\right)(\alpha + \gamma + \delta)\tilde{\phi} + \frac{h_0^2}{\vartheta^2(t)}\widetilde{\lambda}_1\tilde{\phi}\right]\\
\ge {}& \tilde{\phi}\iota e^{-\varepsilon_2 t}\left\{-\varepsilon_2 + \frac{h_0^2}{\vartheta^2(t)}\left[\beta(N + \varepsilon_1) + \widetilde{\lambda}_1\right] - \beta(N + \varepsilon_1)\right\}\\
\ge {}& \tilde{\phi}\iota e^{-\varepsilon_2 t}\left\{-\varepsilon_2 + \frac{h_0^2}{h_0^2(1 + \varepsilon_2)^2}\left[\beta(N + \varepsilon_1) + \widetilde{\lambda}_1\right] - \beta(N + \varepsilon_1)\right\}\\
\ge {}& \tilde{\phi}\iota e^{-\varepsilon_2 t}\left\{-\varepsilon_2 + \frac{1}{(1 + \varepsilon_2)^2}\left[\beta(N + \varepsilon_1) + \widetilde{\lambda}_1\right] - \beta(N + \varepsilon_1)\right\}.
\end{aligned}$$

Here we have used the fact that $\tilde{\phi}_x < 0$ for $x \in (0, h_0]$. It follows that $\bar{I}_t - d\bar{I}_{xx} - \beta\bar{S}\bar{I} + (\alpha + \gamma + \delta)\bar{I} \ge 0$. On the other hand, as $h_0 < h_*$, we can obtain

$$\bar{R}_t - d\bar{R}_{xx} - \gamma\bar{I} + \delta\bar{R} \ge \iota e^{-\varepsilon_2 t}\left(-\varepsilon_2 - \gamma + \frac{d}{8h_0^2}\right) \ge \iota e^{-\varepsilon_2 t}\left(-2\gamma + \frac{d}{8h_0^2}\right) \ge 0.$$

Moreover,

$$-\mu \bar{I}_x(\vartheta(t), t) = -\mu\iota e^{-\varepsilon_2 t}\tilde{\phi}_x(h_0)\frac{h_0}{\vartheta(t)}.$$

If we choose $0 < \iota \le -\varepsilon_2^2 h_0(1 + \frac{\varepsilon_2}{2})/2\mu\tilde{\phi}_x(h_0)$, then

$$\vartheta'(t) \ge -\mu \bar{I}_x(\vartheta(t), t)$$

since $\tilde{\phi}_x(h_0) < 0$. Obviously, $\bar{S}(x, 0) \ge \|S_0\|_\infty$. If $\|I_0\|_\infty \le \iota\phi(\frac{x}{1+\frac{\varepsilon_2}{2}})$ and $\|R_0\|_\infty \le V(\frac{x}{h_0(1+\frac{\varepsilon_2}{2})})$ for $x \in [0, h_0]$, then $I_0(x) \le \bar{I}(x, 0)$ and $R_0(x) \le \bar{R}(x, 0)$ for $x > 0$. This proves that $(\bar{S}, \bar{I}, \bar{R}; \vartheta(t))$ is an upper solution of (1.5). Thus, similalrly as in the proof of Theorem 3.2, we can get $h(t) \le \vartheta(t)$, which yields $h_\infty < \lim_{t\to+\infty} \vartheta(t) = h_0(1 + \varepsilon_2) < +\infty$. This completes the proof. □

We provide a sufficient condition on disease spreading to conclude this subsection.

Theorem 3.4. *If $\mathcal{R}_0 > 1$ and $h_0 > h^* := \sqrt{\frac{d\delta\pi^2}{4\beta\Lambda(1-\frac{1}{\mathcal{R}_0})}}$, then $h_\infty = +\infty$.*

Proof. By way of contradiction, we assume that $h_\infty < +\infty$. It follows from Lemma 3.1 that $\lim_{t\to+\infty} \|I(\cdot, t)\|_{C([0,h(t)])} = 0$. Moreover, $\lim_{t\to+\infty} S(x, t) = \frac{\Lambda}{\delta}$ uniformly in any bounded subset of $[0, +\infty)$.

Since $h_0 > h^*$ and $\mathcal{R}_0 > 1$, we have $\lambda_1 < 0$, where λ_1 is the principal eigenvalue of the eigenvalue problem (3.2). Choose $\iota > 0$ such that $\lambda_1 + \beta\iota < 0$ and $\mathcal{R}_0 > 1 + \frac{\beta\iota}{\alpha+\delta+\gamma}$ (which implies that $\beta(\frac{\Lambda}{\delta} - \iota) - \delta - \alpha - \gamma > 0$). For this ι, there exists $T^* > 0$ such that $S(x, t) \ge \frac{\Lambda}{\delta} - \iota$ and $I(x, t) < 1$ for $x \in [0, h(t)]$ and $t > T^*$. Then $I(x, t)$ satisfies

$$\begin{cases} I_t - dI_{xx} \ge \left[\beta(\dfrac{\Lambda}{\delta} - \iota) - \delta - \alpha - \gamma\right] I(1 - I), & 0 < x < h_0, t > T^*, \\ I_x(0, t) = 0, \quad I(h_0, t) \ge 0, & t > T^*, \\ I(x, T^*) > 0, & 0 \le x < h_0. \end{cases}$$

It is easy to see that $I(x, t) \ge \underline{I}(x, t)$, where $\underline{I}(x, t)$ satisfies

$$\begin{cases} \underline{I}_t - d\underline{I}_{xx} = \left[\beta(\dfrac{\Lambda}{\delta} - \iota) - \delta - \alpha - \gamma\right] \underline{I}(1 - \underline{I}), & 0 < x < h_0, t > T^*, \\ \underline{I}_x(0, t) = 0, \quad \underline{I}(h_0, t) = 0, & t > T^*, \\ \underline{I}(x, T^*) = I(x, T^*), & 0 \le x < h_0. \end{cases} \tag{3.3}$$

Consider the following eigenvalue problem

$$\begin{cases} d\phi_{xx} + \left[\beta(\dfrac{\Lambda}{\delta} - \iota) - \delta - \alpha - \gamma\right]\phi + \lambda\phi = 0, & 0 < x < h_0, \\ \phi_x(0) = \phi(h_0) = 0, \end{cases}$$

whose principal eigenvalue is

$$\widehat{\lambda}_1 = \frac{d\pi^2}{4h_0^2} - \left[\beta\left(\frac{\Lambda}{\delta} - \iota\right) - \delta - \alpha - \gamma\right] = \lambda_1 + \beta\iota < 0.$$

Employing Proposition 3.2 and Proposition 3.3 of [16], we obtain that $\lim_{t\to+\infty} \underline{I}(t,x) = \underline{I}(x)$ uniformly in $x \in [0, h_0]$, where $\underline{I}(x) > 0$ satisfies

$$\begin{cases} -d\underline{I}_{xx} = \left[\beta(\frac{\Lambda}{\delta} - \iota) - \delta - \alpha - \gamma\right]\underline{I}(1 - \underline{I}), & 0 < x < h_0, \\ \underline{I}_x(0) = 0, \quad \underline{I}(h_0) = 0. \end{cases}$$

It follows that $\liminf_{t\to+\infty} I(x,t) \geq \lim_{t\to+\infty} \underline{I}(x,t) = \underline{I}(x) > 0$ uniformly in $x \in [0, h_0]$, which contradicts with $\lim_{t\to+\infty} \|I(\cdot,t)\|_{C([0,h(t)])} = 0$. Therefore, we have proved $h_\infty = +\infty$. □

Remark 3.1. *Obviously, $h_0 > h^*$ is equivalent to $d < d^* \triangleq \frac{4h_0^2\beta\Lambda(1-\frac{1}{\mathcal{R}_0})}{\delta\pi^2}$. As a result, if $\mathcal{R}_0 > 1$ and $0 < d < d^*$, then $h_\infty = +\infty$.*

3.3. The case where $\eta > 0$

In this case, the basic reproduction number $\widetilde{\mathcal{R}}_0$ of problem (1.3) is given by

$$\widetilde{\mathcal{R}}_0 = \frac{\beta\Lambda(\delta + \eta)}{\delta[\gamma\delta + (\delta + \eta)(\alpha + \delta)]}.$$

As in the case where $\eta = 0$, we start with disease vanishing.

Theorem 3.5. *If $\widetilde{\mathcal{R}}_0 \leq 1$, then $\lim_{t\to+\infty} \|I(\cdot,t)\|_{C([0,h(t)])} = 0$. Moreover, $\lim_{t\to+\infty} \|R(\cdot,t)\|_{C([0,h(t)])} = 0$ and $\lim_{t\to+\infty} S(x,t) = \frac{\Lambda}{\delta}$ uniformly in any bounded subset of* $[0, +\infty)$.

Proof. Consider the following system of ordinary differential equations,

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta S(t)I(t) - \delta S(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - (\alpha + \gamma + \delta)I(t) + \eta R(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\delta + \eta)R(t), \end{cases} \tag{3.4}$$

with $(S(0), I(0), R(0)) = (\|S_0\|_\infty, \|I_0\|_\infty, \|R_0\|_\infty)$. As in the proof of Theorem 3.2, a result similar as [12, Lemma 4.1] implies that $S(x,t) \leq S(t)$ for $(x,t) \in [0,+\infty)\times(0,+\infty)$, and $I(x,t) \leq I(t)$ and $R(x,t) \leq R(t)$ for $(x,t) \in \{(x,t) : x \in [0,h(t)], t \in (0,+\infty)\}$.

Obviously, (3.4) has a disease-free equilibrium $E_0 = (\frac{\Lambda}{\delta}, 0, 0)$, which is globally asymptotically stable. Indeed, consider $V : \mathbb{R}_+^3 \to \mathbb{R}$ defined by

$$V(S,I,R) = (\delta + \eta)\left(S - S^0 - S^0 \ln\frac{S}{S^0}\right) + (\delta + \eta)I + \eta R. \tag{3.5}$$

It is clear that $V(S,I,R)$ reaches its global minimum in $\mathbb{R}_+^3$ only at E_0. Moreover, the derivative of (3.5) with respect to t along solutions of (3.4) is

$$\frac{d}{dt}V(S,I,R) = (\delta + \eta)\frac{S - S^0}{S}\frac{dS}{dt} + (\delta + \eta)\frac{dI}{dt} + \eta\frac{dR}{dt}$$

$$\begin{aligned} &= (\delta+\eta)\frac{S-S^0}{S}(\Lambda-\beta SI-\delta S) \\ &\quad +(\delta+\eta)[\beta SI-(\alpha+\gamma+\delta)I+\eta R]+\eta[\gamma I-(\delta+\eta)R] \\ &= (\delta+\eta)\frac{S-S^0}{S}(\Lambda-\beta SI-\delta S) \\ &\quad +(\delta+\eta)[\beta SI-(\alpha+\gamma+\delta)I+\eta R]+\eta[\gamma I-(\delta+\eta)R]. \end{aligned}$$

Using the expression

$$\beta SI\frac{(S-S^0)}{S^0}=\beta I\frac{(S-S^0)^2}{S^0}+\beta I(S-S^0),$$

we obtain

$$\begin{aligned} \frac{d}{dt}V(S,I,R) &= (\delta+\eta)\frac{S-S^0}{S}(\Lambda-\beta SI-\delta S) \\ &\quad +(\delta+\eta)[\beta SI-(\alpha+\gamma+\delta)I+\eta R]+\eta[\gamma I-(\delta+\eta)R] \\ &= -(\delta+\eta)\frac{(S-S^0)^2}{S} \\ &\quad +[\gamma\delta+(\delta+\eta)(\alpha+\delta)]I\left[\frac{(\delta+\eta)S^0\beta}{(\gamma\delta+\delta+\eta)(\alpha+\delta)}-1\right] \\ &= -(\eta+\delta)\frac{(S-S^0)^2}{S}-[\gamma\delta+(\delta+\eta)(\alpha+\delta)]I(1-\widetilde{\mathcal{R}}_0). \end{aligned}$$

Since $\widetilde{\mathcal{R}}_0 \le 1$, we have $\frac{d}{dt}V(S,I,R) \le 0$ for $S > 0$. Moreover, if $\frac{d}{dt}V(S,I,R) = 0$ holds then $S = S^0$. It is easy to verify from this that the disease-free equilibrium E_0 is the largest invariant set in the set where $\frac{d}{dt}V(S,I,R) = 0$. Therefore, by LaSalle's invariance principle [17], E_0 is globally asymptotically stable. This, combined with the above estimates, gives us

$$\begin{aligned} &\limsup_{t\to+\infty} S(x,t) \le \lim_{t\to+\infty} S(t) = \frac{\Lambda}{\delta} \quad \text{uniformly for } x\in[0,+\infty), \\ &\limsup_{t\to\infty} I(x,t) \le \lim_{t\to\infty} I(t) = 0 \quad \text{uniformly in any bounded subset of } [0,h_\infty), \\ &\limsup_{t\to\infty} R(x,t) \le \lim_{t\to\infty} R(t) = 0 \quad \text{uniformly in any bounded subset of } [0,h_\infty), \end{aligned}$$

which implies that

$$\lim_{t\to+\infty}\|I(\cdot,t)\|_{C([0,h(t)])}=\lim_{t\to+\infty}\|R(\cdot,t)\|_{C([0,h(t)])}=0.$$

Then it follows from Lemma 3.1 that $\lim_{t\to+\infty} S(x,t) = \frac{\Lambda}{\delta}$ uniformly in any bounded subset of $[0,+\infty)$ and this completes the proof. □

Now we provide a sufficient condition on disease spreading.

Theorem 3.6. *If* $\widetilde{\mathcal{R}}_0 > \mathcal{R}_0 > 1$ *and* $h_0 > h^* := \sqrt{\frac{d\delta\pi^2}{4\beta\Lambda(1-\frac{1}{\mathcal{R}_0})}}$, *then* $h_\infty = +\infty$.

Proof. We know that $(S(x,t), I(x,t), R(x,t); h(t))$ satisfies

$$\begin{cases} S_t(x,t) = dS_{xx}(x,t) + \Lambda - \beta S(x,t)I(x,t) - \delta S(x,t), & x > 0, t > 0, \\ I_t(x,t) \geq dI_{xx}(x,t) + \beta S(x,t)I(x,t) - (\alpha+\gamma+\delta)I(x,t), & 0 < x < h(t), t > 0, \\ R_t(x,t) = dR_{xx}(x,t) + \gamma I(x,t) - (\delta+\eta)R(x,t), & 0 < x < h(t), t > 0, \\ S_x(0,t) = I_x(0,t) = R_x(0,t) = 0, & t > 0, \\ I(h(t),t) = R(h(t),t) = 0, & x \geq h(t),\ t > 0, \\ h'(t) = -\mu I_x(h(t),t), & t > 0, \\ h(0) = h_0, & \\ S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq 0,\ R(x,0) = R_0(x) \geq 0, & x \geq 0. \end{cases}$$

A result similar as [12, Lemma 4.1] for lower solutions gives $S(x,t) \geq \underline{S}(x,t)$ for $0 < x < +\infty$ and $t > 0$; $I(x,t) \geq \underline{I}(x,t)$ and $R(x,t) \geq \underline{R}(x,t)$ for $0 < x < \underline{h}(t)$ and $t > 0$; and $h(t) \geq \underline{h}(t)$ for $t > 0$, where $(\underline{S}(x,t), \underline{I}(x,t), \underline{R}(x,t); \underline{h}(t))$ satisfies

$$\begin{cases} \underline{S}_t(x,t) = d\underline{S}_{xx} + \Lambda - \beta \underline{S}(x,t)\underline{I}(x,t) - \delta \underline{S}(x,t), & x > 0, t > 0, \\ \underline{I}_t(x,t) = d\underline{I}_{xx} + \beta \underline{S}(x,t)\underline{I}(x,t) - (\alpha+\gamma+\delta)\underline{I}(x,t), & 0 < x < \underline{h}(t), t > 0, \\ \underline{R}_t(x,t) = d\underline{R}_{xx} + \gamma \underline{I}(x,t) - (\delta+\eta)\underline{R}(x,t), & 0 < x < \underline{h}(t), t > 0, \\ \underline{S}_x(0,t) = \underline{I}_x(0,t) = \underline{R}_x(0,t) = 0, & t > 0, \\ \underline{I}(h(t),t) = \underline{R}(h(t),t) = 0, & x \geq \underline{h}(t), t > 0, \\ \underline{h}'(t) = -\mu I_x(\underline{h}(t),t), & t > 0, \\ \underline{h}(0) = h_0, & \\ \underline{S}(x,0) = S_0(x) \geq 0,\ \underline{I}(x,0) = I_0(x) \geq 0, \underline{R}(x,0) = R_0(x) \geq 0, & x \geq 0. \end{cases}$$

It follows from Theorem 3.4 that if $\widetilde{\mathcal{R}}_0 > \mathcal{R}_0 > 1$ and $h_0 > h^*$ then $\underline{h}_\infty = +\infty$, which implies $h_\infty = +\infty$. □

4. Conclusion and discussion

In this paper, we proposed and analyzed a free boundary problem of a reaction-diffusion SIRI model with the bilinear incidence rate. We first obtained the existence and uniqueness of global solutions. Then we established several criteria on disease vanishing and spreading. Roughly speaking, for the case without relapse, the disease will vanish if one of the following three conditions holds. (a) The basic reproduction number $\mathcal{R}_0 < 1$; (b) $\mathcal{R}_0 > 1$ and the initial infected area h_0 and the boundary moving rate μ are small enough; (c) $\mathcal{R}_0 > 1$ together with the initial values $||I_0||_\infty$, $||R_0||_\infty$, and h_0 being small enough. The disease will spread to the whole area if $\mathcal{R}_0 > 1$ and either h_0 is large enough or the diffusion rate d is small enough. For the case with relapse, the disease will die out if the basic reproduction number $\widetilde{\mathcal{R}}_0 \leq 1$ whereas the disease will spread to the whole area if $\widetilde{\mathcal{R}}_0 > \mathcal{R}_0 > 1$ and h_0 is large enough. Unfortunately, we have not considered the case where $\widetilde{R}_0 > 1 > \mathcal{R}_0$. In this case, the disease transmission is complex, which we are working on. Moreover, when the free boundaries can extend to the whole area, we also gave an estimate on the spreading speed.

Compared with the ordinary differential equation model (1.1), the model we studied with free boundary allows more reasonable sufficient conditions on the disease spreading and vanishing. With the main results obtained, we can better understand the phenomenon of relapse. To illustrate this, we demonstrate how the basic reproduction numbers rely on the relapse rate η. For system (1.3), fix other parameters except η, we see that $\mathcal{R}_0^*(\eta) = \widetilde{\mathcal{R}}_0 = \frac{\beta\Lambda(\delta+\eta)}{\delta(\gamma\delta+(\delta+\eta)(\alpha+\delta))}$, which is a strictly increasing function of η. Thus there exists an $\eta^* \in [0, +\infty)$ such that $\mathcal{R}_0^*(\eta) \geq 1$ when $\eta \geq \eta^*$ and $\mathcal{R}_0^*(\eta) < 1$ when $\eta < \eta^*$. Then the relapse rate η plays an important role in $\mathcal{R}_0^*(\eta)$. In other words, when η varies, disease spreading and vanishing will change. Since $\mathcal{R}_0^*(\eta) > \mathcal{R}_0$ always holds, with relapse the disease will be more easily spread to the whole area than without relapse.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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