



Research article

Continuous dependence of an invariant measure on the jump rate of a piecewise-deterministic Markov process

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Abstract: We investigate a piecewise-deterministic Markov process, evolving on a Polish metric space, whose deterministic behaviour between random jumps is governed by some semi-flow, and any state right after the jump is attained by a randomly selected continuous transformation. It is assumed that the jumps appear at random moments, which coincide with the jump times of a Poisson process with intensity λ . The model of this type, although in a more general version, was examined in our previous papers, where we have shown, among others, that the Markov process under consideration possesses a unique invariant probability measure, say ν_λ^* . The aim of this paper is to prove that the map $\lambda \mapsto \nu_\lambda^*$ is continuous (in the topology of weak convergence of probability measures). The studied dynamical system is inspired by certain stochastic models for cell division and gene expression.

Keywords: invariant measure; piecewise-deterministic Markov process; random dynamical system; jump rate; continuous dependence

1. Introduction

Piecewise-deterministic Markov processes (PDMPs) originate with M.H.A. Davis [1]. They constitute an important class of Markov processes that is complementary to those defined by stochastic differential equations. PDMPs are encountered as suitable mathematical models for processes in the physical world around us, e.g., in resource allocation and service provisioning (queing, cf. [1]) or biology: as stochastic models for gene expression and autoregulation [2, 3], cell division [4], excitable membranes [5] or population dynamics [6, 7].

Mathematical research on PDMPs has been conducted over the years in various directions. Applications in control and optimization have been just one direction. The fundamentals of existence and uniqueness of invariant probability measures for Markov operators and semigroups of Markov operators associated with PDMPs, as well as their asymptotic properties, have attracted much

attention. See e.g., [8–11], where the considered underlying state space is locally compact. The theory for the general case of non-locally compact Polish state space is less developed yet. It is considered e.g., in [3, 5, 12–15]. Another direction is that of establishing the validity of the Strong Law of Large Numbers (SLLN), the Central Limit Theorem (CLT) and the Law of the Iterated Logarithm (LIL) for non-stationary PDMPs (cf. [16–19]), which has interest in itself for non-stationary processes in general [20].

In this paper, we are concerned with a special case of the PDMP described in [13, 14], whose deterministic motion between jumps depends on a single continuous semi-flow, and any post-jump location is attained by a continuous transformation of the pre-jump state, randomly selected (with a place-dependent probability) among all possible ones. The jumps in this model occur at random time points according to a homogeneous Poisson process. The random dynamical systems of this type constitute a mathematical framework for certain particular biological models, such as those for gene expression [2] or cell division [4].

The aim of the paper is to establish the continuous (in the Fortet-Mourier distance, cf. [21, Section 8.3]) dependence of the invariant measure on the rate of a Poisson process determining the frequency of jumps. While the SLLN and the CLT provide the theoretical foundation for successful approximation of the invariant measure by means of observing or simulating (many) sample trajectories of the process, this result asserts the stability of this procedure, at least locally in parameter space. It is a prerequisite for the development of a bifurcation theory. Moreover, even stronger regularity of this dependence on parameter (i.e., differentiability in a suitable norm on the space of measures) would be needed for applications in control theory or for parameter estimation (see e.g., [22]).

The outline of the paper is as follows. In Section 2, several facts on integrating measure-valued functions and basic definitions from the theory of Markov operators have been compiled. Section 3 deals with the structure and assumptions of the model under study. In Section 4, we establish certain auxiliary results on the transition operator of the Markov chain given by the post-jump locations. More specifically, we show that the operator is jointly continuous (in the topology of weak convergence of measures) as a function of measure and the jump-rate parameter, and that the weak convergence of the distributions of the chain to its unique stationary distribution must be uniform. Section 5 is the essential part of the paper. Here, we establish the announced results on the continuous dependence of the invariant measure on the jump frequency for both, the discrete-time system, constituted by the post jump-locations, and for the PDMP itself.

2. Preliminaries

Let X be a closed subset of some separable Banach space $(H, \|\cdot\|)$, endowed with the σ -field \mathcal{B}_X consisting of its Borel subsets. Further, let $(BM(X), \|\cdot\|_\infty)$ stand for the Banach space of all bounded Borel-measurable functions $f : X \rightarrow \mathbb{R}$ with the supremum norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$. By $BC(X)$ and $BL(X)$ we shall denote the subspaces of $BM(X)$ consisting of all continuous and all Lipschitz continuous functions, respectively. Let us further introduce

$$\|f\|_{BL} := \max \left\{ \|f\|_\infty, |f|_{Lip} \right\} \quad \text{for any } f \in BL(X),$$

where

$$\|f\|_{Lip} := \sup \left\{ \frac{|f(x) - f(y)|}{\|x - y\|} : x, y \in X, x \neq y \right\}.$$

It is well-known (cf. [23, Proposition 1.6.2]) that $\|\cdot\|_{BL}$ defines a norm in $BL(X)$, for which it is a Banach space.

In what follows, we will write $(\mathcal{M}_{sig}(X), \|\cdot\|_{TV})$ for the Banach space of all finite, countably additive functions (signed measures) on \mathcal{B}_X , endowed with the total variation norm $\|\cdot\|_{TV}$, which can be expressed as

$$\|\mu\|_{TV} := |\mu|(X) = \sup \{ |\langle f, \mu \rangle| : f \in BM(X), \|f\|_\infty \leq 1 \} \quad \text{for } \mu \in \mathcal{M}_{sig}(X),$$

where

$$\langle f, \mu \rangle := \int_X f(x) \mu(dx)$$

and $|\mu|$ stands for the absolute variation of μ (cf. e.g., [24]). The symbols $\mathcal{M}_+(X)$ and $\mathcal{M}_1(X)$ will be used to denote the subsets of $\mathcal{M}_{sig}(X)$, consisting of all non-negative and all probability measures on \mathcal{B}_X , respectively. Moreover, we will write $\mathcal{M}_{1,1}(X)$ for the set of all measures $\mu \in \mathcal{M}_1(X)$ with finite first moment, i.e., satisfying $\langle \|\cdot\|, \mu \rangle < \infty$.

Let us now define, for any $\mu \in \mathcal{M}_{sig}(X)$, the linear functional $I_\mu : BL(X) \rightarrow \mathbb{R}$ given by

$$I_\mu(f) = \langle f, \mu \rangle \quad \text{for } f \in BL(X).$$

It is easy to show that $I_\mu \in BL(X)^*$ for every $\mu \in \mathcal{M}_{sig}(X)$, where $BL(X)^*$ stands for the dual space of $(BL(X), \|\cdot\|_{BL})$ with the operator norm $\|\cdot\|_{BL}^*$ given by

$$\|\varphi\|_{BL}^* := \sup \{ |\varphi(f)| : f \in BL(X), \|f\|_{BL} \leq 1 \} \quad \text{for any } \varphi \in BL(X)^*.$$

Moreover, we have $\|I_\mu\|_{BL}^* \leq \|\mu\|_{TV}$ for any $\mu \in \mathcal{M}_{sig}(X)$.

Furthermore, it is well known (see [25, Lemma 6]), that the mapping

$$\mathcal{M}_{sig}(X) \ni \mu \mapsto I_\mu \in BL(X)^*$$

is injective, and thus the space $(\mathcal{M}_{sig}(X), \|\cdot\|_{TV})$ may be embedded into $(BL(X)^*, \|\cdot\|_{BL}^*)$. This enables us to identify each measure $\mu \in \mathcal{M}_{sig}(X)$ with the functional $I_\mu \in BL(X)^*$. Note that $\|\cdot\|_{BL}^*$ induces a norm on $\mathcal{M}_{sig}(X)$, called the Fortet-Mourier (or bounded Lipschitz, cf. e.g., [26, 27]) norm and denoted by $\|\cdot\|_{FM}$. Consequently, we can write

$$\|\mu\|_{FM} := \|I_\mu\|_{BL}^* = \sup \{ |\langle f, \mu \rangle| : f \in BL(X), \|f\|_{BL} \leq 1 \} \quad \text{for any } \mu \in \mathcal{M}_{sig}(X).$$

As we have already seen, generally $\|\mu\|_{FM} = \|I_\mu\|_{BL}^* \leq \|\mu\|_{TV}$ for any $\mu \in \mathcal{M}_{sig}(X)$. However, for positive measures the norms coincide, i.e., $\|\mu\|_{FM} = \mu(X) = \|\mu\|_{TV}$ for all $\mu \in \mathcal{M}_+(X)$ (cf. [25]).

Let us now write $\mathcal{D}(X)$ and $\mathcal{D}_+(X)$ for the linear space and the convex cone, respectively, generated by the set $\{\delta_x : x \in X\} \subset BL(X)^*$ of functionals of the form

$$\delta_x(f) := f(x) \quad \text{for any } f \in BL(X), x \in X,$$

which can be also viewed as Dirac measures. It is not hard to check that the $\|\cdot\|_{BL}^*$ -closure of $\mathcal{D}(X)$ is a separable Banach subspace of $BL(X)^*$. Moreover, one can show (cf. [27, Theorems 2.3.8–2.3.19]) that $\mathcal{M}_+(X) = \text{cl } \mathcal{D}_+(X)$ (using the completeness of X), which in turn implies that $\mathcal{M}_{sig}(X)$ is a $\|\cdot\|_{BL}^*$ -dense subspace of $\text{cl } \mathcal{D}(X)$, i.e., $\text{cl } \mathcal{M}_{sig}(X) = \text{cl } \mathcal{D}(X)$. The key idea underlying the proof of this result is to show that every measure $\mu \in \mathcal{M}_+(X)$ can be represented by the Bochner integral (for definition see e.g., [28]) of the continuous map $X \ni x \mapsto \delta_x \in \text{cl } \mathcal{D}(X)$, i.e.,

$$\mu = \int_X \delta_x \mu(dx) \in \text{cl } \mathcal{D}_+(X).$$

In particular, it follows that $(\text{cl } \mathcal{M}_{sig}(X), \|\cdot\|_{BL}^*|_{\text{cl } \mathcal{D}(X)})$ is a separable Banach space.

What is more, according to [27, Theorem 2.3.22], the dual space of $\text{cl } \mathcal{M}_{sig}(X) = \text{cl } \mathcal{D}(X)$ with the operator norm

$$\|\kappa\|_{\text{cl } \mathcal{D}}^{**} := \sup\{|\kappa(\varphi)| : \varphi \in \text{cl } \mathcal{D}(X), \|\varphi\|_{BL}^* \leq 1\}, \quad \kappa \in [\text{cl } \mathcal{D}(X)]^*,$$

is isometrically isomorphic with the space $(BL(X), \|\cdot\|_{BL})$, and each functional $\kappa \in [\text{cl } \mathcal{D}(X)]^*$ can be represented by some $f \in BL(X)$, in the sense that $\kappa(\varphi) = \varphi(f)$ for $\varphi \in \text{cl } \mathcal{D}(X)$. In particular, we then have $\kappa(\mu) = I_\mu(f) = \langle f, \mu \rangle$ whenever $\mu \in \mathcal{M}_{sig}(X)$ (by identifying μ with I_μ).

In view of the above observations, the norm $\|\cdot\|_{BL}^*$ is convenient for integrating (in the Bochner sense) measure-valued functions $p : E \rightarrow \mathcal{M}_{sig}(X)$, where E is an arbitrary measure space. The Pettis measurability theorem (see e.g., [28, Chapter II, Theorem 2]), together with the separability of $\text{cl } \mathcal{M}_{sig}(X)$, ensures that p is strongly measurable as a map with values in $\text{cl } \mathcal{M}_{sig}(X)$ (i.e., it is a pointwise a.e. limit of simple functions) if and only if, for any $f \in BL(X)$, the functional $E \ni t \mapsto \langle f, p(t) \rangle \in \mathbb{R}$ is measurable. Moreover, we have at our disposal the following result (see [27, Propositions 3.2.3–3.2.5] or [29, Proposition C.2]), which provides a tractable condition guaranteeing the integrability of p and ensuring that the integral is an element of $\mathcal{M}_{sig}(X)$:

Theorem 2.1. *Let (E, Σ) be a measurable space endowed with a σ -finite measure ν , and let $p : E \rightarrow \mathcal{M}_{sig}(X)$ be a strongly measurable function. Suppose that there exists a real-valued function $g \in \mathcal{L}^1(E, \Sigma, \nu)$ such that*

$$\|p(t)\|_{TV} \leq g(t) \quad \text{for a.e. } t \in E.$$

Then the following conditions hold:

- (i) *The function p is Bochner ν -integrable as a map acting from (E, Σ) to $(\text{cl } \mathcal{M}_{sig}(X), \|\cdot\|_{BL}^*|_{\text{cl } \mathcal{D}(X)})$. Moreover, we have*

$$\left\| \int_E p(t) \nu(dt) \right\|_{TV} \leq \int_E \|p(t)\|_{TV} \nu(dt).$$

- (ii) *The Bochner integral $\int_E p(t) \nu(dt) \in \text{cl } \mathcal{M}_{sig}(X)$ belongs to $\mathcal{M}_{sig}(X)$ and satisfies*

$$\left(\int_E p(t) \nu(dt) \right) (A) = \int_E p(t)(A) \nu(dt) \quad \text{for any } A \in \mathcal{B}_X.$$

Another crucial observation is that the restriction of the weak topology on $\mathcal{M}_{sig}(X)$, generated by $BC(X)$, to $\mathcal{M}_+(X)$ equals to the topology induced by the norm $\|\cdot\|_{FM}|_{\mathcal{M}_+(X)}$ (cf. [25, Theorem 18] or [21, Theorem 8.3.2]). In particular, the following holds:

Theorem 2.2. Let $\mu_n, \mu \in \mathcal{M}_+(X)$ for every $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} \|\mu_n - \mu\|_{FM} = 0$ if and only if $\mu_n \xrightarrow{w} \mu$, as $n \rightarrow \infty$, that is,

$$\lim_{n \rightarrow \infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle \quad \text{for any } f \in BC(X).$$

Let us now recall several basic definitions concerning Markov operators acting on measures. First of all, a function $P : X \times \mathcal{B}_X \rightarrow [0, 1]$ is called a stochastic kernel if, for any fixed $A \in \mathcal{B}_X$, $x \mapsto P(x, A)$ is a Borel-measurable map on X , and, for any fixed $x \in X$, $A \mapsto P(x, A)$ is a probability Borel measure on \mathcal{B}_X . We can consider two operators corresponding to a stochastic kernel P , namely

$$\mu P(A) = \int_X P(x, A) \mu(dx) \quad \text{for } \mu \in \mathcal{M}_{sig}(X), A \in \mathcal{B}_X \quad (2.1)$$

and

$$Pf(x) = \int_X f(y) P(x, dy) \quad \text{for } f \in BM(X), x \in X. \quad (2.2)$$

The operator $(\cdot)P : \mathcal{M}_{sig}(X) \rightarrow \mathcal{M}_{sig}(X)$, given by (2.1), is called a regular Markov operator. It is easy to check that

$$\langle f, \mu P \rangle = \langle Pf, \mu \rangle \quad \text{for any } f \in BM(X), \mu \in \mathcal{M}_{sig}(X),$$

and, therefore, $P(\cdot) : BM(X) \rightarrow BM(X)$, defined by (2.2), is said to be the dual operator of $(\cdot)P$.

A regular Markov operator $(\cdot)P$ is said to be Feller if its dual operator $P(\cdot)$ preserves continuity, that is, $Pf \in BC(X)$ for every $f \in BC(X)$. A measure $\mu^* \in \mathcal{M}_+(X)$ is called an invariant measure for $(\cdot)P$ whenever $\mu^* P = \mu^*$.

We will say that the operator $(\cdot)P$ is exponentially ergodic in the Fortet-Mourier distance if there exists a unique invariant measure $\mu^* \in \mathcal{M}_1(X)$ of $(\cdot)P$, for which there is $q \in [0, 1)$ such that, for any $\mu \in \mathcal{M}_{1,1}(X)$ and some constant $C(\mu)$, we have

$$\|\mu P^n - \mu^*\|_{FM} \leq C(\mu) q^n \quad \text{for any } n \in \mathbb{N}.$$

The measure μ^* is then usually called exponentially attracting.

A regular Markov semigroup $(P(t))_{t \in \mathbb{R}_+}$ is a family of regular Markov operators $(\cdot)P(t) : \mathcal{M}_{sig}(X) \rightarrow \mathcal{M}_{sig}(X)$, $t \in \mathbb{R}_+ := [0, \infty)$, which form a semigroup (under composition) with the identity transformation $(\cdot)P(0)$ as the unity element. Provided that $(\cdot)P(t)$ is a Markov-Feller operator for every $t \in \mathbb{R}_+$, the semigroup $(P(t))_{t \in \mathbb{R}_+}$ is said to be Markov-Feller, too. If, for some $\nu^* \in \mathcal{M}_{sig}(X)$, $\nu^* P(t) = \nu^*$ for every $t \in \mathbb{R}_+$, then ν^* is called an invariant measure of $(P(t))_{t \in \mathbb{R}_+}$.

3. Description of the model

Recall that X is a closed subset of some separable Banach space $(H, \|\cdot\|)$, and let $(\Theta, \mathcal{B}_\Theta, \vartheta)$ be a topological measure space with a σ -finite Borel measure ϑ . With a slight abuse of notation, we will further write $d\theta$ only, instead of $\vartheta(d\theta)$.

Let us consider a PDMP $(X(t))_{t \in \mathbb{R}_+}$, evolving on the space X through random jumps occurring at the jump times τ_n , $n \in \mathbb{N}$, of a homogeneous Poisson process with intensity $\lambda > 0$. The state right after the jump is attained by a transformation $w_\theta : X \rightarrow X$, randomly selected from the set $\{w_\theta : \theta \in \Theta\}$. The probability of choosing w_θ is determined by a place-dependent density function $\theta \mapsto p(x, \theta)$, where x describes the state of the process just before the jump. It is required that the maps $(x, \theta) \mapsto p(x, \theta)$

and $(x, \theta) \mapsto w_\theta(x)$ are continuous. Between the jumps, the process is deterministically driven by a continuous (with respect to each variable) semi-flow $S : \mathbb{R}_+ \times X \rightarrow X$. The flow property means, as usual, that $S(0, x) = x$ and $S(s + t, x) = S(s, S(t, x))$ for any $x \in X$ and any $s, t \in \mathbb{R}_+$.

Let us now move on to the formal description of the model. Given $\lambda > 0$ and $\mu \in \mathcal{M}_1(X)$, on some suitable probability space, we first define a discrete-time stochastic process $(X_n)_{n \in \mathbb{N}_0}$ with initial distribution μ , so that

$$X_{n+1} = w_{\theta_{n+1}}(S(\Delta\tau_{n+1}, X_n)) \quad \text{for every } n \in \mathbb{N}_0,$$

with $\Delta\tau_{n+1} := \tau_{n+1} - \tau_n$, where $(\tau_n)_{n \in \mathbb{N}_0}$ and $(\theta_n)_{n \in \mathbb{N}}$ are sequences of random variables with values in \mathbb{R}_+ and Θ , respectively, defined in such a way that $\tau_0 = 0$, $\tau_n \rightarrow \infty$ \mathbb{P}_μ -a.s., as $n \rightarrow \infty$, and

$$\begin{aligned} \mathbb{P}_\mu(\Delta\tau_{n+1} \leq t | W_n) &= 1 - e^{-\lambda t} \quad \text{for any } t \in \mathbb{R}_+, n \in \mathbb{N}_0, \\ \mathbb{P}_\mu(\theta_{n+1} \in B | S(\Delta\tau_{n+1}, X_n) = x, W_n) &= \int_B p(x, \theta) d\theta \quad \text{for any } x \in X, B \in \mathcal{B}_\Theta, n \in \mathbb{N}_0, \end{aligned}$$

with $W_0 := X_0$ and $W_n := (W_0, \tau_1, \dots, \tau_n, \theta_1, \dots, \theta_n)$ for $n \in \mathbb{N}$. We also demand that, for any $n \in \mathbb{N}_0$, the variables $\Delta\tau_{n+1}$ and θ_{n+1} are conditionally independent given W_n .

A standard computation shows that $(X_n)_{n \in \mathbb{N}_0}$ is a time-homogeneous Markov chain with transition law $P_\lambda : X \times \mathcal{B}_X \rightarrow [0, 1]$ given by

$$P_\lambda(x, A) = \int_0^\infty \lambda e^{-\lambda t} \int_\Theta p(S(t, x), \theta) \mathbb{1}_A(w_\theta(S(t, x))) d\theta dt \quad \text{for } x \in X, A \in \mathcal{B}_X, \quad (3.1)$$

that is,

$$P_\lambda(x, A) = P(X_{n+1} \in A | X_n = x) \quad \text{for any } x \in X, A \in \mathcal{B}_X, n \in \mathbb{N}_0.$$

On the same probability space, we now define a Markov process $(X(t))_{t \in \mathbb{R}_+}$, as an interpolation of the chain $(X_n)_{n \in \mathbb{N}_0}$, namely

$$X(t) = S(t - \tau_n, X_n) \quad \text{for } t \in [\tau_n, \tau_{n+1}), n \in \mathbb{N}_0.$$

By $(P_\lambda(t))_{t \in \mathbb{R}_+}$ we shall denote the Markov semigroup associated with the process $(X(t))_{t \in \mathbb{R}_+}$, so that, for any $t \in \mathbb{R}_+$, $P_\lambda(t)$ is the Markov operator corresponding to the stochastic kernel satisfying

$$P_\lambda(t)(x, A) = \mathbb{P}_\mu(X(s+t) \in A | X(s) = x) \quad \text{for any } A \in \mathcal{B}_X, x \in X, s \in \mathbb{R}_+. \quad (3.2)$$

We further assume that there exist a point $\bar{x} \in X$, a Borel measurable function $J : X \rightarrow [0, \infty)$ and constants $\alpha \in \mathbb{R}$, $L, L_w, L_p, \lambda_{\min}, \lambda_{\max}, \bar{p} > 0$, such that

$$LL_w + \frac{\alpha}{\lambda} < 1 \quad \text{for each } \lambda \in [\lambda_{\min}, \lambda_{\max}], \quad (3.3)$$

and, for any $x, y \in X$, the following conditions hold:

$$\varkappa := \sup_{x \in X} \int_0^\infty e^{-\lambda_{\min} t} \int_\Theta p(S(t, x), \theta) \|w_\theta(S(t, \bar{x}))\| d\theta dt < \infty, \quad (3.4)$$

$$\|S(t, x) - S(t, y)\| \leq L e^{\alpha t} \|x - y\| \quad \text{for } t \in \mathbb{R}_+, \quad (3.5)$$

$$\|S(t, x) - S(s, x)\| \leq (t - s) e^{\max\{\alpha s, \alpha t\}} J(x) \quad \text{for } 0 \leq s \leq t, \quad (3.6)$$

$$\int_{\Theta} p(x, \theta) \|w_{\theta}(x) - w_{\theta}(y)\| d\theta \leq L_w \|x - y\|, \quad (3.7)$$

$$\int_{\Theta} |p(x, \theta) - p(y, \theta)| d\theta \leq L_p \|x - y\|, \quad (3.8)$$

$$\int_{\Theta(x,y)} \min\{p(x, \theta), p(y, \theta)\} d\theta \geq \bar{p}, \quad \text{where } \Theta(x, y) := \{\theta \in \Theta : \|w_{\theta}(x) - w_{\theta}(y)\| \leq L_w \|x - y\|\}. \quad (3.9)$$

Note that, upon assuming (3.3), we have $\lambda > \max\{0, \alpha\}$ for any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$. In what follows, we will write shortly

$$\bar{\alpha} := \max\{0, \alpha\}. \quad (3.10)$$

Moreover, let us introduce

$$\mathcal{M}_{sig,J}(X) = \{\mu \in \mathcal{M}_{sig}(X) : \langle J, |\mu| \rangle < \infty\},$$

where J is given in (3.6).

Note that, if $(H, \langle \cdot | \cdot \rangle)$ is a Hilbert space and $A : X \rightarrow H$ is an α -dissipative operator with $\alpha \leq 0$, i.e.,

$$\langle Ax - Ay | x - y \rangle \leq \alpha \|x - y\|^2 \quad \text{for any } x, y \in X,$$

which additionally satisfies the so-called range condition, that is, for some $T > 0$,

$$X \subset \text{Range}(\text{id}_X - tA) \quad \text{for } t \in (0, T),$$

then, for any $x \in X$, the Cauchy problem of the form

$$\begin{cases} y'(t) = A(y(t)) \\ y(0) = x \end{cases}$$

has a unique solution $t \mapsto S(t, x)$ such that the semi-flow S enjoys conditions (3.5), with $L = 1$, and (3.6), with $J(x) = \|Ax\|$ (cf. [30, Theorem 5.3 and Corollary 5.4], as well as [13, Section 3]).

Moreover, upon assuming compactness of Θ , condition (3.4) can be derived from the conjunction of (3.6) and (3.7) at least in two cases: whenever p does not depend on the pre-jump state, i.e., $p(y, \theta) = \bar{p}(\theta)$ for some continuous density function $\bar{p} : \Theta \rightarrow \mathbb{R}_+$, or if all the transformations w_{θ} , $\theta \in \Theta$, are Lipschitz continuous with a common Lipschitz constant L_w (see [13, Corollary 3.4] for the proof).

Furthermore, note that conditions formulated in a manner similar to (3.7)–(3.9) are commonly required while examining the asymptotic properties of random iterated function systems (see [26, 31, 32]), which are covered by the discrete-time model discussed here (in the case where $S(t, x) = x$). In this connection, it is also worth mentioning that the example described in [33] indicates that the condition of type (3.8) cannot be omitted even in the simplest cases. More precisely, the system $\{(w_1, p), (w_2, 1 - p)\}$, consisting of two contractive maps w_1, w_2 and a positive continuous probability function p , may admit more than one invariant probability measure (unless at least the Dini continuity of p is assumed).

Finally, let us indicate that conditions (3.3)–(3.9) are naturally satisfied by a few particular biological models, such as e.g., the model for gene expression [2] (cf. also [13, Section 5]), the model of autoregulated gene expression [3] or the one for cell division [4, 15] (see also [34]).

4. Some properties of the operator P_λ

Consider the abstract model introduced in Section 3. In order to simplify notation, for any $t \in \mathbb{R}_+$, let us introduce the function $\Pi_{(t)} : X \times \mathcal{B}_X \rightarrow [0, 1]$ given by

$$\Pi_{(t)}(x, A) := \int_{\Theta} p(S(t, x), \theta) \mathbb{1}_A(w_\theta(S(t, x))) d\theta \quad \text{for } x \in X, A \in \mathcal{B}_X. \quad (4.1)$$

Note that $\Pi_{(t)}$ is a stochastic kernel, and that the corresponding Markov operator is Feller, due to the continuity of $p(\cdot, \theta)$, $S(t, \cdot)$ and w_θ , $\theta \in \Theta$. Moreover, observe that, for an arbitrary $\lambda > 0$, we have

$$\begin{aligned} \mu P_\lambda(A) &= \int_X \int_0^\infty \lambda e^{-\lambda t} \Pi_{(t)}(x, A) dt \mu(dx) = \int_0^\infty \lambda e^{-\lambda t} \int_X \Pi_{(t)}(x, A) \mu(dx) dt \\ &= \int_0^\infty \lambda e^{-\lambda t} \mu \Pi_{(t)}(A) dt \quad \text{for any } \mu \in \mathcal{M}_{sig}(X), A \in \mathcal{B}_X. \end{aligned} \quad (4.2)$$

Lemma 4.1. *Suppose that conditions (3.6)–(3.8) hold. Then, for any $\lambda > 0$ and any $\mu \in \mathcal{M}_{sig, J}(X)$, the function $t \mapsto e^{-\lambda t} \mu \Pi_{(t)}$ is Bochner integrable as a map from \mathbb{R}_+ to $(cl \mathcal{M}_{sig}(X), \|\cdot\|_{BL}^*|_{cl \mathcal{M}_{sig}(X)})$, and we have*

$$\mu P_\lambda = \int_0^\infty \lambda e^{-\lambda t} \mu \Pi_{(t)} dt.$$

Proof. Let $\lambda > 0$ and $\mu \in \mathcal{M}_{sig}(X)$. Note that condition (3.6) implies that

$$\|S(t, x) - S(s, x)\| \leq J(x) e^{\bar{\alpha}(t+s)} |t - s| \quad \text{for any } s, t \in \mathbb{R}_+, x \in X,$$

where $\bar{\alpha}$ is given by (3.10). Hence, applying (3.7) and (3.8), we see that, for every $f \in BL(X)$,

$$\begin{aligned} |\langle f, \mu \Pi_{(t)} \rangle - \langle f, \mu \Pi_{(s)} \rangle| &= |\langle \Pi_{(t)} f - \Pi_{(s)} f, \mu \rangle| \\ &\leq \int_X \int_{\Theta} p(S(t, x), \theta) |f(w_\theta(S(t, x))) - f(w_\theta(S(s, x)))| d\theta |\mu|(dx) \\ &\quad + \int_X \int_{\Theta} |p(S(t, x), \theta) - p(S(s, x), \theta)| |f(w_\theta(S(s, x)))| d\theta |\mu|(dx) \\ &\leq (|f|_{Lip} L_w + \|f\|_\infty L_p) \int_X \|S(t, x) - S(s, x)\| |\mu|(dx) \\ &\leq \|f\|_{BL} (L_w + L_p) \langle J, |\mu| \rangle e^{\bar{\alpha}(t+s)} |t - s| \quad \text{for any } s, t \in \mathbb{R}_+. \end{aligned}$$

This shows that the map $t \mapsto \langle f, e^{-\lambda t} \mu \Pi_{(t)} \rangle$ is continuous for any $f \in BL(X)$, and thus it is Borel measurable. Consequently, it now follows from the Pettis measurability theorem (cf. [28]) that the map $t \mapsto e^{-\lambda t} \mu \Pi_{(t)}$ is strongly measurable. Furthermore, we have

$$\|e^{-\lambda t} \mu \Pi_{(t)}\|_{TV} \leq \|\mu\|_{TV} e^{-\lambda t} \quad \text{for any } t \in \mathbb{R}_+,$$

which, due to Theorem 2.1, yields that $t \mapsto e^{-\lambda t} \mu \Pi_{(t)} \in cl \mathcal{M}_{sig}(X)$ is Bochner integrable (with respect to the Lebesgue measure) on \mathbb{R}_+ , and that the integral is a measure in $\mathcal{M}_{sig}(X)$, which satisfies

$$\left(\int_0^\infty \lambda e^{-\lambda t} \mu \Pi_{(t)} dt \right) (A) = \int_0^\infty \lambda e^{-\lambda t} \mu \Pi_{(t)}(A) dt \quad \text{for any } A \in \mathcal{B}_X.$$

The assertion of the lemma now follows from (4.2). \square

Lemma 4.2. Let $f \in BL(X)$. Upon assuming (3.5), (3.7) and (3.8), we have

$$\|\mu\Pi_{(t)}\|_{FM} \leq \left(1 + (L_w + L_p)Le^{\alpha t}\right)\|\mu\|_{FM} \quad \text{for any } \mu \in \mathcal{M}_{sig}(X), t \in \mathbb{R}_+.$$

Proof. Let $f \in BL(X)$ be such that $\|f\|_{BL} \leq 1$. Obviously, $\|\Pi_{(t)}f\|_{\infty} \leq 1$ for every $t \in \mathbb{R}_+$. Moreover, from conditions (3.5), (3.7), (3.8) it follows that $\Pi_{(t)}f \in BL(X)$, and

$$\|\Pi_{(t)}f\|_{Lip} \leq (L_w + L_p)Le^{\alpha t} \quad \text{for any } t \in \mathbb{R}_+,$$

since

$$\begin{aligned} |\Pi_{(t)}f(x) - \Pi_{(t)}f(y)| &= \left| \int_{\Theta} p(S(t, x), \theta) f(w_{\theta}(S(t, x))) d\theta - \int_{\Theta} p(S(t, y), \theta) f(w_{\theta}(S(t, y))) d\theta \right| \\ &\leq (L_w + L_p)\|S(t, x) - S(t, y)\| \leq (L_w + L_p)Le^{\alpha t}\|x - y\| \quad \text{for all } x, y \in X, t \in \mathbb{R}_+. \end{aligned}$$

Therefore, for any $\mu \in \mathcal{M}_{sig}(X)$ and any $t \in \mathbb{R}_+$, we obtain

$$|\langle f, \mu\Pi_{(t)} \rangle| = |\langle \Pi_{(t)}f, \mu \rangle| \leq \|\Pi_{(t)}f\|_{BL} \|\mu\|_{FM},$$

which gives the desired conclusion. \square

Lemma 4.3. For any $\lambda_1, \lambda_2 > 0$, we have

$$\int_0^{\infty} |\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}| dt \leq |\lambda_1 - \lambda_2| \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right).$$

Proof. Without loss of generality, we may assume that $\lambda_1 < \lambda_2$. Since $1 - e^{-x} \leq x$ for every $x \in \mathbb{R}$, we obtain

$$\begin{aligned} \int_0^{\infty} |\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}| dt &\leq \lambda_1 \int_0^{\infty} |e^{-\lambda_1 t} - e^{-\lambda_2 t}| dt + (\lambda_2 - \lambda_1) \int_0^{\infty} e^{-\lambda_2 t} dt \\ &= \lambda_1 \int_0^{\infty} e^{-\lambda_1 t} (1 - e^{-(\lambda_2 - \lambda_1)t}) dt + \frac{\lambda_2 - \lambda_1}{\lambda_2} \\ &\leq \lambda_1 (\lambda_2 - \lambda_1) \int_0^{\infty} e^{-\lambda_1 t} t dt + \frac{(\lambda_2 - \lambda_1)}{\lambda_2} = |\lambda_1 - \lambda_2| \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right), \end{aligned}$$

which completes the proof. \square

Lemma 4.4. Let $\mathcal{M}_{sig}(X)$ be endowed with the topology induced by the norm $\|\cdot\|_{FM}$, and suppose that conditions (3.5)–(3.8) hold. Then, the map $(\bar{\alpha}, \infty) \times \mathcal{M}_{sig,J}(X) \ni (\lambda, \mu) \mapsto \mu P_{\lambda} \in \mathcal{M}_{sig}(X)$, where $\bar{\alpha}$ is given by (3.10), is jointly continuous.

Proof. Let $\lambda_1, \lambda_2 > \bar{\alpha}$ and $\mu_1, \mu_2 \in \mathcal{M}_{sig,J}(X)$. Note that, due to Lemma 4.1, we have

$$\begin{aligned} \|\mu_1 P_{\lambda_1} - \mu_2 P_{\lambda_2}\|_{FM} &= \left\| \int_0^{\infty} (\lambda_1 e^{-\lambda_1 t} \mu_1 \Pi_{(t)} - \lambda_2 e^{-\lambda_2 t} \mu_2 \Pi_{(t)}) dt \right\|_{FM} \\ &\leq \|\mu_1\|_{TV} \int_0^{\infty} |\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}| dt + \int_0^{\infty} \lambda_2 e^{-\lambda_2 t} \|\mu_1 \Pi_{(t)} - \mu_2 \Pi_{(t)}\|_{FM} dt, \end{aligned}$$

where the inequality follows from statement (i) of Theorem 2.1 and the fact that $\|\mu_1 \Pi_{(t)}\|_{TV} \leq \|\mu_1\|_{TV}$. Further, applying Lemmas 4.2 and 4.3, we obtain

$$\|\mu_1 P_{\lambda_1} - \mu_2 P_{\lambda_2}\|_{FM} \leq \|\mu_1\|_{TV} |\lambda_1 - \lambda_2| \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) + \|\mu_1 - \mu_2\|_{FM} \left(1 + (L_w + L_p) L \frac{\lambda_2}{\lambda_2 - \alpha} \right).$$

We now see that $\|\mu_1 P_{\lambda_1} - \mu_2 P_{\lambda_2}\|_{FM} \rightarrow 0$, as $|\lambda_1 - \lambda_2| \rightarrow 0$ and $\|\mu_1 - \mu_2\|_{FM} \rightarrow 0$, which completes the proof. \square

Suppose that conditions (3.4), (3.5) and (3.7)–(3.9) hold. Then, according to [13, Theorem 4.1] (or [14, Theorem 4.1]), for any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ satisfying $LL_w + \alpha\lambda^{-1} < 1$, there exist a unique invariant measure $\mu_\lambda^* \in \mathcal{M}_{1,1}(X)$ for P_λ and constants $q_\lambda \in (0, 1)$, $C_\lambda \in \mathbb{R}_+$ such that

$$\|\mu P_\lambda^n - \mu_\lambda^*\|_{FM} \leq q_\lambda^n C_\lambda (1 + \langle V, \mu \rangle + \langle V, \mu_\lambda^* \rangle) \quad \text{for any } \mu \in \mathcal{M}_{1,1}(X) \text{ and any } n \in \mathbb{N}, \tag{4.3}$$

where $V : X \rightarrow [0, \infty)$ is given by $V(x) = \|x - \bar{x}\|$.

Following the proof of [13, Theorem 4.1], we may conclude that q_λ and C_λ depend only on the jump rate of the PDMP and other constants appearing in conditions (3.3)–(3.5) and (3.7)–(3.9) (note that they do not depend on the structure of the model, that is the definitions of S , w_θ and p).

Upon assuming (3.3)–(3.5) and (3.7)–(3.9), there exists $C_0 > 0$ such that

$$\langle V, \mu_\lambda^* \rangle \leq C_0 \quad \text{for any } \lambda \in [\lambda_{\min}, \lambda_{\max}]. \tag{4.4}$$

Indeed, let us first define

$$a := \frac{\lambda_{\max} LL_w}{\lambda_{\min} - \alpha} \quad \text{and} \quad b := \lambda_{\max} \varkappa,$$

where \varkappa is given in (3.4), and observe that $a < 1$, due to (3.3). Proceeding similarly as in Step I of the proof of [13, Theorem 4.1], we see that conditions (3.5) and (3.7) imply the following:

$$P_\lambda V(x) \leq aV(x) + b \quad \text{for any } x \in X \text{ and any } \lambda \in [\lambda_{\min}, \lambda_{\max}],$$

which further gives

$$P_\lambda^n V(x) \leq a^n V(x) + \frac{b}{1 - a} \quad \text{for any } n \in \mathbb{N} \text{ and any } \lambda \in [\lambda_{\min}, \lambda_{\max}].$$

Now, let $C_0 := b(1 - a)^{-1}$. Then, using the fact that μ_λ^* is an invariant measure of P_λ , we get

$$\langle V, \mu_\lambda^* \rangle = \langle V, \mu_\lambda^* P_\lambda^n \rangle = \langle P_\lambda^n V, \mu_\lambda^* \rangle \leq a^n \langle V, \mu_\lambda^* \rangle + C_0 \quad \text{for any } n \in \mathbb{N} \text{ and any } \lambda \in [\lambda_{\min}, \lambda_{\max}].$$

Going with n to infinity, we obtain the desired estimation (4.4). As a consequence, we may write (4.3) in the following form:

$$\|\mu P_\lambda^n - \mu_\lambda^*\|_{FM} \leq q_\lambda^n \tilde{C}_\lambda (1 + \langle V, \mu \rangle) \quad \text{for any } \mu \in \mathcal{M}_{1,1}(X) \text{ and any } n \in \mathbb{N}, \tag{4.5}$$

where $\tilde{C}_\lambda := C_\lambda(1 + C_0)$.

Lemma 4.5. *Suppose that conditions (3.4), (3.5) and (3.7)–(3.9) hold with constants satisfying (3.3), and, for any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, let μ_λ^* stand for the unique invariant probability measure of P_λ . Then, the convergence $\|\mu P_\lambda^n - \mu_\lambda^*\|_{FM} \rightarrow 0$ (as $n \rightarrow \infty$) is uniform with respect to λ , whenever $\mu \in \mathcal{M}_{1,1}(X)$.*

Proof. In view of [13, Theorem 4.1], it is sufficient to prove that the convergence is uniform with respect to λ .

Let us consider the case where $\alpha \leq 0$. Choose an arbitrary $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, and note that, by substituting $t = \lambda_{\max} \lambda^{-1} u$, the operator P_λ can be expressed in the following form:

$$\begin{aligned} \mu P_\lambda(A) &= \int_X \int_0^\infty \lambda e^{-\lambda t} \int_\Theta p(S(t, x), \theta) \mathbb{1}_A(w_\theta(S(t, x))) \, d\theta \, dt \, \mu(dx) \\ &= \int_X \int_0^\infty \lambda_{\max} e^{-\lambda_{\max} u} \int_\Theta p(S_\lambda(u, x), \theta) \mathbb{1}_A(w_\theta(S_\lambda(u, x))) \, d\theta \, du \, \mu(dx) \end{aligned}$$

for any $\mu \in \mathcal{M}_1(X)$, $A \in \mathcal{B}_X$, where

$$S_\lambda(u, x) := S\left(\frac{\lambda_{\max}}{\lambda} u, x\right) \quad \text{for } u \in \mathbb{R}_+, x \in X.$$

Moreover, the semi-flow S_λ enjoys condition (3.5), since, for any $t \in \mathbb{R}_+$ and any $x, y \in X$, we have

$$\|S_\lambda(t, x) - S_\lambda(t, y)\| \leq L e^{\alpha \lambda_{\max} \lambda^{-1} t} \|x - y\| \leq L e^{\alpha t} \|x - y\|.$$

Hence, we can write $P_\lambda = \tilde{P}_{\lambda_{\max}}$, where $\tilde{P}_{\lambda_{\max}}$ stands for the Markov operator corresponding to the instance of our system with the jump intensity λ_{\max} and the flow S_λ in place of S . Taking into account the above observation, it is evident that such a modified system still satisfies conditions (3.4)–(3.5) and (3.7)–(3.9) with constants determined by the primary model, which additionally satisfy $LL_w + \alpha \lambda_{\max}^{-1} < 1$. Consequently, μ_λ^* is then an invariant measure of $\tilde{P}_{\lambda_{\max}}$, and hence we can denote it by $\tilde{\mu}_{\lambda_{\max}}^*$. Finally, keeping in mind (4.5), we can conclude that there exist $q_{\lambda_{\max}} \in (0, 1)$ and $\tilde{C}_{\lambda_{\max}} \in \mathbb{R}_+$ such that

$$\|\mu P_\lambda^n - \mu_\lambda^*\|_{FM} = \|\mu \tilde{P}_{\lambda_{\max}}^n - \tilde{\mu}_{\lambda_{\max}}^*\|_{FM} \leq q_{\lambda_{\max}}^n \tilde{C}_{\lambda_{\max}} (1 + \langle V, \mu \rangle) \quad \text{for any } \mu \in \mathcal{M}_{1,1}(X), n \in \mathbb{N}.$$

In the case where $\alpha > 0$, the proof is similar to the one conducted above (except that this time we substitute $t := \lambda_{\min} \lambda^{-1} u$), so we omit it. □

5. Main results

Before we formulate and prove the main theorems of this paper, let us first quote the result provided in [35, Theorem 7.11].

Lemma 5.1. *Let (Y, ϱ) and (Z, d) be some metric spaces, and let E be an arbitrary subset of Y . Suppose that $(f_n)_{n \in \mathbb{N}_0}$ is a sequence of functions, defined on E , with values in Z , which converges uniformly on E to some function $f : E \rightarrow Z$. Further, let \bar{y} be a limit point of E , and assume that*

$$a_n := \lim_{y \rightarrow \bar{y}} f_n(y)$$

exists and is finite for every $n \in \mathbb{N}_0$. Then, f has a finite limit at \bar{y} , and the sequence $(a_n)_{n \in \mathbb{N}_0}$ converges to it, that is,

$$\lim_{n \rightarrow \infty} \left(\lim_{y \rightarrow \bar{y}} f_n(y) \right) = \lim_{y \rightarrow \bar{y}} \left(\lim_{n \rightarrow \infty} f_n(y) \right).$$

We are now in a position to state the result concerning the continuous dependence of an invariant measure μ_λ^* of P_λ on the parameter λ . In the proof we will refer to the lemmas provided in Section 4, as well as to Lemma 5.1.

Theorem 5.2. *Suppose that conditions (3.4)–(3.9) hold with constants satisfying (3.3), and, for any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, let μ_λ^* stand for the unique invariant probability measure of P_λ . Then, for every $\bar{\lambda} \in [\lambda_{\min}, \lambda_{\max}]$, we have $\mu_\lambda^* \xrightarrow{w} \mu_{\bar{\lambda}}^*$, as $\lambda \rightarrow \bar{\lambda}$.*

Proof. Let $\bar{\lambda} \in [\lambda_{\min}, \lambda_{\max}]$. Due to Lemma 4.5, we know that, for every $\mu \in \mathcal{M}_1(X)$ and any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, we have $\|\mu P_\lambda^n - \mu_\lambda^*\|_{FM} \rightarrow 0$, as $n \rightarrow \infty$, and the convergence is uniform with respect to λ .

Further, since $\mathcal{M}_1(X) \subset \mathcal{M}_{sig,J}(X)$, Lemma 4.4 yields that $(\bar{\alpha}, \infty) \times \mathcal{M}_1(X) \ni (\lambda, \mu) \mapsto \mu P_\lambda \in \mathcal{M}_1(X)$ is jointly continuous, provided that $\mathcal{M}_1(X)$ is equipped with the topology induced by the Fortet-Mourier norm. Hence, for any $\mu \in \mathcal{M}_1(X)$ and any $n \in \mathbb{N}_0$, it follows that $\|\mu P_\lambda^n - \mu P_{\bar{\lambda}}^n\|_{FM} \rightarrow 0$, as $\lambda \rightarrow \bar{\lambda}$. Finally, according to Lemma 5.1, we get

$$\lim_{\lambda \rightarrow \bar{\lambda}} \mu_\lambda^* = \lim_{\lambda \rightarrow \bar{\lambda}} \left(\lim_{n \rightarrow \infty} \mu P_\lambda^n \right) = \lim_{n \rightarrow \infty} \left(\lim_{\lambda \rightarrow \bar{\lambda}} \mu P_\lambda^n \right) = \lim_{n \rightarrow \infty} P_{\bar{\lambda}}^n \mu = \mu_{\bar{\lambda}}^*,$$

where the limits are taken in $(\mathcal{M}_{sig}(X), \|\cdot\|_{FM})$. This, together with Theorem 2.2, gives the desired conclusion. \square

In the final part of the paper we will study the properties of the Markov semigroup $(P_\lambda(t))_{t \in \mathbb{R}_+}$, defined by (3.2). In order to apply the relevant results of [13], in what follows, we additionally assume that the measure ϑ , given on the set Θ , is finite. Then, according to [13, Theorem 4.4], for any $\lambda > 0$, there is a one-to-one correspondence between invariant measures of the operator P_λ and those of the semigroup $(P_\lambda(t))_{t \in \mathbb{R}_+}$. More precisely, if $\mu_\lambda^* \in \mathcal{M}_1(X)$ is a unique invariant probability measure of P_λ , then $\nu_\lambda^* := \mu_\lambda^* G_\lambda \in \mathcal{M}_1(X)$, where

$$\mu G_\lambda(A) = \int_X \int_0^\infty \lambda e^{-\lambda t} \mathbb{1}_A(S(t, x)) dt \mu(dx) \quad \text{for any } \mu \in \mathcal{M}_1(X), \quad A \in \mathcal{B}_X,$$

is a unique invariant probability measure of $(P_\lambda(t))_{t \in \mathbb{R}_+}$.

The main result concerning the continuous-time model, which is formulated and proven below, ensures the continuity of the map $\lambda \mapsto \nu_\lambda^*$.

Theorem 5.3. *Let ϑ be a finite Borel measure on Θ . Further, suppose that conditions (3.4)–(3.9) hold with constants satisfying (3.3), and, for any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, let ν_λ^* stand for the unique invariant probability measure of $(P_\lambda(t))_{t \in \mathbb{R}_+}$. Then, for any $\bar{\lambda} \in [\lambda_{\min}, \lambda_{\max}]$, we have $\nu_\lambda^* \xrightarrow{w} \nu_{\bar{\lambda}}^*$, as $\lambda \rightarrow \bar{\lambda}$.*

Proof. Let $\bar{\lambda} \in [\lambda_{\min}, \lambda_{\max}]$, and let $f \in BL(X)$ be such that $\|f\|_{BL} \leq 1$. For any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, we have

$$\langle f, \nu_\lambda^* \rangle = \langle f, \mu_\lambda^* G_\lambda \rangle = \int_X \int_0^\infty \lambda e^{-\lambda t} f(S(t, x)) dt \mu_\lambda^*(dx),$$

whence

$$\left| \langle f, v_\lambda^* - v_{\bar{\lambda}}^* \rangle \right| \leq \int_0^\infty |\lambda e^{-\lambda t} - \bar{\lambda} e^{-\bar{\lambda} t}| dt + \left| \int_0^\infty \bar{\lambda} e^{-\bar{\lambda} t} \langle f \circ S(t, \cdot), \mu_\lambda^* - \mu_{\bar{\lambda}}^* \rangle dt \right|.$$

Note that, due to (3.5), $f \circ S(t, \cdot) \in BL(X)$ and $\|f \circ S(t, \cdot)\|_{BL} \leq 1 + Le^{\alpha t}$, and therefore

$$\begin{aligned} \left| \int_0^\infty \bar{\lambda} e^{-\bar{\lambda} t} \langle f \circ S(t, \cdot), \mu_\lambda^* - \mu_{\bar{\lambda}}^* \rangle dt \right| &\leq \|\mu_\lambda^* - \mu_{\bar{\lambda}}^*\|_{FM} \int_0^\infty \bar{\lambda} e^{-\bar{\lambda} t} (1 + Le^{\alpha t}) dt \\ &= \|\mu_\lambda^* - \mu_{\bar{\lambda}}^*\|_{FM} \left(1 + \frac{L\bar{\lambda}}{\bar{\lambda} - \alpha} \right). \end{aligned}$$

Combining this and Lemma 4.3, finally gives

$$\|v_\lambda^* - v_{\bar{\lambda}}^*\|_{FM} \leq |\lambda - \bar{\lambda}| \left(\frac{1}{\lambda} + \frac{1}{\bar{\lambda}} \right) + c \|\mu_\lambda^* - \mu_{\bar{\lambda}}^*\|_{FM}$$

with $c := 1 + L\bar{\lambda}(\bar{\lambda} - \alpha)^{-1}$. Hence, referring to Theorems 5.2 and 2.2, we obtain

$$\lim_{\lambda \rightarrow \bar{\lambda}} \|v_\lambda^* - v_{\bar{\lambda}}^*\|_{FM} = 0,$$

and the proof is completed. \square

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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