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Research article

Renormalization group analysis of heat transfer in the presence of endothermic and exothermic chemical reactions

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Abstract: In the present paper, renormalization group methods are used to develop a macroscopic turbulence model for thermal diffusivity in turbulent fluid flow under conditions of endothermic and exothermic chemical reactions in flow. The temperature field is divided into slow (large-scale) and fast (small-scale) modes. With the help of the renormalization procedure, energy equations for the large-scale modes and relations for effective turbulent thermal diffusivity were obtained. It was shown how the type of the chemical reaction affects turbulent thermal diffusivity. In addition, the conditions were identified where effective thermal diffusivity undergoes a sharp growth.

Keywords: renormalization group; two-phase flow; turbulence model

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1. Introduction

The present paper focuses on the further development of the renormalization group (RNG) theory, which has been widely used to produce turbulence models for pure one-phase fluids as applied to porous media. The definition of a renormalization group includes a set of symmetry transformations, which operate on a space of parameters. The RNG theory has been initially developed for quantum field research and served as a standard method that helps eliminating singularities [1]. Afterwards the RNG theory was used by Wilson [2] to study critical phenomena.

Renormalized perturbation methods were employed by Kraichnan [3] for theoretical studies of turbulence phenomena. As a result, Kraichnan [3] proposed a theory of direct interaction approximation. The RNG theory was applied by Forster et al. [4] for investigations of turbulence problems. This work further developed the RNG theory for power-law forcing for situations where the small scales affect the large scales like eddy viscosity. The RNG model was also utilized by DeDominicis and Martin [5] for turbulence studies, which, however, involved the field formulation. Fournier and Frisch [6] demonstrated the possibility of using the RNG model to obtain amplitude coefficients, which are universal constants independent of the parameters of the system in the small scales area, for instance, molecular viscosity.

Yakhot and Orszag [7] employed the RNG theory to develop a closed model of turbulence. Based on the findings of the work [4], they proposed a model for large scales; simultaneously, a modified transport coefficient described the effect of small scales. The equations for the large scales were obtained through averaging over an infinitesimally narrow range of small scales. Afterwards, the small scales were eliminated from the further consideration. The elimination was performed iteratively and resulted in a differential equation for effective transport coefficients. After the elimination of small scales, the equation for large scales remained unchanged. Using the RNG theory, Yakhot and Orszag [7] obtained relations for turbulent viscosity, equations for a passive scalar, the turbulent Prandtl number, the Batchelor constant, turbulence kinetic energy and the dissipation rate. Kraichnan [8] discussed and commented the theory developed by Yakhot and Orszag [7]. Smith and Reynolds [9] validated the theory of Yakhot and Orszag [7] and further improved and refined the derived relations for the skewness and the dissipation rate. Yakhot and Orszag [7] developed not only the main model, but also a RNG subgrid scale turbulence model for large-eddy simulation. Using the RNG theory, Yaknot et al. [10] obtained an additional term in the equation for the dissipation rate, which plays an important role for flows with anisotropic turbulence. The RNG theory was further used in in-depth studies of turbulence by in the works [11–15].

Avramenko and Kuznetsov [16] developed a macroscopic (large-scale) RNG turbulence model for incompressible flow in porous media. The model considers the large-distance and large-time performance of velocity correlations arising in the momentum equation for a randomly stirred and incompressible flow. The renormalization model resulted in obtaining transport equations for large-scale modes and relations for effective transport coefficients. Avramenko and Kuznetsov [16] obtained also relations for renormalized turbulent viscosity taking into account the ultraviolet subrange of the turbulent kinetic energy spectrum. RNG technique was also used by Avramenko et al. [17] to study nonlinear instability of turbulent flow in porous media.

Avramenko et al. [18] used renormalization group methods to develop a macroscopic turbulence model for incompressible two-phase turbulent flows. The authors obtained expressions for effective turbulent viscosity, which revealed that the presence of the second phase causes decreased turbulent viscosity.

Direct numerical simulations of reacting isotropic turbulence decay under conditions typical of a hypersonic turbulent boundary layer flow were performed in the work [19]. The feedback in the Reynolds stress occurs through the pressure-strain term, which depends on how much heat is released. Authors of the work [20] demonstrated that chemical reactions affect the onset of oscillatory instability, which is more likely to occur for the case of liquids, while for gases only stationary instability is possible for a practical range of parameter values. Maleque [21] paid attention only to laminar heat and mass transfer in the boundary layer flow in the presence of chemical reactions. In the work [22], effects of chemical reactions were studied on the development of turbulent plume.

Another important group of solutions for turbulent boundary layer flows with heat, momentum and mass transfer is represented by the approach developed first by Avramenko et al. [23]. Boundary layer equations were reduced to a self-similar form using the Lie group approach based on the symmetry properties of the transport equations. The resulting self-similar equations were further solved numerically. To enable obtaining the similarity solution, turbulent viscosity was assumed to follow the mixing-length model in combination with the Lie group technique. This methodology was further applied by Avramenko et al. [24] to model a turbulent boundary layer of a nanofluid over a flat plate. The model of Avramenko et al. [24] accounts also for the dependences of physical properties (viscosity, thermal conductivity, and diffusion coefficients) on the nanofluid concentration and temperature. This model was further developed by Avramenko et al. [25] to simulate stable film boiling of nanofluids over a vertical surface. In spite of the mathematical elegance and novelty, the approaches and models [23–25], because of their relative simplicity, do not enable to reveal and model complex turbulence phenomena such as effects of the second phase (particles or droplets) or chemical reactions on the parameters of turbulence, such as velocity and temperature pulsations.

To conclude, there are indications in the literature that chemical reactions affect the onset of instability and Reynolds stresses. However, the effects of chemical reactions on thermal diffusivity in one-phase or two-phase flows have not been studied either experimentally, or theoretically (analytically and numerically).

2. Problem formulation

Turbulent flows with chemical reactions are important parts of transport processes in many industrial and environmental settings. For example, flows which are developed at pool fire, if the vaporization rate of fuel is large enough. Similar processes occur in industrial burners, burners of power stations, combustion chambers etc. Correct modeling of turbulence is required for adequate simulation of such processes.

The main objective of the present investigation consisted in developing a theoretical macroscopic turbulence model for thermal diffusivity in turbulent fluid flow under conditions of endothermic and exothermic chemical reactions. The RNG technique will be employed to obtain an analytical solution for turbulent thermal diffusivity. The temperature field will be divided into slow (long-wavelength) and fast (short-wavelength) modes. The renormalization procedure will enable obtaining energy equations for the long-wavelength modes and relations for effective turbulent thermal diffusivity. It will be demonstrated that the type of the chemical reaction affects turbulent thermal diffusivity.

The present simulation will be performed under the following assumptions: the physical properties of the medium (density, viscosity, thermal conductivity, heat capacity) are assumed to be constant; the radiative heat transfer is not taken into account; the external random force assumed to be Gaussian, solenoidal, isotropic in space, and homogeneous white noise in time; turbulence energy is described by the Kolmogorov law «$\kappa^{-5/3}$».

As it is known [7], the equation for the dissipation rate in the RNG model includes an additional term that considers the influence of small values of the Reynolds numbers, so that this model is not limited to the certain range of the Reynolds numbers. Also, no restrictions are imposed on the numerical values of Prandtl and Schmidt numbers.

3. Basic equations

We will focus on three-dimensional unsteady incompressible turbulent flow with the streamwise pressure gradient in the presence of chemical reactions. The mathematical model including the term for the random force and chemical reactions can be written as

$$\left(\frac{\partial}{\partial t}-\nu_0\nabla^2\right)u_n = f_n-\frac{1}{\rho}\frac{\partial p}{\partial x_n}-\frac{\partial u_n u_m}{\partial x_m}, \tag{1}$$

$$\frac{\partial u_n}{\partial x_n}=0 \tag{2}$$

$$\left(\frac{\partial}{\partial t}-\alpha_0\nabla^2-Q\right)T+\frac{\partial(Tu_n)}{\partial x_n}=0, \tag{3}$$

where f_n is an external random force assumed to be Gaussian, solenoidal [7], isotropic in space, and homogeneous white noise in time; p is pressure; t is time; u_n are projections of the velocity vector of fluid on the orthogonal coordinate axes x_n; x_n are the orthogonal coordinates; ν_0 is kinematic viscosity of the fluid; α_0 is the thermal diffusivity of the fluid (the subscript "0" means that the renormalization process will begin from this value); ρ is density of the fluid; Q is the thermal effect of reaction. Value Q is negative or positive depending upon whether reaction is endothermic or exothermic, respectively.

4. Fourier decomposition of the basic equations

The subsequent analysis will be performed in the Fourier space. The Fourier transformation of the velocity, pressure, random force and temperature fields can be performed as follows:

$$u_n=\frac{1}{(2\pi)^{d+1}}\int_{\kappa\le\kappa_c} d^d\kappa\int d\omega \mathrm{U}_n(\boldsymbol{\kappa},\omega)\exp(i\boldsymbol{\kappa}\cdot\mathbf{x}-i\omega t), \tag{4}$$

$$p = \frac{1}{(2\pi)^{d+1}} \int_{\kappa \le \kappa_c} d^d\kappa \int d\omega \mathrm{P}(\boldsymbol{\kappa},\omega) \exp(i\boldsymbol{\kappa}\cdot\mathbf{x} - i\omega t), \tag{5}$$

$$T = \frac{1}{(2\pi)^{d+1}} \int_{\kappa \le \kappa_c} d^d\kappa \int d\omega \mathrm{T}(\vec{k},\omega) \exp(i\vec{k}\cdot\vec{x} - i\omega t). \tag{6}$$

$$u_n u_m = \frac{1}{(2\pi)^{d+1}} \int_{\kappa \le \kappa_c} d^d\kappa \int d\omega \mathrm{W}_{nm}(\boldsymbol{\kappa},\omega) \exp(i\boldsymbol{\kappa}\cdot\mathbf{x} - i\omega t), \tag{7}$$

$$u_n T = \frac{1}{(2\pi)^{d+1}} \int_{\kappa \le \kappa_c} d^d\kappa \int d\omega \Theta(\boldsymbol{\kappa},\omega) \exp(i\boldsymbol{\kappa}\cdot\mathbf{x} - i\omega t), \tag{8}$$

$$f_n = \frac{1}{(2\pi)^{d+1}} \int_{\kappa \le \kappa_c} d^d\kappa \int d\omega \mathrm{F}_n(\boldsymbol{\kappa},\omega) \exp(i\boldsymbol{\kappa}\cdot\mathbf{x} - i\omega t), \tag{9}$$

where U_n, P, T, W_{nm}, Θ, F_n, are the Fourier transforms of u_n, p, T, $u_n u_m$, $u_n T$ and f_n, respectively; κ is a wavenumber (a component of the vector **κ**); **κ** is the d-dimensional wavevector; ω is frequency; **x** is the d-dimensional position coordinate vector; i is the imaginary unit, and κ_c is the ultraviolet cutoff wavenumber (it is assumed that velocity modes vanish when $\kappa > \kappa_c$ [7].

In the Fourier space, momentum equation has the following form [7]

$$\begin{aligned} &G_0^{-1}(\boldsymbol{\kappa},\omega)\mathrm{U}_n(\boldsymbol{\kappa},\omega) = \mathrm{F}_n(\boldsymbol{\kappa},\omega) - \\ &\lambda_0 M_{nml}(\boldsymbol{\kappa}) \int_{\sigma \le \sigma_c} \frac{d^d\sigma}{(2\pi)^d} \int \frac{d\varpi}{2\pi} \mathrm{U}_m(\boldsymbol{\sigma},\varpi)\mathrm{U}_l(\boldsymbol{\kappa}-\boldsymbol{\sigma},\omega-\varpi) \end{aligned}, \tag{10}$$

where

$$M_{nml} = \frac{1}{2i}\left(\kappa_m M_{nl} + \kappa_l M_{nm}\right), \tag{11}$$

$$M_{nl} = \delta_{nl} - \frac{\kappa_n \kappa_l}{\kappa^2}, \tag{12}$$

σ is the wavenumber (components of the vector **σ**); **σ** are the d-dimensional wavevectors; and ϖ is the frequency, $\lambda_0 = 1$ is introduced here to enable the perturbation analysis that will follow later on and

$$G_0 = \left(-i\omega + \nu_0\kappa^2\right)^{-1} \tag{11}$$

is the zero-order propagator.

After application of the Fourier transformation to energy equation (3), one can obtain

$$\left(-i\omega+\alpha_0\kappa^2-Q\right)\mathrm{T}=-i\kappa_n\Theta\,. \tag{12}$$

In order to eliminate Θ from Eq 12, the convolution theorem [26] is used. With its help, the function Θ can be written as:

$$\Theta=\int_{\kappa\le\kappa_c}\frac{d^d\sigma}{(2\pi)^d}\int\frac{d\varpi}{2\pi}\mathrm{U}_n(\vec{\sigma},\omega)\mathrm{T}(\vec{\kappa}-\vec{\sigma},\omega-\varpi)\,. \tag{13}$$

In view of this, Eq 12 can be recast as

$$G_{T0}^{-1}(\kappa)\mathrm{T}(\vec{\kappa},\omega)=-i\lambda_0\kappa_n\int_{\kappa\le\kappa_c}\frac{d^d\sigma}{(2\pi)^d}\int\frac{d\varpi}{2\pi}\mathrm{U}_n(\vec{\sigma},\omega)\mathrm{T}(\vec{\kappa}-\vec{\sigma},\omega-\varpi), \tag{14}$$

where again $\lambda_0 = 1$ is introduced. Here

$$G_{T0}=\left(-i\omega+\alpha_0\kappa^2-Q\right)^{-1}\,. \tag{15}$$

is the temperature propagator.

5. Renormalization procedure and renormalization of the energy equation

To apply the renormalization procedure [4], velocity and temperature fields are separated into long-wavelength and short-wavelength modes, such as

$$\mathrm{U}(\kappa,\omega)=\begin{cases}\mathrm{U}^{<}(\boldsymbol{\kappa},\omega) & 0<\kappa<\kappa_c\exp(-\tau)\\ \mathrm{U}^{>}(\boldsymbol{\kappa},\omega) & \kappa_c\exp(-\tau)<\kappa<\kappa_c\end{cases}, \tag{16}$$

$$\mathrm{T}(\kappa,\omega)=\begin{cases}\mathrm{T}^{<}(\kappa,\omega) & 0<\kappa<\kappa_c\exp(-\tau)\\ \mathrm{T}^{>}(\kappa,\omega) & \kappa_c\exp(-\tau)<\kappa<\kappa_c\end{cases}, \tag{17}$$

where is a positive parameter; $\mathrm{U}^{<}$ and $\mathrm{T}^{<}$ denote the long-wavelength modes (or, in other words, slow modes); and $\mathrm{U}^{>}$, and $\mathrm{T}^{>}$ denote the short-wavelength modes (or, in other words, fast modes).

Substitution of Eqs. 16 and 17 into the transformed energy equation (14) yields

$$\begin{aligned}G_{T0}^{-1}(\kappa,\omega)\mathrm{T}(\kappa,\omega)&=-i\kappa_l\lambda_0\int_{\kappa\le\kappa_c}\frac{d^d\sigma}{(2\pi)^d}\int\frac{d\varpi}{2\pi}\Big[\mathrm{U}_l^{<}(\sigma,\varpi)\mathrm{T}^{<}(\kappa-\sigma,\omega-\varpi)\\&+\mathrm{U}_l^{>}(\sigma,\varpi)\mathrm{T}^{<}(\kappa-\sigma,\omega-\varpi)++\mathrm{U}_l^{<}(\vec{\sigma},\varpi)\mathrm{T}^{>}(\kappa-\sigma,\omega-\varpi)+\mathrm{U}_l^{>}(\kappa,\varpi)\mathrm{T}^{>}(\kappa-\sigma,\omega-\varpi)\Big]\end{aligned} \tag{18}$$

Equation 18 includes fast and slow modes. Further, we exclude from Eq 18 fast modes of temperature and velocity by introducing a formal series expansion of the velocity and temperature functions

$$\mathrm{U}^{>}(\tilde{\kappa})=\sum_{s=0}^{\infty}\lambda_0^s\mathrm{U}_s^{>}(\tilde{\kappa}), \tag{19}$$

$$\mathrm{T}^{>}(\tilde{\kappa})=\sum_{s=0}^{\infty}\lambda_0^s\mathrm{T}_s^{>}(\tilde{\kappa}). \tag{20}$$

where $\tilde{\kappa}=(\boldsymbol{\kappa},\omega)$.

Substituting Eqs. 19 and 20 into both parts of Eq 18 and equating the terms with the same powers of λ_0, one can obtain expressions for the terms of the series expansions (20). As a result, we have for $s=0$

$$\mathrm{T}_0^{>}(\tilde{\kappa})=0, \tag{21}$$

whereas for $s=1$ a term of the series expansion has the following form

$$\mathrm{T}_1^{>}(\tilde{\kappa})=-G_{T0}(\tilde{\kappa})i\kappa_l\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}\left[\mathrm{U}_l^{<}(\tilde{\sigma})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma})+\mathrm{U}_{0l}^{>}(\tilde{\sigma})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma})\right]. \tag{22}$$

Let us substitute the series expansions (19) and (20), with allowance for Eqs. 21 and 22, into Eq 18. Next, we collect the terms for the same powers of λ_0, taking into account the splitting of the modes of wave numbers into the fast and slow wave modes. As a result, we obtain an equation for slow modes of the temperature function

$$\begin{gathered}G_{T0}^{-1}(\tilde{\kappa})\mathrm{T}^{<}(\tilde{\kappa})=-i\kappa_l\lambda_0\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}\left[\mathrm{U}_l^{<}(\tilde{\sigma})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma})+\mathrm{U}_{l0}^{>}(\vec{\sigma},\omega)\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma})\right]-\\ i\kappa_l\lambda_0^2\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}[\mathrm{U}_{l1}^{>}(\tilde{\sigma})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma})+\mathrm{U}_l^{<}(\tilde{\sigma})\mathrm{T}_1^{>}(\tilde{\kappa}-\tilde{\sigma})+\mathrm{U}_{l0}^{>}(\tilde{\sigma})\mathrm{T}_1^{>}(\tilde{\kappa}-\tilde{\sigma})]+O(\lambda_0^3).\end{gathered} \tag{23}$$

Let us estimate the magnitude of the terms included in Eq 23 using the averaging rules

$$\langle\mathrm{T}^{<}\rangle=\mathrm{T}^{<},\ \langle\mathrm{T}^{>}\rangle=0. \tag{24}$$

In doing so, one can obtain that the second term at λ_0 and the first two terms at λ_0^2 are equal to zero. Let us consider the last term with account for Eq 22

$$
\begin{aligned}
&-i\kappa_l\lambda_0^2\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}\mathrm{U}_{l0}^{>}(\tilde{\sigma})\mathrm{T}_1^{>}(\tilde{\kappa}-\tilde{\sigma})=-\kappa_l\kappa_n\lambda_0^2\\
&\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}G_{T0}(\tilde{\kappa}-\tilde{\sigma})\int\frac{d\tilde{\upsilon}}{(2\pi)^{d+1}}\left[\mathrm{U}_{l0}^{>}(\tilde{\sigma})\mathrm{U}_{n}^{<}(\tilde{\upsilon})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma}-\tilde{\upsilon})+\mathrm{U}_{l0}^{>}(\tilde{\sigma})\mathrm{U}_{n0}^{>}(\tilde{\upsilon})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma}-\tilde{\upsilon})\right]
\end{aligned}
\tag{25}
$$

Here again the first term in Eq 25 is equal to zero, and we need to estimate the second one. In view of this, it is necessary to remind that [7]

$$\mathrm{U}_{n0}^{>}(\tilde{\kappa})=G_0(\tilde{\kappa})\mathrm{F}_{n0}^{>}(\tilde{\kappa}) \tag{26}$$

and also to recall that the random force *f* is assumed to be Gaussian and white noise in time. The random force specified by the following Fourier transformation of its two-point correlation function [4]:

$$\left\langle \mathrm{F}_n(\boldsymbol{\kappa},\omega)\mathrm{F}_m(\boldsymbol{\kappa}',\omega')\right\rangle=\frac{2(2\pi)^{d+1}}{\kappa^{d-4+\varepsilon^*}}D_0M_{nm}(\boldsymbol{\kappa})\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}')\delta(\omega+\omega'), \tag{27}$$

where $\varepsilon^* = 4$, whereas the parameter D_0 is related to the dissipation rate ε by the following equation

$$2D_0\frac{S_d}{(2\pi)^d}=\gamma\varepsilon=1.575\varepsilon\,, \tag{28}$$

where

$$S_d=\frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{29}$$

is the surface area of a *d* – dimensional unit sphere, and Γ is the gamma-function.

Using Eqs. 26 and 27 one can obtain from Eq 25

$$
\begin{aligned}
&=-\kappa_l\kappa_n\lambda_0^2\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}G_{T0}(\tilde{\kappa}-\tilde{\sigma})\int\frac{d\tilde{\upsilon}}{(2\pi)^{d+1}}\mathrm{U}_{l0}^{>}(\tilde{\sigma})\mathrm{U}_{n0}^{>}(\tilde{\upsilon})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma}-\tilde{\upsilon})\\
&=-2D_0\kappa_l\kappa_n\lambda_0^2\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}\frac{G_0(\tilde{\sigma})G_{T0}(\tilde{\kappa}-\tilde{\sigma})}{\sigma^y}M_{nl}(\sigma)\int d\tilde{\upsilon}G_0(\tilde{\sigma})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma}-\tilde{\upsilon})\delta(\tilde{\sigma}+\tilde{\upsilon}).
\end{aligned}
\tag{30}
$$

Next, using the properties of the delta function $\delta(\tilde{\sigma}+\tilde{\upsilon})$, one can transform the last integral to the following form

$$
\begin{aligned}
&-2D_0\kappa_l\kappa_n\lambda_0^2\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}\frac{G_0(\tilde{\sigma})G_{T0}(\tilde{\kappa}-\tilde{\sigma})}{\sigma^y}M_{nl}(\sigma)\int d\tilde{\upsilon}G_0(\tilde{\sigma})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma}-\tilde{\upsilon})\delta(\tilde{\sigma}+\tilde{\upsilon}) \\
&=-2D_0\mathrm{T}^{<}(\tilde{\kappa})\kappa_l\kappa_n\lambda_0^2\int\frac{d\tilde{\sigma}}{(2\pi)^{d+1}}\frac{G_{T0}(\tilde{\kappa}-\tilde{\sigma})}{\sigma^y}\left|G_0(\tilde{\sigma})\right|^2M_{nl}(\sigma).
\end{aligned}
\tag{31}
$$

Now we can rewrite Eq 23 so that

$$
\begin{aligned}
&G_{T0}^{-1}(\tilde{\kappa})\mathrm{T}^{<}(\tilde{\kappa})=-i\lambda_0'\kappa_n\int\frac{d^{d+1}\tilde{\sigma}}{(2\pi)^{d+1}}\mathrm{U}_n^{<}(\tilde{\sigma})\mathrm{T}^{<}(\tilde{\kappa}-\tilde{\sigma})- \\
&2\lambda_0'^{\,2}D_0\kappa_n\kappa_m\int\frac{d^{d+1}\tilde{\sigma}}{(2\pi)^{d+1}}\left|G_0(\tilde{\sigma})\right|^2\frac{G_{T0}(\tilde{\kappa}-\tilde{\sigma})}{\sigma^{d-4+\varepsilon^*}}M_{nm}(\overset{\rho}{\sigma}).
\end{aligned}
\tag{32}
$$

According to the renormalization procedure, Eq 32 can be transformed into the initial equation (14), if on the left-hand side of (32) the zero-order propagator G_{T0} given by Eq 15 is replaced by the renormalized propagator

$$
G_T(\kappa)=\left[-i\omega+\kappa^2(\alpha_0+\Delta\alpha)-Q\right]^{-1},
\tag{33}
$$

where

$$
\Delta\alpha=2\lambda_0{}^2D_0\frac{\kappa_n\kappa_m}{\kappa^2}\int\frac{d^{d+1}\tilde{\sigma}}{(2\pi)^{d+1}}\left|G_0(\tilde{\sigma})\right|^2\frac{G_{T0}(\tilde{\kappa}-\tilde{\sigma})}{\sigma^{d-4+\varepsilon^*}}M_{nm}(\overset{\rho}{\sigma}),
\tag{34}
$$

is the renormalizing thermal diffusivity.

Let us compute the integral (34) in the limiting case of κ → 0, ω → 0. To do it, one can represent this integral in the following form

$$
\begin{aligned}
\Delta\alpha&=\frac{\kappa_n\kappa_m}{\kappa^2}\frac{2\lambda_0{}^2D_0}{(2\pi)^{d+1}}\int\left|G_0(\tilde{\sigma})\right|^2G_0(\tilde{\kappa}-\tilde{\sigma})\sigma^{\text{-y}}M_{nm}(\sigma)d\tilde{\sigma}= \\
&=\frac{\kappa_n\kappa_m}{\kappa^2}\frac{2\lambda_0{}^2D_0}{(2\pi)^{d+1}}\int\limits_{-\infty}^{\infty}\int\limits_{\kappa_c\exp(-\tau)}^{\kappa_c}\frac{\sigma^{-y}M_{nm}(\sigma)d^d\sigma\, d\varpi}{\left(\sigma^4\nu_0^2+\varpi^2\right)\left(a_0(\kappa-\sigma)^2+i(\varpi-\omega)-Q\right)}
\end{aligned}
,
\tag{35}
$$

where

$$
y=d-4+\varepsilon^*.
\tag{36}
$$

At first, we take the integral over the frequencies using the Cauchy theorem. This integral is equal to the sum of the residues at the poles of the upper half-plane

$$
\varpi_1=i\sigma^2\nu_0,\quad \varpi_2=\omega+i\left((\kappa-\sigma)^2\alpha_0-Q\right),
\tag{37}
$$

multiplied by the factor of $2\pi i$. As a result, taking into account the expansion of the integrand into a binomial series with respect to the wave number, one can obtain

$$\Delta\alpha = \frac{\kappa_n\kappa_m}{\kappa^2}\frac{2\lambda_0{}^2 D_0}{(2\pi)^{d+1}\nu_0}\int\limits_{\kappa_c\exp(-\tau)}^{\kappa_c}\frac{\sigma^{-y-2}M_{nm}(\sigma)}{(\alpha_0+\nu_0)\sigma^2-Q}\left(1-\frac{2a_0\kappa\sigma}{(\alpha_0+\nu_0)\sigma^2-Q}\right)d^d\sigma\,. \tag{38}$$

Neglecting the second term in the integrand, we find

$$\Delta\alpha = \frac{\kappa_n\kappa_m}{\kappa^2}\frac{2\lambda_0{}^2 D_0}{(2\pi)^{d+1}\nu_0}\int\limits_{\kappa_c\exp(-\tau)}^{\kappa_c}\left(\frac{\delta_{nm}\sigma^{-y-2}}{(\alpha_0+\nu_0)\sigma^2-Q}-\frac{\sigma^{-y-4}\sigma_n\sigma_m}{(\alpha_0+\nu_0)\sigma^2-Q}\right)d^d\sigma\,. \tag{39}$$

Now it is necessary to take integrals over the wavenumbers. For this purpose, one can use the following relations [27]

$$\int f(\sigma)d^d\sigma = S_d\int f(\sigma)\sigma^{d-1}d\sigma\,, \tag{40}$$

$$\int f(\sigma)\sigma_m\sigma_n d^d\sigma = \frac{S_d}{d}\delta_{mn}\int f(\sigma)\sigma^{d+1}d\sigma\,. \tag{41}$$

As result, we have

$$\Delta\alpha = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{\lambda_0{}^2 D_0}{\nu_0\kappa_c^{\varepsilon^*-2}} \frac{{}_2\mathrm{F}_1\left(1{,}1-\frac{\varepsilon}{2},2-\frac{\varepsilon}{2},\frac{\kappa_c^2(\nu_0+\alpha_0)}{Q}\right)-\exp((\varepsilon^*-2)\tau)\,{}_2\mathrm{F}_1\left(1{,}1-\frac{\varepsilon}{2},2-\frac{\varepsilon}{2},\frac{\exp(-2\tau)\kappa_c^2(\nu_0+\alpha_0)}{Q}\right)}{Q(\varepsilon^*-2)} \tag{42}$$

where ${}_2\mathrm{F}_1$ is the hypergeometric function.

Consequently, the effective renormalized thermal diffusivity can be written as

$$\alpha = \alpha_0 + \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{\lambda_0{}^2 D_0}{\nu_0\kappa_c^{\varepsilon^*-2}} \frac{{}_2\mathrm{F}_1\left(1{,}1-\frac{\varepsilon}{2},2-\frac{\varepsilon}{2},\frac{\kappa_c^2(\nu_0+\alpha_0)}{Q}\right)-\exp(\varepsilon^*-2)\tau\,{}_2\mathrm{F}_1\left(1{,}1-\frac{\varepsilon}{2},2-\frac{\varepsilon}{2},\frac{\exp(-2\tau)\kappa_c^2(\nu_0+\alpha_0)}{Q}\right)}{Q(\varepsilon^*-2)} \tag{43}$$

Let us differentiate this expression with respect to τ in the limit of τ → 0. As a result, we obtain a differential equation

$$\frac{d\alpha}{d\tau} = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{\lambda_0{}^2 D_0}{\nu^2\kappa_c^{\varepsilon^*}}\frac{\nu\kappa_c^2}{(\nu+\alpha)\kappa_c^2-Q}\,. \tag{44}$$

The subscript “0” by ν_0 and α_0 has been discarded, because Eq 44 is valid for every step of the renormalization procedure, and the condition of $\tau \to 0$ leads to $\nu(\kappa_c) \to \nu(\kappa'_c)$ and $\alpha(\kappa_c) \to \alpha(\kappa'_c)$, where κ'_c is the current value of wavenumber.

Unfortunately, differential equation 44 cannot be integrated exactly. That is why let us do make a simplified analysis of this equation. To do it, one can transform equation 44 to following form

$$\frac{d\alpha}{d\alpha_{Q=0}} = \frac{\kappa_c^2(\nu+\alpha)}{\kappa_c^2(\nu+\alpha)-Q} = \frac{\nu\kappa_c^2\left(1+\mathrm{Pr}_t^{-1}\right)}{\nu\kappa_c^2\left(1+\mathrm{Pr}_t^{-1}\right)-Q}, \tag{45}$$

where Pr_t is the turbulent Prandtl number and $\alpha_{Q=0}$ is thermal diffusivity under condition of absent of chemical reactions. Taking into account the equation for viscosity [7]

$$\nu = \left(\frac{3A_d D_0}{\varepsilon^* \kappa_c^{\varepsilon^*}}\right)^{1/3}, \tag{46}$$

one can transform Eq 45 to an approximation form ($\varepsilon^*=4$)

$$\alpha \approx \alpha_{Q=0} \frac{\left(1+\mathrm{Pr}_t^{-1}\right)\sqrt{\dfrac{3A_d D_0}{4\nu}}}{\left(1+\mathrm{Pr}_t^{-1}\right)\sqrt{\dfrac{3A_d D_0}{4\nu}}-Q}, \tag{47}$$

where

$$A_d = \frac{S_d}{(2\pi)^d}\frac{d-1}{2(d+2)}. \tag{48}$$

As one can see from Eq 47, the release of heat during a chemical reaction leads to intensification of turbulent heat transfer, i.e. addition of heat increases turbulent pulsation. One can interpret this phenomenon in such a way that in the present process the thermal energy transforms to chaotic kinetic energy of stochastic temperature pulsations. Contrary, the absorption of heat leads to suppression of turbulent heat transfer. It follows from Eq 47 that under condition

$$Q \to \left(1+\mathrm{Pr}_t^{-1}\right)\sqrt{\frac{3A_d D_0}{4\nu}} \tag{49}$$

a significant increase in effective thermal diffusivity occurs. It should be pointed out that Eq 42, together with Eq 47, is correct under the condition

$$(\kappa-\sigma)^2\alpha_0 > Q, \tag{50}$$

i.e. for a restricted range of the function Q.
When

$$(\kappa-\sigma)^2\alpha_0 < Q \tag{51}$$

we have just one pole in the upper half-plane ϖ_1 (37). In this case, one can obtain from Eq 35

$$\Delta\alpha = \frac{\kappa_n\kappa_m}{\kappa^2}\frac{2\lambda_0{}^2 D_0}{(2\pi)^{d+1}\nu_0}\int_{\kappa_c\exp(-\tau)}^{\kappa_c}\frac{\sigma^{-y-2}M_{nm}(\sigma)}{(\nu_0-\alpha_0)\sigma^2-Q}\left(1-\frac{2a_0\kappa\sigma}{(\nu_0-\alpha_0)\sigma^2-Q}\right)d^d\sigma\,. \tag{52}$$

After integration we have

$$\Delta\alpha = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{\lambda_0{}^2 D_0}{\nu_0\kappa_c^{\varepsilon^*-2}} \frac{{}_2\mathrm{F}_1\left(1,1-\frac{\varepsilon}{2},2-\frac{\varepsilon}{2},\frac{\kappa_c^2(\nu_0-\alpha_0)}{Q}\right)-\exp((\varepsilon^*-2)\tau)\,{}_2\mathrm{F}_1\left(1,1-\frac{\varepsilon}{2},2-\frac{\varepsilon}{2},\frac{\exp(-2\tau)\kappa_c^2(\nu_0-\alpha_0)}{Q}\right)}{Q(\varepsilon^*-2)} \tag{53}$$

Again, after differentiation of Eq 53 with respect to τ in the limit $\tau \to 0$ we have

$$\frac{d\alpha}{d\tau} = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{\lambda_0{}^2 D_0}{\nu^2\kappa_c^{\varepsilon^*}}\frac{\nu\kappa_c^2}{(\nu-\alpha)\kappa_c^2-Q}\,. \tag{54}$$

The resulting relation for thermal diffusivity looks as

$$\alpha \approx \alpha_{Q=0}\frac{\left(1+\mathrm{Pr}_t^{-1}\right)\sqrt{\frac{3A_d D_0}{4\nu}}}{\left(1-\mathrm{Pr}_t^{-1}\right)\sqrt{\frac{3A_d D_0}{4\nu}}-Q}\,. \tag{55}$$

One can ascertain that this equation is not physically realistic, because for the case without chemical reactions it does not reduce to the basic relation for $\alpha_{Q=0}$.

6. Conclusion

The paper focused on a study of turbulent thermal diffusivity in the presence of endothermic and exothermic chemical reactions with the help of the renormalization group approach. As a result, a relation for turbulent thermal diffusivity was obtained, which takes into account the presence of the chemical reactions in the flow. This relation shows that turbulent thermal diffusivity is a function of turbulent viscosity, a parameter proportional to dissipation rate of turbulent energy, as well as the thermal effect of chemical reactions. It was demonstrated that heat release due to chemical reactions leads to enhancement of turbulent heat transfer, i.e. addition of heat amplifies turbulent pulsations. In this case, thermal energy transforms to chaotic kinetic energy of stochastic temperature pulsations. Contrary, the absorption of heat in chemical reactions leads to suppression of turbulent heat transfer. Also, conditions were revealed where effective thermal diffusivity demonstrates a sharp increase.

Conflict of interest

All authors declare that there is no conflict of interests in this paper.

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