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Research article

Global dynamics of an age-structured malaria model with prevention

Zhong-Kai Guo[1], **Hai-Feng Huo**[1,2,*] **and Hong Xiang**[2]

[1] College of Electrical and Information Engineering, Lanzhou University of Technology, Lanzhou, Gansu, 730050, Peoples Republic of China

[2] Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu, 730050, Peoples Republic of China

* **Correspondence:** Email: hfhuo@lut.cn, huohf1970@163.com; Tel: +869312976360.

Abstract: In this paper, we formulate a new age-structured malaria model, which incorporates the age of prevention period of susceptible people, the age of latent period of human and the age of latent period of female Anopheles mosquitoes. We show that there exists a compact global attractor and obtain a sufficient condition for uniform persistence of the solution semiflow. We obtain the basic reproduction number $\mathcal{R}_0$ and show that $\mathcal{R}_0$ completely determines the global dynamics of the model, that is, if $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable, if $\mathcal{R}_0 > 1$, there exists a unique endemic equilibrium that attracts all solutions for which malaria transmission occurs. Finally, we perform some numerical simulations to illustrate our theoretical results and give a brief discussion.

Keywords: Malaria model; prevention age; latent age; basic reproduction number; global stability

1. Introduction

Malaria, an infectious disease caused by the malaria parasite, is one of the most severe public health problems in the world. About half of the world population live in areas at risk of malaria transmission. According to the World Malaria Report 2016 [1], in 2015, there were 212 million new clinical cases of the disease and 429,000 deaths. The majority of malaria cases and deaths are concentrated in Africa. Malaria transmission between people mainly involves malaria parasites infecting successively two types of hosts: humans and female Anopheles mosquitoes. In humans, the parasites (sporozoites) grow and multiply first in the liver cells. After a period of time, the parasites are found in the red cells of the blood. The blood stage parasites are those that cause the symptoms of malaria. In the blood, the parasites grow inside the red cells and then destroy them, releasing more daughter parasites that continue the cycle by invading other red cells. A proportion of the blood stage parasites develop into the gametocytes at some point of this process. When gametocytes are ingested by a female Anopheles mosquito during a blood meal, they start another different cycle of growth and multiplication in the mosquito. After 10-18 days, the parasites (sporozoites) are found in salivary glands of the mosquito. When the anopheles mosquito takes a blood meal on another human, the sporozoites are injected with the mosquito's saliva and start another human infection when they parasitize the liver cells. Thus, the mosquito carries the disease from one human to another. Differently from the human host, the mosquito does not suffer from the presence of the parasites [2].

Mathematical modeling of malaria transmission has always provided useful insights into malaria transmission mechanism since the pioneer work of Ross [3] and Macdonald [4], it has also become an important tool in understanding the dynamics of malaria transmission. Over the years, the Ross-Macdonald model has been extended by many researchers [5, 6, 7, 8, 9, 10, 11, 12, 13]. Ruan et al. [7] introduced modified Ross-Macdonald model to include two discrete time delays which represent incubation periods of parasites within the human and the mosquito. They verified that extending the incubation periods in either humans or mosquitos were beneficial to the malaria control. Cai et al. [8] investigated the effect of distributed delays on the vector-host disease dynamics, they showed that incubation periods can play significant role in affecting the disease transmission. Gao et al. [12] presented a multi-patch malaria model to study the impact of mobility of vector and host populations on malaria transmission. By using analysis and numerical simulation, they found that human movement was a critical factor in the spatial spread of malaria around the world. Li et al. [13] investigated two malaria models with relapse. They separated the dynamics into the fast time dynamics and the slow time dynamics and showed the full dynamics were determined by the slow systems. Other models about epidemic model or age-structured can be found in [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 35].

Prevention and treatment may be the effective way to reduce malaria transmission. In most malaria-endemic countries, four interventionsłcase management (diagnosis and treatment), insecticide-treated nets (ITNs), intermittent preventive treatment of malaria in pregnant women (IPTp) and indoor residual spraying (IRS) łmake up the essential package of malaria interventions. These measures are considered to have made a major contribution to the reduction in malaria burden since 2000. However, global investment for malaria has barely changed in recent years, which has affected malaria control to a certain extent. In 2015, 43% of the population of sub-Saharan Africa were not covered by ITNs or IRS, 69% of pregnant women did not receive three doses of IPTp and 36% of children with fever were not taken for care [1]. Prevention of malaria may be more cost-effective than treatment of the disease in the long run. In March 2010, the newest intervention recommended by the World Health Organization is intermittent preventive treatment during infancy (IPTi). Studies show that IPTi can significantly reduce clinical malaria and anemia in the first year of life, as well as hospital admissions associated with malaria infection or for any cause [1]. Intermittent preventive treatment of malaria in pregnant women (IPTp) is also a malaria intervention. WHO has recommended that pregnant women be given at least three doses during each pregnancy [1]. Vaccination is also a means of prevention. RTS,S(developed by PATH Malaria Vaccine Initiative (MVI) and GlaxoSmithKline (GSK) with support from the Bill and Melinda Gates Foundation) is the most recently developed recombinant vaccine. Phase III clinical trial indicated that RTS,S reduced the number of cases among young children by almost 50 percent and among infants by around 25 percent. But, overall efficacy seem to wane with time [25]. According to this, we introduce age structure in the prevention period of susceptible population in our model.

Motivated by the above, in this paper, we not only introduce age structure to explain the prevention period of susceptible population but also involve age structures to account for the latent periods in humans and mosquitoes. We show that there exists a compact global attractor and obtain a sufficient condition for uniform persistence of the solution semiflow. We also identify the basic reproduction number $\mathcal{R}_0$ and show that $\mathcal{R}_0$ completely determines the global dynamics of our model.

This paper is organized as follows. In Section 2, we introduce the malaria model and present some basic properties. In Section 3, we define the basic reproductive number and prove the local stability of the disease-free equilibrium and the unique endemic equilibrium. In Section 4, we present the uniform persistence and prove the global stability of the disease-free equilibrium and the unique endemic equilibrium. In Section 5, we perform some numerical simulations. In Section 6, we give a brief discussion.

2. Malaria model and basic properties

2.1. The model

We divide the total human population at time t into five mutually-exclusive subgroups: susceptible individuals $S_h(t)$, protected individuals $P(t,a)$, where the parameter a denotes the preventive age of the susceptible individuals, exposed individuals $E_h(t,\theta)$, where θ denotes the latent age of the exposed individuals, infective individuals $I_h(t)$ and removed individuals $R_h(t)$. We assume that the removed individuals are given special protection, they will no longer be involved in the transmission process. We divide the total vector population at time t into three mutually-exclusive subgroups: susceptible vectors $S_m(t)$, exposed vectors $E_m(t,\tau)$, where the parameter τ denotes the latent age of the exposed vectors, infective vectors $I_m(t)$. The flow among those subgroups is shown in the following flowchart (Figure 1).

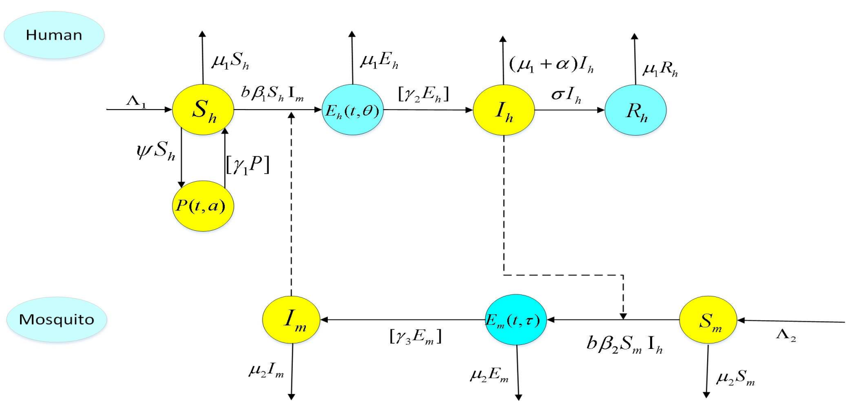

Figure 1. Flowchart of the malaria transmission between mosquitoes and humans, where $[\gamma_1 P]$, $[\gamma_2 E_h]$ and $[\gamma_3 E_m]$ represent $\int_0^{+\infty} \gamma_1(a)P(t,a)da$, $\int_0^{+\infty} \gamma_2(\theta)E_h(t,\theta)d\theta$ and $\int_0^{+\infty} \gamma_3(\tau)E_m(t,\tau)d\tau$, respectively.

By the flowchart (Figure 1) and noting that the removed individuals are decoupled, we can formulate the malaria model as follows:

$$\begin{cases} \dfrac{dS_h(t)}{dt} = \Lambda_1 - (\mu_1 + \psi)S_h(t) - b\beta_1 S_h(t)I_m(t) + \displaystyle\int_0^{+\infty} \gamma_1(a)P(t,a)da, \\ \dfrac{dS_m(t)}{dt} = \Lambda_2 - \mu_2 S_m(t) - b\beta_2 S_m(t)I_h(t), \\ \dfrac{dI_h(t)}{dt} = \displaystyle\int_0^{+\infty} \gamma_2(\theta)E_h(t,\theta)d\theta - (\mu_1 + \alpha + \delta)I_h(t), \\ \dfrac{dI_m(t)}{dt} = \displaystyle\int_0^{+\infty} \gamma_3(\tau)E_m(t,\tau)d\tau - \mu_2 I_m(t), \\ \dfrac{\partial P(t,a)}{\partial t} + \dfrac{\partial P(t,a)}{\partial a} = -(\mu_1 + \gamma_1(a))P(t,a), \\ \dfrac{\partial E_h(t,\theta)}{\partial t} + \dfrac{\partial E_h(t,\theta)}{\partial \theta} = -(\mu_1 + \gamma_2(\theta))E_h(t,\theta), \\ \dfrac{\partial E_m(t,\tau)}{\partial t} + \dfrac{\partial E_m(t,\tau)}{\partial \tau} = -(\mu_2 + \gamma_3(\tau))E_m(t,\tau), \end{cases} \tag{2.1}$$

with the following initial and boundary conditions

$$\begin{cases} P(t,0) = \psi S_h(t), \quad E_h(t,0) = b\beta_1 S_h(t)I_m(t), \quad E_m(t,0) = b\beta_2 S_m(t)I_h(t), \\ P(0,a) = P_0(a), \quad E_h(0,\theta) = E_{h0}(\theta), \quad E_m(0,\tau) = E_{m0}(\tau), \\ S_h(0) = s_{h0}, \quad I_h(0) = i_{h0}, \quad S_m(0) = s_{m0}, \quad I_m(0) = i_{m0}, \end{cases} \tag{2.2}$$

where $P_0(a), E_{h0}(\theta), E_{m0}(\tau) \in L^1_+(0, +\infty)$ and $s_{h0}, i_{h0}, r_{h0}, s_{m0}, i_{m0} \in \mathbb{R}_+$. The meanings of the parameters in (2.1) are explained in Table 1.

Table 1. Description of parameters of the model (2.1).

Parameters	Description
Λ_1	the recruitment rate of the human population
μ_1	the natural death rate of the human population
b	the average number of bites per mosquito per unit time
β_1	the probability of transmission from female anopheles mosquito to human
ψ	the rate of prevention of the susceptible individuals
α	the disease-induced death rate
$\gamma_1(a)$	the prevention wane rate depends on preventive age
$\gamma_2(\theta)$	the rate at which latent individuals progress into infectious class
δ	the remove rate
Λ_2	the recruitment rate of the mosquito population
μ_2	the death rate of mosquito population
β_2	the probability of transmission from human to female anopheles mosquito
$\gamma_3(\tau)$	the rate at which latent individuals progress into infectious class

Throughout this paper, we make the following assumptions and notations.

$(A1):\Lambda_i,\mu_i,\beta_i,\delta,\alpha>0,(i=1,2),\ \psi\geq 0;$
$(A2):\gamma_i\in L^{\infty}_{+}(0,+\infty)$ $(i=1,2,3)$ with essential upper bounds $\bar{\gamma}_i>0$, respectively;
$(A3):P(0,0)=\psi S_h(0),\ E_h(0,0)=b\beta_1S_h(0)I_m(0),\ E_m(0,0)=b\beta_2S_m(0)I_h(0).$
For $a,\theta,\tau\geq 0$, we denote
$k_1(a)=e^{-\int_0^a(\mu_1+\gamma_1(s))da},\quad \mathscr{K}_1=\int_0^{+\infty}\gamma_1(a)k_1(a)da,\quad \phi(a)=\psi\gamma_1(a)k_1(a),$
$k_2(\theta)=e^{-\int_0^\theta(\mu_1+\gamma_2(s))d\theta},\quad \mathscr{K}_2=\int_0^{+\infty}\gamma_2(\theta)k_2(\theta)d\theta,$
$k_3(\tau)=e^{-\int_0^\tau(\mu_2+\gamma_3(s))d\tau},\quad \mathscr{K}_3=\int_0^{+\infty}\gamma_3(\tau)k_3(\tau)d\tau,$
$\mathscr{Y}=\mathbb{R}^4_+\times(\text{Ł}^1_+(0,+\infty))^3$ with norm

$$\|\ (x_1,x_2,x_3,x_4,x_5,x_6,x_7)\ \|_{\mathscr{Y}}=\sum_{i=1}^{4}\mid x_i\mid+\sum_{i=5}^{7}\int_0^{+\infty}\mid x_i(s)\mid ds.$$

2.2. Well-posedness

Base on the above assumptions, we can verify the local existence of unique and nonnegative solution of the model (2.1) with the nonnegative initial conditions (see Webb [26] and Iannelli [27]), thus obtain the following proposition.

Proposition 2.1. *Let $x_0\in\mathscr{Y}$, then there exists $\epsilon>0$ and neighborhood $B_0\subset\mathscr{Y}$ with $x_0\in B_0$ such that there exists a unique continuous function, $\Phi:[0,\epsilon]\times B_0\to\mathscr{Y}$, where $\Phi(t,x_0)$ is the solution to* (2.1) *with $\Phi(0,x_0)=x_0$.*

For $t\in[0,\epsilon]$,

$$\|\ \Phi(t,x_0)\ \|_{\mathscr{Y}}=S_h(t)+S_m(t)+I_h(t)+I_m(t)+\int_0^{+\infty}P(t,a)da+\int_0^{+\infty}E_h(t,\theta)d\theta+\int_0^{+\infty}E_m(t,\tau)d\tau$$

and setting $\mu=min\{\mu_1,\mu_2\}$, we deduce that $\|\ \Phi(t,x_0)\ \|_{\mathscr{Y}}$ satisfies the following inequality:

$$\frac{d}{dt}\|\ \Phi(t,x_0)\ \|_{\mathscr{Y}}\leq\Lambda_1+\Lambda_2-\mu\ \|\ \Phi(t,x_0)\ \|_{\mathscr{Y}},$$

therefore

$$\|\ \Phi(t,x_0)\ \|_{\mathscr{Y}}\leq\frac{\Lambda_1+\Lambda_2}{\mu}-e^{-\mu t}(\frac{\Lambda_1+\Lambda_2}{\mu}-\|\ x_0\ \|_{\mathscr{Y}}),\tag{2.3}$$

which yields

$$\|\ \Phi(t,x_0)\ \|_{\mathscr{Y}}\leq max\{\frac{\Lambda_1+\Lambda_2}{\mu},\|\ x_0\ \|_{\mathscr{Y}}\}.\tag{2.4}$$

Boundedness is a direct consequence of nonnegativity of solutions. Then we have following proposition.

Proposition 2.2. *Let $x_0\in\mathscr{Y}$, then there exists a unique continuous semiflow, $\Phi:\mathbb{R}_+\times\mathscr{Y}\to\mathscr{Y}$, where $\Phi(t,x_0)$ is the solution to* (2.1) *with $\Phi(0,x_0)=x_0$, and* (2.3), (2.4) *are satisfied for $t\in\mathbb{R}_+$. The following set is positively invariant for system* (2.1)

$$\Omega=\{x=(S_h(t),S_m(t),I_h(t),I_m(t),P(t,a),E_h(t,\theta),E_m(t,\tau))\in\mathscr{Y}:\ \|\ x\ \|_{\mathscr{Y}}\leq\frac{\Lambda_1+\Lambda_2}{\mu}\}.$$

From *Proposition* 2.2. and (2.3), we obtain the following proposition.

Proposition 2.3. *(1) The solution of* (2.1), $\Phi(t,\cdot)$*, is point dissipative and* Ω *attracts all points in* $\mathscr{Y}$*;*

(2) Let $B \subset \mathscr{Y}$ *be bounded, then* $\Phi(t,B)$ *is bounded;*

(3) If $x_0 \in \mathscr{Y}$ *and* $\| x_0 \|_{\mathscr{Y}} \leq A$ *for some* $A \geq \frac{\Lambda_1+\Lambda_2}{\mu}$*, then* $S_i(t),\ I_i(t), \| P(t,\cdot) \|_{L^1_+}, \| E_i(t,\cdot) \|_{L^1_+} \leq A, (i = h, m)$.

2.3. Asymptotic smoothness

Integrating the equations for P, E_h, E_m in (2.1) along the characteristic lines, $t-a$=const., $t-\theta$=cost., $t-\tau$=const., respectively, we have

$$P(t,a) = \begin{cases} P(t-a,0)k_1(a), & 0 \leq a \leq t, \\ P_0(a-t)\dfrac{k_1(a)}{k_1(a-t)}, & 0 \leq t < a, \end{cases} \qquad E_h(t,\theta) = \begin{cases} E_h(t-\theta,0)k_2(\theta), & 0 \leq \theta \leq t, \\ E_{h0}(\theta-t)\dfrac{k_2(\theta)}{k_2(\theta-t)}, & 0 \leq t < \theta, \end{cases}$$

$$E_m(t,\tau) = \begin{cases} E_m(t-\tau,0)k_3(\tau), & 0 \leq \tau \leq t, \\ E_{m0}(\tau-t)\dfrac{k_3(\tau)}{k_3(\tau-t)}, & 0 \leq t < \tau. \end{cases} \tag{2.5}$$

Continuous semiflow $\{\Phi(t,\cdot)\}_{t\geq 0}$ is said to be asymptotically smooth, if each positively invariant bounded closed set is attracted by a nonempty compact set.

We will use the following two lemmas [28] to prove the asymptotic smoothness of the semiflow.

Lemma 2.1. *The semiflow* $\Phi : \mathbb{R}_+ \times \mathscr{Y} \to \mathscr{Y}$ *is asymptotically smooth if there are maps* $U_1, U_2 : \mathbb{R}_+ \times \mathscr{Y} \to \mathscr{Y}$ *such that* $\Phi(t,x) = U_1(t,x) + U_2(t,x)$*, and the following hold for any bounded closed set* $\mathscr{A} \subset \mathscr{Y}$ *that is forward invariant under* Φ*:*

(1) $\lim\limits_{t\to+\infty} diam U_2(t,\mathscr{A}) = 0$*;*

(2) there exists $t_{\mathscr{A}} \geq 0$ *such that* $U_1(t,\mathscr{A})$ *has compact closure for each* $t \geq t_{\mathscr{A}}$.

Since $\mathscr{Y}$ is an infinite dimensional space, infinite dimensional space $L^1_+(0,+\infty)$ is a component of $\mathscr{Y}$. For infinite dimensional space, we cannot deduce precompactness only from boundedness. We apply following lemma.

Lemma 2.2. *Let* $\mathscr{B}$ *be a bounded subset of* $L^1_+(0,+\infty)$*. Then* $\mathscr{B}$ *has compact closure if and only if the following conditions hold:*

(i) $\sup\limits_{f\in\mathscr{B}} \int_0^{+\infty} | f(s) | ds < +\infty$*;*

(ii) $\lim\limits_{h\to+\infty} \int_h^{+\infty} | f(s) | ds = 0$ *uniformly in* $f \in \mathscr{B}$*;*

(iii) $\lim\limits_{h\to0^+} \int_0^{+\infty} | f(s+h) - f(s) | ds = 0$ *uniformly in* $f \in \mathscr{B}$*;*

(iv) $\lim\limits_{h\to0^+} \int_0^{h} | f(s) | ds = 0$ *uniformly in* $f \in \mathscr{B}$.

We are now ready to prove a result on the semiflow Φ generated by system (2.1) is asymptotically smooth.

Theorem 2.3. *The semiflow* Φ *generated by system* (2.1) *is asymptotically smooth.*

The proof of this theorem is in the appendix section.

Combining Proposition 2.3 and Φ is asymptotically smooth, as well as the existence theory of global attractors, the following result is immediate from Theorem 2.6 in [29] and Theorem 2.4 in [30].

Theorem 2.4. *The semiflow Φ has a global attractor $\mathcal{A}$ in $\mathscr{Y}$, which attracts any bounded subset of $\mathscr{Y}$.*

3. Equilibria and their local stability

3.1. Existence of equilibria

System (2.1) always has the disease free equilibrium $E_0 = (S_h^0, S_m^0, 0, 0, P^0(a), 0_{L^1}, 0_{L^1})$, where

$$S_h^0 = \frac{\Lambda_1}{\mu_1 + \psi(1 - \mathscr{K}_1)}, \quad S_m^0 = \frac{\Lambda_2}{\mu_2}, \quad P^0(a) = \psi S_h^0 k_1(a).$$

Define the basic reproduction number by

$$\mathscr{R}_0 = \frac{b^2\beta_1\beta_2\mathscr{K}_2\mathscr{K}_3\Lambda_1\Lambda_2}{(\mu_1 + \alpha + \delta)(\mu_1 + \psi(1 - \mathscr{K}_1))\mu_2^2}.$$

$\mathscr{R}_0$ represents the average number of secondary infectious human cases produced by a primary infectious human case that is introduced into two entirely susceptible populations: humans and female anopheles mosquitoes. The biological relevance of the threshold $\mathscr{R}_0$ can be interpreted as follows. There are

$$\frac{\Lambda_1}{\mu_1 + \psi(1 - \mathscr{K}_1)}$$

susceptible people and $\frac{\Lambda_2}{\mu_2}$ susceptible female anopheles mosquitoes. A primary infectious human case has a removal rate $\mu_1 + \alpha + \delta$, the average infectious period is

$$\frac{1}{\mu_1 + \alpha + \delta}.$$

During this time, the average number of mosquito bites from the susceptible mosquitoes is

$$\frac{b}{\mu_1 + \alpha + \delta},$$

so that the average number of infected but not infectious mosquitoes from the infectious human case will be

$$\frac{b\beta_2\Lambda_2}{(\mu_1 + \alpha + \delta)\mu_2}.$$

Then

$$\frac{b\beta_2\Lambda_2}{(\mu_1 + \alpha + \delta)\mu_2}\mathscr{K}_3$$

represents the total number of infectious female anopheles mosquitoes produced by infected but not infectious mosquitoes. The infectious mosquitoes have a removal rate μ_2, the average infectious period is $\frac{1}{\mu_2}$. During this time, the average number of mosquito bites from the susceptible mosquitoes is $\frac{b}{\mu_2}$, so that the average number of infected but not infectious human cases from the infectious mosquitoes will be

$$\frac{b\beta_2\Lambda_2\mathcal{K}_3 b\beta_1\Lambda_1}{(\mu_1+\alpha+\delta)\mu_2\mu_2(\mu_1+\psi(1-\mathcal{K}_1))}.$$

Then

$$\frac{b\beta_2\Lambda_2\mathcal{K}_3 b\beta_1\Lambda_1\mathcal{K}_2}{(\mu_1+\alpha+\delta)\mu_2\mu_2(\mu_1+\psi(1-\mathcal{K}_1))}$$

represents the average number of infectious human cases produced by infected but not infectious human cases. Therefore, the average number of secondary infectious human cases from a primary infectious human case is

$$\frac{b\beta_2\Lambda_2\mathcal{K}_3 b\beta_1\Lambda_1\mathcal{K}_2}{(\mu_1+\alpha+\delta)\mu_2\mu_2(\mu_1+\psi(1-\mathcal{K}_1))},$$

which is $\mathcal{R}_0$.

Next, we investigate the endemic equilibria of system (2.1). Any endemic equilibrium $(S_h^*, S_m^*, I_h^*, I_m^*, P^*(a), E_h^*(\theta), E_m^*(\tau))$ of system (2.1) should satisfy the following equations:

$$\begin{cases} \Lambda_1-(\mu_1+\psi)S_h^*-b\beta_1 S_h^* I_m^*+\displaystyle\int_0^{+\infty}\gamma_1(a)P^*(a)da=0,\\ \Lambda_2-\mu_2 S_m^*-b\beta_2 S_m^* I_h^*=0,\\ \displaystyle\int_0^{+\infty}\gamma_2(\theta)E_h^*(\theta)d\theta=(\mu_1+\alpha+\delta)I_h^*,\\ \displaystyle\int_0^{+\infty}\gamma_3(\tau)E_m^*(\tau)d\tau=\mu_2 I_m^*,\\ \dfrac{dP^*(a)}{da}=-(\mu_1+\gamma_1(a))P^*(a),\\ \dfrac{dE_h^*(\theta)}{d\theta}=-(\mu_1+\gamma_2(\theta))E_h^*(\theta),\\ \dfrac{dE_m^*(\tau)}{d\tau}=-(\mu_2+\gamma_3(\tau))E_m^*(\tau),\\ P^*(0)=\psi S_h^*,\ \ E_h^*(0)=b\beta_1 S_h^* I_m^*,\ \ E_m^*(0)=b\beta_2 S_m^* I_h^*. \end{cases} \tag{3.1}$$

From (3.1) we can easily find that if $\mathcal{R}_0 > 1$, system (2.1) has a unique endemic equilibrium $E^* = (S_h^*, S_m^*, I_h^*, I_m^*, P^*(a), E_h^*(\theta), E_m^*(\tau))$, where

$$I_h^*=\frac{\Lambda_1(1-\frac{1}{\mathcal{R}_0})}{\frac{\mu_2(\mu_1+\alpha+\delta)(\mu_1+\psi(1-\mathcal{K}_1))}{b\beta_1\mathcal{K}_2\mathcal{K}_3\Lambda_2}+\frac{\mu_1+\alpha+\delta}{\mathcal{K}_2}},\quad S_m^*=\frac{\Lambda_2}{b\beta_2 I_h^*+\mu_2},\quad I_m^*=\frac{b\beta_2 S_m^* I_h^*\mathcal{K}_3}{\mu_2},$$

$$S_h^*=\frac{(\mu_1+\alpha+\delta)I_h^*}{b\beta_1\mathcal{K}_2 I_m^*},\ \ P^*(a)=\psi S_h^* k_1(a),\ \ E_h^*(\theta)=b\beta_1 S_h^* I_m^* k_2(\theta),\ \ E_m^*(\tau)=b\beta_2 S_m^* I_h^* k_3(\tau).$$

Summarizing the discussions above, we have the following theorem.

Theorem 3.1. *System* (2.1) *always has the disease free equilibrium E_0. Moreover, apart from E_0, if $\mathcal{R}_0 > 1$, system* (2.1) *has a unique endemic equilibrium E^*.*

3.2. Local stability of the equilibria

Now we consider the linearized system of (2.1) at an equilibrium

$$\widetilde{E} = (\widetilde{S_h}, \widetilde{S_m}, \widetilde{I_h}, \widetilde{I_m}, \widetilde{P}(a), \widetilde{E_h}(\theta), \widetilde{E_m}(\tau)).$$

Let $\overline{S_h}(t) = S_h(t) - \widetilde{S_h}$, $\overline{S_m}(t) = S_m(t) - \widetilde{S_m}$, $\overline{I_h}(t) = I_h(t) - \widetilde{I_h}$, $\overline{I_m}(t) = I_m(t) - \widetilde{I_m}$, $\overline{P}(a,t) = P(a,t) - \widetilde{P}(a)$, $\overline{E_h}(t,\theta) = E_m(t,\theta) - \widetilde{E_h}(\theta)$, $\overline{E_m}(t,\tau)) = E_m(t,\tau) - \widetilde{E_m}(\tau)$, then removing the bar, we obtain the following linearized system:

$$\begin{cases} \dfrac{dS_h(t)}{dt} = -(\mu_1 + \psi)S_h(t) - b\beta_1\widetilde{S_h}I_m(t) - b\beta_1\widetilde{I_m}S_h(t) + \displaystyle\int_0^{+\infty} \gamma_1(a)P(t,a)da, \\ \dfrac{dS_m(t)}{dt} = -\mu_2 S_m(t) - b\beta_2\widetilde{S_m}I_h(t) - b\beta_2\widetilde{I_h}S_h(t), \\ \dfrac{dI_h(t)}{dt} = \displaystyle\int_0^{+\infty} \gamma_2(\theta)E_h(t,\theta)d\theta - (\mu_1 + \alpha + \delta)I_h(t), \\ \dfrac{dI_m(t)}{dt} = \displaystyle\int_0^{+\infty} \gamma_3(\tau)E_m(t,\tau)d\tau - \mu_2 I_m(t), \\ \dfrac{\partial P(t,a)}{\partial t} + \dfrac{\partial P(t,a)}{\partial a} = -(\mu_1 + \gamma_1(a))P(t,a), \\ \dfrac{\partial E_h(t,\theta)}{\partial t} + \dfrac{\partial E_h(t,\theta)}{\partial \theta} = -(\mu_1 + \gamma_2(\theta))E_h(t,\theta), \\ \dfrac{\partial E_m(t,\tau)}{\partial t} + \dfrac{\partial E_m(t,\tau)}{\partial \tau} = -(\mu_2 + \gamma_3(\tau))E_m(t,\tau), \end{cases} \tag{3.2}$$

with the following initial and boundary conditions

$$P(t,0) = \psi S_h(t), \quad E_h(t,0) = b\beta_1\widetilde{S_h}I_m(t) + b\beta_1\widetilde{I_m}S_h(t), \quad E_m(t,0) = b\beta_2\widetilde{S_m}I_h(t) + b\beta_2\widetilde{I_h}S_m(t).$$

Let

$$\mathscr{K}_i(\lambda) = \int_0^{+\infty} \gamma_i(u)e^{-\int_0^u(\lambda+\mu_1+\gamma_i(s))ds}du, \ (i = 1,2), \quad \mathscr{K}_3(\lambda) = \int_0^{+\infty} \gamma_3(\tau)e^{-\int_0^\tau(\lambda+\mu_2+\gamma_3(s))ds}d\tau.$$

For (3.2), let $S_h(t) = S_h^0 e^{\lambda t}$, $S_m(t) = S_m^0 e^{\lambda t}$, $I_h(t) = I_h^0 e^{\lambda t}$, $I_m(t) = I_m^0 e^{\lambda t}$, $P(t,a) = P^0(a)e^{\lambda t}$, $E_h(t,\theta) = E_h^0(\theta)e^{\lambda t}$, $E_m(t,\tau) = E_m^0(\tau)e^{\lambda t}$, we have

$$\begin{cases} (\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda)) + b\beta_1\widetilde{I_m})S_h^0 = -b\beta_1\widetilde{S_h}I_m^0, \\ (\lambda + \mu_2 + b\beta_2\widetilde{I_h})S_m^0 = -b\beta_2\widetilde{S_m}I_h^0, \\ (\lambda + \mu_1 + \alpha + \delta)I_h^0 = \displaystyle\int_0^{+\infty} \gamma_2(\theta)E_h^0(\theta)d\theta, \\ (\lambda + \mu_2)I_m^0 = \displaystyle\int_0^{+\infty} \gamma_3(\tau)E_m^0(\tau)d\tau, \\ \dot{P}^0(a) = -(\lambda + \mu_1 + \gamma_1(a))P^0(a), \\ \dot{E}_h^0(\theta) = -(\lambda + \mu_1 + \gamma_2(\theta))E_h^0(\theta), \\ \dot{E}_m^0(\tau) = -(\lambda + \mu_2 + \gamma_3(\tau))E_m^0(\tau), \end{cases} \tag{3.3}$$

with initial conditions

$$\begin{cases} P^0(0) = \psi S_h^0, \\ E_h^0(0) = b\beta_1 \widetilde{S_h} I_m^0 + b\beta_1 \widetilde{I_m} S_h^0, \\ E_m^0(0) = b\beta_2 \widetilde{S_m} I_h^0 + b\beta_2 \widetilde{I_h} S_m^0. \end{cases} \tag{3.4}$$

We obtain from the system (3.3) that

$$S_h^0 = \frac{-b\beta_1 \widetilde{S_h}}{(\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda)) + b\beta_1 \widetilde{I_m})} I_m^0, \quad S_m^0 = \frac{-b\beta_2 \widetilde{S_m}}{(\lambda + \mu_2 + b\beta_2 \widetilde{I_h})} I_h^0.$$

On substituting the above two equations into the second and third equations of system (3.4), and by simple calculation, we have

$$E_h^0(0) = \frac{b\beta_1 \mathscr{K}_3(\lambda) \widetilde{S_h} (\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda))}{(\lambda + \mu_2)(\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda)) + b\beta_1 \widetilde{I_m})} E_m^0(0)$$

and

$$E_m^0(0) = \frac{b\beta_2 \mathscr{K}_2(\lambda) \widetilde{S_m} (\lambda + \mu_2)}{(\lambda + \mu_1 + \alpha + \delta)(\lambda + \mu_2 + b\beta_2 \widetilde{I_h})} E_h^0(0).$$

We derive that

$$E_h^0(0) = \frac{b\beta_1 \mathscr{K}_3(\lambda) \widetilde{S_h} (\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda))}{(\lambda + \mu_2)(\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda)) + b\beta_1 \widetilde{I_m})} \frac{b\beta_2 \mathscr{K}_2(\lambda) \widetilde{S_m} (\lambda + \mu_2)}{(\lambda + \mu_1 + \alpha + \delta)(\lambda + \mu_2 + b\beta_2 \widetilde{I_h})} E_h^0(0).$$

We obtain the characteristic equation of model (2.1) at an equilibrium $\widetilde{E}$ as follows:

$$f(\lambda) = 1,$$

where $f(\lambda) = \dfrac{b^2 \beta_1 \beta_2 \mathscr{K}_2(\lambda) \mathscr{K}_3(\lambda) \widetilde{S_h} \widetilde{S_m} (\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda))}{(\lambda + \mu_1 + \alpha + \delta)(\lambda + \mu_2 + b\beta_2 \widetilde{I_h})(\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda)) + b\beta_1 \widetilde{I_m})}$.

Theorem 3.2. *(Local stability)*

(i) The disease-free equilibrium E_0 of system (2.1) *is locally stable if $\mathscr{R}_0 < 1$ and unstable if $\mathscr{R}_0 > 1$.*

(ii) The endemic equilibrium E^* of system (2.1) *is locally stable if $\mathscr{R}_0 > 1$.*

Proof. First consider the local stability of the disease-free steady state E_0.

$$f(\lambda) = \frac{b^2 \beta_1 \beta_2 \mathscr{K}_2(\lambda) \mathscr{K}_3(\lambda) \Lambda_1 \Lambda_2}{(\lambda + \mu_1 + \alpha + \delta)(\lambda + \mu_2)\mu_2(\mu_1 + \psi(1 - \mathscr{K}_1))}.$$

Clearly, we have $f(0) = \mathscr{R}_0$, $f'(\lambda) < 0$ and $\lim_{\lambda \to \infty} f(\lambda) = 0$. Hence, if $\mathscr{R}_0 > 1$, then $f(\lambda) = 1$ has a unique positive real root. Thus, if $\mathscr{R}_0 > 1$, the disease-free steady state E_0 is unstable.

If $\mathscr{R}_0 < 1$, the disease-free equilibrium E_0 is locally stable. Otherwise, $f(\lambda_0) = 1$ has at least one root $\lambda_0 = a_1 + ib_1$ satisfying $a_1 \geq 0$. But

$$| f(\lambda_0) | \leq \frac{b^2 \beta_1 \beta_2 \mathscr{K}_2 \mathscr{K}_3 \Lambda_1 \Lambda_2}{(\mu_1 + \alpha + \delta)\mu_2^2(\mu_1 + \psi(1 - \mathscr{K}_1))} = \mathscr{R}_0 < 1.$$

Hence, if $\mathscr{R}_0 < 1$, all roots of $f(\lambda) = 1$ have negative real parts, then the disease-free equilibrium E_0 is locally stable.

Next, we consider the local stability of the endemic equilibrium E^*.

$$f(\lambda) = \frac{b^2\beta_1\beta_2\mathscr{K}_2(\lambda)\mathscr{K}_3(\lambda)S_h^*S_m^*(\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda))}{(\lambda + \mu_1 + \alpha + \delta)(\lambda + \mu_2 + b\beta_2 I_h^*)(\lambda + \mu_1 + \psi(1 - \mathscr{K}_1(\lambda)) + b\beta_1 I_m^*)}.$$

If $\mathscr{R}_0 > 1$, the endemic equilibrium E^* is locally stable. Otherwise, $f(\lambda^*) = 1$ has at least one root $\lambda^* = a_2 + ib_2$ satisfying $a_2 \geq 0$. But

$$\mid f(\lambda^*) \mid < \frac{b^2\beta_1\beta_2\mathscr{K}_2\mathscr{K}_3 S_h^* S_m^*}{(\mu_1 + \alpha + \delta)\mu_2}.$$

Notice, $S_h^*S_m^* = \dfrac{(\mu_1 + \alpha + \delta)\mu_2}{b^2\beta_1\beta_2\mathscr{K}_2\mathscr{K}_3}$, we know $\mid f(\lambda^*) \mid < 1$, Hence, if $\mathscr{R}_0 > 1$, all roots of $f(\lambda) = 1$ have negative real parts, then the endemic equilibrium E^* is locally stable. This completes the proof. □

4. Uniform persistence and global stability

4.1. Uniform persistence

In this subsection, our purpose is to show that system (2.1) is uniformly persistent when $\mathscr{R}_0 > 1$. Define

$$M_0 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T \in \mathscr{Y} \mid \exists\, t_1,\ t_2 \in \mathbb{R}_+ :\ x_3 + x_4 + \int_0^{+\infty} \gamma_2(\theta + t_1)x_6(\theta)d\theta + \int_0^{+\infty} \gamma_3(\tau + t_2)x_7(\tau)d\tau > 0\},$$

let $\partial M_0 = \mathscr{Y} \setminus M_0$, then we have $\mathscr{Y} = M_0 \cup \partial M_0$.

Theorem 4.1. *The sets M_0 and ∂M_0 are forward invariant under the semiflow $\Phi(t,\cdot)$. Also, the disease-free equilibrium E_0 of system* (2.1) *is globally asymptotically stable for the semiflow $\Phi(t,\cdot)$ restricted to ∂M_0.*

The proof of this theorem is in the appendix section. Theorem 4.1 states that if the number of people and mosquitoes initially infected by plasmodium belongs to ∂M_0, then malaria will eventually die out. But if the number of people and mosquitoes initially infected by plasmodium belongs to M_0, then whether malaria is eventually extinct, we need to verify by the following Theorem 4.2, Theorem 4.3 and Theorem 4.5.

Theorem 4.2. *If $\mathscr{R}_0 > 1$, then semiflow $\{\Phi(t,\cdot)\}_{t\geq 0}$ generated by system* (2.1) *is uniformly persistent with respect to the decomposition $(M_0,\ \partial M_0)$. Moreover, there is a compact subset $\mathscr{A}_0 \subset M_0$, which is a global attractor for $\{\Phi(t,\cdot)\}_{t\geq 0}$ in M_0.*

Details of this proof are in appendix section. Theorem 4.2 shows that if $\mathscr{R}_0$ is greater than 1, when the number of people and mosquitoes initially infected by plasmodium belongs to M_0, malaria will always exist at a certain scale.

4.2. Global stablility

Let $x \in \mathcal{A}$, we can find a complete orbit $\{\Phi(t,\cdot)\}_{t\in\mathbb{R}}$ through x in $\mathcal{A}$. By similar analytical method used in [[31], subsection 3.2], system (2.1) can be written as

$$\begin{cases} \dfrac{dS_h(t)}{dt} = \Lambda_1 - (\mu_1+\psi)S_h(t) - b\beta_1 S_h(t)I_m(t) + \displaystyle\int_0^{+\infty} \gamma_1(a)k_1(a)\psi S_h(t-a)da, \\ \dfrac{dS_m(t)}{dt} = \Lambda_2 - \mu_2 S_m(t) - b\beta_2 S_m(t)I_h(t), \\ \dfrac{dI_h(t)}{dt} = \displaystyle\int_0^{+\infty} \gamma_2(\theta)E_h(t,\theta)d\theta - (\mu_1+\alpha+\delta)I_h(t), \\ \dfrac{dI_m(t)}{dt} = \displaystyle\int_0^{+\infty} \gamma_3(\tau)E_m(t,\tau)d\tau - \mu_2 I_m(t), \\ P(t,a) = k_1(a)\psi S_h(t-a), \\ \dfrac{\partial E_h(t,\theta)}{\partial t} + \dfrac{\partial E_h(t,\theta)}{\partial \theta} = -(\mu_1+\gamma_2(\theta))E_h(t,\theta), \\ E_h(t,0) = b\beta_2 S_h(t)I_m(t),\ (or\ E_h(t,\theta) = k_2(\theta)b\beta_1 S_h(t-\theta)I_m(t-\theta)), \\ \dfrac{\partial E_m(t,\tau)}{\partial t} + \dfrac{\partial E_m(t,\tau)}{\partial \tau} = -(\mu_2+\gamma_3(\tau))E_m(t,\tau), \\ E_m(t,0) = b\beta_2 S_m(t)I_h(t),\ (or\ E_m(t,\tau) = k_3(\tau)b\beta_2 S_m(t-\tau)I_h(t-\tau)), \\ (S_h(0),\ S_m(0),\ I_h(0),\ I_m(0),\ P(0,a),\ E_h(0,\theta),\ E_m(0,\tau)) \in \mathcal{A}. \end{cases} \tag{4.1}$$

Theorem 4.3. *The disease-free equilibrium E_0 of system* (2.1) *is globally asymptotically stable if $\mathcal{R}_0 <$ 1.*

The proof of this theorem is in the appendix section. Theorem 4.3 shows that if $\mathcal{R}_0$ is less than 1, malaria will be extinct regardless of whether the number of people and mosquitoes initially infected by plasmodium belongs to M_0 or ∂M_0.

If $\mathcal{R}_0 > 1$, we have know that the system (2.1) is uniformly persistent and has a global attractor $\mathcal{A}_0$ in M_0, meanwhile, let $(S_h(0),\ S_m(0),\ I_h(0),\ I_m(0),\ P(0,a),\ E_h(0,\theta),\ E_m(0,\tau)) \in \mathcal{A}_0$, system (2.1) can be written as (4.1) (replace $\mathcal{A}$ with $\mathcal{A}_0$). In order to study the global stability of E^*, we first prove the following lemma.

Lemma 4.4. *There exist $\epsilon, M > 0$ such that all solutions in $\mathcal{A}_0$ for $t \in \mathbb{R}$, the following inequalities are satisfied*

$$\epsilon \le S_h(t),\ S_m(t),\ I_h(t),\ I_m(t) \le M,\ \ \psi\epsilon k_1(a) \le P(t,a) \le \psi M k_1(a),$$

$$b\beta_1\epsilon^2 k_2(\theta) \le E_h(t,\theta) \le b\beta_1 M^2 k_2(\theta),\ \ b\beta_2\epsilon^2 k_3(\tau) \le E_m(t,\tau) \le b\beta_2 M^2 k_3(\tau).$$

Proof. Let $(S_h(t), S_m(t), I_h(t), I_m(t), P(t,a), E_h(t,\theta), E_m(t,\tau)) \in \mathcal{A}_0$.

First we prove that $S_h(t) > 0,\ S_m(t) > 0$ for $t \in \mathbb{R}$. We assume that there exists $t_1 \in \mathbb{R}$ such that $S_m(t_1) = 0$. From (4.1) we have $\dfrac{dS_m(t_1)}{dt} \ge \Lambda_1 > 0$, then $\exists\ \eta > 0$ such that $S_m(t_1-\eta) < 0$. This contradicts the $\mathcal{A}_0 \subset M_0$. Thus, $S_m(t) > 0$ for $t \in \mathbb{R}$. Similarly, we can get $S_h(t) > 0$ for $t \in \mathbb{R}$.

Next we prove that $I_h(t) > 0,\ I_m(t) > 0$ for $t \in \mathbb{R}$. First we assume that there exists $t_0 \in \mathbb{R}$ such that $I_h(t_0) = 0$ and $I_m(t_0) = 0$, then according to $I_h(t),\ I_m(t)$ equations in (4.1), we can deduce that

$I_h(t) = 0,\ I_m(t) = 0$ for $t \le t_0$. Next form (4.1) we know $\int_0^{+\infty} E_h(t,\theta)d\theta = \int_0^{+\infty} k_2(\theta)b\beta_1 S_h(t-\theta)I_m(t-\theta))d\theta = 0$, $\int_0^{+\infty} E_m(t,\tau)d\tau = \int_0^{+\infty} k_3(\tau)b\beta_2 S_m(t-\tau)I_h(t-\tau))d\tau = 0$ for all $t \le t_0$, then obviously, $\int_0^{+\infty} \gamma_2(\theta + s_1)E_h(t,\theta)d\theta = 0$, $\int_0^{+\infty} \gamma_3(\tau + s_2)E_m(t,\tau)d\tau = 0$ for all $s_1,\ s_2 \ge 0$. This contradicts the $\mathcal{A}_0 \subset M_0$. On the other hand, we assume that there exists $t_0 \in \mathbb{R}$ such that one of $I_h(t_0),\ I_m(t_0)$ is positive, so without loss of generality, we assume that $I_h(t_0) = 0,\ I_m(t_0) > 0$. Similarly, we have $I_h(t) = 0$, $\int_0^{+\infty} \gamma_3(\tau + s)E_m(t,\tau)d\tau = 0$ for all $s \ge 0, t \le t_0$. Then $\frac{dI_m(t)}{dt} = -\mu_2 I_m(t)$ for all $t \le t_0$, we deduce that $I_m(t_0) = I_m(t)e^{-\mu(t_0-t)}$, obviously, $I_m(t) \to +\infty$ as $t \to -\infty$. This contradicts the compactness of $\mathcal{A}_0$. Hence, $I_h(t) > 0,\ I_m(t) > 0$ for all $t \in \mathbb{R}$. In addition, from (4.1), we deduce that $P(t,a) > 0, E_h(t,\theta) > 0,\ E_m(t,\tau) > 0$ for $t \in \mathbb{R}, a,\theta,\tau \in \mathbb{R}_+$.

According to the compactness of $\mathcal{A}_0$, we know there exist $\epsilon, M > 0$ such that

$$\epsilon \le S_h(t),\ S_m(t),\ I_h(t),\ I_m(t) \le M,\ \ \psi\epsilon k_1(a) \le P(t,a) \le \psi M k_1(a),$$

$$b\beta_1\epsilon^2 k_2(\theta) \le E_h(t,\theta) \le b\beta_1 M^2 k_2(\theta),\ \ b\beta_2\epsilon^2 k_3(\tau) \le E_m(t,\tau) \le b\beta_2 M^2 k_3(\tau).$$

The proof is complete. □

Theorem 4.5. *If $\mathcal{R}_0 > 1$, then endemic equilibrium E^* of system* (2.1) *is globally asymptotically stable in M_0.*

Details of this proof are in appendix section. Theorem 4.5 shows that if $\mathcal{R}_0$ is greater than 1, when the number of people and mosquitoes initially infected by plasmodium belongs to M_0, the number of people and mosquitoes will eventually stabilize at E^*.

5. Simulations for the malaria model

In this section, we present some numerical simulations to confirm the above theoretical results (i.e., Theorems 4.3 and 4.5). Some of the parameters are taken from[8] $b = 0.2, \beta_1 = 0.015, \delta = 0.05, \mu_1 = 0.00004$. Other parameters are estimated. we estimate other parameters as follows $\Lambda_1 = 0.2, \Lambda_2 = 20, \beta_2 = 0.015, \mu_2 = 0.03, \alpha = 0.001$. Moreover, we assume that the prevention wane rate and the removal rate from latent class take the form

$$\gamma_i(s) = \begin{cases} 0, & 0 < s < t_i, \\ \gamma_i, & t_i \le t, \end{cases}$$

where $i = 1,2,3$, $t_1 = 12, t_2 = 7, t_3 = 10$ (time units are days), $\gamma_1 = 0.001, \gamma_2 = 0.11, \gamma_3 = 0.1$. By solving $\mathcal{R}_0 = 1$, we obtain that $\psi_0 = 0.4872$. Obviously, $\mathcal{R}_0$ decreases as the degree of prevention ψ increases. Thus, first, we let $\psi = 0.9 > \psi_0$, we can compute $\mathcal{R}_0 = 0.5414 < 1$, then the disease-free equilibrium $E_0 = (0.2426, 666.667, 0, 0, 0.2183k_1(a), 0, 0)$ is globally stable, as depicted in Fig.2, Fig.3. Next, we let $\psi = 0.3 < \psi_0$, we can compute $\mathcal{R}_0 = 1.6239 > 1$, then the endemic equilibrium $E^* = (0.5025, 594.5913, 1.2122, 41.0726, 0.1507k_1(a), 0.0619k_2(\theta), 2.1623k_3(\tau))$ is globally stable, as depicted in Fig.4, Fig.5. From the numerical simulations we can conjecture that increasing ψ, the rate of prevention of the susceptible people, can eliminate malaria. This is consistent with the report [1] of the WHO that prevention is considered to have made a major contribution to the reduction in malaria burden since 2000.

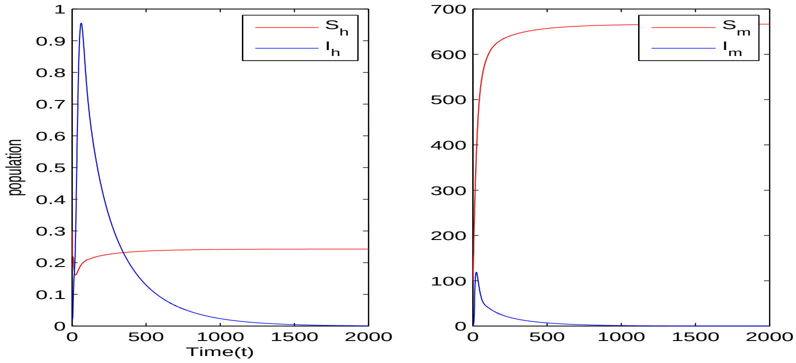

Figure 2. The trajectory of $S_h(t), I_h(t), S_m(t)$ *and* $I_h(t)$ versus time with the initial condition $(0.3, 100, 0.02, 2, 0.1e^{-0.5a}, 0.1e^{-0.5a}, 100e^{-0.5a})$.

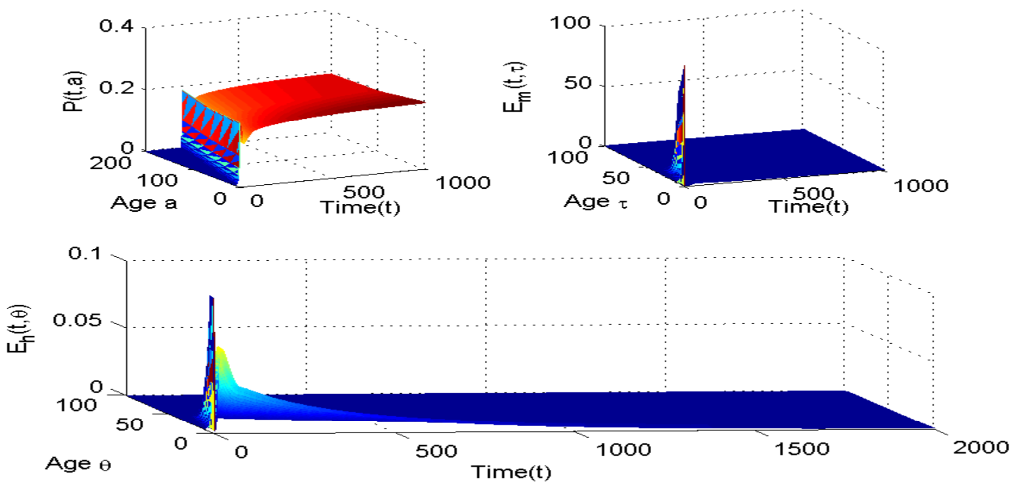

Figure 3. The surface of $P(t, a), E_h(t, \theta)$ *and* $E_m(t, \tau)$ versus time and age with the initial condition $(0.3, 100, 0.02, 2, 0.1e^{-0.5a}, 0.1e^{-0.5t}, 100e^{-0.5t})$.

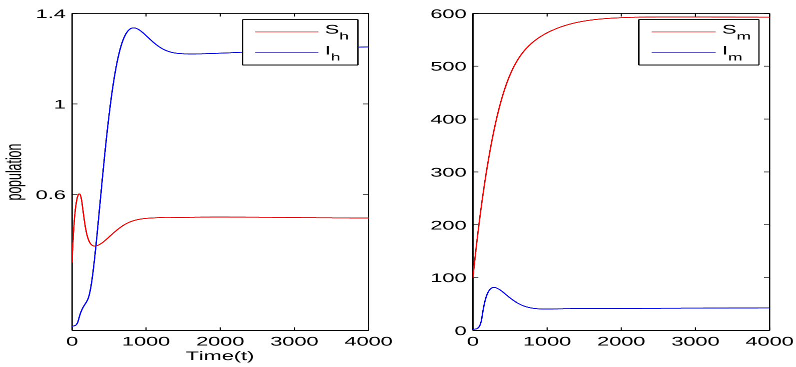

Figure 4. The trajectory of $S_h(t), I_h(t), S_m(t)$ *and* $I_h(t)$ versus time with the initial condition $(0.3, 100, 0.02, 2, 0.1e^{-0.5a}, 0.1e^{-0.5a}, 100e^{-0.5a})$.

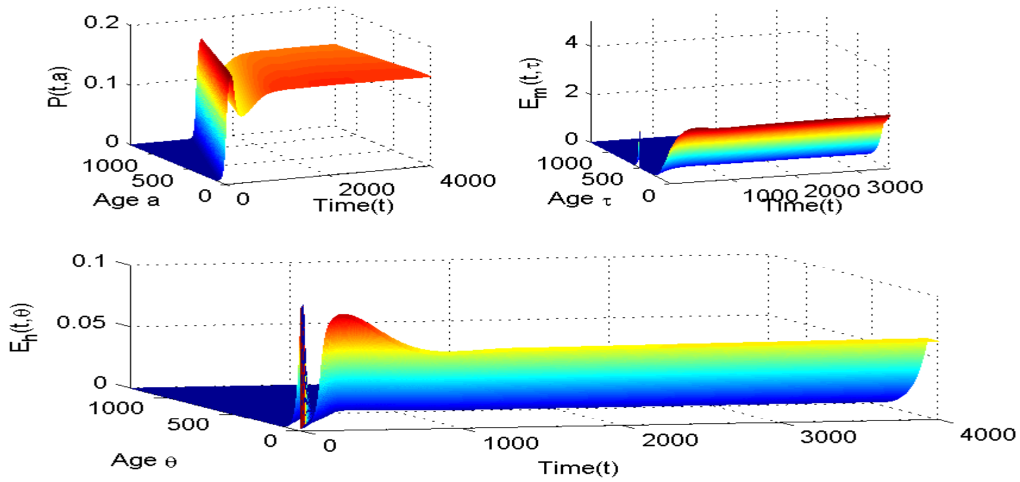

Figure 5. The surface of $P(t, a), E_h(t, \theta)$ *and* $E_m(t, \tau)$ versus time and age with the initial condition $(0.3, 100, 0.02, 2, 0.1e^{-0.5a}, 0.1e^{-0.5a}, 100e^{-0.5a})$.

6. Discussion and conclusion

Malaria is a dangerous and sometimes fatal disease caused by malaria parasite that commonly infect female anopheles mosquitoes which take a blood meal on humans. It is one of the greatest challenges in the field of global public health. The cost of malaria to individuals, families, communities, nations is huge. This issue may become more serious due to lack of prevention. In our paper, we considered an age-structured malaria epidemic model with prevention period of susceptible people and latent period of two types of infected hosts. The basic reproduction number $\mathcal{R}_0$ of our model (2.1) has been found by the definition. Based on regional division of state space and selection of appropriate Lyapunov functions, we proved that disease-free equilibrium E_0 is globally asymptotically stable if $\mathcal{R}_0 < 1$, while endemic equilibrium E^* is globally asymptotically stable in M_0 if $\mathcal{R}_0 > 1$.

We are easy to see that $\mathcal{R}_0$ decreases as the rate of prevention ψ increases, that is, increasing the degree of prevention can lead to malaria extinct. Thus, the prevention is very important for the control of malaria. If we don't take into account the prevention period and latent period of two types of infected hosts , and according to the dynamics of the ODE model in [32], we can easily know that the basic reproduction number $\mathcal{R}_0$ is reduced to $\overline{\mathcal{R}}_0 = \frac{b^2\beta_1\beta_2\Lambda_1\Lambda_2}{(\mu_1+\alpha+\delta)\mu_1\mu_2^2}$, in addition, notice that $\mathcal{K}_1 < 1$, $\mathcal{K}_2 < 1$, $\mathcal{K}_3 < 1$, we can find that $\mathcal{R}_0 < \overline{\mathcal{R}}_0$, so considering the prevention period and latent period of two types of infected hosts can reduce the basic reproduction number, which are beneficial to understanding the dynamics of malaria control.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix

Proof of Theorem 2.3

Proof. We first decompose Φ into the following two parts: U_1, U_2 defined respectively by

$$U_1(t,x) = (S_h(t), S_m(t), I_h(t), I_m(t), \widetilde{P}(t,\cdot), \widetilde{E_h}(t,\cdot), \widetilde{E_m}(t,\cdot)),$$

$$U_2(t,x) = (0,0,0,0,\psi_P(t,\cdot), \psi_{Eh}(t,\cdot), \psi_{Em}(t,\cdot)),$$

where

$$\psi_P(t,a) = \begin{cases} 0, & 0 \le a \le t, \\ P_0(a-t)\dfrac{k_1(a)}{k_1(a-t)}, & 0 \le t < a, \end{cases} \qquad \psi_{Eh}(t,\theta) = \begin{cases} 0, & 0 \le \theta \le t, \\ E_{h0}(\theta-t)\dfrac{k_2(\theta)}{k_2(\theta-t)}, & 0 \le t < \theta, \end{cases}$$

$$\psi_{Em}(t,\tau) = \begin{cases} 0, & 0 \le \tau \le t, \\ E_{m0}(\tau-t)\dfrac{k_3(\tau)}{k_3(\tau-t)}, & 0 \le t < \tau, \end{cases} \qquad \widetilde{P}(t,a) = \begin{cases} P(t-a,0)k_1(\theta), & 0 \le a \le t, \\ 0, & 0 \le t < a, \end{cases}$$

$$\widetilde{E_h}(t,\theta) = \begin{cases} E_h(t-\theta,0)k_2(\theta), & 0 \le \theta \le t, \\ 0, & 0 \le t < \theta, \end{cases} \qquad \widetilde{E_m}(t,\tau) = \begin{cases} E_m(t-\tau,0)k_3(\tau), & 0 \le \tau \le t, \\ 0, & 0 \le t < \tau, \end{cases}$$

for $x = (S_h(0), S_m(0), I_h(0), I_m(0), P_0(a), E_{h0}(\theta), E_{m0}(\tau))$, clearly, we have $\Phi(t,x) = U_1(t,x) + U_2(t,x)$.

Let $\mathscr{A} \subset \mathscr{Y}$, r is a constant greater than $\frac{\Lambda_1+\Lambda_2}{\mu}$, for each $x \in \mathscr{A}$, we have $\| x \|_{\mathscr{Y}} \le r$.

$$\begin{aligned}
\| U_2(t,x) \|_{\mathscr{Y}} &= \int_t^{+\infty} P_0(a-t)\frac{k_1(a)}{k_1(a-t)}da + \int_t^{+\infty} E_{h0}(\theta-t)\frac{k_2(\theta)}{k_2(\theta-t)}d\theta \\
&\quad + \int_t^{+\infty} E_{m0}(\tau-t)\frac{k_3(\tau)}{k_3(\tau-t)}d\tau \\
&= \int_0^{+\infty} P_0(s)\frac{k_1(s+t)}{k_1(s)}ds + \int_0^{+\infty} E_{h0}(s)\frac{k_2(s+t)}{k_2(s)}ds + \int_0^{+\infty} E_{m0}(s)\frac{k_3(s+t)}{k_3(s)}ds \\
&= \int_0^{+\infty} P_0(s)e^{-\int_s^{s+t}(\mu_1+\gamma_1(l))dl}ds + \int_0^{+\infty} E_{h0}(s)e^{-\int_s^{s+t}(\mu_1+\gamma_2(l))dl}ds \\
&\quad + \int_0^{+\infty} E_{m0}(s)e^{-\int_s^{s+t}(\mu_2+\gamma_3(l))dl}ds \le e^{-\mu t} \| x \|_{\mathscr{Y}} \le re^{-\mu t}.
\end{aligned}$$

Thus, $\lim\limits_{t\to+\infty} diam\ U_2(t,\mathscr{A}) = 0$. In the following we show that $U_1(t,\mathscr{A})$ has compact closure for each $t \ge 0$.

From Proposition 2.3, we know that $S_i(t), I_i(t)$ $(i = h, m)$ remain in the compact set $[0,r]$ for all $t \ge 0$. Next, we will show that $\widetilde{P}(t,a), \widetilde{E_h}(t,\theta)$ *and* $\widetilde{E_m}(t,\tau)$ remain in a precompact subset of $L_1^+(0,+\infty)$ which is independent of x.

$$0 \le \widetilde{E_m}(t,\tau) = \begin{cases} E_m(t-\tau,0)k_3(\tau), & 0 \le \tau \le t, \\ 0, & 0 \le t < \tau. \end{cases}$$

It is easy to show that

$$\widetilde{E_m}(t,\tau) \le b\beta_2 r^2 e^{-\mu\tau}.$$

Therefore, the conditions (i),(ii) and (iv) of Lemma 2.2 are satisfied. Now, we only to check the condition (iii) of Lemma 2.2.

$$\int_0^{+\infty} | \widetilde{E_m}(t,\tau+h) - \widetilde{E_m}(t,\tau) | d\tau = \int_0^{t-h} | \widetilde{E_m}(t,\tau+h) - \widetilde{E_m}(t,\tau) | d\tau + \int_{t-h}^{t} | \widetilde{E_m}(t,\tau) | d\tau$$

$$= \int_0^{t-h} | \widetilde{E_m}(t-\tau-h,0)k_3(\tau+h) - \widetilde{E_m}(t-\tau,0)k_3(\tau) | d\tau + \int_{t-h}^{t} | \widetilde{E_m}(t-\tau,0)k_3(\tau) | d\tau$$

$$\leq \int_0^{t-h} | \widetilde{E_m}(t-\tau-h,0) || k_3(\tau+h) - k_3(\tau) | d\tau + \int_0^{t-h} | \widetilde{E_m}(t-\tau-h,0) - \widetilde{E_m}(t-\tau,0) || k_3(\tau) | d\tau + b\beta_2 r^2 h.$$

$$\int_0^{t-h} | \widetilde{E_m}(t-\tau-h,0) || k_3(\tau+h) - k_3(\tau) | d\tau \leq b\beta_2 r^2 \Big(\int_0^{t-h} k_3(\tau)d\tau - \int_0^{t-h} k_3(\tau+h)d\tau\Big)$$

$$= b\beta_2 r^2 \Big(\int_0^{t-h} k_3(\tau)d\tau - \int_h^{t} k_3(s)ds\Big)$$

$$= b\beta_2 r^2 \Big(\int_0^{t-h} k_3(\tau)d\tau - \int_h^{t-h} k_3(s)ds - \int_{t-h}^{t} k_3(s)ds\Big)$$

$$= b\beta_2 r^2 \Big(\int_0^{h} k_3(\tau)d\tau - \int_{t-h}^{t} k_3(s)ds\Big) \leq b\beta_2 r^2 h.$$

Notice that $\left| \dfrac{dI_h(t)}{dt} \right| \leq \bar{\gamma}_2 r + (\mu_1+\alpha+\delta)r$, $\left| \dfrac{dS_m(t)}{dt} \right| \leq \Lambda_1 + (\mu_1+\varphi)r + b\beta_1 r^2 + \bar{\gamma}_1 r$, we have

$$| \widetilde{E_m}(t-\tau-h,0) - \widetilde{E_m}(t-\tau,0) | = b\beta_1 | S_m(t-\tau-h)I_h(t-\tau-h) - S_m(t-\tau)I_h(t-\tau) |$$

$$= b\beta_1(| S_m(t-\tau-h) || I_h(t-\tau-h) - I_h(t-\tau) | + | I_h(t-\tau) || S_m(t-\tau-h) - S_m(t-\tau) |)$$

$$\leq b\beta_1 r(\bar{\gamma}_2 r + (\mu_1+\alpha+\delta)r)h + b\beta_1 r(\Lambda_1 + (\mu_1+\varphi)r + b\beta_1 r^2 + \bar{\gamma}_1 r)h.$$

Then

$$\int_0^{t-h} | \widetilde{E_m}(t-\tau-h,0) - \widetilde{E_m}(t-\tau,0) || k_3(\tau) | d\tau$$

$$\leq b\beta_1 r((\bar{\gamma}_1 + (\mu_1+\delta))r + (\Lambda_1 + (\mu_1+\varphi)r + b\beta_1 r^2 + \bar{\gamma}_1 r))h \int_0^{t-h} e^{-\mu s} ds.$$

$$\leq \frac{b\beta_1 r}{\mu}((\bar{\gamma}_2 + \mu_1 + \alpha + \delta)r + (\Lambda_1 + (\mu_1+\varphi)r + b\beta_1 r^2 + \bar{\gamma}_1 r))h.$$

Hence,

$$\int_0^{+\infty} | \widetilde{E_m}(t,\tau+h) - \widetilde{E_m}(t,\tau) | d\tau \leq \Big(2b\beta_2 r^2 + \frac{b\beta_1 r}{\mu}((\bar{\gamma}_2+\mu_1+\alpha+\delta)r + (\Lambda_1 + (\mu_1+\varphi)r + b\beta_1 r^2 + \bar{\gamma}_1 r))\Big)h.$$

Thus, condition (iii) of Lemma 2.2 holds, then we can get that $\widetilde{E_m}(t,\tau)$ satisfies the conditions of Lemma 2.2. In a similar way, $\widetilde{P}(t,a)$, $\widetilde{E_h}(t,\theta)$ also satisfy the conditions of Lemma 2.2. Therefore, we obtain $U_1(t,\mathscr{A})$ has compact closure for each $t \geq 0$, Using Lemma 2.1, we know semiflow Φ is asymptotically smooth. This completes the proof. □

Proof of Theorem 4.1

Proof. First we prove M_0 is forward invariant under the semiflow $\Phi(t,\cdot)$. Let $\Phi(0,x_0)\in M_0$, if $I_h(0)>0$ or $I_m(0)>0$, From (2.1), it's easy to know $I_h(t)\geq I_h(0)e^{-(\mu_1+\alpha+\delta)t}>0$ or $I_m(t)\geq I_m(0)e^{-\mu_2 t}>0$, then M_0 is forward invariant. Otherwise, if $I_h(0)=I_m(0)=0$, then without loss of generality, $\exists\ t_1\in\mathbb{R}_+$ such that $\int_0^{+\infty}\gamma_2(\theta+t_1)E_h(0,\theta)d\theta>0$ ($x_0\in M_0$). $\forall\ t\in[0,t_1]$, $\exists\ s=t_1-t\geq 0$ such that

$$\int_0^{+\infty}\gamma_2(\theta+s)E_h(t,\theta)d\theta\geq\int_t^{+\infty}\gamma_2(\theta+s)E_h(t,\theta)d\theta=\int_0^{+\infty}\gamma_2(\theta+t+s)E_h(t,\theta+t)d\theta$$

$$=\int_0^{+\infty}\gamma_2(\theta+t_1)E_h(0,\theta)\frac{k_2(\theta+t)}{k_2(\theta)}d\theta\geq e^{-(\mu_1+\bar{\gamma}_2)t}\int_0^{+\infty}\gamma_2(\theta+t_1)E_h(0,\theta)d\theta>0.$$

If $\exists\ t_2\in(0,t_1]$ such that $I_h(t_2)>0$, then $I_h(t)>0,\ \forall t>t_2$, otherwise, since

$$\frac{dI_h(t_1)}{dt}=\int_0^{+\infty}\gamma_2(\theta)E_h(t_1,\theta)d\theta>0,$$

we know $I_h(t)>0$ for all $t>t_1$. Hence, $\Phi(t,M_0)\subset M_0$.

This completes the fact that $\Phi(t,M_0)\subset M_0$, i.e. M_0 is forward invariant under the semiflow $\Phi(t,\cdot)$.

Next, we will prove ∂M_0 is forward invariant under the semiflow $\Phi(t,\cdot)$. Let $\Phi(0,x_0)\in\partial M_0$, we consider the following system

$$\begin{cases}\dfrac{dI_h(t)}{dt}=\displaystyle\int_0^{+\infty}\gamma_2(\theta)E_h(t,\theta)d\theta-(\mu_1+\alpha+\delta)I_h(t),\\ \dfrac{dI_m(t)}{dt}=\displaystyle\int_0^{+\infty}\gamma_3(\tau)E_m(t,\tau)d\tau-\mu_2 I_m(t),\\ \dfrac{\partial E_h(t,\theta)}{\partial t}+\dfrac{\partial E_h(t,\theta)}{\partial\theta}=-(\mu_1+\gamma_2(\theta))E_h(t,\theta),\\ \dfrac{\partial E_m(t,\tau)}{\partial t}+\dfrac{\partial E_m(t,\tau)}{\partial\tau}=-(\mu_2+\gamma_3(\tau))E_m(t,\tau),\\ E_h(t,0)=b\beta_1 S_h(t)I_m(t),\quad E_m(t,0)=b\beta_2 S_m(t)I_h(t),\\ E_h(0,\theta)=E_{h0}(\theta),\quad E_m(0,\tau)=E_{m0}(\tau),\quad I_h(0)=0,\quad I_m(0)=0.\end{cases}\tag{6.1}$$

Since $S_h(t),\ S_m(t)\leq\Delta$, where $\Delta=max\{\frac{\Lambda_1+\Lambda_2}{\mu},\|\ x_0\ \|_{\mathscr{Y}}\}$, it follows that

$$I_i(t)\leq\hat{I}_i(t),\quad E_i(t,s)\leq\hat{E}_i(t,s),\ (i=h,m),\tag{6.2}$$

where

$$\begin{cases}\dfrac{d\hat{I}_h(t)}{dt}=\displaystyle\int_0^{+\infty}\gamma_2(\theta)\hat{E}_h(t,\theta)d\theta-(\mu_1+\alpha+\delta)\hat{I}_h(t),\\ \dfrac{d\hat{I}_m(t)}{dt}=\displaystyle\int_0^{+\infty}\gamma_3(\tau)\hat{E}_m(t,\tau)d\tau-\mu_2\hat{I}_m(t),\\ \dfrac{\partial\hat{E}_h(t,\theta)}{\partial t}+\dfrac{\partial\hat{E}_h(t,\theta)}{\partial\theta}=-(\mu_1+\gamma_2(\theta))\hat{E}_h(t,\theta),\\ \dfrac{\partial\hat{E}_m(t,\tau)}{\partial t}+\dfrac{\partial\hat{E}_m(t,\tau)}{\partial\tau}=-(\mu_2+\gamma_3(\tau))\hat{E}_m(t,\tau),\end{cases}\tag{6.3}$$

with the following initial and boundary conditions

$$\begin{cases} \hat{E}_h(t,0) = b\beta_1\Delta\hat{I}_m(t), \quad \hat{E}_m(t,0) = b\beta_2\Delta\hat{I}_h(t), \\ \hat{E}_h(0,\theta) = E_{h0}(\theta), \quad \hat{E}_m(0,\tau) = E_{m0}(\tau), \quad \hat{I}_h(0) = 0, \quad \hat{I}_m(0) = 0. \end{cases} \tag{6.4}$$

By use of Volterra formulation, we have

$$\hat{E}_h(t,\theta) = \begin{cases} \hat{E}_h(t-\theta,0)k_2(\theta), & 0 \le \theta \le t, \\ E_{h0}(\theta - t)\dfrac{k_2(\theta)}{k_2(\theta - t)}, & 0 \le t < \theta, \end{cases} \qquad \hat{E}_m(t,\tau) = \begin{cases} \hat{E}_m(t-\tau,0)k_3(\tau), & 0 \le \tau \le t, \\ E_{m0}(\tau - t)\dfrac{k_3(\tau)}{k_3(\tau - t)}, & 0 \le t < \tau. \end{cases} \tag{6.5}$$

Substituting (6.5) into the first two equations of (6.3) yield

$$\begin{cases} \dfrac{d\hat{I}_h(t)}{dt} = b\beta_1\Delta\displaystyle\int_0^t \gamma_2(\theta)\hat{I}_m(t-\theta)k_2(\theta)d\theta + F_1(t) - (\mu_1 + \alpha + \delta)\hat{I}_h(t), \\ \dfrac{d\hat{I}_m(t)}{dt} = b\beta_2\Delta\displaystyle\int_0^t \gamma_3(\tau)\hat{I}_h(t-\tau)k_3(\tau)d\tau + F_2(t) - \mu_2\hat{I}_m(t), \\ \hat{I}_h(0) = 0, \quad \hat{I}_m(0) = 0, \end{cases} \tag{6.6}$$

where

$$F_1(t) = \int_t^{+\infty} \gamma_2(\theta)E_{h0}(\theta - t)\frac{k_2(\theta)}{k_2(\theta - t)}d\theta, \quad F_2(t) = \int_t^{+\infty} \gamma_3(\tau)E_{m0}(\tau - t)\frac{k_3(\tau)}{k_3(\tau - t)}d\tau.$$

Since

$$F_1(t) \le \int_t^{+\infty} \gamma_2(\theta)E_{h0}(\theta - t)d\theta = \int_0^{+\infty} \gamma_2(\theta + t)E_{h0}(\theta)d\theta,$$

due to $\Phi(0,x_0) \in \partial M_0$, then $F_1(t) \equiv 0$ for $t \ge 0$. In a similar way, we have $F_2(t) \equiv 0$ for $t \ge 0$. Accordingly, (6.6) has a unique solution $\hat{I}_i(t) \equiv 0, (i = h, m)$ for $t \ge 0$.
From (6.5), we know that $\hat{E}_i(t,s) = 0, (i = h, m)$ for $0 \le s \le t$. Thus, for all $u \ge 0$,

$$\| \gamma_2(\theta + u)\hat{E}_h(t,\theta) \|_{L^1_+} = \int_t^{+\infty} \gamma_2(\theta + u)E_{h0}(\theta - t)\frac{k_2(\theta)}{k_2(\theta - t)}d\theta \le \| \gamma_2(s + u + t)E_{h0}(s) \|_{L^1_+} = 0.$$

Similarly, we have $\| \gamma_3(\tau + u)\hat{E}_m(t,\tau) \|_{L^1_+} = 0$.

By using (6.2) we can obtain that $I_h(t) = 0$, $I_m(t) = 0, \| \gamma_2(\theta + u_1)E_h(t,\theta) \|_{L^1_+} = 0, \| \gamma_3(\tau + u_2)E_m(t,\tau) \|_{L^1_+} = 0$ for $t, u_1, u_2 \ge 0$, and $E_h(t,\theta)$, $E_m(t,\tau) \to 0$ as $t \to +\infty$. Thus, ∂M_0 is forward invariant under the semiflow $\Phi(t,\cdot)$.

Finally, we prove the disease-free equilibrium E_0 of system (2.1) is globally asymptotically stable for the semiflow $\Phi(t,\cdot)$ restricted to ∂M_0. In ∂M_0, system (2.1) can be divided into the following two subsystems

$$\begin{cases} \dfrac{dS_h(t)}{dt} = \Lambda_1 - (\mu_1 + \psi)S_h(t) + \displaystyle\int_0^{+\infty} \gamma_1(a)P(t,a)da, \\ \dfrac{\partial P(t,a)}{\partial t} + \dfrac{\partial P(t,a)}{\partial a} = -(\mu_1 + \gamma_1(a))P(t,a), \\ P(t,0) = \psi S_h(t), \ P(0,a) = P_0(a), \end{cases} \tag{6.7}$$

$$\frac{dS_m(t)}{dt} = \Lambda_2 - \mu_2 S_m(t). \tag{6.8}$$

Obviously, $\lim_{t\to+\infty} S_m(t) = \frac{\Lambda_2}{\mu_2}$. (6.7) has unique equilibrium $(S_h^0,\ P^0(a)) = (\frac{\Lambda_1}{\mu_1 + \psi(1-\mathscr{K}_1)}, \psi S_h^0 k_1(a))$, and $S_h(t) > 0$, $\int_0^{+\infty} P(t,a)da > 0$ for $t > 0$. Similar to the analysis of the section 2, (6.7) has a global attractor $\mathcal{A}_1$ of bounded sets in $(0,+\infty)\times(L_+^1(0,+\infty)\setminus 0_{L^1})$.

Let $x \in \mathcal{A}_1$, we can find a complete orbit $\{\Phi(t,\cdot)\}_{t\in\mathbb{R}}$ through x in $\mathcal{A}_1$. By similar analytical method used in [[31],subsection 3.2], system (6.7) can be written as

$$\begin{cases} \dfrac{dS_h(t)}{dt} = \Lambda_1 - (\mu_1+\psi)S_h(t) + \displaystyle\int_0^{+\infty} \gamma_1(a)P(t,a)da, \\ P(t,a) = \psi k_1(a) S_h(t-a), \\ (S_h(0), P(0,a)) \in \mathcal{A}_1. \end{cases} \tag{6.9}$$

Meanwhile, we easily find that (6.9) can be written as

$$\begin{cases} \dfrac{dS_h(t)}{dt} = \Lambda_1 - (\mu_1+\psi)S_h(t) + \displaystyle\int_0^{+\infty} \phi(a)S_h(t-a)da, \\ S_h(0) = S_{h0}. \end{cases} \tag{6.10}$$

Using similar to the proof of lemma 4.2[33], we can get that any solution in $\mathcal{A}_1$ is satisfied that $S_h(t) > 0$ for $t \in \mathbb{R}$. Once the stability of system (6.10) is obtained, in combination with (6.9), we can obtain the stability of system (6.7). In the following, we will define Lyapunov functionals on $\mathcal{A}_1$. Define

$$V_0(t) = S_h(t) - S_h^0 - S_h^0 \ln\frac{S_h(t)}{S_h^0} + \int_0^{+\infty} \phi(a)\int_0^a (S_h(t-r) - S_h^0 - S_h^0 \ln\frac{S_h(t-r)}{S_h^0})drda.$$

It is not difficult to find that $V_0(t)$ is positive-definite with S_h^0 as its global minimum point. Since compactness of $\mathcal{A}_1$, we can easily deduce $V_0(t)$ is bounded on $\mathcal{A}_1$. Calculating the derivative of $V_0(t)$ along the solution of system (6.10), we obtain have

$$\begin{aligned} \frac{dV_0}{dt} = \ & (1 - \frac{S_h^0}{S_h(t)})(\Lambda_1 - (\mu_1+\psi)S_h(t) + \int_0^{+\infty}\phi(a)S_h(t-a)da) \\ & + \int_0^{+\infty}\phi(a)(S_h(t) - S_h(t-a))da + S_h^0\int_0^{+\infty}\phi(a)\ln\frac{S_h(t-a)}{S_h(t)}da. \end{aligned}$$

In addition, we have noted that $\Lambda_1 = (\mu_1 + \psi)S_h^0 - \int_0^{+\infty} \phi(a)S_h^0 da$, it follows that

$$\begin{aligned}
\frac{dV_0}{dt} = & -\frac{(\mu_1+\psi)}{S_h(t)}(S_h(t)-S_h^0)^2 + \int_0^{+\infty}\phi(a)S_h(t-a)da - S_h^0\int_0^{+\infty}\phi(a)\frac{S_h(t-a)}{S_h(t)}da \\
& -\psi\mathscr{K}_1\frac{S_h^0}{S_h(t)}(S_h(t)-S_h^0) + \psi\mathscr{K}_1 S_h(t) - \int_0^{+\infty}\phi(a)S_h(t-a)da \\
& +S_h^0\int_0^{+\infty}\phi(a)\ln\frac{S_h(t-a)}{S_h(t)}da \\
= & -\frac{(\mu_1+\psi)}{S_h(t)}(S_h(t)-S_h^0)^2 - \psi\mathscr{K}_1 S_h^0 + S_h^0\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)}+\ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da \\
& +\psi\mathscr{K}_1 S_h(t) - \frac{\psi\mathscr{K}_1 S_h^0}{S_h(t)}(S_h(t)-S_h^0) \\
= & -\frac{(\mu_1+\psi)}{S_h(t)}(S_h(t)-S_h^0)^2 + S_h^0\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)}+\ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da \\
& +\frac{\psi\mathscr{K}_1}{S_h(t)}(S_h(t)-S_h^0)^2 \\
= & -\frac{(\mu_1+\psi(1-\mathscr{K}_1))}{S_h(t)}(S_h(t)-S_h^0)^2 + S_h^0\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)}+\ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da.
\end{aligned}$$

Notice that $\mathscr{K}_1 < 1$, then $\frac{dV_0}{dt} \le 0$ holds. Furthermore, the strict equality holds only if $S_h(t) = S_h^0$. Consequently, S_h^0 of (6.10) is globally asymptotically stable. In combination with (6.9), we have $(S_h^0,\ P^0(a))$ of (6.7) is globally asymptotically stable. Hence, the equilibrium E_0 is globally asymptotically stable restricted to ∂M_0. The proof of Theorem 4.1 is complete. □

Proof of Theorem 4.2

Proof. Since the disease-free equilibrium E_0 is globally asymptotically stable restricted to ∂M_0, applying Theorem 4.2 in [34], we need only to prove

$$W_s(E_0) \cap M_0 = \emptyset,$$

where $W_s(E_0) = \{x \in \mathscr{Y} : \lim_{t\to+\infty} \Phi(t,x) = E_0\}$. By way of contradiction, we assume that there exists a $x_0 \in M_0$ such that $x_0 \in W_s(E_0)$. Then we can find a list of $\{x_n\} \subset M_0$ such that

$$\| \Phi(t,x_n) - E_0 \|_{\mathscr{Y}} < \frac{1}{n}, t \ge 0.$$

Denote $\Phi(t,x_n) = (S_{hn}(t), S_{mn}(t), I_{hn}(t), I_{mn}(t), P_n(t,\cdot), E_{hn}(t,\cdot), E_{mn}(t,\cdot))$. Then for all $t \ge 0$, we have

$$S_h^0 - \frac{1}{n} < S_{hn}(t) < S_h^0 + \frac{1}{n},\ \ S_m^0 - \frac{1}{n} < S_{mn}(t) < S_m^0 + \frac{1}{n}$$

and $\Phi(t,x_n) \subset M_0$. Similar to the first part of theorem 4.1 proof, we know that there exists $t_0 \ge 0$ such that $I_{hn}(t) > 0$ for $t \ge t_0$ or $I_{mn}(t) > 0$ for $t \ge t_0$, so without loss of generality, we can take $t_0 = 0$ and $I_{hn}(0) > 0$. Since $\mathcal{R}_0 > 1$, we can choose large enough n such that $S_h^0 > \frac{1}{n}$, $S_m^0 > \frac{1}{n}$ and

$$f(n) = \frac{b^2\beta_1\beta_2\mathscr{K}_2\mathscr{K}_3(S_h^0 - \frac{1}{n})(S_m^0 - \frac{1}{n})}{\mu_2(\mu_1+\alpha+\delta)} > 1.$$

Now we construct the following system

$$\begin{cases} \dfrac{d\hat{I}_h(t)}{dt} = \displaystyle\int_0^{+\infty} \gamma_2(\theta)\hat{E}_h(t,\theta)da - (\mu_1 + \alpha + \delta)\hat{I}_h(t), \\ \dfrac{d\hat{I}_m(t)}{dt} = \displaystyle\int_0^{+\infty} \gamma_3(\tau)\hat{E}_m(t,\tau)da - \mu_2\hat{I}_m(t), \\ \dfrac{\partial\hat{E}_h(t,\theta)}{\partial t} + \dfrac{\partial\hat{E}_h(t,\theta)}{\partial\theta} = -(\mu_1 + \gamma_2(\theta))\hat{E}_h(t,\theta), \\ \dfrac{\partial\hat{E}_m(t,\tau)}{\partial t} + \dfrac{\partial\hat{E}_m(t,\tau)}{\partial\tau} = -(\mu_2 + \gamma_3(\tau))\hat{E}_m(t,\tau), \\ \hat{E}_h(t,0) = b\beta_1(S_h^0 - \dfrac{1}{n})\hat{I}_m(t), \\ \hat{E}_m(t,0) = b\beta_2(S_m^0 - \dfrac{1}{n})\hat{I}_h(t), \\ \hat{E}_h(0,\theta) = E_{hn}(0,\theta), \ \ \hat{E}_m(0,\tau) = E_{mn}(0,\tau), \ \ \hat{I}_h(0) = I_{hn}(0), \ \ \hat{I}_m(0) = I_{mn}(0). \end{cases} \tag{6.11}$$

Using similar analysis of Sect.2, we can get existence, uniqueness, and nonnegative of solution to system (6.11). By the comparison principle, we know

$$I_{in}(t) \geq \hat{I}_i(t), \ \ E_{in}(t,s) \geq \hat{E}_i(t,s), \ (i = h, m). \tag{6.12}$$

By use of Volterra formulation, we have

$$\hat{E}_h(t,\theta) = \begin{cases} \hat{E}_h(t-\theta,0)k_2(\theta), & 0 \leq \theta \leq t, \\ E_{h0}(\theta - t)\dfrac{k_2(\theta)}{k_2(\theta - t)}, & 0 \leq t < \theta, \end{cases} \qquad \hat{E}_m(t,\tau) = \begin{cases} \hat{E}_m(t-\tau,0)k_3(\tau), & 0 \leq \tau \leq t, \\ E_{m0}(\tau - t)\dfrac{k_3(\tau)}{k_3(\tau - t)}, & 0 \leq t < \tau. \end{cases} \tag{6.13}$$

Substituting (6.13) into the first two equations of (6.11) yield

$$\begin{cases} \dfrac{d\hat{I}_h(t)}{dt} \geq b\beta_1(S_h^0 - \dfrac{1}{n}) \displaystyle\int_0^t \gamma_2(\theta)\hat{I}_m(t-\theta)k_2(\theta)d\theta - (\mu_1 + \alpha + \delta)\hat{I}_h(t), \\ \dfrac{d\hat{I}_m(t)}{dt} \geq b\beta_2(S_m^0 - \dfrac{1}{n}) \displaystyle\int_0^t \gamma_3(\tau)\hat{I}_h(t-\tau)k_3(\tau)d\tau - \mu_2\hat{I}_m(t), \\ \hat{I}_h(0) = I_{hn}(0), \ \ \hat{I}_m(0) = I_{mn}(0). \end{cases} \tag{6.14}$$

We can claim that at least one of $\hat{I}_h(t), \hat{I}_m(t)$ is unbounded. Otherwise, we can use Laplace transform in the first two inequations of (6.14) yield

$$\begin{cases} -\hat{I}_h(0) + \lambda\mathcal{L}[\hat{I}_h](\lambda) \geq \mathcal{L}[u_1](\lambda)\mathcal{L}[\hat{I}_m](\lambda) - (\mu_1 + \alpha + \delta)\mathcal{L}[\hat{I}_h](\lambda), \\ -\hat{I}_m(0) + \lambda\mathcal{L}[\hat{I}_m](\lambda) \geq \mathcal{L}[u_2](\lambda)\mathcal{L}[\hat{I}_h](\lambda) - \mu_2\mathcal{L}[\hat{I}_m](\lambda), \end{cases} \tag{6.15}$$

where $\mathcal{L}[\hat{I}_i](\lambda) = \int_0^{+\infty} e^{-\lambda t}\hat{I}_i(t)dt, \ (i = h, m)$ for $(\lambda > 0)$, $\mathcal{L}[u_1](\lambda) = \int_0^{\infty} b\beta_1(S_h^0 - \frac{1}{n})\gamma_2(\theta)k_2(\theta)e^{-\lambda\theta}d\theta$ and $\mathcal{L}[u_2](\lambda) = \int_0^{\infty} b\beta_2(S_m^0 - \frac{1}{n})\gamma_3(\tau)k_3(\tau)e^{-\lambda\tau}d\tau$ for $(\lambda \geq 0)$.
After a simple calculation, we obtain

$$\frac{\mathcal{L}[u_1](\lambda)\mathcal{L}[u_2](\lambda)}{\lambda + \mu_2}\left(\frac{(\lambda + \mu_2)(\lambda + \mu_1 + \alpha + \delta)}{\mathcal{L}[u_1](\lambda)\mathcal{L}[u_2](\lambda)} - 1\right)\mathcal{L}[\hat{I}_h](\lambda) \geq I_{hn}(0) + \frac{\mathcal{L}[u_1](\lambda)}{(\lambda + \mu_2)}I_{mn}(0) > 0. \tag{6.16}$$

$\mathcal{L}[u_1](\lambda) \to \mathcal{L}[u_1](0), \mathcal{L}[u_2](\lambda) \to \mathcal{L}[u_2](0)$ as $\lambda \to 0$ by the Dominated Convergence Theorem. Since

$$\frac{(\lambda + \mu_2)(\lambda + \mu_1 + \alpha + \delta)}{\mathcal{L}[u_1](\lambda)\mathcal{L}[u_2](\lambda)} |_{\lambda=0} = \frac{1}{f(n)} < 1,$$

then there exists $\varepsilon > 0$ such that

$$\frac{(\lambda + \mu_2)(\lambda + \mu_1 + \alpha + \delta)}{\mathcal{L}[u_1](\lambda)\mathcal{L}[u_2](\lambda)} - 1 < 0$$

for all $\lambda \in [0, \varepsilon)$. According to (6.16), we have $\mathcal{L}[\hat{I}_h](\lambda) < 0$ for all $\lambda \in (0, \varepsilon)$. But this contradicts the nonnegative of $\hat{I}_h(t)$, $(t \geq 0)$. Hence, at least one of $\hat{I}_h(t), \hat{I}_m(t)$ is unbounded. Since $I_{in}(t) \geq \hat{I}_i(t), (i = h, m)$, we get that at least one of $I_{hn}(t), I_{mn}(t)$ is unbounded. This contradicts the proposition 2.3. Therefore, $W_s(E_0) \cap M_0 = \emptyset$. By Theorem 4.2 [34] , we get that semiflow $\{\Phi(t, \cdot)\}_{t\geq 0}$ generated by system (2.1) is uniformly persistent. By Theorem 3.7 [29], we get that there exists a compact subset $\mathcal{A}_0 \subset M_0$ which is a global attractor for $\{\Phi(t, \cdot)\}_{t\geq 0}$ in M_0. □

Proof of Theorem 4.3

Proof. Using similar to the proof of lemma 4.2[33], we can get that any solution in $\mathcal{A}$ is satisfied that $S_h(t),\ S_m(t) > 0$ for $t \in \mathbb{R}$.

Let $g(x) = x - \ln x - 1$, note that $g(x)$ is non-negative and continuous on $(0, +\infty)$ with a unique root at $x = 1$. Next, we construct the following Lyapunov function $L = L_1 + L_2 + L_3$ on the global attractor $\mathcal{A}$, since compactness of $\mathcal{A}$, we can easily deduce L is bounded on $\mathcal{A}$, where

$$L_1 = \frac{b\beta_1\mathscr{K}_3}{\mu_2} S_m^0 g(\frac{S_m}{S_m^0}) + g(\frac{S_h}{S_h^0}) + \int_0^{+\infty} \phi(a) \int_0^a g(\frac{S_h(t-r)}{S_h^0}) dr da,$$

$$L_2 = \int_0^{+\infty} F_1(\theta) E_h(t, \theta) d\theta + \int_0^{+\infty} F_2(\tau) E_m(t, \tau) d\tau, \quad L_3 = \frac{1}{\mathscr{K}_2 S_h^0} I_h + \frac{b\beta_1}{\mu_2} I_m,$$

$$F_1(\theta) = \frac{1}{\mathscr{K}_2 S_h^0} \int_\theta^{+\infty} \gamma_2(u) e^{-\int_\theta^u (\mu_1 + \gamma_2(s)) ds} du, \quad F_2(\tau) = \frac{b\beta_1}{\mu_2} \int_\tau^{+\infty} \gamma_3(u) e^{-\int_\tau^u (\mu_2 + \gamma_3(s)) ds} du.$$

Calculating the derivative of L_1, L_2, L_3 along solutions of system (4.1), respectively. In the process of calculate the time derivative of L_1, we used $\Lambda_2 = \mu_2 S_m^0$, $\Lambda_1 = (\mu_1 + \psi) S_h^0 - \psi \mathscr{K}_1 S_h^0$, $\int_0^{+\infty} \phi(a) \int_0^a g(\frac{S_h(t-r)}{S_h^0}) dr da = \int_0^{+\infty} \phi(a) \int_{t-a}^t g(\frac{S_h(r)}{S_h^0}) dr da$ and $\int_0^{+\infty} \phi(a) da = \psi \mathscr{K}_1$. We can deduce

$$
\begin{aligned}
\frac{dL_1}{dt} = & -b\beta_1\mathscr{K}_3\frac{(S_m-S_m^0)^2}{S_m}-\frac{b^2\beta_1\beta_2\mathscr{K}_3}{\mu_2}(S_m-S_m^0)I_h-\frac{(S_h-S_h^0)}{S_h^0}b\beta_1 I_m-\frac{\mu_1+\psi}{S_hS_h^0}(S_h-S_h^0)^2\\
&+(1-\frac{S_h^0}{S_h})\int_0^{+\infty}\phi(a)\frac{S_h(t-a)}{S_h^0}da-\mathscr{K}_1\psi(1-\frac{S_h^0}{S_h})+\int_0^{+\infty}\frac{\phi(a)}{S_h^0}(S_h-S_h(t-a))da\\
&+\int_0^{+\infty}\phi(a)\ln\frac{S_h(t-a)}{S_h(t)}da\\
= & -b\beta_1\mathscr{K}_3\frac{(S_m-S_m^0)^2}{S_m}-\frac{b^2\beta_1\beta_2\mathscr{K}_3}{\mu_2}(S_m-S_m^0)I_h-\frac{(S_h-S_h^0)}{S_h^0}b\beta_1 I_m-\frac{\mu_1+\psi}{S_hS_h^0}(S_h-S_h^0)^2\\
&-\psi\mathscr{K}_1+\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)}+\ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da-\mathscr{K}_1\psi(1-\frac{S_h^0}{S_h})+\int_0^{+\infty}\phi(a)\frac{S_h}{S_h^0}da\\
= & -b\beta_1\mathscr{K}_3\frac{(S_m-S_m^0)^2}{S_m}-\frac{b^2\beta_1\beta_2\mathscr{K}_3}{\mu_2}(S_m-S_m^0)I_h-\frac{(S_h-S_h^0)}{S_h^0}b\beta_1 I_m-\frac{\mu_1+\psi}{S_hS_h^0}(S_h-S_h^0)^2\\
&+\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)}+\ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da+\frac{\mathscr{K}_1\psi}{S_hS_h^0}(S_h-S_h^0)^2\\
= & -b\beta_1\mathscr{K}_3\frac{(S_m-S_m^0)^2}{S_m}-\frac{b^2\beta_1\beta_2\mathscr{K}_3}{\mu_2}(S_m-S_m^0)I_h-\frac{(S_h-S_h^0)}{S_h^0}b\beta_1 I_m\\
&-\frac{\mu_1+\psi(1-\mathscr{K}_1)}{S_hS_h^0}(S_h-S_h^0)^2+\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)}+\ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da.
\end{aligned}
$$

$$
\begin{aligned}
\frac{dL_2}{dt} = & -\int_0^{+\infty}F_1(\theta)((\mu_1+\gamma_2(\theta))E_h(t,\theta)+\frac{\partial E_h}{\partial\theta})d\theta-\int_0^{+\infty}F_2(\tau)((\mu_2+\gamma_3(\tau))E_m(t,\tau)+\frac{\partial E_m}{\partial\tau})d\tau\\
= & F_1(0)E_h(t,0)+F_2(0)E_m(t,0)+\int_0^{+\infty}(F_1'(\theta)-(\mu_1+\gamma_2(\theta))F_1(\theta))E_h(t,\theta)d\theta\\
&+\int_0^{+\infty}(F_2'(\tau)-(\mu_2+\gamma_3(\tau))F_2(\tau))E_m(t,\tau)d\tau\\
= & \frac{b\beta_1S_hI_m}{S_h^0}+\frac{b^2\beta_1\beta_2\mathscr{K}_3S_mI_h}{\mu_2}-\int_0^{+\infty}\frac{1}{\mathscr{K}_2S_h^0}\gamma_2(\theta)E_h(t,\theta)d\theta-\int_0^{+\infty}\frac{b\beta_1}{\mu_2}\gamma_3(\tau)E_m(t,\tau)d\tau.
\end{aligned}
$$

$$
\frac{dL_3}{dt} = \frac{1}{\mathscr{K}_2S_h^0}(\int_0^{+\infty}\gamma_1(\theta)E_h(t,\theta)d\theta-(\mu_1+\alpha+\delta)I_h)+\frac{b\beta_1}{\mu_2}(\int_0^{+\infty}\gamma_3(\tau)E_m(t,\tau)d\tau-\mu_2I_m).
$$

Therefore,

$$
\begin{aligned}
\frac{dL}{dt} = & -\frac{\mu_1+\psi(1-\mathscr{K}_1)}{S_hS_h^0}(S_h-S_h^0)^2+\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)}+\ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da\\
&-b\beta_1\mathscr{K}_3\frac{(S_m-S_m^0)^2}{S_m}+\frac{b^2\beta_1\beta_2\mathscr{K}_3S_m^0I_h}{\mu_2}-\frac{(\mu_1+\alpha+\delta)}{\mathscr{K}_2S_h^0}I_h\\
= & -\frac{\mu_1+\psi(1-\mathscr{K}_1)}{S_hS_h^0}(S_h-S_h^0)^2+\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)}+\ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da\\
&-b\beta_1\mathscr{K}_3\frac{(S_m-S_m^0)^2}{S_m}+\frac{b^2\beta_1\beta_2\mathscr{K}_3S_m^0}{\mu_2}(1-\frac{1}{\mathcal{R}_0})I_h.
\end{aligned}
$$

Notice that $\mathscr{K}_1<1$, if $\mathcal{R}_0<1$, then $\frac{dL}{dt}\le 0$ holds. Let T is the largest invariant subset of $\{\frac{dL}{dt}\mid_{(4.1)}=0\}$, the equality holds only if $S_m(t)=S_m^0$, $S_h(t)=S_h^0$, $I_h(t)=0$. In T, $S_m(t)=S_m^0$ for all t, then we have $\frac{dS_h(t)}{dt}\equiv 0$. Combining this with (4.1), it follows that $I_m(t)=0$ for all t, moreover, $P(t,a)=$

$\psi S_h^0 k_1(a)$, $E_m(t,\tau) = 0$, $E_h(t,\theta) = 0$ for all $t \in \mathbb{R}$, a, θ, $\tau \in \mathbb{R}_+$. Hence, $T = \{E_0\}$. Assume that there exists $x \in \mathcal{A}\backslash T$ and we can find $\Phi(t,x) \subset \mathcal{A}$ through x at $t = 0$, with alpha limit set $\alpha(x)$. Since $x \neq E_0$, we can deduce that $t \to L(\Phi(t,x))$ is a non-increasing and bounded function, then L is a constant on $\alpha(x)$. Since $\alpha(x)$ is invariant in $\mathcal{A}$, we know $\alpha(x) = T$. From Theorem 3.2 we know that the disease free equilibrium E_0 is locally asymptotically stable, which implies $x = E_0$, this contradicts the $x \neq E_0$. Hence, $\mathcal{A} = \{E_0\}$. This proves that E_0 is globally asymptotically stable. This completes the proof. □

Proof of Theorem 4.5

Proof. We define the following Lyapunov function $V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6$ on the global attractor $\mathcal{A}_0$, using lemma 4.4, we can easily deduce V is bounded on $\mathcal{A}_0$, where

$$V_1 = \frac{1}{b\beta_1 I_m^*}(g(\frac{S_h}{S_h^*}) + \int_0^{+\infty}\phi(a)\int_0^a g(\frac{S_h(t-r)}{S_h^*})drda),\ V_2 = \frac{1}{\mathcal{K}_2}\int_0^{+\infty} G_1(\theta)E_h^*(\theta)g(\frac{E_h(t,\theta)}{E_h^*(\theta)})d\theta,$$

$$V_3 = \frac{I_h^*}{\mathcal{K}_2 E_h^*(0)}g(\frac{I_h}{I_h^*}),\ \ V_4 = \frac{1}{b\beta_2 I_h^*}g(\frac{S_m}{S_m^*}),\ \ V_5 = \frac{1}{\mathcal{K}_3}\int_0^{+\infty} G_2(\tau)E_m^*(\tau)g(\frac{E_m(t,\tau)}{E_m^*(\tau)})d\tau,$$

$$V_6 = \frac{I_m^*}{\mathcal{K}_3 E_m^*(0)}g(\frac{I_m}{I_m^*}),$$

$$G_1(\theta) = \frac{1}{E_h^*(0)}\int_\theta^{+\infty}\gamma_2(u)e^{-\int_\theta^u(\mu_1+\gamma_2(s))ds}d\theta,\ \ G_2(\tau) = \frac{1}{E_m^*(0)}\int_\tau^{+\infty}\gamma_3(u)e^{-\int_\tau^u(\mu_2+\gamma_3(s))ds}d\tau.$$

Calculating the derivative of V along a solution in $\mathcal{A}_0$. In the process of calculate the time derivative of V_1, we notice that $\Lambda_1 = b\beta_1 S_h^* I_m^* + (\mu_1+\psi)S_h^* - \int_0^{+\infty}\phi(a)S_h^* da$.

$$\begin{aligned}
\frac{dV_1}{dt} =\ & \frac{1}{b\beta_1 I_m^*}\frac{(S_h - S_h^*)}{S_h S_h^*}((\mu_1+\psi)(S_h^* - S_h) + b\beta_1(S_h^* I_m^* - S_h I_m) + \int_0^{+\infty}\phi(a)(S_h(t-a) - S_h^*)da) \\
& + \frac{1}{b\beta_1 I_m^*}(\int_0^{+\infty}\frac{\phi(a)}{S_h^*}(S_h - S_h(t-a))da + \int_0^{+\infty}\phi(a)\ln\frac{S_h(t-a)}{S_h}da) \\
=\ & (1 - \frac{S_h^*}{S_h} - \frac{S_h I_m}{S_h^* I_m^*} + \frac{I_m}{I_m^*}) - \frac{(\mu_1 + \psi(1-\mathcal{K}_1))(S_h - S_h^*)^2}{b\beta_1 I_m^* S_h S_h^*} \\
& + \frac{1}{b\beta_1 I_m^*}\int_0^{+\infty}(1 - \frac{S_h(t-a)}{S_h(t)} + \ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da.
\end{aligned}$$

$$\frac{dV_2}{dt} = \frac{1}{\mathcal{K}_2}\int_0^{+\infty} G_1(\theta)E_h^*(\theta)(1 - \frac{E_h^*(\theta)}{E_h(t,\theta)})\frac{\frac{\partial E_h(t,\theta)}{\partial t}}{E_h^*(\theta)}d\theta.$$

By using

$$\frac{\partial E_h}{\partial t} = -\frac{\partial E_h}{\partial\theta} - (\mu_1 + \gamma_2(\theta))E_h,\ \ \frac{\partial}{\partial\theta}g(\frac{E_h(t,\theta)}{E_h^*(\theta)}) = \frac{1}{E_h^*(\theta)}(1 - \frac{E_h^*(\theta)}{E_h(t,\theta)})(\frac{\partial E_h}{\partial\theta} + (\mu_1+\gamma_2(\theta))E_h),$$

we have

$$\begin{aligned}\frac{dV_2}{dt} &= -\frac{1}{\mathcal{K}_2}\int_0^{+\infty} G_1(\theta)E_h^*(\theta)\frac{\partial}{\partial\theta}g(\frac{E_h(t,\theta)}{E_h^*(\theta)})d\theta\\ &= \frac{G_1(0)E_h^*(0)}{\mathcal{K}_2}g(\frac{E_h(t,0)}{E_h^*(0)}) + \frac{1}{\mathcal{K}_2}\int_0^{+\infty} g(\frac{E_h(t,\theta)}{E_h^*(\theta)})(G_1'E_h^* + E_h^{*'}(\theta)G_1)d\theta\\ &= g(\frac{E_h(t,0)}{E_h^*(0)}) - \frac{1}{\mathcal{K}_2}\int_0^{+\infty} g(\frac{E_h(t,\theta)}{E_h^*(\theta)})\frac{\gamma_2(\theta)E_h^*(\theta)}{E_h^*(0)}d\theta\\ &= (\frac{S_hI_m}{S_h^*I_m^*} - 1 - \ln\frac{S_hI_m}{S_h^*I_m^*}) - \frac{1}{\mathcal{K}_2E_h^*(0)}\int_0^{+\infty}\gamma_2(\theta)E_h^*(\theta)(\frac{E_h(t,\theta)}{E_h^*(\theta)} - 1 - \ln\frac{E_h(t,\theta)}{E_h^*(\theta)})d\theta.\end{aligned}$$

$$\begin{aligned}\frac{dV_3}{dt} &= \frac{I_h^*}{\mathcal{K}_2E_h^*(0)}(1-\frac{I_h^*}{I_h})\frac{1}{I_h^*}(\int_0^{+\infty}\gamma_2(\theta)E_h(t,\theta)d\theta - (\mu_1+\alpha+\delta)I_h)\\ &= \frac{1}{\mathcal{K}_2E_h^*(0)}\int_0^{+\infty}\gamma_2(\theta)E_h(t,\theta)d\theta - \frac{I_h^*}{\mathcal{K}_2E_h^*(0)I_h}\int_0^{+\infty}\gamma_2(\theta)E_h(t,\theta)d\theta\\ &\quad + \frac{(\mu_1+\alpha+\delta)I_h^*}{\mathcal{K}_2E_h^*(0)} - \frac{(\mu_1+\alpha+\delta)I_h}{\mathcal{K}_2E_h^*(0)}.\end{aligned}$$

Recalling that $I_h^* = \dfrac{\mathcal{K}_2E_h^*(0)}{\mu_1+\alpha+\delta}$, then

$$\frac{dV_3}{dt} = \frac{1}{\mathcal{K}_2E_h^*(0)}\int_0^{+\infty}\gamma_2(\theta)E_h(t,\theta)d\theta - \frac{I_h^*}{\mathcal{K}_2E_h^*(0)I_h}\int_0^{+\infty}\gamma_2(\theta)E_h(t,\theta)d\theta + 1 - \frac{I_h}{I_h^*}.$$

Thus,

$$\begin{aligned}\frac{d\sum_{i=1}^{3}V_i}{dt} &= (1-\frac{S_h^*}{S_h} - 1 - \ln\frac{S_hI_m}{S_h^*I_m^*} + \frac{I_m}{I_m^*}) - \frac{(\mu_1+\psi(1-\mathcal{K}_1))(S_h-S_h^*)^2}{b\beta_1I_m^*S_hS_h^*}\\ &\quad + \frac{1}{b\beta_1I_m^*}\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)} + \ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da\\ &\quad + \frac{1}{\mathcal{K}_2E_h^*(0)}\int_0^{+\infty}\gamma_2(\theta)E_h^*(\theta)(-\frac{I_h^*E_h(t,\theta)}{I_hE_h^*(\theta)} + 1 + \ln\frac{E_h(t,\theta)}{E_h^*(\theta)})d\theta + 1 - \frac{I_h}{I_h^*}.\end{aligned}$$

Notice that $\Lambda_2 = b\beta_2S_m^*I_h^* + \mu_2S_m^*$, we can get

$$\frac{dV_4}{dt} = (1-\frac{S_m^*}{S_m} - \frac{S_mI_h}{S_m^*I_h^*} + \frac{I_h}{I_h^*}) - \frac{\mu_2(S_m-S_m^*)^2}{b\beta_2I_h^*S_mS_m^*}.$$

Notice that $\frac{\partial}{\partial\tau}g(\frac{E_m(t,\tau)}{E_m^*(\tau)}) = \frac{1}{E_m^*(\tau)}(1-\frac{E_m^*(\tau)}{E_m(t,\tau)})(\frac{\partial E_m}{\partial\tau} + (\mu_2+\gamma_3(\tau))E_m)$, we have

$$\begin{aligned}\frac{dV_5}{dt} &= -\frac{1}{\mathcal{K}_3}\int_0^{+\infty} G_2(\tau)E_m^*(\tau)\frac{\partial}{\partial\tau}g(\frac{E_m(t,\tau)}{E_m^*(\tau)})d\tau\\ &= \frac{G_2(0)E_m^*(0)}{\mathcal{K}_3}g(\frac{E_m(t,0)}{E_m^*(0)}) + \frac{1}{\mathcal{K}_3}\int_0^{+\infty} g(\frac{E_m(t,\tau)}{E_m^*(\tau)})(G_2'E_m^* + E_m^{*'}(\tau)G_2)d\tau\\ &= g(\frac{E_m(t,0)}{E_m^*(0)}) - \frac{1}{\mathcal{K}_3}\int_0^{+\infty} g(\frac{E_m(t,\tau)}{E_m^*(\tau)})\frac{\gamma_3(\tau)E_m^*(\tau)}{E_m^*(0)}d\tau\\ &= (\frac{S_mI_h}{S_m^*I_h^*} - 1 - \ln\frac{S_mI_h}{S_m^*I_h^*}) - \frac{1}{\mathcal{K}_3E_m^*(0)}\int_0^{+\infty}\gamma_3(\tau)E_m^*(\tau)(\frac{E_m(t,\tau)}{E_m^*(\tau)} - 1 - \ln\frac{E_m(t,\tau)}{E_m^*(\tau)})d\tau.\end{aligned}$$

Notice that $I_m^* = \dfrac{\mathscr{K}_3 E_m^*(0)}{\mu_2}$, we have

$$\begin{aligned}\frac{dV_6}{dt} &= \frac{I_m^*}{\mathscr{K}_3 E_m^*(0)}(1-\frac{I_m^*}{I_m})\frac{1}{I_m^*}(\int_0^{+\infty}\gamma_3(\tau)E_m(t,\tau)d\tau - \mu_2 I_m)\\ &= \frac{1}{\mathscr{K}_2 E_m^*(0)}\int_0^{+\infty}\gamma_3(\tau)E_m(t,\tau)d\tau - \frac{I_m^*}{\mathscr{K}_3 E_m^*(0)I_m}\int_0^{+\infty}\gamma_3(\tau)E_m(t,\tau)d\tau + \frac{\mu_2 I_m^*}{\mathscr{K}_3 E_m^*(0)}\\ &\quad -\frac{\mu_2 I_m}{\mathscr{K}_3 E_m^*(0)}\\ &= \frac{1}{\mathscr{K}_3 E_m^*(0)}\int_0^{+\infty}\gamma_3(\tau)E_m(t,\tau)d\tau - \frac{I_m^*}{\mathscr{K}_3 E_m^*(0)I_m}\int_0^{+\infty}\gamma_3(\tau)E_m(t,\tau)d\tau + 1 - \frac{I_m}{I_m^*}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d\sum_{i=4}^{6}V_i}{dt} &= (1-\frac{S_m^*}{S_m} - 1 - \ln\frac{S_m I_h}{S_m^* I_h^*} + \frac{I_h}{I_h^*}) - \frac{\mu_2(S_m - S_m^*)^2}{b\beta_2 I_h^* S_m S_m^*}\\ &\quad +\frac{1}{\mathscr{K}_3 E_m^*(0)}\int_0^{+\infty}\gamma_3(\tau)E_m^*(\tau)(-\frac{I_m^* E_m(t,\tau)}{I_m E_m^*(\tau)} + 1 + \ln\frac{E_m(t,\tau)}{E_h^*(\tau)})d\tau + 1 - \frac{I_m}{I_m^*}.\end{aligned}$$

By using $\dfrac{\int_0^{+\infty}\gamma_2(\theta)E_h^*(\theta)d\theta}{\mathscr{K}_2 E_h^*(0)} = 1$, $\dfrac{\int_0^{+\infty}\gamma_3(\tau)E_m^*(\tau)d\tau}{\mathscr{K}_3 E_m^*(0)} = 1$, we can get

$$\begin{aligned}\frac{dV}{dt} = \frac{d\sum_{i=1}^{6}V_i}{dt} &= (1-\frac{S_h^*}{S_h} + \ln\frac{S_h^*}{S_h}) + (1-\frac{S_m^*}{S_m} + \ln\frac{S_m^*}{S_m}) - \frac{(\mu_1 + \psi(1-\mathscr{K}_1))(S_h - S_h^*)^2}{b\beta_1 I_m^* S_h S_h^*}\\ &\quad -\frac{\mu_2(S_m - S_m^*)^2}{b\beta_2 I_h^* S_m S_m^*} + \frac{1}{b\beta_1 I_m^*}\int_0^{+\infty}(1-\frac{S_h(t-a)}{S_h(t)} + \ln\frac{S_h(t-a)}{S_h(t)})\phi(a)da\\ &\quad +\frac{1}{\mathscr{K}_3 E_m^*(0)}\int_0^{+\infty}\gamma_3(\tau)E_m^*(\tau)(-\frac{I_m^* E_m(t,\tau)}{I_m E_m^*(\tau)} + 1 + \ln\frac{I_m^* E_m(t,\tau)}{I_m E_m^*(\tau)})d\tau\\ &\quad +\frac{1}{\mathscr{K}_2 E_h^*(0)}\int_0^{+\infty}\gamma_2(\theta)E_h^*(\theta)(-\frac{I_h^* E_h(t,\theta)}{I_h E_h^*(\theta)} + 1 + \ln\frac{I_h^* E_h(t,\theta)}{I_h E_h^*(\theta)})d\theta.\end{aligned}$$

It follows from the non-negativity of g, we know that $\dfrac{dV}{dt} \le 0$, and

$$\frac{dV}{dt} = 0 \;\Leftrightarrow\; S_h(t) = S_h^*,\; S_m(t) = S_m^*,\; \frac{I_m^* E_m(t,\tau)}{I_m E_m^*(\tau)} = 1,\; \frac{I_h^* E_h(t,\theta)}{I_h E_h^*(\theta)} = 1. \tag{6.17}$$

Let T^* is the largest invariant subset of $\{\frac{dV}{dt}\mid_{(4.1)} = 0\}$, then we have $\frac{dS_h(t)}{dt} \equiv 0$, $\frac{dS_m(t)}{dt} \equiv 0$. Combining this with (4.1), we can obtain that $I_h(t) = I_h^*$ for all t, $P(t,a) = \psi S_h^* k_1(a)$. Further, $I_m(t) = I_m^*$ for all t, in combining with (6.17), we can get $E_h(t,\theta) = E_h^*(\theta)$, $E_m(t,\tau) = E_m^*(\tau)$. Hence, $T^* = \{E^*\}$. According to the proof of the theorem 4.3, we can easily obtain that $\mathscr{A}_0 = \{E^*\}$. This proves that E^* is globally asymptotically stable. The proof is complete. □