

ON THE INITIAL VALUE PROBLEM FOR A CLASS OF DISCRETE VELOCITY MODELS

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ABSTRACT. In this paper we investigate the initial value problem for a class of hyperbolic systems relating the mathematical modeling of a class of complex phenomena, with emphasis on vehicular traffic flow. Existence and uniqueness for large times of solutions, a basic requisite both for models building and for their numerical implementation, are obtained under weak hypotheses on the terms modeling the interaction among agents. The results are then compared with the existing literature on the subject.

1. Introduction. In recent years the mathematical modeling based on suitable generalizations of kinetic theory has shown its ability to describe many of the features characterizing the phenomenon of complexity [14]. These features include, among others, the emergence of collective behaviors that are not immediately deducible from the mutual microscopic interactions among the individual constituents. The systems exhibiting aspects typical of complexity that have been successfully approached by kinetic modeling include social [1] and biological systems [6, 12, 15], as well as pedestrian crowds [7] and vehicular traffic flow [4, 5, 8]. The phenomenology of the latter reveals a number of features, such as stop-and-go waves and ghost queues [17], that makes vehicular traffic a prototype of complex systems.

Models referring to generalized kinetic theory describe the system under investigation using a distribution function $f(t, x, v)$ such that $f(t, x, v)dx dv$ represents the number of vehicles at time t in an infinitesimal neighborhood $(x + dx, v + dv)$ in the phase space, x being the spatial position and v the velocity. The evolution of f is then obtained as solution to a partial differential equation describing the balance in time between inflow and outflow in the neighborhood $(x, x + dx) \times (v, v + dv)$ as resulting from interactions among vehicles. The granular aspects of vehicular traffic naturally suggest the use of discrete velocities kinetic models [13, 11], in which the speed v can only assume a finite number of values, corresponding to vehicles collected according their “typical” velocity. In the discrete velocities modeling the distribution function becomes a vector function $f = (f_1(t, x), \dots, f_n(t, x))$, each component $f_i(t, x)$ related to the corresponding velocity class of vehicles.

This paper deals with the existence and uniqueness of solutions to a large class of discrete velocity kinetic models related the mathematical description of vehicular traffic flow. The question of well-posedness is of fundamental relevance both for a

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proper selection of mathematically meaningful class of models and for the robustness of their numerical implementation and simulation. A basic requirement for any admissible mathematical model of vehicular flow is that it should possess global (*i.e.*, for large times) existence and uniqueness of solutions *in any traffic condition*, provided the interaction terms are “sufficiently reasonable”. It is worth observing that the mathematical problem related to the well-posedness for large times of discrete velocities traffic models is a difficult task due to the nonlinearity of interactions and to their hyperbolic multi-characteristics aspect. This issue has been addressed in [2], while the recent, more comprehensive, results are given in [3], which is the starting point of the present work. In the latter the Authors obtain global in time existence of solutions under general hypotheses. Their approach, however, requires a key tool the introduction of an *a priori* cut off on the interaction terms that freezes the interactions when the traffic density reaches a critical value. As the Authors of [3] themselves emphasize in their concluding remarks, such an assumption, though mimicking the phenomenological observations that show the existence of different phases in the traffic flow corresponding to different densities [17], should possibly appear as a byproduct of the model rather than being introduced into it from the very beginning in order to achieve the well-posedness.

In this paper we prove global existence and uniqueness under hypotheses similar to that in [3], but under weaker assumptions on the structure of the interaction terms. Specifically, we prove global in time existence and uniqueness of solution without assuming that the interaction terms as the density of vehicles becomes large. In Sect. 2 the mathematical problem is formulated and the main hypotheses on the interaction terms are introduced. The main results of this paper are given in Sect. 3 where existence and uniqueness of solutions for any time is proven in the case of non-local interactions. Finally, Sect. 4 reports a toy model that explicitly shows how the global existence results of Sects. 3 are in fact more general than those available in the existing literature, and ends with some concluding remarks and perspectives.

2. Statement of the problem. In this paper we study the initial-value problem for a class of discrete velocities models of the form:

$$\begin{cases} \partial_t f_i(t, x) + v_i \partial_x f_i(t, x) = J_i(f)(t, x), \\ f_i(0, x) = \bar{f}_i(x), \end{cases} \quad i = 1, \dots, n. \quad (1)$$

Such a class of models, here applied to the study of vehicles traffic flow, find applications also in the context of mathematical biology.

Here the vector distribution function $f = (f_1, \dots, f_n)$ is such that:

$$f_i = f_i(t, x) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+, \quad i = 1 \dots, n,$$

and the velocity can assume only a finite set $\{v_1, \dots, v_n\}$ of values, with $v_i < v_{i+1}$ for $i = 1, \dots, n - 1$, while $\bar{f}_i(x)$ are the values of the distribution function components at $t = 0$. Introducing the number density:

$$\rho(t, x) := \sum_{i=1}^n f_i(t, x), \quad (2)$$

such that $\rho(x, t)dx$ represents the number of cars at time t in the spatial neighborhood $(x, x + dx)$, the terms J_i at right hand side of (1) describing the interactions

between the vehicles of the i -class and the others read:

$$J_i(f) = \Gamma_i(f) - f_i \Lambda_i(f), \quad (3)$$

where:

$$\Gamma_i(f)(t, x) := \sum_{h,k=1}^n f_h(t, x) \int_{D_x} \eta(\rho(t, y)) B_{hk}^i(\rho(t, y)) f_k(t, y) dy, \quad (4)$$

is a gain term that describes the inflow of the i -th velocity class, while:

$$\Lambda_i(f)(t, x) := \sum_{h=1}^n \int_{D_x} \eta(\rho(t, y)) f_h(t, y) dy. \quad (5)$$

is the corresponding loss term, i.e., the flow of cars leaving the i -th velocity class. The presence of an integral in the r.h.s. of (4), (5) takes into account that vehicles exhibit non-local interactions in a domain related to a certain drivers' visibility zone. In other words, a vehicle positioned in x interacts with other cars within the interaction domain $D_x \subseteq \mathbb{R}$, and it does not interact with other vehicles which are outside from interaction domain, in general

$$D_x = [x - \Delta^-, x + \Delta^+],$$

where $\Delta^\pm \geq 0$ and $(\Delta^+, \Delta^-) \neq (0, 0)$. Moreover:

- η is the encounter rate that gives the number of interactions per unit time. We assume that it depends on the local number density ρ .
- B_{hk}^i is a transition probability density that gives the probability that a vehicle in the h -th velocity class changes its velocity to v_i after an interaction with a vehicle with velocity v_k .

From the definition of B_{hk}^i itself, the following properties:

$$B_{hk}^i(\rho) \geq 0, \quad \sum_{i=1}^n B_{hk}^i(\rho) = 1, \quad \forall \rho \geq 0, \forall h, k = 1, \dots, n. \quad (6)$$

hold true. Thanks to (6) we have that:

$$\sum_{i=1}^n J_i(f) = 0. \quad (7)$$

This relation reflects the absence in the system of proliferative and destructive effects, and will lead to the conservation of the total number of vehicles, as we shall see later on.

Now we state the basic assumptions we need in order to get local and global existence of solutions for (1).

- The encounter rate η is Lipschitz continuous and bounded, i.e., there exist $L_\eta > 0$ and $C_\eta > 0$ such that:

$$|\eta(\rho_1) - \eta(\rho_2)| \leq L_\eta |\rho_1 - \rho_2|, \quad \forall \rho_1, \rho_2 \in \mathbb{R}, \quad (8)$$

$$|\eta(\rho)| \leq C_\eta, \quad \forall \rho \in \mathbb{R}.$$

- Transition probability density B_{hk}^i is locally Lipschitz for all $h, k, i = 1, \dots, n$, i.e, for all $r > 0$ there exists a $C_{B_{hk}^i, r} > 0$ such that:

$$|B_{hk}^i(\rho_1) - B_{hk}^i(\rho_2)| \leq C_{B_{hk}^i, r} |\rho_1 - \rho_2|, \quad \forall \rho_1, \rho_2 \in \mathbb{R} \text{ with } |\rho_1| < r, |\rho_2| < r, \quad (9)$$

and verifies (6).

These hypotheses are crucial in order to have global solutions to system (1) in a suitable space.

3. Global existence and uniqueness of solutions. The well-posedness of problem (1) in the spatially homogeneous case is well known. We refer to the paper [13], in which the result is achieved under similar hypotheses. Here we study the non-homogeneous problem, pointing out that (1) is a system of n semilinear first-order hyperbolic equations. Thanks to the fact that the velocities v_i are constant we get the existence of characteristics for all $t \geq 0$ [18]. If we fix the point (t, x) in the $t - x$ plane the i -th characteristic passing through (t, x) is:

$$\xi = \gamma_i(\tau, t, x) = x + v_i(\tau - t).$$

Finding no problem for existence of characteristics we can introduce the idea of mild solution, presented in [16] for kinetic equations, that is related integrability properties of f ,

Definition 3.1. Let $T > 0$ be a positive number, a mild solution of (1) is a function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^n$ verifying

$$f_i(t, x) = \bar{f}_i(\gamma_i(0, t, x)) + \int_0^t J_i(f(\tau, \gamma_i(\tau, t, x))) d\tau, \quad (10)$$

for all $i = 1, \dots, n$ and for all $(t, x) \in [0, T] \times \mathbb{R}$.

We put:

$$X = (L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))^n,$$

and define:

$$\|u\|_X := \|u\|_1 + \|u\|_\infty, \quad u \in X,$$

where:

$$\|u\|_\infty := \max_i \|u_i\|_{L^\infty}, \quad \text{and} \quad \|u\|_1 := \sum_{i=1}^n \|u_i\|_{L^1}.$$

It is easy to verify that $(X, \|\cdot\|_X)$ is a Banach space.

Lemma 3.2. Let η and B_{hk}^i be functions verifying (6), (8) and (9), then function J , defined in (3)-(5), maps X into itself.

Proof. We have to prove that if $f \in X$ then:

$$\|J(f)\|_X = \|J(f)\|_1 + \|J(f)\|_\infty < +\infty.$$

We start analyzing the L^1 norm $\|\cdot\|_1$. We have:

$$\int_{\mathbb{R}} |J_i(f(x))| dx \leq \int_{\mathbb{R}} |\Gamma_i(f(x))| dx + \int_{\mathbb{R}} |f_i(x) \Lambda_i(f(x))| dx.$$

Evaluating the gain term, we find

$$\begin{aligned} \int_{\mathbb{R}} |\Gamma_i(f(x))| dx &\leq \sum_{h,k=1}^n \int_{\mathbb{R}} |f_h(x)| \left(\int_{D_x} |\eta(\rho(y)) B_{hk}^i(\rho(y)) f_k(y)| dy \right) dx \\ &\leq \sum_{h,k=1}^n \|f_h\|_{L^1} \int_{\mathbb{R}} |\eta(\rho(y)) B_{hk}^i(\rho(y)) f_k(y)| dy, \end{aligned}$$

and using the boundedness of η and B_{hk}^i we find

$$\int_{\mathbb{R}} |\Gamma_i(f(x))| dx \leq \sum_{h,k=1}^n C_\eta \|f_h\|_{L^1} \int_{\mathbb{R}} |f_k(y)| dy = \sum_{h,k=1}^n C_\eta \|f_h\|_{L^1} \|f_k\|_{L^1} < +\infty.$$

Similarly, we have:

$$\begin{aligned} \int_{\mathbb{R}} |f_i(x) \Lambda_i(f(x))| dx &\leq \|f_i\|_{L^1} \sum_{h=1}^n \int_{\mathbb{R}} |\eta(\rho(y)) f_h(y)| dy \\ &\leq C_\eta \|f_i\|_{L^1} \sum_{h=1}^n \int_{\mathbb{R}} |f_h(y)| dy \leq C_\eta \|f_i\| \sum_{h=1}^n \|f_h\| < +\infty. \end{aligned}$$

Coming to analyze the L^∞ part, we find:

$$\begin{aligned} |\Gamma_i(f(x))| &\leq \sum_{h,k=1}^n \|f_h\|_{L^\infty} \int_{D_x} |\eta(\rho(y)) B_{hk}^i(\rho(y)) f_k(y)| dy \\ &\leq C_\eta \sum_{h,k=1}^n \|f_h\|_{L^\infty} \|f_k\|_{L^1} < +\infty, \end{aligned}$$

which gives the boundedness of $\|\Gamma_i(f)\|_{L^\infty}$. Furthermore:

$$|f_i(x) \Lambda_i(f(x))| \leq C_\eta \|f_i\|_{L^\infty} \sum_{h,k=1}^n \|f_h\|_{L^1} < +\infty,$$

which ends the proof. \square

Lemma 3.3. *Let η and B_{hk}^i be functions verifying (6), (8) and (9), then function J is locally Lipschitz on X .*

Proof. Let $r > 0$ and $f, g \in X$ such that $\|f\|_X, \|g\|_X < r$. Then:

$$|J_i(f(x)) - J_i(g(x))| \leq |\Gamma_i(f(x)) - \Gamma_i(g(x))| + |f_i(x) \Lambda_i(f(x)) - g_i(x) \Lambda_i(g(x))|.$$

Now:

$$\begin{aligned} |\Gamma_i(f(x)) - \Gamma_i(g(x))| &\leq \sum_{h,k=1}^n \left| f_h(x) \int_{D_x} \eta(\rho_f(y)) B_{hk}^i(\rho_f(y)) f_k(y) dy \right. \\ &\quad \left. - g_h(x) \int_{D_x} \eta(\rho_g(y)) B_{hk}^i(\rho_g(y)) g_k(y) dy \right| \\ &\leq \sum_{h,k=1}^n |f_h(x) - g_h(x)| \int_{\mathbb{R}} |\eta(\rho_f(y)) B_{hk}^i(\rho_f(y)) f_k(y)| dy \\ &\quad + \sum_{h,k=1}^n |g_h(x)| \int_{\mathbb{R}} |\eta(\rho_f(y)) B_{hk}^i(\rho_f(y)) f_k(y) - \eta(\rho_g(y)) B_{hk}^i(\rho_g(y)) g_k(y)| dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

Reminding that, by 9, the quantities B_{hk}^i are locally Lipschitz for all i, h, k with Lipschitz constants $C_{B_{hk}^i, r}$ we find the following inequalities

$$I_1(x) \leq C_\eta |f_h(x) - f_k(x)| \int_{\mathbb{R}} f_k(y) dy$$

$$\leq C_\eta \sum_{h,k=1}^n \|f_k\|_{L^1} |f_h(x) - g_h(x)| \leq C_\eta n r \sum_{h=1}^n |f_h(x) - g_h(x)|,$$

from which we have

$$\|I_1\|_{L^1} \leq C_\eta n r \|f - g\|_1, \quad (11)$$

$$\|I_1\|_{L^\infty} \leq C_\eta n^2 r \|f - g\|_\infty. \quad (12)$$

For I_2 ,

$$\begin{aligned} I_2(x) &\leq \sum_{h,k=1}^n |g_h(x)| \left(\int_{\mathbb{R}} |\eta(\rho_f(y)) B_{hk}^i(\rho_f(y)) - \eta(\rho_g(y)) B_{hk}^i(\rho_g(y))| |f_k(y)| dy \right. \\ &\quad \left. + \int_{\mathbb{R}} |B_{hk}^i(\rho_g(y)) \eta(\rho_g(y)) (f_k(y) - g_k(y))| dy \right) \\ &\leq \sum_{h,k=1}^n |g_h(x)| \left(\|f_k\|_{L^\infty} \int_{\mathbb{R}} (L_\eta + L_{B_{hk}^i} C_\eta) |\rho_f(y) - \rho_g(y)| dy \right. \\ &\quad \left. + C_\eta \int_{\mathbb{R}} |f_k(y) - g_k(y)| dy \right) \leq r n (L_\eta + (1 + L_{\beta r}) C_\eta) \|f - g\|_1 \sum_{h=1}^n |g_h(x)|, \end{aligned}$$

with $C_{\beta r} = \max_{i,h,k} C_{B_{hk}^i}$, from which follows,

$$\|I_2\|_{L^1} \leq r^2 n (L_\eta + (1 + L_{\beta r}) C_\eta) \|f - g\|_1, \quad (13)$$

$$\|I_2\|_{L^\infty} \leq r^2 n^2 (L_\eta + (1 + L_{\beta r}) C_\eta) \|f - g\|_1. \quad (14)$$

In a similar way we have

$$\begin{aligned} |f_i(x) \Lambda_i(f)(x) - g_i(x) \Lambda_i(g)(x)| &\leq \sum_{h=1}^n |f_i(x) - g_i(x)| \int_{\mathbb{R}} |\eta(\rho_f(y)) f_h(y)| dy \\ &\quad + \sum_{h=1}^n |g_i(x)| \int_{\mathbb{R}} |\eta(\rho_f(y)) f_h(y) - \eta(\rho_g(y)) g_h(y)| dy, \end{aligned}$$

which leads to the following inequalities

$$\|f_i \Lambda_i(f) - g_i \Lambda_i(g)\|_{L^\infty} \leq C_\eta r \|f - g\|_\infty + (C_\eta r + L_\eta r^2) \|f - g\|_1, \quad (15)$$

and

$$\|f_i \Lambda_i(f) - g_i \Lambda_i(g)\|_{L^1} \leq (2C_\eta r + L_\eta r^2) \|f - g\|_1. \quad (16)$$

Combining the previous inequalities we conclude the proof. \square

The previous lemma is crucial in order to use a fixed point argument in the following theorem.

Theorem 3.4 (Local existence and uniqueness). *Let η and B_{hk}^i be functions verifying (6), (8), (9) and let $\bar{f} \in X$. Then, there exists $T > 0$ and an unique $f \in C([0, T], X)$ mild solution of (1).*

Proof. Let $\delta > 0$ such that $\|\bar{f}\|_X \leq \delta$. We take $r := 1 + \delta$ and $0 < a \leq 1$ and define:

$$D_r(a) := \left\{ u \in C([0, a], X) \mid \|u\|_* := \sup_{t \in [0, a]} \|u(t)\|_X \leq r \right\}.$$

Let $u \in C([0, a], X)$, We introduce the map:

$$\Phi(u)_i(t, x) := \bar{f}_i(\gamma_i(0, t, x)) + \int_0^t J_i(u(\tau, \gamma_i(\tau, t, x)))(\tau, \gamma_i(\tau, t, x)) dt. \quad (17)$$

Clearly $\Phi(u) \in C([0, a], X)$. If we take $u, v \in C([0, a], X)$ we easily find the following estimate:

$$\|\Phi(u)(t)\|_X \leq \delta + \int_0^t (\|J(u(\tau)) - J(0)\|_X + \|J(0)\|_X) ds,$$

and thus, reminding that $J(0) = 0$,

$$\|\Phi(u)(t)\|_X \leq \delta + aL_r,$$

with $L_r > 0$ the Lipschitz constant of J . Moreover:

$$\|\Phi(u)(t) - \Phi(v)(t)\|_X \leq \int_0^t \|J(u(\tau)) - J(v(\tau))\|_X d\tau \leq aL_r \|u - v\|_*,$$

and thus:

$$\|\Phi(u) - \Phi(v)\|_* \leq aL_r \|u - v\|_*. \quad (18)$$

Then, for every $a \in (0, L_r/2]$, we have that $\Phi(u) \in D_r(a)$ and that Φ is Lipschitz in $D_r(a)$ with Lipschitz constant less than or equal to $1/2$. Since $D_r(a)$ is a complete metric space with respect to the metric induced by $\|\cdot\|_{L^\infty}$, we have the existence and uniqueness of a fixed point $f = \Phi(f) \in D_r(a)$, which is also solution of (1). \square

Theorem 3.4 states that if the initial datum \bar{f} belongs to X , then there is a mild solution of (1), overlooking that we are interested to positive initial data. Now, let us suppose that the initial datum $\bar{f} \in X$ is positive, i.e. $\bar{f}_i(x) \geq 0$ for all $x \in \mathbb{R}$ and for $i = 1, \dots, n$. Let us defined the total number of vehicles at time t as:

$$N(t) := \int_{\mathbb{R}} \rho(t, x) dx = \int_{\mathbb{R}} \left(\sum_{i=1}^n f_i(t, x) \right) dx, \quad (19)$$

Integrating on x the n equations of (1), summing over i and taking account of (7), we have:

$$\sum_{i=1}^n \int_{\mathbb{R}} (\partial_t f_i(t, x) + v_i \partial_x f_i(t, x)) dx = 0, \quad (20)$$

from which:

$$\sum_{i=1}^n \left(\int_{\mathbb{R}} \partial_t f_i(t, x) dx + v_i [f_i(t, x)]_{-\infty}^{+\infty} \right) = \sum_{i=1}^n \int_{\mathbb{R}} \partial_t f_i(t, x) dx = 0, \quad (21)$$

and finally:

$$\frac{dN}{dt}(t) = 0, \quad (22)$$

that is, the total vehicles number is conserved and is equal to its initial value N_0 .

The following proposition shows that if the system has non-negative initial data then the solution remains non-negative.

Lemma 3.5. *Let $f \in C([0, T], X)$ be a mild solution to problem (1) corresponding to a non-negative initial datum. Then, f remains non-negative for all $0 \leq t \leq T$.*

Proof. If \bar{f} is a non-negative initial datum, then $\bar{f}(x) \geq 0$ for all $x \in \mathbb{R}$. Given a $x \in \mathbb{R}$ such that $\bar{f}_i(x) > 0$ for all indexes then the solution components along the characteristics remain non-negative for a certain time interval. If, on the contrary, $\bar{f}_i(x) = 0$ for $i \in L \subseteq \{1, \dots, n\}$, then, for these indexes,

$$J_i(\bar{f}(x)) = \Gamma_i(\bar{f}(x)) \geq 0,$$

and thus along characteristics they are non-decreasing functions, while remaining components remains non-negative for a certain time interval, always along characteristics. If f is a solution of (1) we can write it as in (10). If we have a point $(\bar{\tau}, \bar{\xi}) \in [0, T] \times \mathbb{R}$ such that $f_i(\bar{\tau}, \bar{\xi}) = 0$, for $i \in L \subseteq \{1, \dots, n\}$, then the integral in r.h.s. of (10big) is greater than or equal to zero, and f_i has to remain greater than or equal to zero in an interval (\bar{t}, t^*) . \square

The previous proposition, together with the conservation of the total number of vehicles, ensures that:

$$\|f(t)\|_1 = \sum_{i=1}^n \int_{\mathbb{R}} |f_i(t, x)| dx = \sum_{i=1}^n \int_{\mathbb{R}} f_i(t, x) dx = N_0. \quad (23)$$

In Theorem 3.4 we established existence and uniqueness of local in-time solutions to (1). Putting:

$$T_{\bar{f}} := \sup(T > 0 \mid \exists u \in C([0, T], X) \text{ solution of (1)}),$$

then, stitching together the solutions, we obtain the existence of a unique maximal mild solution $u \in C([0, T_{\bar{f}}], X)$. Our next goal is to prove that for system (1) with positive initial data we actually have $T_{\bar{f}} = +\infty$.

Lemma 3.6. *Let $f \in C([0, T_{\bar{f}}], X)$ the unique mild solution of (1). If $T_{\bar{f}} < +\infty$ then:*

$$\lim_{t \rightarrow T_{\bar{f}}} \|f(t)\|_X = +\infty$$

Proof. First of all we observe that the length T of the existence time interval $[0, T]$ given by Theorem 3.4 is only related to the norm δ of initial data. Let $T_{\bar{f}} < \infty$ and $\{t_n\}$ a sequence of instants such that $t_n < T_{\bar{f}}$ and $t_n \rightarrow T_{\bar{f}}$. Assume that $C := \sup_n \|u(t_n)\|_X < \infty$. Then, from Theorem 3.4 there exists a $T_C > 0$ such that we have a unique solution $f \in C([0, T_C], X)$ to (1) for all initial data with $\|\bar{f}\| \leq C$. Now we fix an index $n \in \mathbb{N}$ such that $t_n + T_C > T_{\bar{f}}$ and consider the solution $f \in C([0, t_n], X)$ to (1) as well as the solution $f_n \in C([0, T_C], X)$ corresponding to the initial value $f(t_n)$. Pasting together these latter we obtain a solution defined in $[0, t_n + T_C]$, contradicting the property of $T_{\bar{f}}$. Then we necessarily have that $\|f(t)\|_X \rightarrow \infty$ when $t \rightarrow T_{\bar{f}}$. \square

Thanks to the previous Lemma we have only to verify that:

$$\lim_{t \rightarrow T_{\bar{f}}} \|u(t)\|_X < +\infty.$$

Moreover, since (23) says that the norm $\|\cdot\|_1$ is constant in time, we restrict our considerations to find a bound for the norm $\|\cdot\|_\infty$, reminding that η and B_{hk}^i are both bounded functions. We have:

$$\begin{aligned} |f_i(t, x)| &= f_i(t, x) = \bar{f}_i(\gamma_i(0, t, x)) + \int_0^t J_i(f(\tau, \gamma_i(\tau, t, x))) d\tau \\ &\leq \bar{f}_i(\gamma_i(0, t, x)) + \int_0^t \Gamma_i(f(\tau, \gamma_i(\tau, t, x))) d\tau \\ &\leq \|\bar{f}_i\|_{L^\infty} + \sum_{h,k=1}^n C_\eta \int_0^t f_h(\tau, \gamma_i(\tau, t, x)) \int_{D_{\gamma_i(\tau, t, x)}} f_k(\tau, y) dy d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \|\bar{f}_i\|_{L^\infty} + \sum_{h=1}^n C_\eta \int_0^t f_h(\tau, \gamma_i(\tau, t, x)) \int_{\mathbb{R}} \sum_{k=1}^n f_k(\tau, y) dy d\tau \\
&\leq \|\bar{f}\|_\infty + C_\eta \int_0^t N(\tau) \sum_{h=1}^n f_h(\tau, \gamma_i(\tau, t, x)) d\tau \\
&\leq \|\bar{f}\|_\infty + C_\eta N_0 \int_0^t \sum_{h=1}^n \|f_h(\tau)\|_{L^\infty} d\tau \\
&\leq \|\bar{f}\|_\infty + nC_\eta N_0 \int_0^t \|f(\tau)\|_\infty d\tau,
\end{aligned}$$

and thus:

$$\|f(t)\|_\infty \leq \|\bar{f}\|_\infty + nC_\eta N_0 \int_0^t \|f(\tau)\|_\infty d\tau.$$

Using the Gronwall's lemma we find:

$$\|f(t)\|_\infty \leq \|\bar{f}\|_\infty \exp(nC_\eta N_0 t),$$

which gives us the bound for $\|f\|_\infty$. This ends the proof of following theorem.

Theorem 3.7 (Global existence and uniqueness). *Let η and B_{hk}^i be functions verifying (6), (8), (9) and let $\bar{f} \in X$ a positive initial data. Then, there exists a unique mild solution $f \in C([0, +\infty), X)$ to (1).*

We conclude discussion about well-posedness of Cauchy problem (1), with the following theorem which gives continuous dependence on initial data.

Theorem 3.8. *Let $f, g \in C([0, T], X)$ be two solution of (1) related to two different initial data, respectively $\bar{f}, \bar{g} \in X$, then for all $b \in [0, T)$ there exists $L \geq 0$ such that*

$$\|f(t) - g(t)\|_X \leq e^{Lt} \|\bar{f} - \bar{g}\|_X,$$

for all $t \in [0, b]$.

Proof. Fixed $b \in [0, T)$, then there exists a $\delta > 0$ such that $B_\delta(0)$ contains the whole trajectories of solutions, i.e.

$$f(t) \in B_\delta(0), g(t) \in B_\delta(0), \text{ for all } t \in [0, b],$$

Reminding that J is locally Lipschitz, let $L \geq 0$ be the Lipschitz constant of J related to $B_\delta(0)$,

$$\begin{aligned}
\|f(t) - g(t)\|_X &\leq \|\bar{f} - \bar{g}\|_X + \int_0^t \|J(f(\tau)) - J(g(\tau))\|_X d\tau \\
&\leq \|\bar{f} - \bar{g}\|_X + L \int_0^t \|f(\tau) - g(\tau)\|_X d\tau,
\end{aligned}$$

using Gronwall's inequality we conclude the proof. \square

4. Discussion and concluding remarks. To the best of our knowledge the most general result so far available concerning the well-posedness of (1) is in [3], where the Authors study the initial value problem:

$$\begin{cases} \partial_t f_i(t, x) + v_i \partial_x f_i(t, x) = \chi(\rho(t, x)) \leq \rho_c J_i(f)(t, x), \\ f_i(0, x) = \bar{f}_i(x), \end{cases} \quad i = 1, \dots, n. \quad (24)$$

Here the interaction terms J_i , that have the same structure as in (3), differ from those defined in (4), (5) by the presence, under the integral sign in Γ_i and Λ_i , of a non negative function $w(x, y)$, with $\int_{D_x} w(x, x+y)dy = 1$, that weigh the interactions over the visibility zone D_x in front of the vehicle in x and whose addition is not relevant in respect to the well-posedness issue. Crucial in the results of [3] is the presence in the r.h.s. of (24) of the function χ , defined as:

$$\chi(\rho \leq \rho_c) = \begin{cases} 1 & \text{if } \rho \leq \rho_c, \\ 0 & \text{if } \rho > \rho_c, \end{cases} \quad (25)$$

where ρ_c is a prescribed critical density. The Authors of [3] justify the introduction of the cut-off function (25) by observing that the vehicles are obliged to stop when the traffic density reaches a critical value, due to overcrowding, as is largely shown in the experimental fundamental diagrams speed-density. On the other hand, it is apparent that when $\rho > \rho_c$ in (24) all the vehicles in each velocity class continue to move at their own speed, i.e., each component $f_i(t, x)$ of the distribution function is transported along the corresponding characteristic, leading to the unrealistic situation in which at high density the vehicles stop to interact.

The aim of this section is to show, by a numerical example, that situations can occur in which the assumption that the interactions freeze at high density seems to be restrictive. In the sequel we furnish a toy model in which, given an arbitrary $\rho_c > 0$, we are able to find B_{hk}^i and η satisfying (8) and (9) (that ensures global well-posedness), an interaction interval $D_x = [x, x + \Delta]$ with $\Delta > 0$ and initial data \bar{f} , with:

$$\|\bar{f}\|_\infty \leq \rho_c, \quad \bar{\rho}(x) = \sum_{i=1}^n \bar{f}_i(x) \leq \rho_c, \quad \forall x \in \mathbb{R},$$

for which there exist $t^* > 0$ and $x^* \in \mathbb{R}$ such that $\rho(t^*, x^*) > \rho_c$. In this way we show that the assumption made in [3] that ρ does not exceed ρ_c is not coherent with the general form of the initial value problem (1). We consider a two velocities system, that is, we assume that every vehicle belongs to one of the two possible velocity classes f_1 , corresponding to $v_1 = 0$, and f_2 that corresponds to $v_2 = 1$. The transition probability densities for this case model are assumed to be:

$$B_{hk}^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_{hk}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad h, k = 1, 2.$$

and $\eta(\rho) = 1$. Hence the first line of (1) become:

$$\begin{cases} \partial_t f_1(t, x) = f_2(t, x) \int_{D_x} f_1(t, y) dy, \\ \partial_t f_2(t, x) + \partial_x f_2(t, x) = -f_2(t, x) \int_{D_x} f_1(t, y) dy. \end{cases} \quad (26)$$

We solve numerically the Cauchy problem for (26) relative to initial data $\bar{f}_1(x)$, $\bar{f}_2(x)$ having compact support and such that if $y \in \text{supp}(\bar{f}_1)$ and $x \in \text{supp}(\bar{f}_2)$ then $x \leq y$. Specifically, we choose the initial data as in Figure 1. A group of vehicles, corresponding to the component $\bar{f}_1(x)$ of the initial distribution function \bar{f} is at rest ahead of a group $\bar{f}_2(x)$ of vehicles that is moving with velocity $v = 1$. The prescribed value of the ‘‘critical’’ density ρ_c is 5. In Table 1 we analyze the role of the size Δ of the interaction domain D_x . The maximum ρ_{t^*} of the equilibrium density increases as Δ decreases, and is reached at a time t^* that increases with increasing Δ .

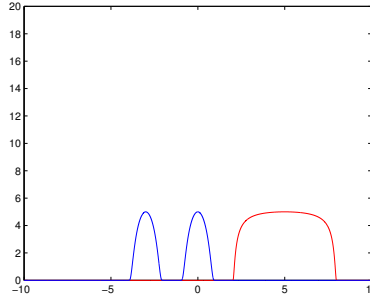


FIGURE 1. Initial data corresponding to $\rho_c = 5$. The class \bar{f}_1 of velocity $v_1 = 0$ is on the right in red color, the class \bar{f}_2 of velocity $v_2 = 1$ is represented by the bimodal distribution on the left in blue color.

Δ	t^*	ρ_{t^*}	Δ	t^*	ρ_{t^*}
1.05	4.5425	5.1890	0.55	5.4985	6.9726
1.00	4.5885	5.2214	0.50	5.5406	7.5731
0.95	4.6365	5.2660	0.45	5.5786	8.3056
0.90	4.6855	5.3269	0.40	5.6106	9.2196
0.85	4.7365	5.4092	0.35	5.6346	10.3606
0.80	4.7925	5.5192	0.30	5.6536	11.7122
0.75	4.8555	5.6652	0.25	5.6706	13.0508
0.70	4.9255	5.8575	0.20	5.6846	13.7962
0.65	5.0025	6.1090	0.15	5.6926	13.3582
0.60	5.4595	6.4658			

TABLE 1. Maximum ρ_{t^*} of the density reached at time t^* as function of the size Δ of the interaction domain D_x . Observe that in any case ρ_{t^*} is greater than ρ_c .

In Figure 2, the final configurations for different values of Δ are plotted for the same initial data as in Figure 1. Again, we stress that, if we take Δ small enough, the critical density is passed.

In summary, the well-posedness results of Section. 3 complete those of [3], furnishing global existence and uniqueness of solutions to (1) in its general form, i.e., in any traffic condition. Along the lines of what the Authors of [3] point out in their concluding remarks, in the present paper we succeeded in overcoming the constraint assumption that the r.h.s. of (1) disappears when the density is higher than a prescribed value ρ_c .

Finally, we point out the generalization of the proof method used in this paper toward the qualitative analysis of models based on the so-called kinetic theory of active particles (KTAP) [6], a more general framework though leading to mathematical structures similar to those here considered, as a challenging research perspective, in view of the ability of KTAP to design models for a wider class of complex systems. However, it is worth to advise that such a goal is far from being a trivial adaptation, due to additional difficulties such as stronger nonlinearity and nonlinear additivity of interactions [10], all aspects related to the inherent complexity of the above mentioned phenomena.

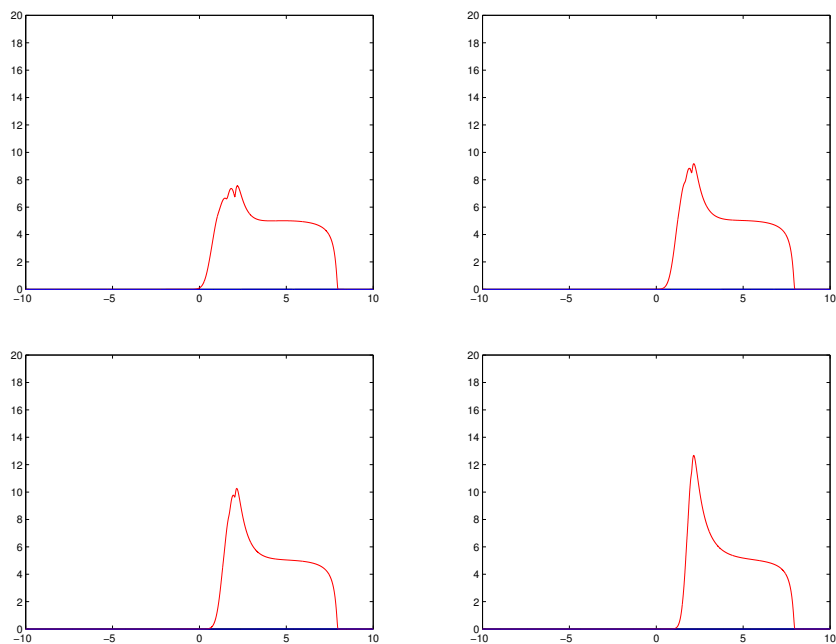


FIGURE 2. Final distribution functions corresponding to $\Delta = 0.5, 0.4, 0.35, 0.25$, and initial data as in Figure 1.

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