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NEWTON'S METHOD FOR NONLINEAR STOCHASTIC WAVE EQUATIONS DRIVEN BY ONE-DIMENSIONAL BROWNIAN MOTION

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Abstract. We consider nonlinear stochastic wave equations driven by one-dimensional white noise with respect to time. The existence of solutions is proved by means of Picard iterations. Next we apply Newton's method. Moreover, a second-order convergence in a probabilistic sense is demonstrated.

1. **Introduction.** In 1960's wave equations subject to random perturbations attracted a lot of attention due to their applications in physics, relativistic quantum mechanics and oceanography to name a few. We give a brief review of problems being discussed in the literature. For the introduction to the theory of stochastic wave equations (SWE) see [7, 21]. Existence results for nonlinear SWE including random field solutions and function-valued solutions are given in [5, 6, 17]. Weak solutions to semilinear SWE are treated in [14]. Various regularity properties of solutions and their densities, e.g. absolute continuity and smoothness of the law, Hölder continuity, Malliavin differentiability, are investigated in [9, 11, 15, 18, 20]. Asymptotic properties of moments are considered in [8]. SWE with polynomial nonlinearities are studied in [4]; SWE with values in Riemannian manifolds in [2]. The case of SWE driven by fractional noise is presented in [3]. Several results for damped SWE are proposed in [13]. In [16] a class of semilinear SWE is solved in the framework of Colombeau generalized stochastic process space. Various numerical methods are applied to SWE in [19, 22, 10].

Newton's methods for stochastic differential equations are studied by Kawabata and Yamada in [12] and Amano in [1]. In [23] we derive further nontrivial generalizations to the case of stochastic functional differential equations with Hale functionals. In [24] and [25] we establish the convergence of Newton's method for stochastic functional partial differential equations of parabolic and first-order hyperbolic types.

Since various phenomena are concerned with the delay dependence on one variable, we employ one-dimensional Brownian motion. In this case the main tool in proving our results is the Doob inequality. The case of two-dimensional Brownian motion requires more advanced techniques.

The paper is organized as follows. In Section 2 we introduce basic notations and formulate the initial value problem for nonlinear stochastic wave equations. The

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existence of solutions is proved by means of successive approximations (Section 4). Next we establish a first-order convergence (Section 5) and a probabilistic second-order convergence (Section 6) of Newton's method. The results in Section 4 and 5 base on two lemmas presented in Section 3: a two-dimensional Gronwall-type inequality and an estimation of solutions to a class of nonlinear stochastic wave equations.

Our results can be applied to periodic boundary value problems. This can be done by means of appropriate extensions of the data onto the real line (reflection principles).

2. **Formulation of the problem.** Fix $T > 0$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $W = (W_t)_{t\in[0,T]}$ the standard Brownian motion, $(\mathcal{F}_t)_{t\in[0,T]}$ - its natural filtration and $\dot{W}_t$ - the respective white noise. The space of all continuous and $(\mathcal{F}_t)$-adapted processes $X : [0,T] \to \mathbb{R}$ is equipped with the seminorms

$$|X|_t^2 = \mathbb{E}\left[\sup_{0\le s\le t} |X_s|^2\right] \text{ for } t \in [0,T].$$

For $(t,x) \in [0,T]\times\mathbb{R}$ let $C_{t,x}$ be the wave cone with vertex (t,x), that is the triangle delimited by the points (t,x), $(0,x+t)$, $(0,x-t)$:

$$C_{t,x} = \{(s,y) : 0 \le s \le t,\ |y-x| \le t-s\}.$$

We say that a function $\Psi : [0,T]\times\mathbb{R} \to \mathbb{R}$ is increasing w.r.t. cones if $\Psi(s,y) \le \Psi(t,x)$ for all $(s,y) \in C_{t,x}$. It means that

$$C_{s,y} \subset C_{t,x} \Rightarrow \Psi(s,y) \le \Psi(t,x).$$

Consider the following initial value problem for the nonlinear stochastic wave equation with nonlocal dependence

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= f(t,x,u(\cdot,x)) + g(t,x,u(\cdot,x))\dot{W}_t \quad \text{for } (t,x) \in [0,T]\times\mathbb{R}, \\ u(0,x) &= \phi(x) \quad \text{for } x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(0,x) &= \psi(x) \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{1}$$

where $u(\cdot,x)$ is understood as being defined on $[0,t]$, functions $\phi(x)$, $\psi(x)$ are continuous, independent of the Brownian motion and such that $\mathbb{E}\left[|\phi|^2\right] < \infty$, $\mathbb{E}\left[|\psi|^2\right] < \infty$, $f(t,x,\cdot), g(t,x,\cdot) : C([0,t]) \to \mathbb{R}$ are continuous, Fréchet differentiable functions that satisfy the Lipschitz condition:

$$|f(t,x,v) - f(t,x,\bar{v})| \le L(t,x) \sup_{0\le\tilde{t}\le t} |v(\tilde{t}) - \bar{v}(\tilde{t})| \tag{2}$$

$$|g(t,x,v) - g(t,x,\bar{v})| \le L(t,x) \sup_{0\le\tilde{t}\le t} |v(\tilde{t}) - \bar{v}(\tilde{t})| \tag{3}$$

for some function $L : [0,T]\times\mathbb{R} \to \mathbb{R}_+$ increasing w.r.t. cones and $v, \bar{v} \in C([0,t])$, where $C([0,t])$ is the space of all continuous real-valued functions on $[0,t]$. Let $\mathcal{U}$ denote the space of continuous and $\mathcal{F}_t$-adapted processes $u : [0,T]\times\mathbb{R} \to \mathbb{R}$ such that $|u(\cdot,x)|_T < \infty$, $u(\cdot,x)$ is a diffusion with respect to W, $u(t,\cdot)$ is a continuous function, $u = u(t,x)$ is measurable w.r.t. the σ-field generated by $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ are the Borel subsets on $\mathbb{R}$.

$u \in \mathcal{U}$ is a solution to (1) if it satisfies the integral equation

$$
\begin{aligned}
u(t,x) &= \frac{\phi(x-t)+\phi(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy \qquad (4)\\
&+ \frac{1}{2}\int_{C_{t,x}} f\left(s,y,u(s,y)\right)dyds + \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} g\left(s,y,u(s,y)\right)dy\right)dW_s,
\end{aligned}
$$

which is based on d'Alembert's formula corresponding to (1) and the stochastic integral is of Itô type. This equation is satisfied $\mathbb{P}$-almost surely. The exceptional set is independent of t, x.

3. **Estimation of solutions.**

We formulate a two-dimensional Gronwall-type lemma.

Lemma 3.1. *Suppose that $\Psi, K : [0,T]\times\mathbb{R}\to\mathbb{R}_+$ are continuous and increasing w.r.t. cones. If $z : [0,T]\times\mathbb{R}\to\mathbb{R}_+$ is continuous and satisfies*

$$
z(t,x) \le \frac{1}{2}\int_{C_{t,x}}\Psi(s,y)dyds + \frac{1}{2}\int_{C_{t,x}} K(s,y)z(s,y)dyds, \quad (t,x)\in[0,T]\times\mathbb{R},
$$

then

$$
z(t,x) \le K_1(t,x)\int_{C_{t,x}}\Psi(s,y)dyds,
$$

where $K_1(t,x) = \frac{1}{2}e^{t^2K(t,x)}$.

Proof. We conduct the proof for a function $z(t,x)$ that is increasing w.r.t. cones. The general case can be reduced to that one by defining

$$
\mathbf{z}(t,x) = \max_{(s,y)\in C_{t,x}} z(s,y).
$$

The function $\mathbf{z}$ satisfies the same integral inequality and $z \le \mathbf{z}$. Let

$$
\hat{z}(t,x) = \frac{1}{2}\int_{C_{t,x}}\Psi(s,y)dyds + \frac{1}{2}\int_{C_{t,x}} K(s,y)z(s,y)dyds.
$$

Then $z(t,x)\le\hat{z}(t,x)$. The function $\hat{z}(t,x)$ is C^2, $\hat{z}(0,x) = \frac{\partial}{\partial t}\hat{z}(0,x) = 0$ and

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\hat{z}(t,x) - \frac{\partial^2}{\partial x^2}\hat{z}(t,x) &= \Psi(t,x) + K(t,x)z(t,x)\\
&\le \Psi(t,x) + K(t,x)\hat{z}(t,x).
\end{aligned}
$$

Fix a cone C_{t_0,x_0} with a vertex $(t_0,x_0)\in[0,T]\times\mathbb{R}$. Since K is increasing w.r.t. cones, we have

$$
\frac{\partial^2}{\partial t^2}\hat{z}(t,x) - \frac{\partial^2}{\partial x^2}\hat{z}(t,x) \le \Psi(t,x) + K(t_0,x_0)\hat{z}(t,x) \qquad (5)
$$

for $(t,x)\in C_{t_0,x_0}$. The solution to (5) is estimated by any solution to the following comparison inequality

$$
\frac{\partial^2}{\partial t^2}\tilde{z}(t,x) - \frac{\partial^2}{\partial x^2}\tilde{z}(t,x) \ge \Psi(t,x) + K(t_0,x_0)\tilde{z}(t,x) \qquad (6)
$$

with zero initial conditions. Our goal is to find a function $\Phi : [0,T]\to\mathbb{R}$ such that

$$
\tilde{z}(t,x) = \frac{1}{2}\int_{C_{t,x}}\Phi(s)\Psi(s,y)dyds
$$

satisfies (6). By d'Alembert's formula it is a solution to the wave equation

$$\frac{\partial^2}{\partial t^2}\tilde{z}(t,x) - \frac{\partial^2}{\partial x^2}\tilde{z}(t,x) = \Phi(t)\Psi(t,x).$$

Hence (6) takes the form

$$\Phi(t)\Psi(t,x) \geq \Psi(t,x) + \frac{1}{2}K(t_0,x_0)\int_{C_{t,x}} \Phi(s)\Psi(s,y)dyds. \tag{7}$$

Utilizing the fact that Ψ is increasing w.r.t. cones, we enlarge the right hand side of (7) and find the solution to a stronger inequality

$$\Phi(t)\Psi(t,x) \geq \Psi(t,x) + K(t_0,x_0)\Psi(t,x)t_0\int_0^t \Phi(s)ds.$$

It suffices to take

$$\Phi(t) = e^{t_0 t K(t_0,x_0)}.$$

Hence

$$z(t,x) \leq \hat{z}(t,x) \quad \leq \quad \frac{1}{2}e^{t_0 t K(t_0,x_0)}\int_{C_{t,x}} \Psi(s,y)dyds$$

for $(t,x) \in C_{t_0,x_0}$, in particular we can take $(t_0,x_0) = (t,x)$. This completes the proof. $\square$

By $C^*([0,t])$ we denote the space of all linear and bounded functionals $\mathcal{T} : C([0,t]) \to \mathbb{R}$ with the norm

$$|\mathcal{T}|_t^* := \sup_v |\mathcal{T}v|,$$

where the supremum is taken over all $v \in C([0,t])$ whose uniform norms do not exceed 1.

In the following lemma we give an estimation of solutions to nonlinear stochastic wave equations.

Lemma 3.2. *Suppose that $\alpha^{(1)}, \alpha^{(2)} : [0,T]\times\mathbb{R} \to \mathcal{U}$ are continuous, $\mathcal{T}^{(1)}(t,x)$, $\mathcal{T}^{(2)}(t,x) : C([0,T]) \to \mathbb{R}$ and there exists a function $L : [0,T]\times\mathbb{R} \to \mathbb{R}_+$ increasing w.r.t. cones such that*

$$\left|\mathcal{T}^{(i)}(t,x)\right|_t^* \leq L(t,x) \quad \textit{for } (t,x) \in [0,T]\times\mathbb{R},\ i = 1,2. \tag{8}$$

If $u \in \mathcal{U}$ satisfies the stochastic wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= \alpha^{(1)} + \mathcal{T}^{(1)}u(\cdot,x) + \left(\alpha^{(2)} + \mathcal{T}^{(2)}u(\cdot,x)\right)\dot{W}_t, \quad (t,x) \in [0,T]\times\mathbb{R} \\ u(0,x) &= 0, \quad \frac{\partial}{\partial t}u(0,x) = 0, \quad x \in \mathbb{R}, \end{aligned}$$

then we have

$$|u(\cdot,x)|_t^2 \leq K_1(t,x)\int_{C_{t,x}} \left(T^2|\alpha^{(1)}(\cdot,y)|_s^2 + 4|\alpha^{(2)}(\cdot,y)|_s^2\right)dyds \tag{9}$$

for $(t,x) \in [0,T]\times\mathbb{R}$, where

$$K_1(t,x) = \frac{1}{2}e^{2t^2(T^2+4)L^2(t,x)} \quad \textit{for} \quad (t,x) \in [0,T]\times\mathbb{R}\,. \tag{10}$$

Proof. By d'Alembert's formula and the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le\tilde t\le t}|u(\tilde t,x)|^2\right] &\le \mathbb{E}\left[\sup_{0\le\tilde t\le t}\left|\int_{C_{\tilde t,x}}\left(\alpha^{(1)}(s,y)+\mathcal{T}^{(1)}(s,y)u(s,y)\right)dyds\right|^2\right]\\
&+ \mathbb{E}\left[\sup_{0\le\tilde t\le t}\left|\int_{C_{\tilde t,x}}\left(\alpha^{(2)}(s,y)+\mathcal{T}^{(2)}(s,y)u(s,y)\right)dydW_s\right|^2\right]\\
&:= I_1+I_2.
\end{aligned}$$

By the Schwarz inequality and (8) we obtain

$$\begin{aligned}
I_1 &\le t^2\,\mathbb{E}\left[\sup_{0\le\tilde t\le t}\int_{C_{\tilde t,x}}\left(\alpha^{(1)}(s,y)+\mathcal{T}^{(1)}(s,y)u(s,y)\right)^2 dyds\right]\\
&\le t^2\,\mathbb{E}\left[\int_{C_{t,x}}\left(\alpha^{(1)}(s,y)+\mathcal{T}^{(1)}(s,y)u(s,y)\right)^2 dyds\right]\\
&\le 2t^2\int_{C_{t,x}}|\alpha^{(1)}(\cdot,y)|_s^2\,dyds+2t^2\int_{C_{t,x}}L^2(s,y)|u(\cdot,y)|_s^2\,dyds.
\end{aligned}$$

By the Doob inequality, the Itô isometry and (8) we have

$$\begin{aligned}
I_2 &= \mathbb{E}\left[\sup_{0\le\tilde t\le t}\left|\int_0^{\tilde t}\left(\int_{x-(\tilde t-s)}^{x+(\tilde t-s)}\left(\alpha^{(2)}(s,y)+\mathcal{T}^{(2)}(s,y)u(s,y)\right)dy\right)dW_s\right|^2\right]\\
&\le 4\,\mathbb{E}\left[\int_0^t\int_{x-(t-s)}^{x+(t-s)}\left(\alpha^{(2)}(s,y)+\mathcal{T}^{(2)}(s,y)u(s,y)\right)^2 dyds\right]\\
&\le 8\int_{C_{t,x}}|\alpha^{(2)}(\cdot,y)|_s^2\,dyds+8\int_{C_{t,x}}L^2(s,y)|u(\cdot,y)|_s^2\,dyds.
\end{aligned}$$

Hence

$$\begin{aligned}
|u(\cdot,x)|_t^2 &\le 2\int_{C_{t,x}}\left[t^2|\alpha^{(1)}(\cdot,y)|_s^2+4|\alpha^{(2)}(\cdot,y)|_s^2\right]\,dyds\\
&+ 2(t^2+4)\int_{C_{t,x}}L^2(s,y)|u(\cdot,y)|_s^2\,dyds.
\end{aligned}$$

Applying Lemma 3.1 we obtain

$$|u(\cdot,x)|_t^2 \le \frac{1}{2}e^{2t^2(T^2+4)L^2(t,x)}\int_{C_{t,x}}\left(T^2|\alpha^{(1)}(\cdot,y)|_s^2+4|\alpha^{(2)}(\cdot,y)|_s^2\right)dyds$$

for $(t,x)\in[0,T]\times\mathbb{R}$. □

Remark 1. If $\mathcal{T}^{(1)}=\mathcal{T}^{(2)}\equiv 0$, then one derives the assertion (9) with $K_1(t,x)=\frac{1}{2}$.

4. **Existence of solutions.** We formulate an iterative scheme for problem (1). Let

$$u^{(0)}(t,x)=\frac{\phi(x-t)+\phi(x+t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy \tag{11}$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}u^{(k+1)} - \frac{\partial^2}{\partial x^2}u^{(k+1)} &= f\left(t,x,u^{(k)}(\cdot,x)\right) + g\left(t,x,u^{(k)}u(\cdot,x)\right)\dot{W}_t, \\
& \quad (t,x)\in[0,T]\times\mathbb{R} \\
u^{(k)}(0,x) &= \phi(x), \quad x\in\mathbb{R}, \\
\frac{\partial}{\partial t}u^{(k)}(0,x) &= \psi(x), \quad x\in\mathbb{R}\,.
\end{aligned} \tag{12}
$$

If we denote the increments $\Delta u^{(k)} = u^{(k+1)} - u^{(k)}$, then we have

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\Delta u^{(k+1)} - \frac{\partial^2}{\partial x^2}\Delta u^{(k+1)} &= f\left(t,x,u^{(k+1)}(\cdot,x)\right) - f\left(t,x,u^{(k)}(\cdot,x)\right) \\
&+ \left[g\left(t,x,u^{(k+1)}(\cdot,x)\right) - g\left(t,x,u^{(k)}(\cdot,x)\right)\right]\dot{W}_t
\end{aligned}
$$

for $(t,x)\in[0,T]\times\mathbb{R}$.

Theorem 4.1. *Under the Lipschitz condition* (2) *and* (3), *the sequence* $\left(u^{(k)}\right)_{k\in\mathbb{N}}$ *defined by* (12) *converges to the solution* u *of equation* (1) *in the following sense*

$$
\lim_{k\to\infty}\left|u^{(k)}(\cdot,x) - u(\cdot,x)\right|_t = 0 \quad \text{for } t\in[0,T].
$$

Proof. We show that the sequence $(u^{(k)})_{k\in\mathbb{N}}$, generated by the above Picard iteration scheme, satisfies the Cauchy condition with respect to the norm $|\cdot|_t$. Applying Lemma 3.2 with

$$
\begin{aligned}
\mathcal{T}^{(1)}(t,x) &= \mathcal{T}^{(2)}(t,x) \equiv 0 \\
\alpha^{(1)}(t,x) &= f\left(t,x,u^{(k+1)}(\cdot,x)\right) - f\left(t,x,u^{(k)}(\cdot,x)\right) \\
\alpha^{(2)}(t,x) &= g\left(t,x,u^{(k+1)}(\cdot,x)\right) - g\left(t,x,u^{(k)}(\cdot,x)\right)
\end{aligned}
$$

together with the Lipschitz condition (2), (3) we obtain

$$
\left|\Delta u^{(k+1)}(\cdot,x)\right|_t^2 \le \frac{1}{2}(T^2+4)\int_{C_{t,x}} L^2(s,y)\left|\Delta u^{(k)}(\cdot,y)\right|_s^2 dyds.
$$

Since $L:[0,T]\times\mathbb{R}\to\mathbb{R}_+$ is increasing w.r.t. cones, we get

$$
\left|\Delta u^{(k+1)}(\cdot,x)\right|_t^2 \le \frac{1}{2}(T^2+4)L^2(t,x)\int_{C_{t,x}} \left|\Delta u^{(k)}(\cdot,y)\right|_s^2 dyds.
$$

Hence

$$
\left|\Delta u^{(k+1)}(\cdot,x)\right|_t^2 \le \frac{\left[\frac{1}{2}(T^2+4)L^2(t,x)\right]^{k+1} t^{2(k+1)}}{(k+1)!}\left|\Delta u^{(0)}(\cdot,x)\right|_t^2, \quad k=0,1,\ldots.
$$

Thus the sequence $\left(u^{(k)}\right)_{k\in\mathbb{N}}$ defined by (12) converges to the solution u of equation (1). This completes the proof. □

Remark 2. The first increment $\Delta u^{(0)}$ in the scheme (11) is estimated in L^2 by some function $C_0(t,x)$.

5. **First-order convergence of Newton's method.** We formulate Newton's scheme for problem (1) which starts from the function $u^{(0)}$ given by (11).

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u^{(k+1)} - \frac{\partial^2}{\partial x^2}u^{(k+1)} =& f\left(t,x,u^{(k)}(\cdot,x)\right) + f_v\left(t,x,u^{(k)}(\cdot,x)\right)\Delta u^{(k)}(\cdot,x)\\ &+\left[g\left(t,x,u^{(k)}(\cdot,x)\right) + g_v\left(t,x,u^{(k)}(\cdot,x)\right)\Delta u^{(k)}(\cdot,x)\right]\dot{W}_t\\ &\text{for}\quad (t,x)\in[0,T]\times\mathbb{R}, \qquad (13)\\ u^{(k)}(0,x) =& \phi(x), \quad \text{for}\quad x\in\mathbb{R},\\ \frac{\partial}{\partial t}u^{(k)}(0,x) =& \psi(x), \quad \text{for}\quad x\in\mathbb{R}.\end{aligned}$$

We have the following differential equation for the increments $\Delta u^{(k+1)}$:

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}\Delta u^{(k+1)} - \frac{\partial^2}{\partial x^2}\Delta u^{(k+1)}\\ =\;& f\left(t,x,u^{(k+1)}(\cdot,x)\right) - f\left(t,x,u^{(k)}(\cdot,x)\right) - f_v\left(t,x,u^{(k)}(\cdot,x)\right)\Delta u^{(k)}(\cdot,x)\\ +\;& f_v\left(t,x,u^{(k+1)}(\cdot,x)\right)\Delta u^{(k+1)}(\cdot,x) \qquad (14)\\ +\;& \left[g\left(t,x,u^{(k+1)}(\cdot,x)\right) - g\left(t,x,u^{(k)}(\cdot,x)\right) - g_v\left(t,x,u^{(k)}(\cdot,x)\right)\Delta u^{(k)}(\cdot,x)\right]\dot{W}_t\\ +\;& g_v\left(t,x,u^{(k+1)}(\cdot,x)\right)\Delta u^{(k+1)}(\cdot,x)\dot{W}_t \quad \text{for}\quad (t,x)\in[0,T]\times\mathbb{R}\end{aligned}$$

with zero initial values.

Theorem 5.1. *Suppose that there exists a function $L:[0,T]\times\mathbb{R}\to\mathbb{R}_+$ increasing w.r.t. cones such that*

$$|f_v(t,x)|_t^* \le L(t,x), \quad |g_v(t,x)|_t^* \le L(t,x) \quad \text{for } (t,x)\in[0,T]\times\mathbb{R}, \qquad (15)$$

which implies the Lipschitz condition (2) *and* (3) *for f and g. Then the Newton sequence $\left(u^{(k)}\right)_{k\in\mathbb{N}}$ defined by* (13) *converges to the unique solution u of equation* (1) *in the following sense*

$$\lim_{k\to\infty}\left|u^{(k)}(\cdot,x) - u(\cdot,x)\right|_t = 0 \qquad \text{for } t\in[0,T].$$

Proof. We show that $(u^{(k)})_{k\in\mathbb{N}}$ satisfies the Cauchy condition with respect to the norm $|\cdot|_t$. We apply Lemma 3.2 with

$$\begin{aligned}\alpha^{(1)}(t,x) &= f\left(t,x,u^{(k+1)}(\cdot,x)\right) - f\left(t,x,u^{(k)}(\cdot,x)\right) - f_v\left(t,x,u^{(k)}(\cdot,x)\right)\Delta u^{(k)}(\cdot,x),\\ \alpha^{(2)}(t,x) &= g\left(t,x,u^{(k+1)}(\cdot,x)\right) - g\left(t,x,u^{(k)}(\cdot,x)\right) - g_v\left(t,x,u^{(k)}(\cdot,x)\right)\Delta u^{(k)}(\cdot,x),\\ \mathcal{T}^{(1)}(t,x) &= f_v\left(t,x,u^{(k+1)}(\cdot,x)\right),\\ \mathcal{T}^{(2)}(t,x) &= g_v\left(t,x,u^{(k+1)}(\cdot,x)\right).\end{aligned}$$

Hence and by (2), (3), (15) we get

$$\begin{aligned}\left|\Delta u^{(k+1)}(\cdot,x)\right|_t^2 &\le 2(T^2+N)K_1(t,x)\int_{C_{t,x}} 2L^2(s,y)\left|\Delta u^{(k)}(\cdot,y)\right|_s^2 dyds\\ &\le 4(T^2+N)K_1(t,x)L^2(t,x)\int_{C_{t,x}}\left|\Delta u^{(k)}(\cdot,y)\right|_s^2 dyds.\end{aligned}$$

Thus

$$\left|\Delta u^{(k+1)}(\cdot,x)\right|_t^2 \le \frac{J^{k+1}(t,x)t^{2(k+1)}}{(k+1)!}\left|\Delta u^{(0)}(\cdot,x)\right|_t^2, \quad k=0,1,\dots,$$

where

$$J(t,x) = 4(T^2+N)K_1(t,x)L^2(t,x)$$

and $K_1(t,x)$ is given by (10). Thus the Newton sequence $\left(u^{(k)}\right)_{k\in\mathbb{N}}$ defined by (13) converges to the solution u of equation (1). □

6. **Probabilistic second-order convergence of Newton's method.** The following theorem establishes a second-order convergence of Newton's method in a probabilistic sense.

Theorem 6.1. *Assume that there exists a function* $L : [0,T]\times\mathbb{R}\to\mathbb{R}_+$ *increasing w.r.t. cones such that*

$$|f_v(t,x)|_t^* \le L(t,x), \quad |g_v(t)|_{t,x}^* \le L(t,x) \quad \text{for } (t,x)\in[0,T]\times\mathbb{R},$$

which implies the Lipschitz condition (2) *and* (3) *for* f *and* g*. Suppose additionally that there exists a function* $M : [0,T]\times\mathbb{R}\to\mathbb{R}_+$ *increasing w.r.t. cones such that*

$$|f_v(t,x,u(\cdot,x)) - f_v(t,x,\bar u(\cdot,x))|_t^* \le M(t,x)\sup_{0\le\tilde t\le t}|u(\tilde t,x)-\bar u(\tilde t,x)|, \tag{16}$$

$$|g_v(t,x,u(\cdot,x)) - g_v(t,x,\bar u(\cdot,x))|_t^* \le M(t,x)\sup_{0\le\tilde t\le t}|u(\tilde t,x)-\bar u(\tilde t,x)|. \tag{17}$$

Then for any $T>0$ *there exists a function* $H : [0,T]\times\mathbb{R}\to\mathbb{R}_+$ *such that*

$$P\left(\sup_{0\le s\le t}|\Delta u^{(k)}(s,x)|\le\rho \quad\Rightarrow\quad \sup_{0\le s\le t}|\Delta u^{(k+1)}(s,x)|\le R\rho^2\right) \ge 1 - H(t,x)R^{-2}$$

for all $R>0$, $0<\rho\le 1$, $k=0,1,2,\dots$

Proof. Let a cone C_{t_0,x_0} be fixed. Define the sets

$$A_{\rho,t}^{(k)} = \left\{\omega : \sup_{0\le\tilde t\le t}|\Delta u^{(k)}(\tilde t,x)|\le\rho,\ 0\le s\le t,\ x_0-(t_0-s)\le x\le x_0+(t_0-s)\right\}$$

for $0<\rho\le 1$, $0\le t\le T$, $k=0,1,2,\dots.$ We consider the sequence $(\Delta u^{(k)})_{k\in\mathbb{N}}$ restricted to the sets $A_{\rho,t}^{(k)}$. For this reason we apply d'Alembert formula to equation (14) and multiply it by $\mathbf{1}_{A_{\rho,t}^{(k)}}$, the characteristic function of the set $A_{\rho,t}^{(k)}$, to obtain

$$\begin{aligned}
&\mathbf{1}_{A_{\rho,t}^{(k)}}\Delta u^{(k+1)}(\cdot,x)\\
&\quad= \frac12\mathbf{1}_{A_{\rho,t}^{(k)}}\int_{C_{t,x}}\left(T_f^{(k)}(s,y)+f_v^{(k+1)}(s,y)\Delta u^{(k+1)}(s,y)\right)dyds\\
&\quad+ \frac12\mathbf{1}_{A_{\rho,t}^{(k)}}\int_0^t\left(\int_{x-(t-s)}^{x+(t-s)}\left(T_g^{(k)}(s,y)+g_v^{(k+1)}(s,y)\Delta u^{(k+1)}(s,y)\right)dy\right)dW_s
\end{aligned}$$

for $(t,x) \in [0,T] \times \mathbb{R}$, where

$$\begin{aligned}
\Delta f^{(k)}(t,x) &= f\left(t,x,u^{(k+1)}(\cdot,x)\right) - f\left(t,x,u^{(k)}(\cdot,x)\right),\\
\Delta g^{(k)}(t,x) &= g\left(t,x,u^{(k+1)}(\cdot,x)\right) - g\left(t,x,u^{(k)}(\cdot,x)\right),\\
f_v^{(k)}(t,x) &= f_v\left(t,x,u^{(k)}(\cdot,x)\right), \quad g_v^{(k)}(t,x) = g_v\left(t,x,u^{(k)}(\cdot,x)\right),\\
T_f^{(k)}(t,x) &= \Delta f^{(k)}(t,x) - f_v^{(k)}(t,x)\Delta u^{(k)}(t,x)\\
T_g^{(k)}(t,x) &= \Delta g^{(k)}(t,x) - g_v^{(k)}(t,x)\Delta u^{(k)}(t,x)
\end{aligned}$$

If

$$\begin{aligned}
F(t,x) &= T_f^{(k)}(t,x) + f_v^{(k+1)}(t,x)\Delta u^{(k+1)}(\cdot,x),\\
G(t,x) &= T_g^{(k)}(t,x) + g_v^{(k+1)}(t,x)\Delta u^{(k+1)}(\cdot,x),
\end{aligned}$$

then we have

$$\begin{aligned}
\left|\mathbf{1}_{A_{\rho,t}^{(k)}}\Delta u^{(k+1)}(\cdot,x)\right|_t^2 &\le \frac{1}{2}\cdot 2\,\mathbb{E}\left[\mathbf{1}_{A_{\rho,t}^{(k)}}\sup_{0\le\tilde{t}\le t}\left|\int_{C_{\tilde{t},x}} F^{(k)}(s,y)dyds\right|^2\right]\\
&+ \frac{1}{2}\cdot 2\,\mathbb{E}\left[\mathbf{1}_{A_{\rho,t}^{(k)}}\sup_{0\le\tilde{t}\le t}\left|\int_0^{\tilde{t}}\int_{x-(\tilde{t}-s)}^{x+(\tilde{t}-s)} G^{(k)}(s,y)dydW_s\right|^2\right]\\
&:= I_1 + I_2.
\end{aligned}$$

Notice that for $s \le t$ we have the monotonicity property

$$A_{\rho,t}^{(k)} \subset A_{\rho,s}^{(k)} \Rightarrow \mathbf{1}_{A_{\rho,t}^{(k)}} = \mathbf{1}_{A_{\rho,t}^{(k)}}\mathbf{1}_{A_{\rho,s}^{(k)}}.$$

Hence

$$\begin{aligned}
I_1 &\le \mathbb{E}\left[\mathbf{1}_{A_{\rho,t}^{(k)}}\sup_{0\le\tilde{t}\le t}\left|\int_{C_{\tilde{t},x}}\mathbf{1}_{A_{\rho,s}^{(k)}}F(s,y)dyds\right|^2\right]\\
&\le \mathbb{E}\left[\sup_{0\le\tilde{t}\le t}\left|\int_{C_{\tilde{t},x}}\mathbf{1}_{A_{\rho,s}^{(k)}}F(s,y)dyds\right|^2\right].
\end{aligned}$$

By the Schwarz inequality we obtain

$$\begin{aligned}
I_1 &\le t^2\,\mathbb{E}\left[\sup_{0\le\tilde{t}\le t}\int_{C_{\tilde{t},x}}\mathbf{1}_{A_{\rho,s}^{(k)}}|F(s,y)|^2dyds\right]\\
&\le 2t^2\,\mathbb{E}\left[\sup_{0\le\tilde{t}\le t}\int_{C_{\tilde{t},x}}\mathbf{1}_{A_{\rho,s}^{(k)}}\left|\Delta f^{(k)}(s,y) - f_v^{(k)}(s,y)\Delta u^{(k)}(s,y)\right|^2dyds\right]\\
&+ 2t^2\,\mathbb{E}\left[\sup_{0\le\tilde{t}\le t}\int_{C_{\tilde{t},x}}\mathbf{1}_{A_{\rho,s}^{(k)}}\left|f_v^{(k+1)}(s,y)\Delta u^{(k+1)}(s,y)\right|^2dyds\right].
\end{aligned}$$

From the fundamental theorem of calculus and by (16) it follows that

$$\begin{aligned}
&\left|\Delta f^{(k)}(t,x) - f_v^{(k)}(t,x)\Delta u^{(k)}(\cdot,x)\right| \\
&\le \sup_{0\le s\le t}\left|\Delta u^{(k)}(s,x)\right| \\
&\quad\times \int_0^1 \left|f_v\left(s,x,u^{(k)}(s,x)+\theta\Delta u^{(k)}(s,x)\right) - f_v\left(s,x,u^{(k)}(s,x)\right)\right|_s^* d\theta \\
&\le \frac{1}{2}M(t,x)\sup_{0\le s\le t}\left|\Delta u^{(k)}(s,x)\right|^2 .
\end{aligned}$$

Hence by (15) we get

$$\begin{aligned}
I_1 &\le 2t^2\frac{1}{4}\,\mathbb{E}\left[\sup_{0\le\tilde t\le t}\int_{C_{\tilde t,x}} \mathbf{1}_{A^{(k)}_{\rho,s}}\, M^2(s,y)\sup_{0\le\tilde s\le s}\left|\Delta u^{(k)}(\tilde s,y)\right|^4 dyds\right] \\
&+ 2t^2\,\mathbb{E}\left[\sup_{0\le\tilde t\le t}\int_{C_{\tilde t,x}} \mathbf{1}_{A^{(k)}_{\rho,s}}\, L^2(s,y)\sup_{0\le\tilde s\le s}\left|\Delta u^{(k+1)}(\tilde s,y)\right|^2 dyds\right] \\
&\le \frac{1}{2}t^2\,\mathbb{E}\left[\int_{C_{t,x}} \mathbf{1}_{A^{(k)}_{\rho,s}}\, M^2(s,y)\sup_{0\le\tilde s\le s}\left|\Delta u^{(k)}(\tilde s,y)\right|^4 dyds\right] \\
&+ 2t^2\,\mathbb{E}\left[\int_{C_{t,x}} \mathbf{1}_{A^{(k)}_{\rho,s}}\, L^2(s,y)\sup_{0\le\tilde s\le s}\left|\Delta u^{(k+1)}(\tilde s,s)\right|^2 dyds\right].
\end{aligned}$$

Recall that $|\Delta u^{(k)}(\tilde s,y)|\le\rho$ on $A^{(k)}_{\rho,s}$ for $0\le\tilde s\le s$. Thus

$$I_1 \le \frac{1}{2}t^2\rho^4\int_{C_{t,x}} M^2(s,y)dyds + 2t^2\int_{C_{t,x}} L^2(s,y)\left|\mathbf{1}_{A^{(k)}_{\rho,s}}\,\Delta u^{(k+1)}(\cdot,y)\right|_s^2 dyds.$$

By the monotonicity property of $\mathbf{1}_{A^{(k)}_{\rho,t}}$ and the Doob inequality we have

$$\begin{aligned}
I_2 &\le \mathbb{E}\left[\sup_{0\le\tilde t\le t}\left|\int_0^{\tilde t}\left(\int_{x-(\tilde t-s)}^{x+(\tilde t-s)} \mathbf{1}_{A^{(k)}_{\rho,s}}\, G(s,y)dy\right)dW_s\right|^2\right] \\
&\le 4\,\mathbb{E}\left[\int_{C_{t,x}} \mathbf{1}_{A^{(k)}_{\rho,s}}\,|G(s,y)|^2\,dyds\right].
\end{aligned}$$

Hereafter we estimate I_2 similarly as I_1 and get

$$I_2 \le \frac{1}{2}\cdot 2\rho^4\int_{C_{t,x}} M^2(s,y)dyds + 2\cdot 4\int_{C_{t,x}} L^2(s,y)\left|\mathbf{1}_{A^{(k)}_{\rho,s}}\,\Delta u^{(k+1)}(\cdot,y)\right|_s^2 dyds.$$

Finally we have the estimate

$$\begin{aligned}
\left|\mathbf{1}_{A^{(k)}_{\rho,t}}\,\Delta u^{(k+1)}(\cdot,x)\right|_t^2 &\le \frac{1}{2}\rho^4(T^2+2)\int_{C_{t,x}} M^2(s,y)dyds \\
&+ 2(T^2+4)\int_{C_{t,x}} L^2(s,y)\left|\mathbf{1}_{A^{(k)}_{\rho,s}}\,\Delta u^{(k+1)}(\cdot,y)\right|_s^2 dyds.
\end{aligned}$$

Applying Lemma 3.2 we get

$$\begin{aligned}\left|\mathbf{1}_{A^{(k)}_{\rho,t}}\,\Delta u^{(k+1)}(\cdot,x)\right|^2_t &\le K_1(t,x)\rho^4(T^2+4)\int_{C_{t,x}} M^2(s,y)dyds \\ &\le \frac{1}{2}\rho^4 T^2(T^2+4)M^2(t,x)\exp(4(T^2+4)t^2L^2(t,x)).\end{aligned}$$

Hence

$$\left|\mathbf{1}_{A^{(k)}_{\rho,t}}\,\Delta u^{(k+1)}(\cdot,x)\right|^2_t \le \frac{1}{2}\rho^4 T^2(T^2+4)M^2(t,x)\exp(4(T^2+4)t^2L^2(t,x)).$$

The Chebyshev inequality yields

$$\begin{aligned}&P\left(\sup_{0\le s\le t}|\Delta u^{(k)}(s,x)|\le\rho \quad\wedge\quad \sup_{0\le s\le t}|\Delta u^{(k+1)}(s,x)|>R\rho^2\right)\\ &= P\left(\mathbf{1}_{A^{(k)}_{\rho,t}}\sup_{0\le s\le t}|\Delta u^{(k+1)}(s,x)|>R\rho^2\right) \le \frac{1}{R^2\rho^4}\left|\mathbf{1}_{A^{(k)}_{\rho,t}}\,\Delta u^{(k+1)}(\cdot,x)\right|^2_t\\ &\le \frac{1}{2}T^2(T^2+4)M^2(t,x)e^{4(T^2+4)t^2L^2(t,x)}R^{-2} = H(t,x)R^{-2}.\end{aligned}$$

Thus we have

$$P\left(\sup_{0\le s\le t}|\Delta u^{(k)}(s,x)|\le\rho \quad\Rightarrow\quad \sup_{0\le s\le t}|\Delta u^{(k+1)}(s,x)|\le R\rho^2\right) \ge 1-H(t,x)R^{-2}$$

for all $R>0$, $0<\rho\le 1$, $k=0,1,2,\ldots$ □

Remark 3. All results of the paper carry over to SWE on an interval with periodic boundary conditions. These results are just simple consequences of our theorems. In the case of uniform Dirichlet boundary conditions ($u(t,0)=u(t,1)=0$) one can extend the initial data $\phi(x)$, $\psi(x)$ to 2-periodic odd functions $\tilde{\phi}(x)$, $\tilde{\psi}(x)$. In the case of Neumann boundary conditions $\left(\frac{\partial}{\partial x}u(t,0)=\frac{\partial}{\partial x}u(t,1)=0\right)$ we use the reflections with respect to lines $x=k$ for $k\in\mathbb{Z}$.

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