

DYNAMICS OF A DIFFUSIVE AGE-STRUCTURED HBV MODEL WITH SATURATING INCIDENCE

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ABSTRACT. In this paper, we propose and investigate an age-structured hepatitis B virus (HBV) model with saturating incidence and spatial diffusion where the viral contamination process is described by the age-since-infection. We first analyze the well-posedness of the initial-boundary values problem of the model in the bounded domain $\Omega \subset \mathbb{R}^n$ and obtain an explicit formula for the basic reproductive number R_0 of the model. Then we investigate the global behavior of the model in terms of R_0 : if $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable, whereas if $R_0 > 1$, then the infected steady state is globally asymptotically stable. In addition, when $R_0 > 1$, by constructing a suitable Lyapunov-like functional decreasing along the traveling waves to show their convergence towards two steady states as t tends to $\pm\infty$, we prove the existence of traveling wave solutions. Numerical simulations are provided to illustrate the theoretical results.

1. Introduction. Hepatitis B is a potentially life-threatening liver infection caused by the hepatitis B virus (HBV). It is a major global health problem as it can cause chronic infection and puts people at high risk of death from cirrhosis and liver cancer [2, 51]. Chronic HBV infection is often the result of exposure early in life, leading to viral persistence in the absence of strong humoral and/or cellular immune responses [12]. The prevalence of chronic HBV infection in areas of high endemicity is at least 8% with 10-15% prevalence in Africa/Far East. As of 2010, China has 120 million infected people, followed by India and Indonesia with 40 million and

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12 million, respectively [26]. According to World Health Organization (WHO), an estimated 600,000 people die every year related to the hepatitis B infection [17]. In an effort to model HBV infection dynamics, Nowak et al. [36] introduced a basic viral infection model within-host. After then, dynamical properties of HBV models have been studied by many authors, see, for example, [3, 42, 37].

Notice that for HBV infection, susceptible host cells and infected host cells are both hepatocytes and cannot move under normal conditions, but viruses can move freely in liver. Based on the basic model of [36, 3, 42], Wang and Wang [48] proposed the following system to simulate HBV infection with spatial dependence:

$$\begin{cases} \frac{\partial u}{\partial t} = \lambda - au(x, t) - \beta u(x, t)v(x, t), \\ \frac{\partial w}{\partial t} = \beta u(x, t)v(x, t) - bw(x, t), \\ \frac{\partial v}{\partial t} = D\Delta v + kw(x, t) - dv(x, t), \end{cases} \quad (1)$$

where $u(x, t)$, $w(x, t)$ and $v(x, t)$ represent the densities of uninfected cells, infected cells and free virus at location x and time t , respectively; susceptible cells are produced at rate λ , die at rate au , and become infected at rate βuv ; infected cells are produced at rate βuv and die at rate bw ; free viruses are produced from infected cells at rate kw and are removed at rate dv . The parameters $\lambda, a, \beta, b, k, d$ are all positive constants. $\Delta v = \sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}$, $n = 1, 2$ or 3 is the Laplacian operator, D is the diffusion coefficient. For system (1), Wang and Wang [48] established the existence of traveling waves.

Considering the fact that there exists a intracellular time delay between infection of a cell and production of new virus particles, Wang *et al.* [49] modified system (1) as a diffusive HBV model with delay (HBV production lags by a delay τ behind the infection of a hepatocyte), and studied its stability when the space is assumed to be homogeneous and inhomogeneous. For the same model, the traveling wave solutions results are established in Gan *et al.* [13]. The readers can also refer to [18, 52, 27, 28] and the references cited therein for related studies.

Note that the rate of infection in most virus infection models is assumed to be bilinear in the free virus v and the uninfected cells u . However, experiments reported in [11] have shown that the infection rate of microparasitic infections is an increasing function of the parasite dose, and is usually sigmoidal in shape. Thus, it is reasonable to assume that the infection rate of HBV is given by the saturation response [45, 53]. Xu and Ma [53] considered a diffusive HBV infection model with time delay and saturating incidence,

$$\begin{cases} \frac{\partial u}{\partial t} = s - \mu u(x, t) - \frac{\beta u(x, t)v(x, t)}{1 + \alpha v(x, t)}, \\ \frac{\partial w}{\partial t} = \frac{\beta u(x, t - \tau)v(x, t - \tau)}{1 + \alpha v(x, t - \tau)} - \delta w(x, t), \\ \frac{\partial v}{\partial t} = D\Delta v + N\delta w(x, t) - dv(x, t). \end{cases} \quad (2)$$

They obtained the global stability results of the infected and the uninfected steady states. More recently, Zhang and Xu [54] considered a diffusive HBV model with delayed Beddington-DeAngelis response, they showed that there exist traveling wave solutions connecting the infected steady state and the uninfected steady state when

the basic reproduction number is larger than unit. McCluskey and Yang [34] proposed a diffusive virus dynamics model with general incidence function and time delay, and they obtained the global stability results of the model.

Recently, the study of age-structured models has attracted many authors' attention. Since in 1927 Kermack and McKendrick [23, 24, 25] introduced the age-since-infection in some epidemic models, then the age variable is more and more widely used to describe either the age of individuals or the age since infection (see for example, [46, 30, 32, 6, 29]). In virus dynamics models, recent observations suggest that the death rate of infected cells should vary over their life span [1, 14] and the virion production rate is initially low and increases with the age of the infected cell [41] (where age is defined as the time that has passed since the infection of the cell). Nelson *et al.* [35] developed an age-structured model of HIV infection, in which the production rate of viral particles and the death rate of productively infected cells are allowed to vary and depend on two general functions of age, respectively. The authors studied the local stability of the model. Its global stability results were proved by Huang *et al.* in [22]. Instead of using the bilinear infection rate kTV in [35], recently, Wang *et al.* [50] proposed a class of age-infection HIV models with nonlinear infection rate type $F(T)G(V)$, they also obtained the global stability results of the model. One can refer to [43, 39, 55] and the references therein for more related studies on age-structured viral infection models. We can see from these studies that the age since infection plays an important role in the study of viral infection models.

Motivated by above works, by incorporating an age-since-infection and saturating incidence into model (1), in this paper, we consider the following model:

$$\begin{cases} \frac{\partial u}{\partial t} = s - \mu u(x, t) - \frac{\beta u(x, t)v(x, t)}{1 + \alpha v(x, t)}, \\ \frac{\partial w(x, \theta, t)}{\partial \theta} + \frac{\partial w(x, \theta, t)}{\partial t} = -\delta(\theta)w(x, \theta, t), \\ \frac{\partial v}{\partial t} = D\Delta v + \int_0^\infty p(\theta)w(x, \theta, t)d\theta - dv(x, t) \end{cases} \quad (3)$$

for $t > 0$, $x \in \Omega$, with homogeneous Neumann boundary conditions

$$\frac{\partial v}{\partial \nu} = 0, t > 0, x \in \partial\Omega, \quad (4)$$

and the initial and boundary conditions:

$$\begin{cases} w(x, 0, t) = \frac{\beta u(x, t)v(x, t)}{1 + \alpha v(x, t)}, x \in \bar{\Omega}, \\ u(x, 0) = \vartheta_{10}(x), w(x, \theta, 0) = \vartheta_{20}(x, \theta), v(x, 0) = \vartheta_{30}(x), x \in \bar{\Omega}. \end{cases} \quad (5)$$

In model (3)-(5), θ is the infection age, the time that has elapsed since an HBV virion has penetrated cell; $w(x, \theta, t)$ is the density of infected cells of infection age θ at location x and at time t . $\beta uv/(1 + \alpha v)$ is the saturating incidence of HBV infection, where $\alpha, \beta > 0$. Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. The boundary conditions in (4) imply that the virus populations do not move across the boundary $\partial\Omega$. The positive constants s and μ are the recruitment and death rate of uninfected cells,

respectively. The functions $p(\theta)$ and $\delta(\theta)$ denote respectively the infection age-specific viral production rate and the age-specific death rate of productively infected cells.

One of the most challenging problems in the analysis of models for viral infection is determining sharp conditions for the global stability of steady states. Yet such results are necessary for derivation of parameter thresholds for clearing infections. On the other hand, for age-structured reaction-diffusion models, traveling wave solutions are important since in many situations they determine the long-term behavior of other solutions and account for propagation of patterns and domain invasion of species in population biology. Recently, there has been some progress on the study of traveling wave solution for age-structured reaction-diffusion equations, see, for example [7, 8, 9, 44, 10]. But to the best of our knowledge, there are few results on the global stability for age-structured reaction-diffusion equations in the literatures.

In the present paper, we are mainly interested in the local stability as well as the global stability of the two steady states and the existence of traveling wave solutions connecting the two steady states of system (3)-(5). The organization of this paper is as follows. In the next section, we present some basic results of model (3)-(5). In Section 3, we first discuss the local stability of the uninfected and infected steady states by analyzing the corresponding characteristic equations. Then by constructing Lyapunov like functionals, we further investigate the global stability of the two steady states. In Section 4, by constructing a pair of upper and lower solutions and following the frameworks established in [13] and [10], we prove the existence of traveling wave solutions for system (3) when $R_0 > 1$, and the non-existence of traveling wave solutions when $R_0 < 1$. Some numerical simulations are performed in Section 5 to illustrate the main results. Finally, a brief discussion is presented in the last section.

2. The well-posedness of system (3). Throughout this work, we make the following assumption.

Assumption 2.1. *The maps $\theta \rightarrow \delta(\theta)$, $\theta \rightarrow p(\theta)$ are almost everywhere bounded, and belong to $L^{\infty}((0, +\infty), \mathbb{R}) \setminus \{0_{L^{\infty}}\}$. Moreover, $\delta(\theta) > \delta_{min}$ for some positive number δ_{min} .*

Since $\delta(\theta)$ denotes the infection age-specific death rate of productively infected cells, it is reasonable to assume that $\delta_{min} \geq \mu$. Then we study the well-posedness for the initial-boundary value problem of system (3)-(5).

Defined a threshold value as

$$R_0 = \frac{\beta s}{\mu d} \int_0^{+\infty} p(\theta) e^{-\int_0^{\theta} \delta(\tau) d\tau} d\theta,$$

R_0 is called the basic reproduction number of system (3)-(5). Denote that

$$\Pi(\theta) = p(\theta) e^{-\int_0^{\theta} \delta(\tau) d\tau},$$

then the total number of viral particles produced by an infected cell in its life span equals

$$\mathcal{K} \equiv \int_0^{\infty} \Pi(\theta) d\theta,$$

which is finite since that $\int_0^{+\infty} p(\theta) d\theta$ is finite.

By a direct computation, we get the following conclusion.

Lemma 2.1. *System (3)-(5) has a uninfected steady state $E^0 = (u^0, 0, 0)$ with $u^0 = s/\mu$. If $R_0 > 1$, then system (3)-(5) has a unique infected steady state $E^* = (u^*, w^*(\theta), v^*)$, where*

$$u^* = \frac{\alpha s \mathcal{K} + d}{\mathcal{K}(\beta + \alpha\mu)}, \quad v^* = \frac{\mu}{\alpha\mu + \beta}(R_0 - 1), \quad w^*(\theta) = \frac{\beta u^* v^*}{1 + \alpha v^*} e^{-\int_0^\theta \delta(\tau) d\tau}.$$

Let $\mathbb{C} := BUC(\bar{\Omega}, \mathbb{R})$ be the set of all bounded and uniformly continuous functions from $\bar{\Omega}$ to \mathbb{R} , $\mathbb{C}_+ := BUC(\bar{\Omega}, \mathbb{R}_+)$. $\mathbb{C}^2(\bar{\Omega})$ denotes the set of all functions in \mathbb{C} whose derivatives up to order 2 and $\mathbb{C}_+^2(\bar{\Omega})$ denotes the set of all functions in \mathbb{C}_+ whose derivatives up to order 2.

$X := \mathbb{C} \times L^1((0, +\infty), \mathbb{C}) \times \mathbb{C}^2(\bar{\Omega})$, and $X_+ = \mathbb{C}_+ \times L^1((0, +\infty), \mathbb{C}_+) \times \mathbb{C}_+^2(\bar{\Omega})$. Then X_+ is a closed cone of X and induces a partial ordering on X . Moreover, we define a norm

$$\|\chi\|_X = \sup_{x \in \bar{\Omega}} |(\chi_1(x), \|\chi_2(x, \cdot)\|_{L^1}, \chi_3(x))|,$$

where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^3 . It then follows that $(X, \|\cdot\|_X)$ is a Banach lattice.

In the following, we reformulate system (3)-(5) as a inhomogeneous Cauchy problem in order to apply integrated semigroup theory. To this end, we structure a space $\widehat{X} := \mathbb{C} \times \mathbb{C} \times L^1((0, +\infty), \mathbb{C}) \times \mathbb{C}^2(\bar{\Omega})$ endowed with the usual supremum norm. Then we set

$$\begin{aligned} \widehat{X}_0 &:= \mathbb{C} \times \{0_{\mathbb{C}}\} \times L^1((0, +\infty), \mathbb{C}) \times \mathbb{C}^2(\bar{\Omega}), \\ \widehat{X}_+ &:= \mathbb{C}_+ \times \mathbb{C}_+ \times L^1_+((0, +\infty), \mathbb{C}) \times \mathbb{C}_+^2(\bar{\Omega}), \\ \widehat{X}_{0+} &:= \widehat{X}_0 \cap \widehat{X}_+. \end{aligned}$$

Let

$$A \begin{pmatrix} u \\ \begin{pmatrix} 0 \\ w \\ v \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\mu u \\ -w(0) \\ -\frac{\partial w}{\partial \theta} - \delta(\theta)w \\ D\Delta v - dv \end{pmatrix},$$

with $\text{Dom}(A) = \mathbb{C} \times \{0_{\mathbb{C}}\} \times W^{1,1}((0, +\infty), \mathbb{C}) \times W^{2,2}(\bar{\Omega}) \subset \widehat{X}$, where $W^{1,1}$ is a Sobolev space and $W^{2,2}(\bar{\Omega})$ denotes the Sobolev space of functions in $BUC(\bar{\Omega}, \mathbb{R}_+)$ whose distributional derivatives up to order 2 are represented by elements in $BUC(\bar{\Omega}, \mathbb{R}_+)$. By Theorem 6.2 in Thieme [47], then the linear operator A generates a nondegenerate integrated semigroup $\mathcal{T}(t)$ on \widehat{X} . We define a map $F : [0, +\infty) \rightarrow \widehat{X}$ by

$$F(t) = \begin{pmatrix} s - \frac{\beta u(\cdot, t)v(\cdot, t)}{1 + \alpha v(\cdot, t)} \\ \begin{pmatrix} \frac{\beta u(\cdot, t)v(\cdot, t)}{1 + \alpha v(\cdot, t)} \\ 0_{L^1} \end{pmatrix} \\ \int_0^{+\infty} p(\theta)w(\cdot, \theta, t)d\theta \end{pmatrix},$$

which is continuous with respect to t and therefore F belongs to $L^1((0, +\infty), \widehat{X})$.

In order to simplify notations, we set $U(t) = \begin{pmatrix} u(\cdot, t) \\ 0 \\ w(\cdot, \cdot, t) \\ v(\cdot, t) \end{pmatrix}$. Then we can

rewrite system (3)-(5) as the following inhomogeneous abstract Cauchy problem

$$\frac{dU(t)}{dt} = AU(t) + F(t); t > 0, \tag{6}$$

with $U(0) = U^0 \in \widehat{X}_{0+}$. From Theorem 6.5 in Thieme [47] or the theory developed in [33], we can have that there exists an unique continuous solution to (6) with initial values in \widehat{X}_{0+} . Furthermore, set $\eta = (\eta_1, \eta_2(\cdot), \eta_3)$ and define

$$\mathcal{D} := \{\eta \in X : \mathbf{0} \leq \eta(x) \leq \mathbf{M}, \forall x \in \Omega\}, \tag{7}$$

with $\mathbf{0} = (0, 0_{L^1}, 0)$ and $\mathbf{M} = (M_1, M_2(\theta), M_3)$, where

$$M_1 = \frac{s}{\mu}, M_2(\theta) = \frac{\beta s}{\mu \alpha} e^{-\int_0^\theta \delta(\tau) d\tau}, M_3 = \frac{R_0}{\alpha \delta_{min}}.$$

Define the solution semiflow $\Psi(t) : X_+ \rightarrow X_+$ of system (3)-(5) by

$$\Psi(t)\phi = (u(\cdot, t, \phi), w(\cdot, \cdot, t, \phi), v(\cdot, t, \phi)), \quad \forall t \geq 0,$$

$\phi = \vartheta_{10}(x), \vartheta_{20}(x, \theta), \vartheta_{30}(x) \in X_+$. Then \mathcal{D} is positively invariant for $\Psi(t)$ in the sense that

$$\Psi(t)(\phi) \in \mathcal{D}, \quad \forall t \geq 0, \phi \in \mathcal{D}.$$

The following lemma then follows from Theorem 3.4 developed by Redlinger [40].

Lemma 2.2. *For any given initial values $(\vartheta_{10}(x), \vartheta_{20}(x, \theta), \vartheta_{30}(x)) \in \mathcal{D}$, system (3)-(5) has exactly one regular solution $U(x, t) = (u(x, t), w(x, \cdot, t), v(x, t))$ satisfying $U(x, t) \in \mathcal{D}$ on $[0, +\infty)$.*

Since the first two equations in system (3)-(5) have no diffusion terms, its solution map $\Psi(t)$ is not compact. In order to deal with this case, we use the Kuratowski measure of noncompactness (see [5]), \mathcal{K} , which is defined by

$$\mathcal{K}(B) := \inf\{r : B \text{ has a finite cover of diameter } < r\}, \tag{8}$$

for any bounded set B . We set $\mathcal{K}(B) = \infty$ whenever B is unbounded. It is easy to see that B is precompact (i.e., \bar{B} is compact) if and only if $\mathcal{K}(B) = 0$. Then we have the following results.

Lemma 2.3. *The solution semiflow $\Psi(t)$ is \mathcal{K} -contracting in the sense that*

$$\lim_{t \rightarrow +\infty} \mathcal{K}(\Psi(t)(B)) = 0$$

for any bounded set $B \subset X_+$.

Proof. See Appendix. □

Theorem 2.4. *The solution semiflow $\Psi(t)$ admits a global attractor on X_+ .*

Proof. By Lemma 2.3, it follows that $\Psi(t)$ is \mathcal{K} -contracting on X_+ . By Lemma 2.2, it follows that $\Psi(t)$ is point dissipative on X_+ , and forward orbits of bounded subsets of X_+ for $\Psi(t)$ are bounded. By Theorem 2.6 in [31], $\Psi(t)$ has a global attractor that attracts each bounded set in X_+ . □

3. Stability of steady states. In this section, we first discuss the local stability of the steady states E^0 and E^* by analyzing the corresponding characteristic equations. Then by use of the method of constructing Lyapunov-like functionals, we investigate the global stability of the two steady states.

3.1. **Local stability analysis of steady states.** Using integration, $w(x, \theta, t)$ satisfies the following Volterra formulation:

$$w(x, \theta, t) = \begin{cases} \frac{\beta u(x, t - \theta)v(x, t - \theta)}{1 + \alpha v(x, t - \theta)} e^{-\int_0^\theta \delta(\tau) d\tau}, & \text{if } t \geq \theta, \\ w(x, \theta - t, 0) e^{-\int_0^t \delta(\tau + \theta - t) d\tau}, & \text{if } \theta \geq t. \end{cases} \tag{9}$$

Then system (3)–(5) can be modified as the following equal system

$$\begin{cases} \frac{\partial u}{\partial t} = s - \mu u(x, t) - \frac{\beta u(x, t)v(x, t)}{1 + \alpha v(x, t)}, \\ w(x, \theta, t) = \begin{cases} \frac{\beta u(x, t - \theta)v(x, t - \theta)}{1 + \alpha v(x, t - \theta)} e^{-\int_0^\theta \delta(\tau) d\tau}, & \text{if } t \geq \theta, \\ w(x, \theta - t, 0) e^{-\int_0^t \delta(\tau + \theta - t) d\tau}, & \text{if } \theta \geq t, \end{cases} \\ \frac{\partial v}{\partial t} = D\Delta v + \int_0^\infty p(\theta)w(x, \theta, t) d\theta - dv(x, t) \end{cases} \tag{10}$$

for $t > 0, x \in \Omega$, with homogeneous Neumann boundary conditions (4). Substituting (9) into the third equation of system (10), we have

$$\begin{cases} \frac{\partial u}{\partial t} = s - \mu u(x, t) - \frac{\beta u(x, t)v(x, t)}{1 + \alpha v(x, t)}, \\ \frac{\partial v}{\partial t} = D\Delta v + \int_0^t \Pi(\theta) \frac{\beta u(x, t - \theta)v(x, t - \theta)}{1 + \alpha v(x, t - \theta)} d\theta - dv(x, t) + F_w(x, t), \end{cases} \tag{11}$$

for $t > 0, x \in \Omega$, where

$$F_w(x, t) = \int_t^{+\infty} p(\theta)w(x, \theta - t, 0) e^{-\int_0^t \delta(\tau + \theta - t) d\tau} d\theta.$$

It follows from Assumption 2.1, $p(\theta)$ belongs to $L^+_\infty((0, +\infty), \mathbb{R}) \setminus \{0_{L^\infty}\}$, that $F_w(x, t)$ equals to zero when t is large enough. Thus system (11) can be written as

$$\begin{cases} \frac{\partial u}{\partial t} = s - \mu u(x, t) - \frac{\beta u(x, t)v(x, t)}{1 + \alpha v(x, t)}, \\ \frac{\partial v}{\partial t} = D\Delta v + \int_0^{+\infty} \Pi(\theta) \frac{\beta u(x, t - \theta)v(x, t - \theta)}{1 + \alpha v(x, t - \theta)} d\theta - dv(x, t), \end{cases} \tag{12}$$

for $t > 0, x \in \Omega$. System (12) always has an uninfected steady state $E_1^0 = (s/\mu, 0)$; If $R_0 > 1$, then system (12) has a unique infected steady state $E_1^* = (u^*, v^*)$.

Let $E(u_0, v_0)$ represent any feasible steady state of system (12). Then the linearization of system (12) at $E(u_0, v_0)$ is of the form

$$\begin{cases} \frac{\partial u}{\partial t} = -(\mu + P)u(x, t) - Qv(x, t), \\ \frac{\partial v}{\partial t} = D\Delta v + P \int_0^{+\infty} \Pi(\theta)u(x, t - \theta) d\theta + Q \int_0^{+\infty} \Pi(\theta)v(x, t - \theta) d\theta - dv(x, t), \end{cases} \tag{13}$$

for $t > 0, x \in \Omega$, where

$$P = \frac{\beta v_0}{1 + \alpha v_0}, \quad Q = \frac{\beta u_0}{(1 + \alpha v_0)^2}.$$

Let $0 = \mu_1 < \mu_2 < \dots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogenous Neumann boundary conditions, and $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\Omega)$. Let $\mathbb{X} = [C^1(\Omega)]^2, \{\phi_{ij}; j = 1, 2, \dots, \dim E(\mu_i)\}$ be an orthogonal basis of $E(\mu_i)$, and $\mathbb{X}_{ij} = \{c\phi_{ij} \mid c \in \mathbb{R}^2\}$. Then

$$\mathbb{X} = \bigoplus_{i=0}^{\infty} \mathbb{X}_i \quad \text{and} \quad \mathbb{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbb{X}_{ij}.$$

Substituting $u(x, t) = e^{\lambda t}\phi(x)$ and $v(x, t) = e^{\lambda t}\psi(x)$ into system (13), we have the associated eigenvalue problem

$$\begin{cases} \lambda\phi(x) = -(\mu + P)\phi(x) - Q\psi(x), \\ \lambda\psi(x) = D\psi''(x) + P\widehat{\Pi}(\lambda)\phi(x) + Q\widehat{\Pi}(\lambda)\psi(x) - d\psi(x), \end{cases} \tag{14}$$

where $\widehat{\Pi}(\lambda)$ represents the Laplace transform of the function $\Pi(\theta)$. It follows from the proof of Lemma 2.2 in [21], we have that the problem (14) has a principal eigenvalue, denoted by λ^* and λ^* satisfies the following equation

$$(\lambda + d + \mu_i D)(\lambda + \mu + P) - (\lambda + \mu)Q\widehat{\Pi}(\lambda) = 0. \tag{15}$$

For E_1^0 , the principal eigenvalue λ^* satisfies the following equation

$$(\lambda + \mu)\left(\lambda + d + \mu_i D - \frac{\beta s}{\mu}\widehat{\Pi}(\lambda)\right) = 0. \tag{16}$$

Since $\lambda_1 = -\mu$ is a root of Eq. (16), we need only consider the roots of the following

$$\lambda + d + \mu_i D - \frac{\beta s}{\mu}\widehat{\Pi}(\lambda) = 0. \tag{17}$$

In fact, if λ is a root of Eq. (17) with $\Re\lambda \geq 0$, then we have

$$|\widehat{\Pi}(\lambda)| = \left| \int_0^\infty p(\theta)e^{-\lambda\theta}e^{-\int_0^\theta \delta(\tau)d\tau}d\theta \right| \leq \mathcal{K}$$

and also

$$\frac{\mu}{\beta s}|\lambda + d + \mu_i D| \geq \frac{\mu(d + \mu_i D)}{\beta s}.$$

It follows from above inequalities and Eq. (17) we have that

$$\mu d \leq \mu(d + \mu_i D) \leq \beta s \mathcal{K}, \quad \text{i.e.,} \quad R_0 \geq 1.$$

So, if $R_0 < 1$, all the roots of Eq. (17), and therefore Eq. (16) have negative real parts. Accordingly, the principal eigenvalue λ^* has negative real part, the uninfected steady state $E_1^0(s/\mu, 0)$ of system (11) is locally asymptotically stable if $R_0 < 1$.

For E_1^* , the principal eigenvalue λ^* satisfies the following equation

$$(\lambda + d + \mu_i D)\left(\lambda + \mu + \frac{\beta v^*}{1 + \alpha v^*}\right) - (\lambda + \mu)\frac{\beta u^*}{(1 + \alpha v^*)^2}\widehat{\Pi}(\lambda) = 0. \tag{18}$$

Whether the value of μ equals that of $(d + \mu_i D)$ for some i can determine the types of the roots to Eq. (18). Now we are in a position to consider two cases.

(i) If $\mu \neq d + \mu_i D$ for all $i \in \mathbb{N}^+$ then $\lambda_1 = -\mu$ is not the root of Eq. (18). So that Eq. (18) can be written as

$$\lambda + d + \mu_i D + \frac{\beta v^*}{1 + \alpha v^*}Z - \frac{\beta u^*}{(1 + \alpha v^*)^2}\widehat{\Pi}(\lambda) = 0, \tag{19}$$

where $Z = \frac{(\lambda+d+\mu_i D)}{(\lambda+\mu)}$. If λ is a root of Eq. (19) with $\Re\lambda \geq 0$, then we have $\Re Z \geq 0$,

$$\left| \lambda + d + \mu_i D + \frac{\beta v^*}{1 + \alpha v^*} Z \right| \geq |d + \mu_i D| \geq d$$

and

$$\left| \frac{\beta u^*}{(1 + \alpha v^*)^2} \widehat{\Pi}(\lambda) \right| \leq \left| \frac{\mathcal{K} \beta u^*}{(1 + \alpha v^*)^2} \right| = \frac{d}{1 + \alpha v^*}.$$

Combing the above inequalities and Eq. (19) we have that

$$d \leq \frac{d}{1 + \alpha v^*},$$

which contradicts $v^* > 0$.

(ii) If $\mu = d + \mu_i D$ for some i , then $\lambda_1 = -\mu$ is a root of Eq. (18). So we need only to consider the following

$$\lambda + d + \mu_i D + \frac{\beta v^*}{1 + \alpha v^*} - \frac{\beta u^*}{(1 + \alpha v^*)^2} \widehat{\Pi}(\lambda) = 0. \tag{20}$$

If λ is a root of Eq. (20) with $\Re\lambda \geq 0$, then it leads to the contradiction:

$$d \leq \left| \lambda + d + \mu_i D + \frac{\beta v^*}{1 + \alpha v^*} \right| = \left| \frac{\beta u^*}{(1 + \alpha v^*)^2} \widehat{\Pi}(\lambda) \right| \leq \left| \frac{\mathcal{K} \beta u^*}{(1 + \alpha v^*)^2} \right| = \frac{d}{1 + \alpha v^*}.$$

To sum up, all the roots of Eq. (18) have negative real parts if $R_0 > 1$. Accordingly, the principal eigenvalue λ^* has negative real part, the infected steady state $E_1^*(u^*, v^*)$ of system (11) is locally asymptotically stable.

Summarizing the above discussions, we can arrive at the following results.

Theorem 3.1. *If $R_0 < 1$, the uninfected steady state E^0 of system (3) is locally asymptotically stable, otherwise it is unstable; if $R_0 > 1$, the infected steady state E^* of system (3) is locally asymptotically stable.*

3.2. Global stability of the steady states. Motivated by the method used in [16], in this subsection, by constructing two Lyapunov functionals and using LaSalle’s invariant principle, we investigate the global stability of the uninfected steady state E^0 and the infected steady state E^* .

The following theorem is about the global stability of the uninfected steady state E^0 .

Theorem 3.2. *If $R_0 \leq 1$, then we have*

$$\lim_{t \rightarrow +\infty} (u(x, t), w(x, \theta, t), v(x, t)) = (u^0, 0_{L^1}, 0)$$

uniformly for $x \in \bar{\Omega}$, namely, the uninfected steady state E^0 of model (3)–(5) is globally asymptotically stable.

Proof. Since the local stability of system (3) has been proved in Theorem 3.1, then we need only to prove the omega limit set contain only the uninfected steady state E^0 . We set function $\mathcal{H}(z) = z - 1 - \ln z$, $z \in \mathbb{R}_+$ and consider the following Lyapunov functional

$$L(t) = \int_{\Omega} \left(u^0 \mathcal{H}\left(\frac{u(x, t)}{u^0}\right) + \frac{1}{\mathcal{K}} \int_0^\infty f(\theta) w(x, \theta, t) d\theta + \frac{1}{\mathcal{K}} v(x, t) \right) dx, \tag{21}$$

where

$$f(\theta) := \int_\theta^\infty p(\tau) e^{-\int_\theta^\tau \delta(\sigma) d\sigma} d\tau. \tag{22}$$

Obviously, the function $f(\theta)$ is bounded and satisfies $f(0) = \mathcal{K}$, $f(\theta) > 0$ for $0 < \theta < +\infty$ and $f'(\theta) = \frac{df(\theta)}{d\theta} = \delta(\theta)f(\theta) - p(\theta)$.

Next, we calculate the time derivative of $L(t)$ along the solution of system (3)–(5)

$$\frac{dL(t)}{dt} = \int_{\Omega} \left(\left(1 - \frac{u^0}{u(x,t)}\right) \frac{\partial u(x,t)}{\partial t} + \frac{1}{\mathcal{K}} \int_0^\infty f(\theta) \frac{\partial w(x,\theta,t)}{\partial t} d\theta + \frac{1}{\mathcal{K}} \frac{\partial v(t)}{\partial t} \right) dx.$$

By use of $u^0 = s/\mu$, we have

$$\begin{aligned} \frac{dL(t)}{dt} &= \int_{\Omega} \left(\left(1 - \frac{u^0}{u(x,t)}\right) \mu (u^0 - u(x,t)) + \frac{\beta v(x,t)u^0}{1 + \alpha v(x,t)} - \frac{\beta v(x,t)u(x,t)}{1 + \alpha v(x,t)} \right) dx \\ &\quad - \frac{1}{\mathcal{K}} \int_{\Omega} \int_0^\infty f(\theta) \left(\frac{\partial w(x,\theta,t)}{\partial \theta} + \delta(\theta)w(x,\theta,t) \right) d\theta dx \\ &\quad + \int_{\Omega} \left(D\Delta v + \frac{1}{\mathcal{K}} \int_0^\infty p(\theta)w(x,\theta,t)d\theta \right) dx \\ &= \int_{\Omega} \left(-\frac{\mu}{u(x,t)}(u^0 - u(x,t))^2 + \frac{\beta v(x,t)u^0}{1 + \alpha v(x,t)} + D\Delta v - \frac{d}{\mathcal{K}}v(x,t) \right) dx \\ &\quad - \int_{\Omega} C(x,t)dx \\ &= \int_{\Omega} \left(-\frac{\mu}{u(x,t)}(u^0 - u(x,t))^2 + D\Delta v + \frac{d(R_0 - 1) - \alpha dv(x,t)}{\mathcal{K}(1 + \alpha v(x,t))}v(x,t) \right) dx \\ &\quad - \int_{\Omega} C(x,t)dx, \end{aligned}$$

where

$$\begin{aligned} C(x,t) &= \frac{1}{\mathcal{K}} \int_0^\infty f(\theta) \frac{\partial w(x,\theta,t)}{\partial \theta} d\theta \\ &\quad + \frac{1}{\mathcal{K}} \int_0^\infty \left(\delta(\theta)f(\theta) + p(\theta) \right) w(x,\theta,t)d\theta - \frac{\beta v(x,t)u(x,t)}{1 + \alpha v(x,t)}. \end{aligned}$$

Since

$$\begin{aligned} \int_0^\infty f(\theta) \frac{\partial w(x,\theta,t)}{\partial \theta} d\theta &= \int_0^\infty f(\theta)dw(x,\theta,t) \\ &= f(\theta)w(x,\theta,t) \Big|_{\theta=0}^{\theta=\infty} - \int_0^\infty w(x,\theta,t)f'(\theta)d\theta \\ &= f(\theta)w(x,\theta,t) \Big|_{\theta=\infty} - \mathcal{K} \frac{\beta u(x,t)v(x,t)}{1 + \alpha v(x,t)} - \int_0^\infty w(x,\theta,t)f'(\theta)d\theta, \end{aligned}$$

we then deduce that

$$C(x,t) = \frac{1}{\mathcal{K}}f(\theta)w(x,\theta,t) \Big|_{\theta=\infty},$$

which is bounded due to Lemma 2.2 and the boundedness of function $f(\theta)$.

Recalling that $\int_{\Omega} D\Delta v dx = 0$, we have

$$\begin{aligned} \frac{dL(t)}{dt} &= - \int_{\Omega} \frac{\mu}{u(x,t)} \left(u^0 - u(x,t) \right)^2 dx - \int_{\Omega} \frac{1}{\mathcal{K}} f(\theta)w(x,\theta,t) \Big|_{\theta=\infty} dx \\ &\quad + \int_{\Omega} \frac{d(R_0 - 1) - \alpha dv(x,t)}{\mathcal{K}(1 + \alpha v(x,t))}v(x,t)dx. \end{aligned}$$

If $R_0 \leq 1$, it is easy to see that $dL(t)/dt \leq 0$ with the equality holding if and only if at E^0 . We conclude that the largest invariant set $\{(u, w, v) \in \mathbb{R}_+ \times L_+^1((0, +\infty), \mathbb{R}) \times$

$\mathbb{R}_+ : dL(t)/dt = 0$ in X_{0+} is the singleton $\{E^0\}$. By LaSalle’s invariant principle (see Theorem 5.3.1 in [15]), E^0 is globally asymptotically stable when $R_0 \leq 1$. This completes the proof of Theorem 3.2. \square

In the following, we prove the global stability of the infected steady state E^* by considering a Lyapunov functional for $R_0 > 1$. For this, it is important to establish some type of positivity of solutions so that the Lyapunov functional will be known to be finite. The following definition characterizes initial conditions that will be used in the future discussion.

Definition 3.3. We say the disease is initially present if

$$\int_{\Omega} v(x, 0)dx > 0.$$

Lemma 3.4. Suppose the disease is initially present. If $R_0 > 1$, there exists $\varepsilon > 0$ such that $u(x, t), v(x, t) > \varepsilon$ for all $x \in \Omega, t > 0$.

Proof. By the first equation of system (3), we have that

$$\frac{\partial u}{\partial t} \geq s - \left(\mu + \frac{\beta}{\alpha}\right) u(x, t)$$

for each $x \in \Omega, t > 0$. Thus, $u(x, \cdot)$ is increasing if it is less than $\frac{\alpha s}{\alpha\mu + \beta}$. Furthermore, by solving this differential inequality for each x , we find that

$$u(x, t) \geq \frac{\alpha s}{\alpha\mu + \beta} + \left(u(x, 0) - \frac{\alpha s}{\alpha\mu + \beta}\right) e^{-(\mu + \frac{\beta}{\alpha})t} \geq \frac{\alpha s}{\alpha\mu + \beta} \left(1 - e^{-(\mu + \frac{\beta}{\alpha})t}\right).$$

Therefore, for any positive t_1 , there exists $\varepsilon_u = \varepsilon_u(t_1) > 0$ such that $u(x, t) > \varepsilon_u$ for all $x \in \Omega, t \geq t_1$.

Let

$$\widehat{v}(t) = \int_{\Omega} v(x, t)dx.$$

We note that \widehat{v} is non-negative. From Definition 3.3, we know that $\widehat{v}(0) > 0$.

Next we show that $\widehat{v}(t)$ is positive for $t > 0$. By use of $\int_{\Omega} D\Delta v dx = 0$, we have that

$$\begin{aligned} \frac{d\widehat{v}}{dt} &= \frac{d}{dt} \int_{\Omega} v(x, t)dx = \int_{\Omega} \frac{\partial v}{\partial t} dx \\ &= \int_{\Omega} \left(D\Delta v(x, t) + \int_0^{\infty} p(\theta)w(x, \theta, t)d\theta - dv(x, t) \right) dx \\ &= \int_{\Omega} (D\Delta v(x, t))dx + \int_{\Omega} dx \int_0^{\infty} p(\theta)w(x, \theta, t)d\theta - d \int_{\Omega} v(x, t)dx \\ &= \int_{\Omega} dx \int_0^{\infty} p(\theta)w(x, \theta, t)d\theta - d\widehat{v}(t). \end{aligned}$$

It follows from $v(x, t) \leq \frac{R_0}{\delta_{min}}$ and (12) that

$$\begin{aligned} \frac{d\widehat{v}}{dt} &= \int_{\Omega} dx \int_0^{+\infty} \Pi(\theta) \frac{\beta u(x, t - \theta)v(x, t - \theta)}{1 + \alpha v(x, t - \theta)} d\theta - d\widehat{v}(t) \\ &\geq \int_0^{+\infty} \Pi(\theta) \frac{\beta \varepsilon_u \delta_{min}}{\delta_{min} + R_0} \int_{\Omega} v(x, t - \theta) dx d\theta - d\widehat{v}(t) \\ &= \int_0^{+\infty} \Pi(\theta) \frac{\beta \varepsilon_u \delta_{min}}{\delta_{min} + R_0} \widehat{v}(t - \theta) d\theta - d\widehat{v}(t). \end{aligned}$$

Since $R_0 > 1$, we have that

$$\int_0^{+\infty} \Pi(\theta) \frac{\beta \varepsilon_u \delta_{min}}{\delta_{min} + R_0} d\theta \geq \frac{\mu d \varepsilon_u \delta_{min}}{s(\delta_{min} + R_0)}.$$

By Lemma 5.1 in [29], $\widehat{v}(t)$ is unbounded. Since Ω is connected, the diffusion term causes the support of $v(\cdot, t)$ to spread instantaneously to all of the $int(\Omega)$, the interior of Ω . Thus, $v(x, t) > 0$ for all $x \in int(\Omega)$, $t > 0$.

We now show that v is positive on $\partial\Omega$. Suppose $v(x_0, t) = 0$ for some $x_0 \in \partial\Omega$, $t > 0$. By Proposition 13.3 in [19] (a version of the Maximum Principle used for the boundary), it follows that $\frac{\partial v}{\partial \nu}|_{x=x_0} < 0$, contradicting the boundary condition (4). Thus, $v(x, t) > 0$ for all $x \in \partial\Omega$, $t > 0$, and therefore for all $x \in \bar{\Omega}$, $t > 0$. \square

Theorem 3.5. *Suppose the disease is initially present. If $R_0 > 1$, then we have*

$$\lim_{t \rightarrow +\infty} (u(x, t), w(x, \theta, t), v(x, t)) = (u^*, w^*(\theta), v^*)$$

uniformly for $x \in \bar{\Omega}$, namely, the infected steady state E^ of model (3)–(5) is globally asymptotically stable.*

Proof. From Theorem 3.1, we know that the infected steady state E^* of system (3) is locally asymptotically stable if $R_0 > 1$. In the following, with $f(\theta)$ in 22 and $\mathcal{H}(z) = z - 1 - \ln z$, we construct the following Lyapunov functional

$$\begin{aligned} L(t) = & \int_{\Omega} \left(u^* \mathcal{H}\left(\frac{u(x, t)}{u^*}\right) + \frac{1}{\mathcal{K}} \int_0^{\infty} f(\theta) w^*(\theta) \mathcal{H}\left(\frac{w(x, \theta, t)}{w^*(\theta)}\right) d\theta \right) dx \\ & + \frac{1}{\mathcal{K}} \int_{\Omega} v^* \mathcal{H}\left(\frac{v(x, t)}{v^*}\right) dx. \end{aligned}$$

We now show that $dL(t)/dt \leq 0$ along the solution of system (3)–(5).

By direct calculation, we have

$$\begin{aligned} \frac{dL(t)}{dt} = & \int_{\Omega} \left(1 - \frac{u^*}{u(x, t)} \right) \left(s - \mu u(x, t) - \frac{\beta u(x, t) v(x, t)}{1 + \alpha v(x, t)} \right) dx \\ & + \frac{1}{\mathcal{K}} \int_{\Omega} \int_0^{\infty} f(\theta) \left(1 - \frac{w^*(\theta)}{w(x, \theta, t)} \right) \frac{\partial}{\partial t} w(x, \theta, t) d\theta dx \\ & + \int_{\Omega} \left(\frac{D}{\mathcal{K}} \left(1 - \frac{v^*}{v(x, t)} \right) \Delta v - \frac{d}{\mathcal{K}} v(x, t) + \frac{d}{\mathcal{K}} v^* \right) dx \\ & + \frac{1}{\mathcal{K}} \int_{\Omega} \left(1 - \frac{v^*}{v(x, t)} \right) \int_0^{\infty} p(\theta) w(x, \theta, t) d\theta dx. \end{aligned}$$

Note that

$$\begin{aligned} & \int_0^{\infty} f(\theta) \left(1 - \frac{w^*(\theta)}{w(x, \theta, t)} \right) \frac{\partial}{\partial t} w(x, \theta, t) d\theta \\ = & - \int_0^{\infty} f(\theta) w^*(\theta) d \left(\frac{w(x, \theta, t)}{w^*(\theta)} - 1 - \ln \frac{w(x, \theta, t)}{w^*(\theta)} \right) \\ = & - f(\theta) w^*(\theta) \left(\frac{w(\theta, t)}{w^*(\theta)} - 1 - \ln \frac{w(x, \theta, t)}{w^*(\theta)} \right) \Big|_{\theta=0}^{\theta=\infty} \\ & + \int_0^{\infty} \left(\frac{w(x, \theta, t)}{w^*(\theta)} - 1 - \ln \frac{w(x, \theta, t)}{w^*(\theta)} \right) \left(f'(\theta) w^*(\theta) + f(\theta) \frac{dw^*(\theta)}{d\theta} \right) d\theta. \end{aligned}$$

It then follows from $f'(\theta)w^*(\theta) + f(\theta)(dw^*(\theta)/d\theta) = -p(\theta)w^*(\theta)$ and

$$\begin{cases} f(0) = \mathcal{K}, \quad w^*(0) = \frac{\beta u^* v^*}{1 + \alpha v^*}, \quad w(x, 0, t) = \frac{\beta u(x, t) v(x, t)}{1 + \alpha v(x, t)}, \\ s = \mu u^* + \frac{\beta u^* v^*}{1 + \alpha v^*}, \quad \frac{\beta u^*}{1 + \alpha v^*} = \frac{d}{\mathcal{K}}, \\ w^*(0) = \frac{1}{\mathcal{K}} \int_0^{+\infty} p(\theta) w^*(\theta) d\theta \end{cases} \tag{23}$$

we have that

$$\begin{aligned} \frac{dL(t)}{dt} = & - \int_{\Omega} \frac{\mu(u(x, t) - u^*)^2}{u(x, t)} dx \\ & - \frac{1}{\mathcal{K}} \int_{\Omega} f(\theta) w^*(\theta) \left(\frac{w(x, \theta, t)}{w^*(\theta)} - 1 - \ln \frac{w(x, \theta, t)}{w^*(\theta)} \right) \Big|_{\theta=0} dx \\ & + \int_{\Omega} \left[\frac{D}{\mathcal{K}} \left(1 - \frac{v^*}{v(x, t)} \right) \Delta v + C_1(x, t) \right] dx, \end{aligned}$$

where

$$\begin{aligned} C_1(x, t) = & - \frac{u^*}{u(x, t)} \left(w^*(0) - w(x, 0, t) \right) - w^*(0) \ln \frac{w(x, 0, t)}{w^*(0)} \\ & - \frac{1}{\mathcal{K}} \int_0^{\infty} p(\theta) w^*(\theta) \left(\frac{w(x, \theta, t)}{w^*(\theta)} - 1 - \ln \frac{w(x, \theta, t)}{w^*(\theta)} \right) d\theta \\ & + \frac{1}{\mathcal{K}} \left(1 - \frac{v^*}{v(x, t)} \right) \int_0^{\infty} p(\theta) w(x, \theta, t) d\theta + w^*(0) - w^*(0) \frac{v(x, t)}{v^*}. \end{aligned}$$

By use of the equalities in (23) and simple calculations we have

$$\begin{aligned} C_1(x, t) = & - \frac{1}{\mathcal{K}} \int_0^{\infty} p(\theta) w^*(\theta) \left(\frac{v^*}{v(x, t)} \frac{w(x, \theta, t)}{w^*(\theta)} - 1 - \ln \frac{w(x, \theta, t)}{w^*(\theta)} \right) d\theta \\ & + w^*(0) \left(1 - \ln \frac{w(x, 0, t)}{w^*(0)} - \frac{v(x, t)}{v^*} + \frac{w(x, 0, t)}{w^*(0)} \frac{u^*}{u(x, t)} - \frac{u^*}{u(x, t)} \right) \\ = & - \frac{1}{\mathcal{K}} \int_0^{\infty} p(\theta) w^*(\theta) \left[\mathcal{H} \left(\frac{u^*}{u(x, t)} \right) + \mathcal{H} \left(\frac{w(x, \theta, t)}{w^*(\theta)} \frac{v^*}{v(x, t)} \right) \right] d\theta \\ & - \frac{1}{\mathcal{K}} \int_0^{\infty} p(\theta) w^*(\theta) \left[\mathcal{H} \left(\frac{v(x, t)}{v^*} \right) - \mathcal{H} \left(\frac{w(x, 0, t)}{w^*(0)} \frac{u^*}{u(x, t)} \right) \right] d\theta. \end{aligned}$$

Due to the monotonicity of function $g(z) = z/(1 + \alpha z)$ in $(0, +\infty)$, combing with (23), we deduce that

$$\begin{cases} \frac{v(x, t)}{v^*} \leq \frac{v(x, t)}{v^*} \frac{1 + \alpha v^*}{1 + \alpha v(x, t)} \left(= \frac{w(x, 0, t)}{w^*(0)} \frac{u^*}{u(x, t)} \right) \leq 1, & \text{if } v(x, t) \leq v^*, \\ \frac{v(x, t)}{v^*} \geq \frac{v(x, t)}{v^*} \frac{1 + \alpha v^*}{1 + \alpha v(x, t)} \left(= \frac{w(x, 0, t)}{w^*(0)} \frac{u^*}{u(x, t)} \right) \geq 1, & \text{if } v(x, t) \geq v^*. \end{cases} \tag{24}$$

Since the function $\mathcal{H}(z)$ is monotonically decreasing in $(0, 1]$ and increasing in $[1, +\infty)$, it then follows from (24) that

$$\mathcal{H} \left(\frac{v(x, t)}{v^*} \right) \geq \mathcal{H} \left(\frac{w(x, 0, t)}{w^*(0)} \frac{u^*}{u(x, t)} \right),$$

and therefore $C_1(x, t) \leq 0$ for all $u, w, v > 0$.

Recalling that $\int_{\Omega} \frac{\Delta v}{v} dx = \int_{\Omega} \frac{\|\nabla v\|^2}{v^2} dx$, we obtain

$$\begin{aligned} \frac{dL(t)}{dt} = & - \int_{\Omega} \frac{\mu(u(x,t) - u^*)^2}{u(x,t)} dx - \frac{Dv^*}{\mathcal{K}} \int_{\Omega} \frac{\|\nabla v\|^2}{v^2} dx + \int_{\Omega} C_1(x,t) dx \\ & - \frac{1}{\mathcal{K}} \int_{\Omega} f(\theta)w^*(\theta) \left(\frac{w(x,\theta,t)}{w^*(\theta)} - 1 - \ln \frac{w(x,\theta,t)}{w^*(\theta)} \right) \Big|_{\theta=\infty} dx. \end{aligned}$$

It follows from $u^*, w^*(\theta), v^* > 0$, if $R_0 > 1$, we have that $dL(t)/dt \leq 0$ for all $u, w, v > 0$ with the equality holding if and only if at E^* . We conclude that the largest invariant set $\{(u, w, v) \in \mathbb{R}_+^2 \times L_+^1((0, +\infty), \mathbb{R}) : dL(t)/dt = 0\}$ in X_{0+} is the singleton $\{E^*\}$. By LaSalle’s invariant principle (see [15], Theorem 5.3.1), E^* is globally asymptotically stable when $R_0 > 1$. This completes the proof of Theorem 3.5. \square

4. Existence of traveling wave solutions. Based on the stability results obtained above, in this section, we will study the existence and non-existence of traveling wave solutions describing the spatial invasion of the hepatitis B virus within a liver, where the infection is initially absent. In Subsection 4.1, we first make some basic preparations and state the main results obtained in this section, the proof of which are deferred to Subsection 4.2.

4.1. Preliminaries and main results. Using the change of variables

$$\begin{aligned} \bar{u}(x,t) &= \frac{\mu}{s}u(x,t), \quad \bar{w}(\theta,x,t) = \frac{\mu}{s}w(x,\theta,t), \quad \bar{v}(x,t) = \frac{\beta}{\mu}v(x,t), \\ \bar{t} &= \mu t, \quad \bar{\theta} = \mu\theta, \quad \bar{D} = \frac{D}{\mu} \end{aligned}$$

and

$$a := \frac{\alpha\mu}{\beta}, \quad \rho_1(\theta) := \frac{1}{\mu}\delta(\theta), \quad \rho_2(\theta) := \frac{\beta s}{\mu^3}p(\theta), \quad \rho_3 := \frac{d}{\mu} \tag{25}$$

and dropping the bars, we obtain from system (3) that

$$\begin{cases} \partial_t u = 1 - u(x,t) - \frac{u(x,t)v(x,t)}{1 + av(x,t)}, \\ \partial_t w + \partial_{\theta} w = -\rho_1(\theta)w(x,\theta,t), \\ w(x,0,t) = \frac{u(x,t)v(x,t)}{1 + av(x,t)}, \\ \partial_t v = D\Delta v + \int_0^{\infty} \rho_2(\theta)w(x,\theta,t)d\theta - \rho_3v(x,t), \\ u(x,0) = u_0(x), \quad w(x,\theta,0) = w_0(x,\theta), \quad v(x,0) = v_0(x). \end{cases} \tag{26}$$

Let $\kappa = \frac{1}{\mu}\delta_{min}$, $\rho(\theta) = \rho_1(\theta) - \kappa$, $\ell(\theta) = e^{-\int_0^{\theta} \rho(s)ds}$, $\sigma(\theta) = \rho_2(\theta)\ell(\theta)$. Then by using the change of variables

$$\bar{w}(x,\theta,t) = \frac{w(x,\theta,t)}{\ell(\theta)}, \quad \bar{u}(x,t) = 1 - u(x,t), \quad \bar{v}(x,t) = v(x,t)$$

and dropping the bars, we obtain from system (26) that

$$\begin{cases} \partial_t u = \frac{(1-u(x,t))v(x,t)}{1+av(x,t)} - u(x,t), \\ \partial_t w + \partial_\theta w = -\kappa w(x,\theta,t), \\ w(x,0,t) = \frac{(1-u(x,t))v(x,t)}{1+av(x,t)}, \\ \partial_t v = D\Delta v + \int_0^\infty \sigma(\theta)w(x,\theta,t)d\theta - \rho_3 v(x,t), \\ u(x,0) = u_0(x), w(x,\theta,0) = w_0(x,\theta), v(x,0) = v_0(x). \end{cases} \tag{27}$$

So, we need only to consider the existence of traveling wave solutions for system (27). Similarly to system (3), we have the following results for system (27).

Denote

$$K = \int_0^{+\infty} \sigma(\theta)e^{-\kappa\theta} d\theta. \tag{28}$$

Then the basic reproduction number takes the form $R_0 = \frac{K}{\rho_3}$. System (27) always has the uninfected steady state $E^0 = (0,0,0)$. If $R_0 > 1$, in addition to $E^0 = (0,0,0)$, system (27) has a unique infected steady state $E^* = (u^*, w^*(\theta), v^*)$, where

$$u^* = \frac{1}{1+a} \left(1 - \frac{1}{R_0}\right), \quad w^*(\theta) = u^* e^{-\kappa\theta}, \quad v^* = \frac{1}{1+a} (R_0 - 1). \tag{29}$$

Set $\eta = (\eta_1, \eta_2(\cdot), \eta_3)$ and define

$$[\mathbf{0}, \widetilde{\mathbf{M}}]_X := \{\eta \in X : \mathbf{0} \leq \eta(x) \leq \widetilde{\mathbf{M}}, \forall x \in \Omega\},$$

with $\mathbf{0} = (0, 0_{L^1}, 0)$ and $\widetilde{\mathbf{M}} = (\widetilde{M}_1, \widetilde{M}_2(\theta), \widetilde{M}_3)$, where $\widetilde{M}_1 = 1$, $\widetilde{M}_2(\theta) = \frac{1}{a}e^{-\kappa\theta}$, $\widetilde{M}_3 = R_0/a$. Then from Lemma 2.2, for any given initial value $(u_0(x), w_0(x, \theta), v_0(x)) \in [\mathbf{0}, \widetilde{\mathbf{M}}]_X$, the solution of system (27) satisfies that

$$0 \leq u(x,t) < 1; \quad 0 \leq w(x,\theta,t) \leq \widetilde{M}_2(\theta); \quad 0 \leq v(x,t) \leq \widetilde{M}_3.$$

Substituting $u(x,t) = \phi(x \cdot e + ct)$, $w(x,\theta,t) = \varphi(\theta, x \cdot e + ct)$, $v(x,t) = \psi(x \cdot e + ct)$ into system (27), and denoting $x \cdot e + ct$ by t , we obtain the corresponding wave equation

$$\begin{cases} -c\phi'(t) - \phi(t) + \frac{(1-\phi(t))\psi(t)}{1+a\psi(t)} = 0, \\ \partial_\theta \varphi(\theta,t) + c\partial_t \varphi(\theta,t) = -\kappa\varphi(\theta,t), \\ \varphi(0,t) = \frac{(1-\phi(t))\psi(t)}{1+a\psi(t)}, \\ D\psi''(t) - c\psi'(t) + \int_0^\infty \sigma(\theta)\varphi(\theta,t)d\theta - \rho_3\psi(t) = 0. \end{cases} \tag{30}$$

Definition 4.1. (Travelling waves). We will say that system (27) has a travelling wave solution if there exists a real number $c > 0$ and a pair of positive functions $(\tilde{u}, \tilde{w}, \tilde{v})$ (i.e. $\tilde{u} > 0$ on \mathbb{R} , $\tilde{w} > 0$ on $[0, +\infty) \times \mathbb{R}$ and $\tilde{v} > 0$ on \mathbb{R}) such that the maps $\tilde{u} \in C_b^0(\mathbb{R}) \cap C^1(\mathbb{R})$, $\tilde{w} \in C_b^0([0, +\infty) \times \mathbb{R}) \cap C^{1,1}((0, +\infty) \times \mathbb{R})$

$L^1((0, +\infty), C_b^0(\mathbb{R}))$, $\tilde{v} \in C_b^0(\mathbb{R}) \cap C^2(\mathbb{R})$ satisfy $\lim_{t \rightarrow -\infty} \tilde{u}(t) = 0$, $\lim_{t \rightarrow -\infty} \tilde{w}(\cdot, t) = 0$ in $L^1((0, +\infty), \mathbb{R})$ and $\lim_{t \rightarrow -\infty} \tilde{v}(t) = 0$, and such that the maps

$$u(t, x) = \tilde{u}(x \cdot e + ct), \quad w(\theta, t, x) = \tilde{w}(\theta, x \cdot e + ct), \quad v(t, x) = \tilde{v}(x \cdot e + ct)$$

is a pair of entire solutions of system (27). Here $C_b^0(\mathbb{R})$ denotes the space of bounded and continuous functions from \mathbb{R} into itself.

For $1 < R_0 < 1 + \alpha^*$, we define the minimal speed by

$$c^* := \sqrt{D\alpha^*},$$

where α^* is the unique solution of the equation

$$\frac{1}{\rho_3} \int_0^{+\infty} \sigma(\theta) e^{-(\kappa + \alpha^*)\theta} d\theta = 1. \tag{31}$$

Now, we are in a position to state the main results in this section.

Theorem 4.2. *Suppose the disease is initially present. Let Assumption 2.1 be satisfied. We have*

- (i) *If $1 < R_0 < 1 + \alpha^*$, then for each $c > c^*$ system (27) has a travelling wave solution. Moreover, whenever such a travelling solution exists, it satisfies*

$$\lim_{t \rightarrow +\infty} \tilde{u}(t) = u^*, \quad \lim_{t \rightarrow +\infty} \tilde{w}(\theta, t) = w^*(\theta) \in L^1((0, +\infty), \mathbb{R}), \quad \text{and} \quad \lim_{t \rightarrow +\infty} \tilde{v}(t) = v^*.$$

- (ii) *If $R_0 < 1$, then system (27) has no travelling wave solution.*

The existence of travelling wave solution of system (27) can help us to understand the viral contamination process. The condition $1 < R_0 < 1 + \alpha^*$ in Theorem 4.2 (i) cannot be modified as $1 < R_0$, since the viral release strategy adopted in system (3) (therefore in system (27)) is “budding”, which means the death of infected cells and release of virions are independent processes (see [36, 48, 49, 38]). To complete the discussion, we need consider another type of viral release strategy, “bursting”, which means there is a coupling of the release of free virions and burst of infected cells (see [53, 54]). Mathematically, $p(\theta) = N\delta(\theta)$ in system (3), then the number of virions will not increase until the lysis of the infected cells and release of virions.

Corollary 4.1. *Suppose the disease is initially present. Let Assumption 2.1 be satisfied and $p(\theta) = N\delta(\theta)$. If $R_0 > 1$, then system (27) has a travelling wave solution for each $c > c^*$.*

Proof. Based on Theorem 4.2, we need only to prove that $R_0 > 1$ implies that $R_0 < 1 + \alpha^*$. From $p(\theta) = N\delta(\theta)$ and (25) we have that $\rho_2(\theta) = \bar{N}\rho_1(\theta)$, where $\bar{N} = \frac{\beta_s N}{\mu^2}$. It follows from $R_0 = \frac{K}{\rho_3}$, (28) and (31) that

$$\begin{aligned} R_0 - 1 &= \frac{1}{\rho_3} \int_0^\infty \rho_2(\theta) e^{-\int_0^\theta \rho_1(\tau) d\tau} d\theta - \frac{1}{\rho_3} \int_0^\infty \rho_2(\theta) e^{-\int_0^\theta \rho_1(\tau) d\tau} e^{-\alpha^* \theta} d\theta \\ &= \frac{\bar{N}}{\rho_3} \left(\int_0^\infty \rho_1(\theta) e^{-\int_0^\theta \rho_1(\tau) d\tau} d\theta - \int_0^\infty \rho_1(\theta) e^{-\int_0^\theta \rho_1(\tau) d\tau} e^{-\alpha^* \theta} d\theta \right) \\ &= \frac{\bar{N}}{\rho_3} \left(-e^{-\int_0^\theta \rho_1(\tau) d\tau} \Big|_{\theta=0}^{\theta=\infty} + \int_0^\infty e^{-\alpha^* \theta} d \left(e^{-\int_0^\theta \rho_1(\tau) d\tau} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{\bar{N}}{\rho_3} \left(1 - e^{-\int_0^\infty \rho_1(\tau) d\tau} + e^{-\int_0^\theta \rho_1(\tau) d\tau} e^{-\alpha^* \theta} \Big|_{\theta=0}^{\theta=\infty} + \alpha^* \int_0^\infty e^{-\alpha^* \theta} e^{-\int_0^\theta \rho_1(\tau) d\tau} d\theta \right) \\
 &< \alpha^* \frac{\bar{N}}{\rho_3} \int_0^\infty e^{-\alpha^* \theta} e^{-\int_0^\theta \rho_1(\tau) d\tau} d\theta \\
 &\leq \alpha^* \frac{\bar{N}}{\rho_3} \int_0^\infty \rho_1(\theta) e^{-\alpha^* \theta} e^{-\int_0^\theta \rho_1(\tau) d\tau} d\theta \\
 &= \alpha^* \frac{1}{\rho_3} \int_0^\infty \rho_2(\theta) e^{-\int_0^\theta \rho_1(\tau) d\tau} e^{-\alpha^* \theta} d\theta = \alpha^*.
 \end{aligned}$$

This completes the proof of Corollary 4.1. □

4.2. The proof of Theorem 4.2.

4.2.1. *Existence result.* In this subsection we prove assertion (i) of Theorem 4.2, the proof of which consists of several steps. We will need the following definition of upper and lower solutions to system (30).

Definition 4.3. A pair of continuous functions $\bar{\xi}(t) = (\bar{\phi}(t), \bar{\varphi}(\cdot, t), \bar{\psi}(t))$ and $\underline{\xi}(t) = (\underline{\phi}(t), \underline{\varphi}(\cdot, t), \underline{\psi}(t))$ are called a pair of upper-lower solutions of system (30), respectively, if there exists a set with finitely many points: $\Sigma := \{t_1, t_2, \dots, t_m\}$ such that $\bar{\xi}(t)$ and $\underline{\xi}(t)$ are twice continuous differentiable in $\mathbb{R} \setminus \Sigma$, and the essentially bounded functions $\bar{\xi}'(t), \bar{\xi}''(t), \underline{\xi}'(t), \underline{\xi}''(t)$ satisfy, for $t \in \mathbb{R} \setminus \Sigma$,

$$-c\bar{\phi}'(t) - \bar{\phi}(t) + \frac{(1 - \bar{\phi}(t))\bar{\psi}(t)}{1 + a\bar{\psi}(t)} \leq 0, \tag{32}$$

$$\begin{cases} \partial_\theta \bar{\varphi}(\theta, t) + c\partial_t \bar{\varphi}(\theta, t) = -\kappa \bar{\varphi}(\theta, t), \\ \bar{\varphi}(0, t) \geq \frac{(1 - \underline{\phi}(t))\bar{\psi}(t)}{1 + a\bar{\psi}(t)}, \end{cases} \tag{33}$$

$$D\bar{\psi}''(t) - c\bar{\psi}'(t) + \int_0^\infty \sigma(\theta)\bar{\varphi}(\theta, t)d\theta - \rho_3\bar{\psi}(t) \leq 0, \tag{34}$$

and

$$-c\underline{\phi}'(t) - \underline{\phi}(t) + \frac{(1 - \underline{\phi}(t))\underline{\psi}(t)}{1 + a\underline{\psi}(t)} \geq 0, \tag{35}$$

$$\begin{cases} \partial_\theta \underline{\varphi}(\theta, t) + c\partial_t \underline{\varphi}(\theta, t) = -\kappa \underline{\varphi}(\theta, t), \\ \underline{\varphi}(0, t) \leq \frac{(1 - \bar{\phi}(t))\underline{\psi}(t)}{1 + a\underline{\psi}(t)}, \end{cases} \tag{36}$$

$$D\underline{\psi}''(t) - c\underline{\psi}'(t) + \int_0^\infty \sigma(\theta)\underline{\varphi}(\theta, t)d\theta - \rho_3\underline{\psi}(t) \geq 0. \tag{37}$$

In addition, we also assume that a pair of upper and lower solutions $\bar{\xi}(t) = (\bar{\phi}(t), \bar{\varphi}(\cdot, t), \bar{\psi}(t))$ and $\underline{\xi}(t) = (\underline{\phi}(t), \underline{\varphi}(\cdot, t), \underline{\psi}(t))$ are given in Definition 4.3 such that

(P₁) $(0, 0, 0) \leq (\underline{\phi}(t), \underline{\varphi}(\cdot, t), \underline{\psi}(t)) \leq (\bar{\phi}(t), \bar{\varphi}(\cdot, t), \bar{\psi}(t)) \leq (\widetilde{M}_1, \widetilde{M}_2(\cdot), \widetilde{M}_3), t \in \mathbb{R}$,
and

(P₂) $\lim_{t \rightarrow -\infty} (\underline{\phi}(t), \underline{\varphi}(\cdot, t), \underline{\psi}(t)) = (0, 0, 0), \lim_{t \rightarrow +\infty} (\bar{\phi}(t), \bar{\varphi}(\cdot, t), \bar{\psi}(t)) = (u^*, w^*(\cdot), v^*)$,
where $(u^*, w^*(\cdot), v^*)$ are given in (29).

Then the solutions of system (30) are all bounded. It follows that the convergence towards the infected steady state E^* can be solved by building a suitable Lyapunov like functional (see for instance, [10]). Therefore, the first thing to do in this subsection is to construct a pair of upper and lower solutions to system (30).

Obviously, if $1 < R_0 < 1 + \alpha^*$ and $c > c^*$, then there exists $\lambda_0 (= \alpha^*/c)$ such that

$$\frac{R_0 - 1}{c} \leq \lambda_0 \leq \frac{c}{D}. \tag{38}$$

Then one can choose three constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ satisfying the following inequalities:

$$\varepsilon_1 > 2, \quad \varepsilon_2 > \widetilde{M}_3 - u^*, \quad \varepsilon_3 > \max \left\{ \widetilde{M}_3 - v^*, \frac{\widetilde{M}_3}{v^*}(\widetilde{M}_3 - v^*) \right\}. \tag{39}$$

For any $\lambda \in (0, \lambda_0]$, it follows from (39) that

$$\frac{\varepsilon_2}{\frac{1}{a} - u^*} > \frac{1}{au^*}, \quad \text{and} \quad \frac{\varepsilon_3}{\widetilde{M}_3 - v^*} > \frac{\widetilde{M}_3}{v^*},$$

which imply that

$$\frac{1}{\lambda_0} \ln \frac{1}{av^*} < \frac{1}{\lambda_0} \ln \frac{1}{au^*} < -\frac{1}{\lambda} \ln \left[\frac{1}{\varepsilon_2} \left(\frac{1}{a} - u^* \right) \right]$$

and

$$\frac{1}{\lambda_0} \ln \frac{\widetilde{M}_3}{v^*} < \frac{1}{\lambda} \ln \frac{\varepsilon_3}{\widetilde{M}_3 - v^*}.$$

For the above constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ satisfying (39), we define two functions $\bar{\xi}(t) = (\bar{\phi}(t), \bar{\varphi}(\cdot, t), \bar{\psi}(t))$ and $\underline{\xi}(t) = (\underline{\phi}(t), \underline{\varphi}(\cdot, t), \underline{\psi}(t))$ as follows

$$\begin{aligned} \underline{\phi}(t) &= \max \left\{ 0, u^* - \varepsilon_1 e^{-\lambda t} \right\}, \\ \underline{\varphi}(\cdot, t) &= \max \left\{ 0, w^*(\cdot) - w^*(\cdot) \frac{e^{c\lambda\theta - \lambda t}}{u^*} \right\}, \\ \underline{\psi}(t) &= \max \left\{ 0, v^* - 2v^* \frac{e^{-\lambda t}}{u^*} \right\}, \end{aligned}$$

and

$$\begin{aligned} \bar{\phi}(t) &= \min \left\{ u^* e^{\lambda_0 t}, \widetilde{M}_1, u^* + \frac{e^{-\lambda t}}{u^*} \right\}, \\ \bar{\varphi}(\cdot, t) &= \min \left\{ v^* e^{\lambda_0 t} e^{-(\kappa + \alpha^*)\theta}, \widetilde{M}_2(\cdot), w^*(\cdot) + \varepsilon_2 e^{-\kappa\theta} e^{c\lambda\theta} e^{-\lambda t} \right\}, \\ \bar{\psi}(t) &= \min \left\{ v^* e^{\lambda_0 t}, \widetilde{M}_3, v^* + \varepsilon_3 e^{-\lambda t} \right\}, \end{aligned}$$

where $\lambda \in (0, \lambda_0]$ is a constant to be chosen later. It is easy to see $\bar{\xi}(t)$ and $\underline{\xi}(t)$ satisfy (P_1) - (P_2) . Next, we will show that $\bar{\xi}(t)$ and $\underline{\xi}(t)$ are respectively the lower and upper solutions of system (30) under some suitable conditions.

Lemma 4.4. $\underline{\xi}(t) = (\underline{\phi}(t), \underline{\varphi}(\cdot, t), \underline{\psi}(t))$ is a lower solution of system (30).

Proof. For $\underline{\phi}(t)$, we consider the following two cases.

Case 1. If $t \leq (1/\lambda) \ln(\varepsilon_1/u^*)$, then $\underline{\phi}(t) = 0$. It follows that

$$-c\underline{\phi}'(t) - \underline{\phi}(t) + \frac{(1 - \underline{\phi}(t))\underline{\psi}(t)}{1 + a\underline{\psi}(t)} = \frac{\underline{\psi}(t)}{1 + a\underline{\psi}(t)} \geq 0.$$

Case 2. If $t \geq (1/\lambda) \ln(\varepsilon_1/u^*)$, then $\underline{\phi}(t) = u^* - \varepsilon_1 e^{\lambda t}$. It follows from $\underline{\psi}(t) \geq v^* - 2v^* e^{-\lambda t}/u^*$ that

$$\begin{aligned} & -c\underline{\phi}'(t) - \underline{\phi}(t) + \frac{(1 - \underline{\phi}(t))\underline{\psi}(t)}{1 + a\underline{\psi}(t)} \\ & \geq -c\varepsilon_1 \lambda e^{\lambda t} - u^* + \varepsilon_1 e^{\lambda t} + \frac{(1 - u^*)(v^* - 2v^* \frac{e^{-\lambda t}}{u^*})}{1 + a(v^* - 2v^* \frac{e^{-\lambda t}}{u^*})} \\ & \geq -c\varepsilon_1 \lambda e^{\lambda t} - u^* + \varepsilon_1 e^{\lambda t} + \frac{(1 - u^*)v^*}{1 + av^*} - \frac{(1 - u^*)2v^* \frac{e^{-\lambda t}}{u^*}}{1 + av^*} \\ & = -c\varepsilon_1 \lambda e^{\lambda t} + (\varepsilon_1 - 2)e^{\lambda t} := I_1(\lambda). \end{aligned}$$

Notice that $I_1(0) = \varepsilon_1 - 2 > 0$. Then, there exists $\lambda_1^* > 0$ such that

$$I_1(\lambda) > 0, \quad \forall \lambda \in (0, \lambda_1^*),$$

and therefore the inequality (4.11) holds.

For $\underline{\varphi}(t)$, we also consider two cases.

Case 3. If $t \leq (1/\lambda) \ln(1/u^*) + c\theta$, then $\underline{\varphi}(\cdot, t) = 0$. It follows that (4.12) holds.

Case 4. If $t \geq (1/\lambda) \ln(1/u^*) + c\theta$, then $\underline{\varphi}(\cdot, t) = w^*(\cdot) - w^*(\cdot) \frac{e^{c\lambda\theta - \lambda t}}{u^*}$ and $\underline{\varphi}(0, t) = u^* - e^{-\lambda t}$. Using $\bar{\phi}(t) = u^* + e^{-\lambda t}/u^*$ and $\underline{\psi}(t) \geq v^* - 2v^* e^{-\lambda t}/u^*$, we have that

$$\begin{aligned} \frac{(1 - \bar{\phi}(t))\underline{\psi}(t)}{1 + a\underline{\psi}(t)} & \geq \frac{(1 - u^* - e^{-\lambda t}/u^*)(v^* - 2v^* e^{-\lambda t}/u^*)}{1 + a(v^* - 2v^* e^{-\lambda t}/u^*)} \\ & \geq \frac{[(1 - u^*) - e^{-\lambda t}/u^*](v^* - 2v^* e^{-\lambda t}/u^*)}{1 + av^*} \\ & = u^* - 2e^{-\lambda t} - \frac{e^{-\lambda t}}{1 - u^*} + \frac{2e^{-2\lambda t}}{u^*(1 - u^*)}. \end{aligned}$$

To verify that $\underline{\varphi}(\cdot, t)$ satisfies (4.12), we need only to prove that

$$u^* - 2e^{-\lambda t} - \frac{e^{-\lambda t}}{1 - u^*} + \frac{2e^{-2\lambda t}}{u^*(1 - u^*)} \geq u^* - e^{-\lambda t} = \underline{\varphi}(0, t),$$

which is equivalent to

$$I_2(\lambda) := -e^{-\lambda t} - \frac{e^{-\lambda t}}{1 - u^*} + \frac{2e^{-2\lambda t}}{u^*(1 - u^*)} > 0.$$

It then follows from

$$I_2(0) = (u^*)^2 - 2u^* + 2 = (u^* - 1)^2 + 1 > 0,$$

that there exists $\lambda_2^* > 0$ such that $I_2(\lambda) > 0, \forall \lambda \in (0, \lambda_2^*)$, and therefore (4.12) holds.

For $\underline{\psi}(t)$, we consider the following two cases.

Case 5. If $t \leq (1/\lambda) \ln(2/u^*)$, then $\underline{\psi}(t) = 0$. It follows that

$$D\underline{\psi}''(t) - c\underline{\psi}'(t) + \int_0^\infty \sigma(\theta)\underline{\varphi}(\theta, t)d\theta - \rho_3\underline{\psi}(t) = \int_0^\infty \sigma(\theta)\underline{\varphi}(\theta, t)d\theta \geq 0.$$

Case 6. If $t \geq (1/\lambda) \ln(2/u^*)$, then $\underline{\psi}(t) = v^* - 2v^*e^{-\lambda t}/u^*$. Since $\underline{\varphi}(\cdot, t) \geq w^*(\cdot) - w^*(\cdot)e^{c\lambda\theta - \lambda t}/u^*$, it then follows that

$$\begin{aligned} & D\underline{\psi}''(t) - c\underline{\psi}'(t) + \int_0^\infty \sigma(\theta)\underline{\varphi}(\theta, t)d\theta - \rho_3\underline{\psi}(t) \\ & \geq -2D\frac{v^*}{u^*}\lambda^2e^{-\lambda t} - 2c\frac{v^*}{u^*}\lambda e^{-\lambda t} \\ & \quad + \int_0^\infty \sigma(\theta) \left(w^*(\theta) - w^*(\theta)\frac{e^{c\lambda\theta - \lambda t}}{u^*} \right) d\theta - \rho_3 \left(v^* - 2v^*\frac{e^{-\lambda t}}{u^*} \right) \\ & = -2D\frac{v^*}{u^*}\lambda^2e^{-\lambda t} - 2c\frac{v^*}{u^*}\lambda e^{-\lambda t} - \int_0^\infty \sigma(\theta)w^*(\theta, t)\frac{e^{c\lambda\theta - \lambda t}}{u^*}d\theta + 2\rho_3v^*\frac{e^{-\lambda t}}{u^*}. \end{aligned}$$

Let

$$I_3(\lambda) = -2D\frac{v^*}{u^*}\lambda^2e^{-\lambda t} - 2c\frac{v^*}{u^*}\lambda e^{-\lambda t} - \int_0^\infty \sigma(\theta)w^*(\theta, t)\frac{e^{c\lambda\theta - \lambda t}}{u^*}d\theta + 2\rho_3v^*\frac{e^{-\lambda t}}{u^*}.$$

Clearly,

$$I_3(0) = \left[- \int_0^\infty \sigma(\theta)w^*(\theta, t)d\theta + 2\rho_3v^* \right] \frac{1}{u^*} = \frac{\rho_3v^*}{u^*} > 0.$$

Hence, there exists $\lambda_3^* > 0$ such that

$$I_3(\lambda) > 0, \forall \lambda \in (0, \lambda_3^*),$$

and therefore the inequality (4.13) holds. Then, taking $\lambda \in (0, \min\{\lambda_1^*, \lambda_2^*, \lambda_3^*\})$ we know that $\underline{\xi}(t) = (\underline{\phi}(t), \underline{\varphi}(\cdot, t), \underline{\psi}(t))$ is an lower solution of system (30). \square

Lemma 4.5. $\bar{\xi}(t) = (\bar{\phi}(t), \bar{\varphi}(\cdot, t), \bar{\psi}(t))$ is an upper solution of system (30).

Proof. For $\bar{\phi}(t)$, we consider the following three cases.

Case 1. If $t \leq (1/\lambda_0) \ln(1/u^*)$, then $\bar{\phi}(t) = u^*e^{\lambda_0 t}$. It follows from $\bar{\psi}(t) \leq v^*e^{\lambda_0 t}$ and (38) that

$$-c\bar{\phi}'(t) - \bar{\phi}(t) + \frac{(1 - \bar{\phi}(t))\bar{\psi}(t)}{1 + a\bar{\psi}(t)} = -cu^*\lambda_0e^{\lambda_0 t} - u^*e^{\lambda_0 t} + v^*e^{\lambda_0 t} \leq 0.$$

Case 2. If $(1/\lambda_0) \ln(1/u^*) \leq t \leq -(1/\lambda) \ln(u^*(1 - u^*))$, then $\bar{\phi}(t) = 1$. It follows that

$$-c\bar{\phi}'(t) - \bar{\phi}(t) + \frac{(1 - \bar{\phi}(t))\bar{\psi}(t)}{1 + a\bar{\psi}(t)} = -1 \leq 0.$$

Case 3. If $t \geq -(1/\lambda) \ln(u^*(1 - u^*))$, then $\bar{\phi}(t) = u^* + e^{-\lambda t}/u^*$. It follows from $\bar{\psi}(t) \leq \widetilde{M}_3$ that

$$\begin{aligned} & -c\bar{\phi}'(t) - \bar{\phi}(t) + \frac{(1 - \bar{\phi}(t))\bar{\psi}(t)}{1 + a\bar{\psi}(t)} \\ & \leq c\lambda\frac{e^{-\lambda t}}{u^*} - u^* - \frac{e^{-\lambda t}}{u^*} + \left(1 - u^* - \frac{e^{-\lambda t}}{u^*} \right) \widetilde{M}_3 \\ & = e^{-\lambda t} \left(\frac{c\lambda}{u^*} - u^*e^{\lambda t} - \frac{1}{u^*} + \left(1 - u^* - \frac{e^{-\lambda t}}{u^*} \right) \widetilde{M}_3e^{\lambda t} \right) \\ & < e^{-\lambda t}I_4(\lambda), \end{aligned}$$

where

$$I_4(\lambda) = 1 + \frac{c\lambda}{u^*} - u^* e^{\lambda t} - \frac{1}{u^*} + \left(1 - u^* - \frac{e^{-\lambda t}}{u^*}\right) \widetilde{M}_3 e^{\lambda t}.$$

Since $0 \leq u^* \leq 1$, then

$$I_4(0) = \left(1 - u^* - \frac{1}{u^*}\right) (\widetilde{M}_3 + 1) < 0.$$

Then, there exists $\lambda_4^* > 0$ such that

$$I_4(\lambda) < 0, \quad \forall \lambda \in (0, \lambda_4^*).$$

Hence, it follows from cases 1-3 that inequality (4.8) holds.

Next, for $\overline{\varphi}(\cdot, t)$. Similarly, we also consider the following three cases.

Case 4. If $t \leq (1/\lambda_0) \ln(1/(av^*)) + c\theta$, then $\overline{\varphi}(\cdot, t) = v^* e^{\lambda_0 t} e^{-(\kappa + \alpha^*)\theta}$. It then follows from $0 \leq \overline{\psi}(t) \leq v^* e^{\lambda_0 t}$ that

$$\frac{(1 - \underline{\phi}(t))\overline{\psi}(t)}{1 + a\overline{\psi}(t)} \leq \frac{\overline{\psi}(t)}{1 + a\overline{\psi}(t)} < v^* e^{\lambda_0 t} = \overline{\varphi}(0, t).$$

Case 5. If $(1/\lambda_0) \ln(1/(av^*)) + c\theta \leq t \leq c\theta - (1/\lambda) \ln[(1/a - u^*)/\varepsilon_2]$, then $\overline{\varphi}(\cdot, t) = \widetilde{M}_2(\theta)$. Obviously,

$$\frac{(1 - \underline{\phi}(t))\overline{\psi}(t)}{1 + a\overline{\psi}(t)} < \frac{1}{a} = \overline{\varphi}(0, t).$$

Case 6. If $t \geq c\theta - (1/\lambda) \ln[(1/a - u^*)/\varepsilon_2]$, then $\overline{\varphi}(\cdot, t) = w^*(\cdot) + \varepsilon_2 e^{-\kappa\theta} e^{c\lambda\theta} e^{-\lambda t}$. Since $\overline{\varphi}(0, t) = u^* + \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t) \geq 0$ and $\overline{\psi}(t) \leq \widetilde{M}_3$, we need only to prove that

$$\frac{(1 - \underline{\phi}(t))\overline{\psi}(t)}{1 + a\overline{\psi}(t)} \leq \widetilde{M}_3 \leq u^* + \varepsilon_2 e^{-\lambda t}.$$

Let

$$I_5(\lambda) := u^* + \varepsilon_2 e^{-\lambda t} - \widetilde{M}_3,$$

It then follows from (39) that

$$I_5(0) = u^* + \varepsilon_2 - \widetilde{M}_3 > 0,$$

and therefore there exists $\lambda_5^* > 0$ such that $I_5(\lambda) > 0, \forall \lambda \in (0, \lambda_5^*)$, and therefore (4.9) holds.

Now, we verify that inequality (4.10) holds by considering the following three cases.

Case 7. If $t \leq (1/\lambda_0) \ln(\widetilde{M}_3/v^*)$, then $\overline{\psi}(t) = v^* e^{\lambda_0 t}$ and $\overline{\varphi}(\cdot, t) \leq v^* e^{\lambda_0 t} e^{-(\kappa + \alpha^*)\theta}$. Using (31) and (38) we have

$$\begin{aligned} & D\overline{\psi}''(t) - c\overline{\psi}'(t) + \int_0^\infty \sigma(\theta)\overline{\varphi}(\theta, t)d\theta - \rho_3\overline{\psi}(t) \\ & \leq D\lambda_0^2 v^* e^{\lambda_0 t} - c\lambda_0 v^* e^{\lambda_0 t} + \int_0^\infty \sigma(\theta)v^* e^{\lambda_0 t} e^{-(\kappa + \alpha^*)\theta} d\theta - \rho_3 v^* e^{\lambda_0 t} \\ & = (D\lambda_0 - c)\lambda_0 v^* e^{\lambda_0 t} \leq 0. \end{aligned}$$

Case 8. If $(1/\lambda_0) \ln(\widetilde{M}_3/v^*) \leq t \leq (1/\lambda) \ln(\varepsilon_3/(\widetilde{M}_3 - v^*))$, then $\overline{\psi}(t) = \widetilde{M}_3$ and $\overline{\varphi}(\cdot, t) \leq \widetilde{M}_2(\theta)$. Obviously,

$$D\overline{\psi}''(t) - c\overline{\psi}'(t) + \int_0^\infty \sigma(\theta)\overline{\varphi}(\theta, t)d\theta - \rho_3\overline{\psi}(t) \leq \int_0^\infty \sigma(\theta)\widetilde{M}_2(\theta)d\theta - \rho_3\widetilde{M}_3 = 0.$$

Case 9. If $t \geq (1/\lambda) \ln(\varepsilon_3/(\widetilde{M}_3 - v^*))$, then $\overline{\psi}(t) = v^* + \varepsilon_3 e^{-\lambda t}$ and $\overline{\varphi}(\cdot, t) \leq \widetilde{M}_2(\theta)$. Therefore,

$$\begin{aligned} & D\overline{\psi}''(t) - c\overline{\psi}'(t) + \int_0^\infty \sigma(\theta)\overline{\varphi}(\theta, t)d\theta - \rho_3\overline{\psi}(t) \\ & \leq D\varepsilon_3\lambda^2 e^{-\lambda t} + c\varepsilon_3\lambda e^{-\lambda t} + \int_0^\infty \sigma(\theta)\widetilde{M}_2(\theta)d\theta - \rho_3(v^* + \varepsilon_3 e^{-\lambda t}) \\ & := I_6(\lambda). \end{aligned}$$

Using (39) we have that

$$\begin{aligned} I_6(0) &= \int_0^\infty \sigma(\theta)\widetilde{M}_2(\theta)d\theta - \rho_3(v^* + \varepsilon_3) \\ &< \int_0^\infty \sigma(\theta)\widetilde{M}_2(\theta)d\theta - \rho_3\widetilde{M}_3 = 0, \end{aligned}$$

and therefore there exists $\lambda_6^* > 0$ such that $I_6(\lambda) \leq 0, \forall \lambda \in (0, \lambda_6^*)$, and therefore (4.10) holds. \square

In order to complete the proof of Theorem 4.2 (i), it remains to prove the convergence of the solutions to the infected steady state E^* as $t \rightarrow +\infty$. This result can be obtained by using a suitable Lyapunov functional.

Let

$$g(\theta) = \int_\theta^{+\infty} \sigma(\varsigma)e^{-\kappa(\varsigma-\theta)} d\varsigma. \tag{40}$$

From (28) we have $g(0) = K$. By using (29), we define a set

$$\mathcal{C} = \left\{ \begin{array}{l} (\phi, \varphi, \psi) \in C^1(\mathbb{R}^2, (0, \infty)) \times C^{1,1}([0, \infty) \times \mathbb{R}, (0, \infty)) : \\ \phi > 0, \varphi > 0, \psi > 0, \exists M > 0, \frac{\varphi(\theta, t)}{w^*(\theta)} \leq M, \\ g(\cdot)\mathcal{H}\left(\frac{\varphi(\cdot, t)}{w^*(\cdot)}\right) \in L^1((0, \infty), \mathbb{R}), \forall t \in \mathbb{R} \end{array} \right\},$$

and for each $(\phi, \varphi, \psi) \in \mathcal{C}$ consider a functional $V(\phi, \varphi, \psi)(t) : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$V(\phi, \varphi, \psi)(t) = cW(\phi, \varphi, \psi)(t) + D\frac{v^*}{K}\psi'(t) \left(\frac{1}{\psi(t)} - \frac{1}{v^*} \right), \tag{41}$$

where

$$\begin{aligned} W(\phi, \varphi, \psi)(t) &= (1 - u^*)\mathcal{H}\left(\frac{1 - \phi(t)}{1 - u^*}\right) + \frac{v^*}{K}\mathcal{H}\left(\frac{\psi(t)}{v^*}\right) \\ &\quad + \frac{1}{K} \int_0^{+\infty} g(\theta)w^*(\theta)\mathcal{H}\left(\frac{\varphi(\theta, t)}{w^*(\theta)}\right) d\theta. \end{aligned}$$

Then we have the following result.

Lemma 4.6. (Lyapunov properties). Let $(u(t), w(\theta, t), v(t))$ be a positive solution of system (30) such that there exists some constant $M > 1$ and

$$0 \leq u(t) < 1 - \frac{1}{M}, \quad w(\theta, t) \leq Mw^*(\theta), \quad \text{and} \quad \frac{1}{M} \leq v(t) \leq \widetilde{M}_3. \tag{42}$$

Then there exists some constant $m > 0$ (only depending on M) such that

$$-m \leq V(u, w, v)(t) \leq m + cW(u, w, v)(t) < +\infty, \quad \forall t \in \mathbb{R}, \tag{43}$$

and the map $t \rightarrow V(u, w, v)(t)$ is non-increasing. Moreover, if $t \rightarrow V(u, w, v)(t)$ is constant then

$$u \equiv u^*, \quad w \equiv w^*, \quad v \equiv v^*, \quad u' \equiv 0, \quad w_t \equiv 0, \quad \text{and } v' \equiv 0.$$

Proof of Lemma 4.6. From Lemmas 4.4-4.5, we obtain that (42) holds. Since v is bounded, we obtain that v is also bounded in $W^{2,\infty}(\mathbb{R})$. Therefore we have

$$\left| D \frac{v^*}{K} v'(t) \left(\frac{1}{v(t)} - \frac{1}{v^*} \right) \right| \leq D \frac{v^*}{K} M \|v'\|_\infty \left(1 + \frac{\widetilde{M}_3}{v^*} \right).$$

From the definition of function \mathcal{H} we have $0 \leq W(u, w, v)(t)$ for all $t \in \mathbb{R}$. Now we can claim that

$$W(u, w, v)(t) < \infty \quad \text{for all } t \in \mathbb{R}. \tag{44}$$

To this end, we need only to prove that

$$g(\cdot) \mathcal{H} \left(\frac{w(\cdot, t)}{w^*(\cdot)} \right) \in L^1((0, \infty), \mathbb{R}), \quad \forall t \in \mathbb{R}. \tag{45}$$

To check this, let $t_0 > 0$ be given. Since $w(0, t) > 0$ for all $t \in \mathbb{R}$, there exists $\epsilon > 0$ such that

$$w(0, t) \geq \epsilon, \quad \forall t \in [-t_0, t_0].$$

We consider $\lambda_T \in \mathbb{R}$ the eigenvalue and j an associated eigenvector of the following problem:

$$-cj'(t) - \kappa j(t) = \lambda_T j(t), \quad t \in [-t_0, t_0], \quad j(0) = \frac{\epsilon}{2},$$

Moreover, we assume that $0 < j(t) \leq \epsilon$ on $(-t_0, t_0)$ and set

$$i(\theta, t) = e^{\lambda_T \theta} j(t), \quad \theta \geq 0, \quad t \in [-t_0, t_0].$$

It then follows from the comparison principle we have that $i(\theta, t) \leq w(\theta, t)$ for all $\theta \geq 0$ and $t \in [-t_0, t_0]$. As a consequence, we obtain that for each $t \in [-t_0/2, t_0/2]$,

$$\left| \ln \left(\frac{w(\cdot, t)}{w^*(\cdot)} \right) \right| \leq \max \left\{ |\ln M|, \left| \ln \frac{j(t)}{w^*(0)} \right| + (|\lambda_T| + \kappa)\theta \right\}.$$

Therefore, we have that for each $t_0 > 0$, there exist two constants $M_{t_0} > 0$ and $m_{t_0} > 0$ such that

$$\left| \ln \left(\frac{w(\cdot, t)}{w^*(\cdot)} \right) \right| \leq M_{t_0} + m_{t_0} \theta, \quad \forall \theta \geq 0, \quad \forall t \in [-t_0, t_0].$$

From (40) we obtain that the map $\theta \rightarrow (1 + \theta)g(\theta)$ belongs to $L^1((0, \infty), \mathbb{R})$, and then the claim (44) follows. Therefore, (43) holds since $c > 0$.

Let us now show that the map $t \rightarrow V(u, w, v)(t)$ is decreasing.

$$\begin{aligned} \frac{dV(u, w, v)(t)}{dt} &= -cu'(t) \left(1 - \frac{1 - u^*}{1 - u(t)} \right) + \frac{1}{K} \int_0^{+\infty} g(\theta) \left(\frac{w^*(\theta)}{w(\theta, t)} - 1 \right) (-c\partial_t w) d\theta \\ &\quad + D(v''(t) - cv'(t)) \frac{v^*}{K} \left(\frac{1}{v(t)} - \frac{1}{v^*} \right) - D \frac{v^*}{K} \frac{v'^2(t)}{v^2(t)} \\ &= \frac{u^* - u(t)}{1 - u(t)} \left(u(t) - u^* + \frac{(1 - u^*)v^*}{1 + av^*} - \frac{(1 - u(t))v(t)}{1 + av(t)} \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{K} \int_0^{+\infty} g(\theta)w^*(\theta) \left(\frac{1}{w(\theta,t)} - \frac{1}{w^*(\theta)} \right) (\partial_\theta w + \kappa w) d\theta \\
 & + \frac{1}{K} \left(\frac{v^*}{v(t)} - 1 \right) \left(\rho_3 v(t) - \int_0^{+\infty} \sigma(\theta)w(\theta,t) d\theta \right) - D \frac{v^* v'^2(t)}{K v^2(t)} \\
 = & - \frac{(u^* - u(t))^2}{1 - u(t)} + \left(1 - \frac{1 - u^*}{1 - u(t)} \right) \left(\frac{(1 - u^*)v^*}{1 + av^*} - \frac{(1 - u(t))v(t)}{1 + av(t)} \right) \\
 & - \frac{1}{K} \int_0^{+\infty} g(\theta)w^*(\theta) \partial_\theta \left(\mathcal{H} \left(\frac{w(\theta,t)}{w^*(\theta)} \right) \right) d\theta - D \frac{v^* v'^2(t)}{K v^2(t)} \\
 & + \frac{1}{K} \left(\frac{v^*}{v(t)} - 1 \right) \left(\rho_3 v(t) - \int_0^{+\infty} \sigma(\theta)w(\theta,t) d\theta \right).
 \end{aligned}$$

Noticing that

$$\begin{aligned}
 \frac{d}{d\theta} \left(g(\theta)w^*(\theta) \right) & = \sigma(\theta)w^*(\theta), \\
 w(0,t) & = \frac{(1 - u(t))v(t)}{1 + av(t)}, \quad w^*(0) = \frac{(1 - u^*)v^*}{1 + av^*}, \\
 w^*(0) & = \frac{1}{K} \int_0^{+\infty} \sigma(\theta)w^*(\theta) d\theta, \quad \frac{\rho_3}{K} = \frac{1 - u^*}{1 + av^*},
 \end{aligned}$$

by simple calculations, we have that

$$\begin{aligned}
 \frac{dV(u, w, v)(t)}{dt} & = - \frac{(u^* - u(t))^2}{1 - u(t)} - D \frac{v^* v'^2(t)}{K v^2(t)} - \mathcal{H} \left(\frac{v(t)}{v^*} \right) \\
 & - \frac{1}{K} \int_0^{+\infty} \sigma(\theta)w^*(\theta) \mathcal{H} \left(\frac{v^*w(\theta,t)}{v(t)w^*(\theta)} \right) d\theta - w^*(0) \ln \frac{w(0,t)}{w^*(0)} \\
 & - \frac{1 - u^*}{1 - u(t)} \left[w(0,t) \mathcal{H} \left(\frac{w^*(0)}{w(0,t)} \right) + w(0,t) + \ln \frac{w^*(0)}{w(0,t)} \right].
 \end{aligned}$$

It then follows from $0 \leq u(t) \leq 1, 0 \leq u^* \leq 1$ we have that

$$\frac{dV(u, w, v)(t)}{dt} \leq 0,$$

and therefore the map $t \rightarrow V(u, w, v)(t)$ is decreasing. This completes the proof of Lemma 4.6. \square

Now, we shall use Lemma 4.6 to prove the final step in the proof of Theorem 4.2 (i).

Proof of Theorem 4.2 (i). As in Lemma 4.6, we again point out that the condition (42) holds. Now, we denote $\{t_n\}_{n \geq 0}$ as an increasing sequence of positive real numbers, and $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Then we obtain the following sequences of functions:

$$u_n(t) = u(t + t_n), \quad w_n(\theta, t) = w(t + t_n, \theta), \quad v_n(t) = u(t + t_n).$$

Due to exponential and elliptic estimates, one may assume that the sequences $\{u_n\}, \{v_n\}$ and $\{w_n\}$ converge towards some functions \tilde{u}, \tilde{v} and \tilde{w} for the topology of $C_{loc}^1(\mathbb{R}^2, (0, \infty)) \times C_{loc}^{1,1}([0, \infty) \times \mathbb{R}, (0, \infty))$. Then, by use of (42), we have that functions \tilde{u}, \tilde{v} and \tilde{w} satisfy

$$0 \leq \tilde{u} < 1 - \frac{1}{M}, \quad \tilde{w} \leq Mw^*(\theta), \quad \text{and} \quad \frac{1}{M} \leq \tilde{v} \leq \tilde{M}_3. \tag{46}$$

Moreover, since the map $t \rightarrow V(u, w, v)(t)$ is decreasing we obtain that for each $n \geq 0$

$$V(u_n, w_n, v_n)(t) = V(\tilde{u}, \tilde{w}, \tilde{v})(t + t_n) \leq V(u, w, v)(t), \quad \forall t \in \mathbb{R}.$$

From Lemma 4.6, it is bounded and then there exists $\zeta \in \mathbb{R}$ such that

$$\lim_{t \rightarrow +\infty} V(u_n, w_n, v_n)(t) = \zeta, \quad \forall t \in \mathbb{R}.$$

By using (42), (46) and Lebesgue convergence theorem we have

$$\lim_{t \rightarrow +\infty} V(u_n, w_n, v_n)(t) = V(\tilde{u}, \tilde{w}, \tilde{v})(t), \quad \forall t \in \mathbb{R},$$

and therefore $V(\tilde{u}, \tilde{w}, \tilde{v})(t) = \zeta$ for $t \in \mathbb{R}$.

Finally, we can find that $(\tilde{u}, \tilde{w}, \tilde{v})$ is a solution of system (30). It then follows from the last part of Lemma 4.6 that

$$\tilde{u} \equiv u^*, \quad \tilde{w} \equiv w^*, \quad \tilde{v} \equiv v^*, \quad \tilde{u}' \equiv 0, \quad \tilde{w}' \equiv 0, \quad \text{and} \quad \tilde{v}' \equiv 0.$$

That is to say,

$$\lim_{t \rightarrow +\infty} \tilde{u}(t) = u^*, \quad \lim_{t \rightarrow +\infty} \tilde{w}(\theta, t) = w^*(\theta) \in L^1((0, +\infty), \mathbb{R}), \quad \text{and} \quad \lim_{t \rightarrow +\infty} \tilde{v}(t) = v^*.$$

This completes the proof of Theorem 4.2 (i). □

4.2.2. *Non-existence result.* Assume that $R_0 < 1$, and there is a non-negative solution (u, w, v) of system (30) such that

$$\begin{aligned} 0 &\leq u(t) \leq 1, \\ \partial_\theta w(\theta, t) + c\partial_t w(\theta, t) &= -\kappa w(\theta, t), \quad \theta > 0, \quad t \in \mathbb{R}, \\ w(0, t) &= \frac{(1 - u(t))v(t)}{1 + av(t)}, \quad t \in \mathbb{R}, \\ Dv''(t) - cv'(t) + \int_0^\infty \sigma(\theta)w(\theta, t)d\theta - \rho_3v(t) &= 0. \end{aligned}$$

Since $0 \leq u(t) \leq 1$, we have

$$w(0, t) = \frac{(1 - u(t))v(t)}{1 + av(t)} \leq v(t),$$

where $v(t)$ is bounded for $t \in \mathbb{R}$. Using the comparison principle we obtain that for any $\theta \geq 0$

$$\sup_{t \in \mathbb{R}} w(\theta, t) \leq e^{-\kappa\theta} \|w(0, \cdot)\|_\infty \leq e^{-\kappa\theta} v(t).$$

Thus,

$$\begin{aligned} 0 &\leq Dv''(t) - cv'(t) + \int_0^\infty \sigma(\theta)e^{-\kappa\theta}v(t)d\theta - \rho_3v(t) \\ &= Dv''(t) - cv'(t) + \rho_3(R_0 - 1)v(t). \end{aligned}$$

It then follows from $R_0 < 1$ and the maximum principle that $v \equiv 0$. Accordingly,

$$w(0, t) = \frac{(1 - u(t))v(t)}{1 + av(t)} \equiv 0,$$

and therefore $w \equiv 0$.

5. Numerical simulations. In this section, by taking the explicit function forms of $p(\theta)$ and $\delta(\theta)$ used in [35, 20], we perform some numerical simulations to illustrate the theoretical results obtained in Sections 3 and 4. The infection age-specific viral production rate takes the form

$$p(\theta) = \begin{cases} P_{max}(1 - \exp^{-b(\theta-d_1)}), & \theta \geq d_1, \\ 0, & \text{otherwise,} \end{cases} \tag{47}$$

where b controls how rapidly the saturation level, P_{max} , is reached. The term d_1 represents the delay in viral production; that is, it takes time d_1 days after initial infection for the first viral particles to be produced. The death rate of infected cells takes the form

$$\delta(\theta) = \begin{cases} \delta_0 & \theta < d_2, \\ \delta_0 + \delta_m(1 - \exp^{-c(\theta-d_2)}), & \theta \geq d_2 \end{cases} \tag{48}$$

where δ_0 is the background death rate, $\delta_0 + \delta_m$ is the maximal death rate, c controls the time to saturation, and d_2 is the delay between infection and the onset of cell-mediated killings.

From Assumption 2.1, the functions $p(\theta)$ and $\delta(\theta)$ belong to $L^{\infty}_+((0, +\infty), \mathbb{R}) \setminus \{0_{L^{\infty}}\}$, then there exist constants $\bar{\theta}_1, \bar{\theta}_2 > 0$ satisfying

$$\bar{\theta}_1 = \inf \left\{ \theta : \int_{\theta}^{\infty} p(\tau) d\tau = 0 \right\} \text{ and } \bar{\theta}_2 = \inf \left\{ \theta : \int_{\theta}^{\infty} \delta(\tau) d\tau = 0 \right\}. \tag{49}$$

For sake of convenience, we assume that $\bar{\theta}_1 = \bar{\theta}_2 = \bar{\theta}$ in the sequel. We take the parameter values in (47) and (48) as follows:

$$b = 1, d_1 = 0, c = 0.5, \delta_0 = 0.1, \delta_m = 1, d_2 = 0. \tag{50}$$

When the age factor is presented, it is difficult to simulate system (3) directly. Notice that the liver can be separated into several segments, including the left lobe of the liver, spigelian lobe, quadrate lobe of the liver and the right lobe of the liver. For the sake of mathematical simplification, we suppose that the space is homogeneous in each segment and the length of each segment is equivalent. Thus, we can use a lattice differential equation to replace system (3).

Now, we divide the space into four lattices and let $U_i(t)$ denote the variable in the i th lattice at time t , $i = 1, 2, 3, 4$. Since the Neumann boundary conditions (5) describe a situation where viral particles do not leave the domain, they are “reflected” at the boundary. Similarly to [49, 4], we define the discrete Laplacian as

$$(\Delta U)_1 = U_2 - U_1, (\Delta U)_i = U_{i-1} + U_{i+1} - 2u_i, i = 2, 3, (\Delta U)_4 = U_3 - U_4.$$

Therefore, system (3) can be expressed as lattice differential equations (see Eq.(4.4) in [49]), and the dimension of Ω is one in this situation. For simplification, we take the initial conditions as follows:

$$\begin{aligned} u_1(0) &= 3 \times 10^6, w_1(\theta, 0) = 0, v_1(0) = 500, \\ u_i(0) &= 0, w_i(\theta, 0) = 0, v_i(0) = 0, i = 2, 3, 4. \end{aligned} \tag{51}$$

As in [53] we take in system (3)

$$s = 10^7, \beta = 5 \times 10^{10}, \mu = 0.1, \alpha = 0.002, d = 5, \tag{52}$$

and take a day as the unit time.

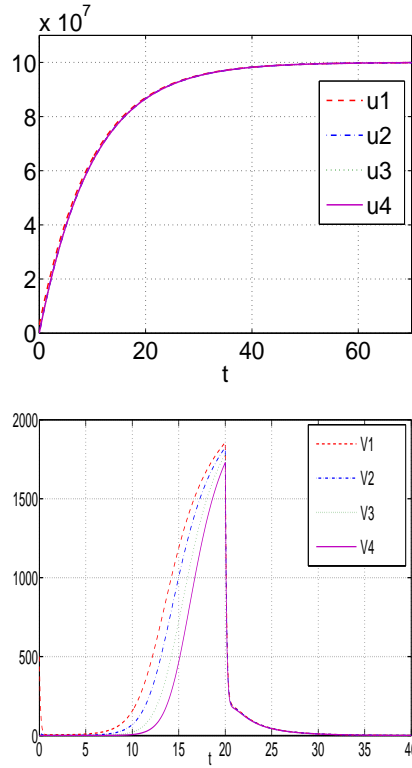


FIGURE 1. $P_{max} = 50$, $R_0 = 0.5785 < 1$, the uninfected steady state E^0 of model (3) is globally stable.

Example 5.1. In system (3), we set $P_{max} = 50$, $\bar{\theta} = 20$ and $D = 0.5$. Then system (3) with parameter values (51)-(52) has an unique uninfected steady state $E^0 = (s/\mu, 0, 0)$. It follows from the direct computation that $R_0 = 0.5785 < 1$. By Theorem 3.1, we see that E^0 is globally asymptotically stable. Numerical simulation illustrates our result (see Fig. 1).

Example 5.2. In system (3), we set $P_{max} = 350$, $\bar{\theta} = 20$ and $D = 0.5$. Then system (3) with parameter values (51)-(52) has a infected steady state $E^* = (u^*, w^*, v^*)$ (defined in Lemma 2.1). It follows from the direct computation that $R_0 = 4.0493 > 1$. By Theorem 3.2, we see that E^* is globally asymptotically stable. It follows from Theorem 4.2 that system (3) always has a traveling wave solution with speed $c > c^*$ connecting E^0 and E^* . Numerical simulation illustrates our result (see Fig. 2).

Next, we will simulate system (3) to study the effect of age factor under different diffusion coefficients. Here, we take the baseline parameter values as in (52) and (52) and let $P_{max} = 350$. Comparing Figs. 3(a)-3(c) with Figs. 3(b)-3(d), we can find that the solutions of (3) tend to the infection steady state when time t is larger than the value of $\bar{\theta}$, no matter what the value of θ is. In addition, comparing Figs. 3(a)-3(b) with Figs. 3(c)-3(d), we can find that the arrived time with small diffusion coefficient is slower than that with the large diffusion coefficient, which is reasonable since the smaller diffusion coefficient means the slower mobility of virus.

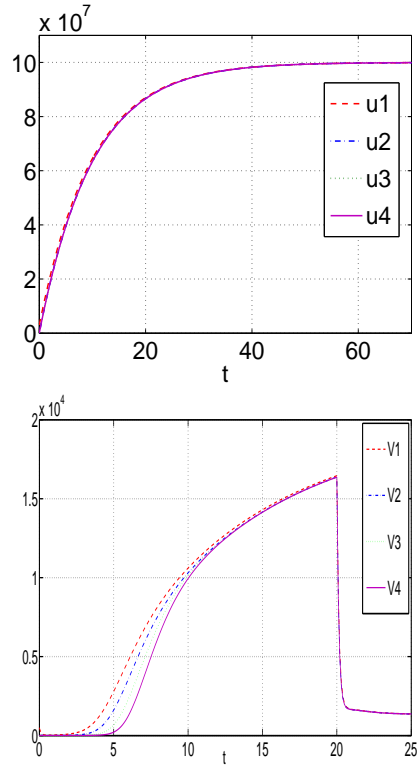


FIGURE 2. $P_{max} = 350$, $R_0 = 4.0493 > 1$, the infected steady state E^* of model (3) is globally stable.

Corollary 4.1 tells us that the condition $1 < R_0 < 1 + \alpha^*$ in Theorem 4.2 can be simplified as $R_0 > 1$ provided $p(\theta) = N\delta(\theta)$. This is true when $p(\theta)$ and $\delta(\theta)$ take respectively the forms of (47)-(48). To show this, we need only to check it by varying the parameter values in the function $p(\theta)$. With the help of Matlab, we can respectively calculate a series of values for α^* and R_0 as the values of P_{max} , c and d_2 vary, and then plot them in Fig. 4, from which we can see that the inequality $R_0 < 1 + \alpha^*$ naturally holds provided $R_0 > 1$.

6. Conclusion and discussion. In this paper, a more general diffusion HBV model is studied in which only the free virus particles have diffusivity. The model incorporates the saturating incidence $\beta uv/(1 + \alpha v)$ ($\alpha > 0$), the infection-age-dependent virion production rate and the death rate of productively infected cells. Mathematically, the results obtained in the present paper can be extended to the limit case when $\alpha = 0$. For this model, we derive the basic reproduction number R_0 . Under the Neumann boundary condition, we discuss the local and global stability of the uninfected steady state and the infected steady state of system (3), respectively. More precisely, the uninfected steady state is globally asymptotically stable when $R_0 \leq 1$, and the infected steady state is globally asymptotically stable when $R_0 > 1$. This implies that the basic reproduction number R_0 plays a sharp threshold role in the HBV infection. On the other hand, by constructing a pair of upper and lower solutions and applying Lemma 2.9 developed by Ducrot *et al.* [10], we prove the

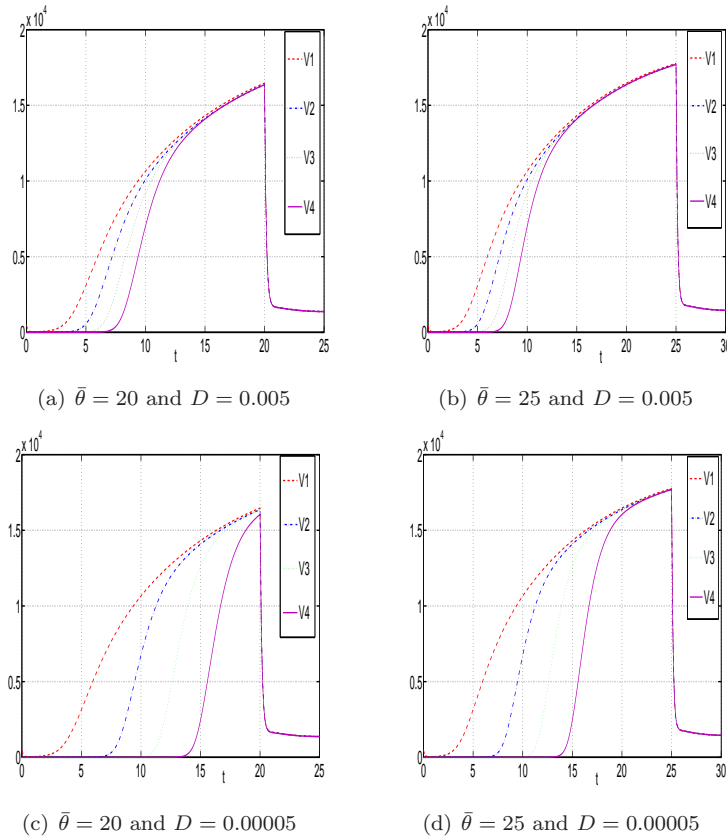
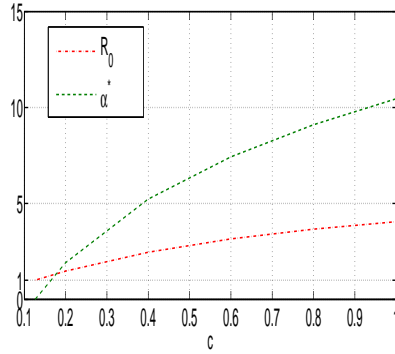


FIGURE 3. The values of $\bar{\theta}$ and D can effect the virus diffusion processes.

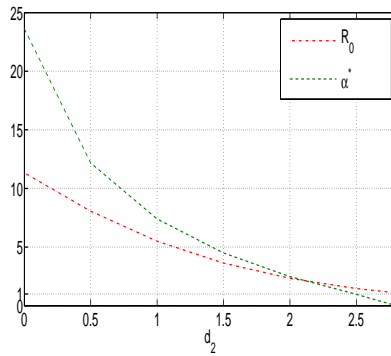
existence of traveling wave solutions for system (27). Specifically, there exists a $c^* := \sqrt{D\alpha^*}$, where α^* is defined in (31), such that system (27) has traveling wave solutions with positive wave speed $c > c^*$ which connecting the two steady states when $1 < R_0 < 1 + \alpha^*$, which reduces to $R_0 > 1$ in the case when $p(\theta) = N\delta(\theta)$.

Theorems 3.2-3.5 show that R_0 may be used to design the control strategies of the virus infection and to estimate the infection level. In the case when $R_0 > 1$, we are concerned with the age distribution and spatial velocity of infection (i.e., spreading speed), and their interactions as viruses move freely in a liver. Theorem 4.2 and Corollary 4.1 confirm the existence of the spreading speed of HBV in system (27). By comparison, it is shown that “bursting” may be more easily result in traveling wave solutions than “budding”. The closely relationship between wave speed and age factor can be clearly seen from the expression of c^* and (31), i.e., the age-since-infection may effect the spatial velocity of infection.

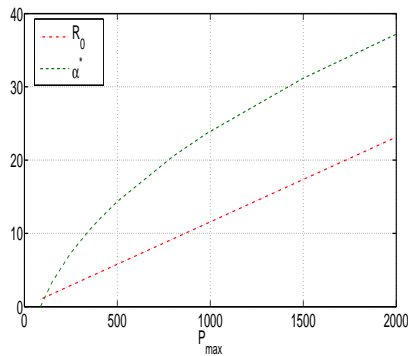
A natural question is whether c^* is the minimal wave speed c_{min} , that is, there does not exist traveling wave solution connecting the two steady states for $0 < c < c_{min}$. Although Ducrot and Magal [10] have obtained some general results on the asymptotic speeds of spread for a class of age-structured epidemic system with diffusion which two speeds indeed coincide, their methods are not valid for system



(a) $P_{max} = 350$ and $d_2 = 0$



(b) $P_{max} = 980$ and $c = 1$



(c) $d_2 = 0, c = 1$

FIGURE 4. $\bar{\theta} = 20$ and other parameters are assumed in (52) and (50)

(27). It is of interest to study the relation of the minimal wave speed and the asymptotic speeds of spread for system (27). We leave it for further investigation.

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Appendix. The proof of Lemma 2.3.

Proof. Let B be a bounded subset in X_+ . Similarly to the proof of Lemma 4.1 in [21], we need only to show that $\Psi(t)$ is asymptotically compact on B in the sense that for any sequences $\phi_n \in B$ and $t_n \rightarrow \infty$, there exist subsequences $\phi_{n_k} \in B$ and $t_{n_k} \rightarrow \infty$ such that $\Psi(t_{n_k})(\phi_{n_k})$ converges in X as $k \rightarrow \infty$. Let $(u_n(x, t), w_n(x, \cdot, t), v_n(x, t)) = \Psi(t)(\phi_n)(x), \forall \phi_n \in X_+, t \geq 0, x \in \bar{\Omega}$. Then, we need only to prove that for any $\varepsilon > 0$, there exists $\zeta > 0$ such that

$$\|u_n(x, t) - u_n(y, t)\| < \varepsilon, \text{ and } \int_0^{+\infty} \|w_n(x, \theta, t) - w_n(y, \theta, t)\| d\theta < \varepsilon, \forall |x - y| < \zeta.$$

For simplification, we define $\bar{u}_n(x, t) := u_n(x, t + t_n), \bar{w}_n(x, \cdot, t) := w_n(x, \cdot, t + t_n)$ and $\bar{v}_n(x, t) := v_n(x, t + t_n), \forall t \geq -t_n, x \in \bar{\Omega}$. In fact, by a direct computation, we have that for all $t \geq -t_n$ and $x, y \in \bar{\Omega}$,

$$\begin{aligned} & \frac{\partial}{\partial t} (\bar{u}_n(x, t) - \bar{u}_n(y, t))^2 \\ &= 2(\bar{u}_n(x, t) - \bar{u}_n(y, t)) \frac{\partial}{\partial t} (\bar{u}_n(x, t) - \bar{u}_n(y, t)) \\ &= 2(\bar{u}_n(x, t) - \bar{u}_n(y, t)) \left(-\mu \bar{u}_n(x, t) - \frac{\beta \bar{u}_n(x, t) \bar{v}_n(x, t)}{1 + \alpha \bar{v}_n(x, t)} + \mu \bar{u}_n(y, t) + \frac{\beta \bar{u}_n(y, t) \bar{v}_n(y, t)}{1 + \alpha \bar{v}_n(y, t)} \right). \end{aligned}$$

By a simple calculation, we have that

$$\begin{aligned} & \left(-\mu \bar{u}_n(x, t) - \frac{\beta \bar{u}_n(x, t) \bar{v}_n(x, t)}{1 + \alpha \bar{v}_n(x, t)} + \mu \bar{u}_n(y, t) + \frac{\beta \bar{u}_n(y, t) \bar{v}_n(y, t)}{1 + \alpha \bar{v}_n(y, t)} \right) \\ &= \left(-\mu \bar{u}_n(x, t) - \frac{\beta \bar{u}_n(x, t) \bar{v}_n(x, t)}{1 + \alpha \bar{v}_n(x, t)} + \mu \bar{u}_n(y, t) + \frac{\beta \bar{u}_n(y, t) \bar{v}_n(x, t)}{1 + \alpha \bar{v}_n(x, t)} \right) \\ & \quad + \left(-\frac{\beta \bar{u}_n(y, t) \bar{v}_n(x, t)}{1 + \alpha \bar{v}_n(x, t)} + \frac{\beta \bar{u}_n(y, t) \bar{v}_n(y, t)}{1 + \alpha \bar{v}_n(y, t)} \right) \\ &= \left(-\mu - \frac{\beta \bar{v}_n(x, t)}{1 + \alpha \bar{v}_n(x, t)} \right) (\bar{u}_n(x, t) - \bar{u}_n(y, t)) \\ & \quad + \beta \bar{u}_n(y, t) \frac{\bar{v}_n(y, t) - \bar{v}_n(x, t)}{(1 + \alpha \bar{v}_n(x, t))(1 + \alpha \bar{v}_n(y, t))}. \end{aligned}$$

Set

$$h_n(t, x, y) := \left\| \frac{\bar{v}_n(y, t) - \bar{v}_n(x, t)}{(1 + \alpha \bar{v}_n(x, t))(1 + \alpha \bar{v}_n(y, t))} \right\|.$$

Note that $\bar{u}_n(x, t)$ and $\bar{v}_n(x, t)$ are uniformly bounded for $n \geq 1, t \geq 0, x \in \bar{\Omega}$ (see also Lemma 2.2). Then it follows that there exists a real number $M > 0$ such that

$$\frac{\partial}{\partial t} \|\bar{u}_n(x, t) - \bar{u}_n(y, t)\|^2 \leq -2\mu \|\bar{u}_n(x, t) - \bar{u}_n(y, t)\|^2 + M h_n(t, x, y).$$

By exchange the position of x and y , we have that

$$\frac{\partial}{\partial t} \|\bar{u}_n(x, t) - \bar{u}_n(y, t)\|^2 \leq -2\mu \|\bar{u}_n(x, t) - \bar{u}_n(y, t)\|^2 + M h_n(t, y, x).$$

Define $H_n(t, x, y) := h_n(t, x, y) + h_n(t, y, x)$. It then follows that

$$\frac{\partial}{\partial t} \|\bar{u}_n(x, t) - \bar{u}_n(y, t)\|^2 \leq -2\mu \|\bar{u}_n(x, t) - \bar{u}_n(y, t)\|^2 + M H_n(t, x, y)$$

for all $t > 0, x, y \in \bar{\Omega}$.

Note that $\{v_n(x, t_n)\}_{n \geq 1}$ is equicontinuous on $\bar{\Omega}$ for all $n \geq 1$. Thus, it suffices to prove that $\{u_n(x, t_n)\}_{n \geq 1}$ is equicontinuous on $\bar{\Omega}$ for all $n \geq 1$.

From the expression (3.1), we have that

$$\begin{aligned}
& \int_0^{+\infty} (w_n(x, \theta, t_n) - w_n(y, \theta, t_n)) d\theta \\
&= \int_0^{t_n} \left[\frac{\beta u_n(x, t_n - \theta) v_n(x, t_n - \theta)}{1 + \alpha v_n(x, t_n - \theta)} - \frac{\beta u_n(y, t_n - \theta) v_n(y, t_n - \theta)}{1 + \alpha v_n(y, t_n - \theta)} \right] e^{-\int_0^\theta \delta(\tau) d\tau} d\theta \\
&\quad + \int_{t_n}^{+\infty} [w_n(x, \theta - t_n, 0) - w_n(y, \theta - t_n, 0)] e^{-\int_0^{t_n} \delta(\tau + \theta - t_n) d\tau} d\theta \\
&= \int_0^{t_n} \frac{\beta v_n(x, t_n - \theta)}{1 + \alpha v_n(x, t_n - \theta)} (u_n(x, t_n - \theta) - u_n(y, t_n - \theta)) e^{-\int_0^\theta \delta(\tau) d\tau} d\theta \\
&\quad + \int_0^{t_n} \beta u_n(y, t_n - \theta) \frac{v_n(y, t_n - \theta) - v_n(x, t_n - \theta)}{(1 + \alpha v_n(x, t_n - \theta))(1 + \alpha v_n(y, t_n - \theta))} e^{-\int_0^\theta \delta(\tau) d\tau} d\theta \\
&\quad + \int_{t_n}^{+\infty} [w_n(x, \theta - t_n, 0) - w_n(y, \theta - t_n, 0)] e^{-\int_0^{t_n} \delta(\tau + \theta - t_n) d\tau} d\theta.
\end{aligned}$$

Note that $\bar{w}_n(x, \cdot, 0)$ is uniformly bounded, therefore, we have that

$$\lim_{n \rightarrow +\infty} \int_{t_n}^{+\infty} [w_n(x, \theta - t_n, 0) - w_n(y, \theta - t_n, 0)] e^{-\int_0^{t_n} \delta(\tau + \theta - t_n) d\tau} d\theta = 0.$$

It then follows that $\{w_n(x, \cdot, t_n)\}_{n \geq 1}$ is equicontinuous on $\bar{\Omega}$ for all $n \geq 1$ in the sense that for any $\varepsilon > 0$, there exists $\zeta > 0$ such that

$$\int_0^{+\infty} \|w_n(x, \theta, t_n) - w_n(y, \theta, t_n)\| d\theta < \varepsilon, \quad \forall n \geq 1, \quad \forall |x - y| < \zeta.$$

Consequently, $\Psi(t)$ is asymptotically compact on B . Thus, from the proof of Lemma 4.1 in [21], we can deduce that the solution semiflow $\Psi(t)$ is \mathcal{K} -contracting with respect to the Kuratowski measure defined in (2.3), i.e.,

$$\lim_{t \rightarrow +\infty} \mathcal{K}(\Psi(t)(B)) = 0,$$

for any bounded set $B \subset X_+$. □

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