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OPTIMAL HARVESTING POLICY FOR THE BEVERTON–HOLT MODEL

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Abstract. In this paper, we establish the exploitation of a single population modeled by the Beverton–Holt difference equation with periodic coefficients. We begin our investigation with the harvesting of a single autonomous population with logistic growth and show that the harvested logistic equation with periodic coefficients has a unique positive periodic solution which globally attracts all its solutions. Further, we approach the investigation of the optimal harvesting policy that maximizes the annual sustainable yield in a novel and powerful way; it serves as a foundation for the analysis of the exploitation of the discrete population model. In the second part of the paper, we formulate the harvested Beverton–Holt model and derive the unique periodic solution, which globally attracts all its solutions. We continue our investigation by optimizing the sustainable yield with respect to the harvest effort. The results differ from the optimal harvesting policy for the continuous logistic model, which suggests a separate strategy for populations modeled by the Beverton–Holt difference equation.

1. **Introduction.** Beverton and Holt introduced their population model in the context of fisheries in 1957 [4], and it still attracts interest in various fields such as biology, economy and social sciences, see [3, 4, 19, 27]. Numerous authors, see [12, 16, 21, 22, 23, 25], investigated the Beverton–Holt equation and its characteristics. In [22, 23], the authors discussed the dynamics of the Beverton–Holt model and in [16] the Beverton–Holt equation with survival rate was introduced. The authors in [25] presented two modifications of the classical Beverton–Holt model while in [21] a special modification of the sigmoid Beverton–Holt equation was studied.

The Beverton–Holt model can be rewritten into a so-called logistic dynamic equation as introduced in [8, p. 30]. The classical logistic differential equation introduced by Verhulst [29] is of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x \left(1 - \frac{x}{K}\right), \tag{1}$$

where x represents the density of the resource population at time t and α, K are positive constants representing the growth rate and the carrying capacity, respectively. In order to reflect the reality more accurately, positive functions $K(t)$ and

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$\alpha(t)$ were introduced to represent the coefficients. In [8], the authors introduced the so-called logistic dynamic equation on time scales as a generalization of (1), as

$$x^{\Delta}(t) = \alpha(t)x(\sigma(t))\left(1 - \frac{x(t)}{K(t)}\right), \quad t \in \mathbb{T},$$

where $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. In the discrete case, i.e., $\mathbb{T} = \mathbb{Z}$, the previous equation becomes

$$\Delta x_n = \alpha_n x_{n+1}\left(1 - \frac{x_n}{K_n}\right), \quad n \in \mathbb{N}_0, \tag{2}$$

where $\alpha_n \in (0,1)$ is the inherent growth rate, $K_n \in \mathbb{R}^+$ is the carrying capacity, and x_n is the population density for $n \in \mathbb{N}_0$.

The optimal management of renewable resources, which is directly related to sustainable development, has been studied by various authors [5, 24, 28, 33]. In 1976, Clark started the discussion of economical and biological aspects of renewable resources for the logistic growth model by including harvest effort [15]. Suppose the population described in (1) is exposed to harvest by the catch-per-unit-effort hypothesis. Then the model becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x\left(1 - \frac{x}{K}\right) - Ex, \tag{3}$$

where E denotes the harvest effort. Fan and Wang obtained in [17] some results about the optimal harvesting policy for the model (3) under the assumption of a one-periodic carrying capacity and one-periodic growth rate. It was proved that (3) has a unique periodic solution which is globally attractive. The authors showed that, if

$$\int_0^1 (\alpha(\tau) - E(\tau))\,\mathrm{d}\tau > 0,$$

then the annual-sustainable yield

$$Y(E) = \int_0^1 E(t)x(t)\,\mathrm{d}t$$

is maximal for

$$E(t) = \frac{1}{2}\alpha(t) - \frac{K'(t)}{K(t)}. \tag{4}$$

This result was obtained by a change of variable, which transforms the annual-sustainable yield into

$$\hat{Y}(Z(t)) = \int_0^1 \frac{-\left(\frac{(Z'(t)+\alpha(t)Z(t))K(t)}{\alpha(t)}\right)}{Z(t)}\,\mathrm{d}t.$$

Using integration by parts and introducing a new variable $W(t) = (\ln(Z(t)))'$, the problem is of the form of a variational calculus optimization problem, namely

$$\tilde{Y}(W) := -\int_0^1 F(t, W, W')\,\mathrm{d}t.$$

The Euler–Lagrange equation

$$\frac{\partial F}{\partial W} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial W'}\right) = 0$$

yields the optimum $W^* = -\frac{1}{2}\alpha$. Resubstitution gives the optimal harvest effort $E^* = \frac{1}{2}\alpha - \frac{K'}{K}$ for which the harvest yield over one period is $Y(E^*) =$

$\frac{1}{4}\int_0^1 \alpha(t)K(t)\mathrm{d}t$.

In [14], the authors discuss the maximum sustainable yield for a class of differential equations of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = g(N(t), K(t)),$$

where the function g satisfies particular conditions. The logistic growth model satisfies these conditions and was therefore discussed as an example. In [14], the optimal harvest effort for the continuous model

$$\frac{\mathrm{d}N}{\mathrm{d}t} = g(N(t), K(t)) - h(t)$$

is compared to periodic impulsive harvesting

$$\Delta N(\tau_j + n\omega) = I_j(N(\tau_j + n\omega)), \quad 0 \le \tau_1 < \tau_2 < \ldots < \tau_m < \omega, n \in \mathbb{N}_0,$$

where $I_j : [0,\infty) \to [0,\infty)$ are continuous functions, $j = 1, 2, \ldots, m$. For the logistic growth model, it reads as

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r(t)N(t)\left(1 - \frac{N(t)}{K(t)}\right),$$
$$N(\tau_k^+) = (1 - E_k)N(\tau_k), \quad k = 1, 2, \ldots,$$

where harvesting takes place at time τ_k for $k = 1, 2, \ldots$. Maximizing the population growth at each segment $(\tau_k, \tau_{k+1}]$ yields the maximum sustainable yield (under a particular condition). In Remark 2, we suggest a pulse harvesting for the logistic dynamic equation for the discrete time setting. In [14], the authors prove that independent of the impulsive strategy, the impulsive sustainable yield is not exceeding the continuous sustainable yield, under specific conditions, which are satisfied for the logistic growth model and the Gompertz model. The authors furthermore discuss a modified model including a constant term that can be interpreted as pollution, noise, or general harm.

Optimizing the sustainable yield has also been considered for diffusive models in nonhomogeneous environments, see for example [1, 13]. To describe the dispersal speed of the population, a diffusion term has been introduced to the model. The model reads then as

$$\frac{\partial u}{\partial t} = D\Delta u + r(t,x)f(u, K(t,x)) - E(t,x)u, \tag{5}$$

where $u = u(t,x)$ is a function of time t and space x. K represents the carrying capacity, r the growth rate, E the harvest effort, and D the constant diffusion term. For the time independent logistic growth, $f(u,K) = u\left(1 - \frac{u}{K}\right)$, and the diffusion term $D\left(\frac{u}{K(t,x)}\right)$, the authors show in [13] that, despite the diffusion term, the optimal harvesting strategy is obtained for $E^*(x) = \frac{r(x)}{2}$. Even if the coefficients are time dependent and periodic, the optimal harvesting is given by

$$E^*(t,x) = \frac{r(t,x)}{2} - \frac{1}{K(t,x)}\frac{\partial K(t,x)}{\partial t},$$

under the assumption $r(t,x)$ such that $E^* \ge 0$. Note the relation to the optimal harvesting effort $E^* = \frac{r}{2} - \frac{K'}{K}$ for the continuous logistic growth model with harvesting. In Remark 3, we present a diffusive model and compare it briefly to the model presented in [13].

In this paper, we dedicate the first part to the analysis of (3). We will provide the reader with an alternative, elementary technique to obtain the optimal harvesting policy for single populations with periodic coefficients and extend the results obtained in [17] to any period ω. This effective method provides the foundation for the novel results regarding the maximum sustainable yield for a population modeled by the discrete Beverton–Holt equation. First, we show that if the coefficients are ω-periodic sequences, then the harvested Beverton–Holt difference equation has a unique ω-periodic solution which globally attracts all positive solutions. The discussion of the optimal harvest policy is initiated by considering first the special case of constant coefficients. The final part focuses on the optimization of the harvest yield for the Beverton–Holt equation with exploitation. In Example 3.14, we will discuss an example of a population that follows a mixed continuous-discrete pattern.

2. **The logistic growth model including exploitation.**

We consider the continuous logistic population model including harvesting given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha(t)x\left(1 - \frac{x}{K(t)}\right) - E(t)x, \tag{6}$$

where x represents the population density at time $t \in \mathbb{R}_0^+$, $\alpha \in (0,1)$ is the inherent growth rate, K is the carrying capacity, and E is the harvest effort.

2.1. **Existence and uniqueness of the logistic growth population model with harvesting.**

The study of (6) can be simplified by applying the change of variable $u = \frac{1}{x}$ which leads to the linear differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -(\alpha(t) - E(t))u + \frac{\alpha(t)}{K(t)} \tag{7}$$

with the solution

$$u(t) = e^{-\int_{t_0}^{t}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}u_0 + \int_{t_0}^{t}\frac{\alpha(s)}{K(s)}e^{-\int_{s}^{t}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}\mathrm{d}s,$$

where $u(t_0) = u_0$.

Theorem 2.1. *Assume $K \in \mathrm{C}(\mathbb{R}, \mathbb{R}^+)$, $\alpha \in \mathrm{C}(\mathbb{R}, (0,1))$, $E \in \mathrm{C}(\mathbb{R}, (0,1))$ are ω-periodic functions and $E(t) < \alpha(t)$ for all $t \in \mathbb{R}$ (to avoid extinction). Then the unique ω-periodic solution of (6) is given by*

$$\bar{x}(t) = \lambda\left(\int_{t}^{t+\omega}\frac{\alpha(s)}{K(s)}e^{-\int_{s}^{t}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}\mathrm{d}s\right)^{-1}, \tag{8}$$

where $\lambda = e^{\int_0^{\omega}(\alpha(\tau)-E(\tau))\mathrm{d}\tau} - 1$. In addition, $\bar{x}$ is globally asymptotically stable, i.e.,

$$\lim_{t\to\infty}|x(t) - \bar{x}(t)| = 0$$

for any solution x with $x(t_0) > 0$.

Proof. We already introduced the relation between (6) and (7). Applying the periodicity of the coefficient functions, we get

$$\begin{aligned} u(t+\omega) &= e^{-\int_{t_0}^{t+\omega}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}u_0 + \int_{t_0}^{t+\omega}\frac{\alpha(s)}{K(s)}e^{-\int_{s}^{t+\omega}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}\,\mathrm{d}s \\ &= e^{-\int_{0}^{\omega}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}e^{-\int_{t_0}^{t}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}u_0 \end{aligned}$$

$$+ e^{-\int_0^\omega (\alpha(\tau)-E(\tau))\mathrm{d}\tau} \int_{t_0}^{t} \frac{\alpha(s)}{K(s)} e^{-\int_s^t (\alpha(\tau)-E(\tau))\mathrm{d}\tau}\,\mathrm{d}s$$

$$+ e^{-\int_0^\omega (\alpha(\tau)-E(\tau))\mathrm{d}\tau} \int_{t}^{t+\omega} \frac{\alpha(s)}{K(s)} e^{-\int_s^t (\alpha(\tau)-E(\tau))\mathrm{d}\tau}\,\mathrm{d}s$$

$$= e^{-\int_0^\omega (\alpha(\tau)-E(\tau))\mathrm{d}\tau} u(t)$$

$$+ e^{-\int_0^\omega (\alpha(\tau)-E(\tau))\mathrm{d}\tau} \int_{t}^{t+\omega} \frac{\alpha(s)}{K(s)} e^{-\int_s^t (\alpha(\tau)-E(\tau))\mathrm{d}\tau}\,\mathrm{d}s.$$

If $\bar{u}$ is an ω-periodic solution of (7), then $\bar{u}(t+\omega) = \bar{u}(t)$ so that

$$\bar{u}(t) = \left(e^{\int_0^\omega (\alpha(\tau)-E(\tau))\mathrm{d}\tau} - 1\right)^{-1} \int_t^{t+\omega} \frac{\alpha(s)}{K(s)} e^{-\int_s^t (\alpha(\tau)-E(\tau))\mathrm{d}\tau}\,\mathrm{d}s. \tag{9}$$

Conversely, if a solution to (7) is of the form (9), then it is easy to show that it is ω-periodic. The unique ω-periodic solution $\bar{x}$ of (6) is then

$$\bar{x}(t) = (e^{\int_0^\omega (\alpha(\tau)-E(\tau))\mathrm{d}\tau} - 1) \left(\int_t^{t+\omega} \frac{\alpha(s)}{K(s)} e^{-\int_s^t (\alpha(\tau)-E(\tau))\mathrm{d}\tau}\,\mathrm{d}s\right)^{-1}.$$

Addressing the global attractivity of the periodic solution, let x be any solution of (6) with $x(t_0) > 0$. For simplicity define $p := \alpha - E$. We have

$$\begin{aligned}
&|x(t) - \bar{x}(t)| \\
&= \left| \frac{1}{\frac{e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau}}{x_0} + \int_{t_0}^t \frac{\alpha(s)e^{-\int_s^t p(\tau)\mathrm{d}\tau}}{K(s)}\mathrm{d}s} - \frac{1}{\frac{e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau}}{\bar{x}_0} + \int_{t_0}^t \frac{\alpha(s)e^{-\int_s^t p(\tau)\mathrm{d}\tau}}{K(s)}\mathrm{d}s} \right| \\
&= \frac{e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau} \left|\frac{1}{\bar{x}_0} - \frac{1}{x_0}\right|}{\left|\left(\frac{e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau}}{x_0} + \int_{t_0}^t \frac{\alpha(s)e^{-\int_s^t p(\tau)\mathrm{d}\tau}}{K(s)}\mathrm{d}s\right)\left(\frac{e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau}}{\bar{x}_0} + \int_{t_0}^t \frac{\alpha(s)e^{-\int_s^t p(\tau)\mathrm{d}\tau}}{K(s)}\mathrm{d}s\right)\right|} \\
&\le \frac{e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau} \left|\frac{1}{\bar{x}_0} - \frac{1}{x_0}\right|}{\left(\int_{t_0}^t \frac{\alpha(s)}{K(s)} e^{-\int_s^t p(\tau)\mathrm{d}\tau}\mathrm{d}s\right)^2} \le ||K||_\infty \frac{e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau} \left|\frac{1}{\bar{x}_0} - \frac{1}{x_0}\right|}{\left(\int_{t_0}^t \alpha(s) e^{-\int_s^t p(\tau)\mathrm{d}\tau}\mathrm{d}s\right)^2} \\
&\le ||K||_\infty \frac{e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau}}{\left(1 - e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau}\right)^2} \left|\frac{1}{\bar{x}_0} - \frac{1}{x_0}\right|,
\end{aligned}$$

where the last terms tends to zero because $e^{-\int_{t_0}^t p(\tau)\mathrm{d}\tau} \to 0$ as $t \to \infty$. This completes the proof. □

2.2. **Optimal harvesting policy for the logistic growth population model.** We are interested in maximizing the sustainable harvest yield without endangering the modeled species. Maximizing the harvest yield over one period serves economical aims, but the aspect of sustainable development of the population follows ecological reasons. It is well known that, under the assumption of a constant inherent growth rate and constant carrying capacity, the optimal harvest policy for a population modeled by (3) is to keep a constant harvest effort of $E = \frac{1}{2}\alpha$ [18]. A more realistic model, such as introduced in (6), assumes a periodic inherent growth rate and periodic carrying capacity due to seasonality and has been studied in [17, 31]. The optimization problem of the harvest yield over one period was discussed in [17] under the assumption of one-periodic coefficients. The optimal harvest effort was derived to be $E^* = \frac{1}{2}\alpha - \frac{K'}{K}$. In [17], the authors introduced transformations to rewrite the optimization into a variational calculus problem as described in Section 1. In the following, we derive the optimal harvest effort for the logistic population model with ω-periodic growth rate using an elementary technique, the weighted Jensen inequality.

Theorem 2.2. *If $\alpha \in \mathrm{C}(\mathbb{R},(0,1))$, $K \in \mathrm{C}^1(\mathbb{R},\mathbb{R}^+)$, $E \in \mathrm{C}(\mathbb{R},(0,1))$ are ω-periodic, $\frac{K'}{K} \le \frac{\alpha}{2}$, and $E < \alpha$, then the optimal harvest effort for (6) that maximizes the harvest yield over one period is given by $E^* = \frac{1}{2}\alpha - \frac{K'}{K}$.*

Proof. Let $\bar{x}$ be the ω-periodic solution of (6) given by (8). We apply the weighted integral form of Jensen's inequality [26] (see also [2]) in the following way:

$$
\begin{aligned}
Y(E) &= \int_0^\omega E(t)\bar{x}(t)\mathrm{d}t = \int_0^\omega E(t)\frac{\lambda}{\int_t^{t+\omega}\frac{\alpha(s)}{K(s)}e^{-\int_s^t(\alpha(\tau)-E(\tau))\mathrm{d}\tau}\mathrm{d}s}\mathrm{d}t \\
&= \lambda\int_0^\omega E(t)\frac{1}{\int_t^{t+\omega}\frac{\alpha(s)}{K(s)}e^{-\frac{1}{2}\int_s^t\alpha(\tau)\mathrm{d}\tau}e^{-\int_s^t\frac{1}{2}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}\mathrm{d}s}\mathrm{d}t \\
&\le \lambda\int_0^\omega E(t)\frac{\int_t^{t+\omega}K(s)\alpha(s)e^{-\frac{1}{2}\int_s^t\alpha(\tau)\mathrm{d}\tau}e^{\int_s^t\frac{1}{2}(\alpha(\tau)-E(\tau))\mathrm{d}\tau}\mathrm{d}s}{\left(\int_t^{t+\omega}\alpha(s)e^{-\frac{1}{2}\int_s^t\alpha(\tau)\mathrm{d}\tau}\mathrm{d}s\right)^2}\mathrm{d}t \\
&= \frac{\lambda}{4\left(e^{\frac{1}{2}\int_0^\omega\alpha(\tau)\mathrm{d}\tau}-1\right)^2}\int_0^\omega\int_t^{t+\omega}E(t)K(s)\alpha(s)e^{\int_t^s E(\tau)\mathrm{d}\tau}\mathrm{d}s\,\mathrm{d}t \\
&= \frac{\lambda}{4\left(e^{\frac{1}{2}\int_0^\omega\alpha(\tau)\mathrm{d}\tau}-1\right)^2}\int_0^\omega K(s)\alpha(s)\int_0^s E(t)e^{\int_t^s E(\tau)\mathrm{d}\tau}\mathrm{d}t\,\mathrm{d}s \\
&\quad + \frac{\lambda}{4\left(e^{\frac{1}{2}\int_0^\omega\alpha(\tau)\mathrm{d}\tau}-1\right)^2}\int_\omega^{2\omega} K(s)\alpha(s)\int_{s-\omega}^\omega E(t)e^{\int_t^s E(\tau)\mathrm{d}\tau}\mathrm{d}t\,\mathrm{d}s \\
&= \frac{\lambda}{4\left(e^{\frac{1}{2}\int_0^\omega\alpha(\tau)\mathrm{d}\tau}-1\right)^2}\int_0^\omega K(s)\alpha(s)\left(e^{\int_0^s E(\tau)\mathrm{d}\tau}-1\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{4\left(e^{\frac{1}{2}\int_0^\omega\alpha(\tau)\mathrm{d}\tau}-1\right)^2}\int_\omega^{2\omega} K(s)\alpha(s)\left(e^{\int_{s-\omega}^s E(\tau)\mathrm{d}\tau}-e^{\int_\omega^s E(\tau)\mathrm{d}\tau}\right)\mathrm{d}s \\
&= \frac{\lambda}{4\left(e^{\frac{1}{2}\int_0^\omega\alpha(\tau)\mathrm{d}\tau}-1\right)^2}\int_0^\omega K(s)\alpha(s)\left(e^{\int_0^s E(\tau)\mathrm{d}\tau}-1\right)\mathrm{d}s \\
&\quad + \frac{\lambda}{4\left(e^{\frac{1}{2}\int_0^\omega\alpha(\tau)\mathrm{d}\tau}-1\right)^2}\int_0^\omega K(s)\alpha(s)\left(e^{\int_s^{s+\omega} E(\tau)\mathrm{d}\tau}-e^{\int_\omega^{s+\omega} E(\tau)\mathrm{d}\tau}\right)\mathrm{d}s
\end{aligned}
$$

$$= \frac{\lambda}{4\left(e^{\frac{1}{2}\int_0^\omega \alpha(\tau)\mathrm{d}\tau} - 1\right)^2} \int_0^\omega K(s)\alpha(s)\left(e^{\int_0^s E(\tau)\mathrm{d}\tau} - 1 \right.$$

$$\left. + e^{\int_0^\omega E(\tau)\mathrm{d}\tau} - e^{\int_0^s E(\tau)\mathrm{d}\tau}\right)\mathrm{d}s$$

$$= \frac{\lambda\left(e^{\int_0^\omega E(\tau)\mathrm{d}\tau} - 1\right)}{4\left(e^{\frac{1}{2}\int_0^\omega \alpha(\tau)\mathrm{d}\tau} - 1\right)^2} \int_0^\omega K(s)\alpha(s)\mathrm{d}s \le \frac{1}{4}\int_0^\omega K(s)\alpha(s)\mathrm{d}s,$$

where we have used the algebraic inequality

$$\frac{\left(e^{\int_0^\omega (\alpha(\tau)-E(\tau))\mathrm{d}\tau} - 1\right)\left(e^{\int_0^\omega E(\tau)\mathrm{d}\tau} - 1\right)}{\left(e^{\frac{1}{2}\int_0^\omega \alpha(\tau)\mathrm{d}\tau} - 1\right)^2} \le 1.$$

This is true because

$$(e^{\int_0^\omega \frac{1}{2}\alpha(\tau)\mathrm{d}\tau} e^{\int_0^\omega p(\tau)\mathrm{d}\tau} - 1)(e^{\int_0^\omega \frac{1}{2}\alpha(\tau)\mathrm{d}\tau} e^{-\int_0^\omega p(\tau)\mathrm{d}\tau} - 1) \le \left(e^{\int_0^\omega \frac{1}{2}\alpha(\tau)\mathrm{d}\tau} - 1\right)^2,$$

i.e.,

$$-e^{\int_0^\omega p(\tau)\mathrm{d}\tau} - e^{-\int_0^\omega p(\tau)\mathrm{d}\tau} \le -2,$$

i.e.,

$$\left(e^{\frac{1}{2}\int_0^\omega p(\tau)\mathrm{d}\tau} - e^{-\frac{1}{2}\int_0^\omega p(\tau)\mathrm{d}\tau}\right)^2 \ge 0,$$

where $p = \frac{1}{2}\alpha - E$. This calculation provides an upper bound for the harvest yield; that is, $\frac{1}{4}\int_0^\omega K(s)\alpha(s)\mathrm{d}s$. It is left to show that this supremum is achieved by $E^* = \frac{1}{2}\alpha - \frac{K'}{K}$:

$$\begin{aligned}
Y(E^*) &= \int_0^\omega E^*(t)\bar{x}(t)\mathrm{d}t \\
&= \int_0^\omega E^*(t)\frac{e^{\int_0^\omega\left(\alpha(\tau)-\frac{1}{2}\alpha(\tau)+\frac{K'(\tau)}{K(\tau)}\right)\mathrm{d}\tau} - 1}{\int_t^{t+\omega}\frac{\alpha(s)}{K(s)}e^{-\int_s^t\left(\alpha(\tau)-\frac{1}{2}\alpha(\tau)+\frac{K'(\tau)}{K(\tau)}\right)\mathrm{d}\tau}\mathrm{d}s}\mathrm{d}t \\
&= \int_0^\omega E^*(t)\frac{\frac{K(\omega)}{K(0)}e^{\int_0^\omega \frac{1}{2}\alpha(\tau)\mathrm{d}\tau} - 1}{\int_t^{t+\omega}\frac{\alpha(s)}{K(s)}e^{-\int_s^t \frac{1}{2}\alpha(\tau)\mathrm{d}\tau}\frac{K(s)}{K(t)}\mathrm{d}s}\mathrm{d}t \\
&= \int_0^\omega E^*(t)K(t)\frac{e^{\int_0^\omega \frac{1}{2}\alpha(\tau)\mathrm{d}\tau} - 1}{\int_t^{t+\omega}\alpha(s)e^{-\int_s^t \frac{1}{2}\alpha(\tau)\mathrm{d}\tau}\mathrm{d}s}\mathrm{d}t \\
&= \left(e^{\int_0^\omega \frac{1}{2}\alpha(\tau)\mathrm{d}\tau} - 1\right)\int_0^\omega E^*(t)K(t)\frac{1}{2\left(e^{\int_0^\omega \frac{1}{2}\alpha(\tau)\mathrm{d}\tau} - 1\right)}\mathrm{d}t \\
&= \frac{1}{2}\int_0^\omega\left(\frac{1}{2}\alpha(t) - \frac{K'(t)}{K(t)}\right)K(t)\mathrm{d}t \\
&= \frac{1}{2}\left(\frac{1}{2}\int_0^\omega \alpha(t)K(t)\mathrm{d}t - K(\omega) + K(0)\right) = \frac{1}{4}\int_0^\omega \alpha(t)K(t)\mathrm{d}t.
\end{aligned}$$

This completes the proof. □

Remark 1. The special case of constant coefficients yields in the previous theorem $E^* = \frac{\alpha}{2}$, which is consistent with the results discussed in [18].

3. **The Beverton–Holt model including exploitation.** The classical Beverton–Holt model with periodic coefficients is of the form [10]

$$x_{n+1} = \frac{\nu_n K_n x_n}{K_n + (\nu_n - 1)x_n}, \tag{10}$$

where the sequence $\{x_n\}$ represents the population density, $\{K_n\}$ the carrying capacity, and $\{\nu_n\}$, $\nu_n > 1$ for all $n \in \mathbb{N}_0$ is the inherent growth rate. While originally introduced with constant coefficients by Ray Beverton and Sidney Holt in 1957 in the context of fisheries [4], the model gains attention in a wide range of applications in population models. Equation (10) and the quantum calculus analogue of (10) were recently studied in [6, 9, 10, 11, 23]. In [10], the transformation $\alpha = \frac{\nu-1}{\nu}$, $0 < \alpha < 1$, was applied to the model (10) to obtain the discrete logistic growth model introduced in [8]

$$\Delta x_n = \alpha_n x_{n+1}\left(1 - \frac{x_n}{K_n}\right).$$

Consider now the discrete population model (10) and introduce exploitation, under the catch-per-unit-effort hypothesis as

$$x_{n+1} = \frac{\nu_n K_n x_n}{K_n + (\nu_n - 1)x_n} - h_n x_{n+1}, \tag{11}$$

where we extend the classical model by a harvest effort $h : \mathbb{N}_0 \to \mathbb{R}_0^+$.

Similarly, let us define $\alpha := \frac{\nu-1}{\nu}$ to obtain an equivalent difference equation of (11) that we will investigate instead:

$$x_{n+1} = \frac{K_n x_n}{(1-\alpha_n)K_n + \alpha_n x_n} - h_n x_{n+1}. \tag{12}$$

As before, $x : \mathbb{N}_0 \to \mathbb{R}^+$ represents the density of the resource population, $K : \mathbb{N}_0 \to \mathbb{R}^+$ is the carrying capacity, $0 < \alpha_n < 1$ for all $n \in \mathbb{N}_0$ represents the inherent growth rate. To avoid extinction of the population, we require

$$0 \le h_n < \frac{\alpha_n}{1-\alpha_n} \quad \text{for all } n \in \mathbb{N}_0. \tag{13}$$

This designed discrete population model with harvesting is related to the existing continuous harvesting model as explicated in the following. By expanding (12), we obtain

$$\begin{aligned} K_n x_{n+1} - \alpha_n K_n x_{n+1} + \alpha_n x_n x_{n+1} \\ = K_n x_n - K_n h_n x_{n+1} + \alpha_n K_n h_n x_{n+1} - \alpha_n h_n x_n x_{n+1}, \end{aligned}$$

i.e.,

$$K_n \Delta x_n = \alpha_n K_n x_{n+1}(1+h_n) - \alpha_n x_n x_{n+1}(1+h_n) - K_n h_n (x_n + \Delta x_n),$$

which is of the form

$$\Delta x_n = \alpha_n x_{n+1}\left(1 - \frac{x_n}{K_n}\right) - E_n x_n, \tag{14}$$

where $E_n = \frac{h_n}{1+h_n}$ for all $n \in \mathbb{N}_0$. Using the relation between E_n and h_n, condition (13) can be rephrased to $0 < E_n < \alpha_n$, which is the discrete analogue of the continuous requirement $E(t) < \alpha(t)$ to avoid extinction.

We now begin our discussion of the discrete Beverton–Holt population model including harvesting by first investigating the existence and uniqueness of a periodic solution.

3.1. **Existence and uniqueness of the Beverton–Holt difference equation with exploitation.** Let us first introduce some necessary definitions and their primary properties that assist us in the further analysis.

Definition 3.1 (See [8, Definition 1.38]). If $c \in \mathbb{R}$ and $p_n \neq -1$ for all $n \in \mathbb{N}_0$, then the unique solution of

$$\Delta x_n = p_n x_n, \quad x_n = c$$

is denoted by $e_p(n, 0)c$.

Note that $e_p(i, j) = \prod_{k=j}^{i-1}(1 + p_k)$ if $j < i$, $e_p(i, i) = 1$, and $e_p(i, j) = \frac{1}{e_p(j,i)}$ if $i < j$ for i, $j \in \mathbb{N}_0$.

Lemma 3.2 (Properties of $e_p(i, j)$). *Assume $p_i \neq -1$ for all $i \in \mathbb{N}_0$ and p is ω-periodic. Then we have*

a) $$e_p(j, i) = \frac{1}{e_p(i, j)}, \tag{15}$$

b) $$e_p(j + \omega, j) = e_p(\omega, 0), \tag{16}$$

c) $$e_p(j + \omega, i + \omega) = e_p(j, i), \tag{17}$$

d) $$e_p(i, j) = e_p(i, m)e_p(m, j), \tag{18}$$

e) $$e_{\frac{\Delta p}{p}}(i, j) = \frac{p_i}{p_j}, \tag{19}$$

for all $i, j, m \in \mathbb{N}_0$.

Proof. For the proof of (15), see [7, Theorem 2.36]. [10, Lemma 2.2] provides the proofs of (16) and (17). The equality (18) is true by [7, Theorem 2.39]. To realize (19), note that for $j < i$

$$e_{\frac{\Delta p}{p}}(i, j) = \prod_{k=j}^{i-1}\left(1 + \frac{\Delta p_k}{p_k}\right) = \prod_{k=j}^{i-1}\frac{p_{k+1}}{p_k} = \frac{p_i}{p_j}. \tag{20}$$

For $i < j$, use (15) and (20) and for $i = j$, the equation is true, since $e_{\frac{\Delta p}{p}}(i, i) = 1$. □

Theorem 3.3 (See [7, Theorem 2.44]). *If the sequence $\{p_n\}$ satisfies $p_n > -1$ for all $n \in \mathbb{N}_0$, then $e_p(i, j) > 0$ for all $i, j \in \mathbb{N}_0$.*

To simplify upcoming notations, let us introduce the circle plus $\oplus$ and circle minus $\ominus$ operations on $\mathbb{Z}$.

Definition 3.4. Define the "circle plus" addition between two sequences $\{p_n\}$ and $\{q_n\}$, $n \in \mathbb{N}_0$ as

$$(p \oplus q)_n = p_n + q_n + p_n q_n,$$

and the "circle minus" subtraction as [7, Definition 2.13]

$$(p \ominus q)_n = \frac{p_n - q_n}{1 + q_n},$$

which is the additive inverse under the operation $\oplus$. $(\mathbb{Z}, \oplus)$ is an Abelian group[7, Theorem 2.7].

Definition 3.5 (See [8, p. 18])**.** The circle dot multiplication $\odot$ of a constant value α and a function $p : \mathbb{N}_0 \to (-1, \infty)$ is defined as

$$(\alpha \odot p)_n = \alpha p_n \int_0^1 (1 + hp_n)^{\alpha - 1} \mathrm{d}h.$$

Example 3.6. Let $p : \mathbb{N}_0 \to (-1, \infty)$ and $\alpha = \frac{1}{2}$. Then

$$\left(\frac{1}{2} \odot p\right)_n = \frac{1}{2} \int_0^1 \frac{p_n}{\sqrt{1 + hp_n}} \mathrm{d}h = \sqrt{1 + p_n} - 1 = \frac{p_n}{1 + \sqrt{1 + p_n}}.$$

Note that by the definition of the dot multiplication,

$$\left(\frac{1}{2} \odot (-\alpha)\right) \oplus \left(\frac{1}{2} \odot (-\alpha)\right) = -\alpha.$$

The following identities are not hard to show.

Corollary 1. *Assume p, $q : \mathbb{N}_0 \to \mathbb{R}$, $p \neq -1$, and $q \neq -1$. Then*

$$a)\quad e_{p \oplus q}(i, j) = e_p(i, j) e_q(i, j) \quad \text{for all } i, j \in \mathbb{N}_0, \tag{21}$$

$$b)\quad e_{\ominus p}(i, j) = e_p(j, i) = \frac{1}{e_p(i, j)} \quad \text{for all } i, j \in \mathbb{N}_0. \tag{22}$$

Lemma 3.7 (See [7, Theorem 2.39])**.** *If $p \neq -1$, then*

$$a)\quad \sum_{i=n}^{m-1} p_i e_p(i, c) = e_p(m, c) - e_p(n, c), \tag{23}$$

$$b)\quad \sum_{i=n}^{m-1} p_i e_p(c, i + 1) = e_p(c, n) - e_p(c, m), \tag{24}$$

for all $n < m$; $n, m, c \in \mathbb{N}_0$.

We can now focus on the existence and uniqueness of a solution of the Beverton–Holt equation with harvesting.

Theorem 3.8. *Assume*

$$\begin{cases} K : \mathbb{N}_0 \to \mathbb{R}^+ \text{ is } \omega\text{-periodic}, \\ \alpha\ : \mathbb{N}_0 \to \mathbb{R}^+ \text{ is } \omega\text{-periodic and } 0 < \alpha_n < 1 \text{ for all } n \in \mathbb{N}_0, \\ h\ : \mathbb{N}_0 \to \mathbb{R}^+ \text{ is } \omega\text{-periodic and } 0 < h_n < \frac{\alpha_n}{1 - \alpha_n} \text{ for all } n \in \mathbb{N}_0. \end{cases} \tag{25}$$

Then (12) has a unique ω-periodic solution which globally attracts all positive solutions.

Proof. Using the relation of (12) and (14) and applying the substitution $x = \frac{1}{u}$ to (14), we obtain the linear difference equation

$$\Delta u_n = -\alpha_n u_n + \frac{\alpha_n}{K_n} + \frac{h_n}{1 + h_n} u_{n+1}.$$

Realizing that $u_{n+1} = u_n + \Delta u_n$ yields

$$\Delta u_n = (h_n \oplus (-\alpha_n)) u_n + \frac{\alpha_n (1 + h_n)}{K_n}. \tag{26}$$

Equation (26) is a nonhomogeneous linear difference equation which has the solution given in [20, Theorem 3.1] by

$$u_n = e_{h\oplus(-\alpha)}(n,0)u_0 + \sum_{j=0}^{n-1} e_{h\oplus(-\alpha)}(n,j+1)\beta_j,$$

where $\beta = \frac{\alpha(1+h)}{K}$, and $u_0 > 0$ is an initial condition. Any ω-periodic solution of (26) satisfies $\bar{u}_n = \bar{u}_{n+\omega}$ for all $n \in \mathbb{N}_0$, so

$$\begin{aligned}
\bar{u}_n = \bar{u}_{n+\omega} &= e_{h\oplus(-\alpha)}(n+\omega,0)\bar{u}_0 + \sum_{j=0}^{n+\omega-1} e_{h\oplus(-\alpha)}(n+\omega,j+1)\beta_j \\
&\overset{(16)}{\underset{(18)}{=}} e_{h\oplus(-\alpha)}(\omega,0)e_{h\oplus(-\alpha)}(n,0)\bar{u}_0 + \sum_{j=0}^{n-1} e_{h\oplus(-\alpha)}(\omega,0)e_{h\oplus(-\alpha)}(n,j+1)\beta_j \\
&\quad + \sum_{j=n}^{n+\omega-1} e_{h\oplus(-\alpha)}(\omega,0)e_{h\oplus(-\alpha)}(n,j+1)\beta_j \\
&= e_{h\oplus(-\alpha)}(\omega,0)\bar{u}_n + e_{h\oplus(-\alpha)}(\omega,0) \sum_{j=n}^{n+\omega-1} e_{h\oplus(-\alpha)}(n,j+1)\beta_j.
\end{aligned}$$

We get

$$\bar{u}_n = \frac{1}{\lambda} \sum_{j=n}^{n+\omega-1} e_{h\oplus(-\alpha)}(n,j+1)\beta_j, \tag{27}$$

where

$$\lambda = e_{h\oplus(-\alpha)}(0,\omega) - 1 \neq 0.$$

Conversely, if a solution of (26) is given by (27), then it is easy to show that the solution is ω-periodic. This yields the unique ω-periodic solution of (12) as

$$\bar{x}_n = \lambda \left(\sum_{j=n}^{n+\omega-1} e_{h\oplus(-\alpha)}(n,j+1)\beta_j \right)^{-1}, \tag{28}$$

where $\beta = \frac{\alpha(1+h)}{K}$. In order to prove the global attractivity of the ω-periodic solution, let x be any solution of (12) with $x_0 > 0$. Define $F(n,j) := e_{h\oplus(-\alpha)}(n,j+1)\beta_j$. Then

$$\begin{aligned}
|x_n - \bar{x}_n| &= \left| \frac{1}{\frac{e_{h\oplus(-\alpha)}(n,0)}{x_0} + \sum_{j=0}^{n-1} F(n,j)} - \frac{1}{\frac{e_{h\oplus(-\alpha)}(n,0)}{\bar{x}_0} + \sum_{j=0}^{n-1} F(n,j)} \right| \\
&= \frac{\left| \frac{1}{\bar{x}_0} - \frac{1}{x_0} \right| e_{h\oplus(-\alpha)}(n,0)}{\left| \left[\frac{e_{h\oplus(-\alpha)}(n,0)}{x_0} + \sum_{j=0}^{n-1} F(n,j) \right] \left[\frac{e_{h\oplus(-\alpha)}(n,0)}{\bar{x}_0} + \sum_{j=0}^{n-1} F(n,j) \right] \right|} \\
&\leq \frac{\left| \frac{1}{\bar{x}_0} - \frac{1}{x_0} \right| e_{h\oplus(-\alpha)}(n,0)}{\left[\sum_{j=0}^{n-1} F(n,j) \right]^2} \leq \|K\|_\infty^2 \frac{\left| \frac{1}{\bar{x}_0} - \frac{1}{x_0} \right| e_{h\oplus(-\alpha)}(n,0)}{\left[\sum_{j=0}^{n-1} e_{h\oplus(-\alpha)}(n,j+1)\alpha_j(1+h_j) \right]^2}
\end{aligned}$$

$$\leq \quad \|K\|_\infty^2 \frac{\left|\frac{1}{\bar{x}_0} - \frac{1}{x_0}\right| e_{h\oplus(-\alpha)}(n,0)}{(e_{h\oplus(-\alpha)}(n,0)-1)^2},$$

where we have used (24). The last term tends to zero as $n \to \infty$ since $\frac{h}{1+h} < \alpha$, i.e., $h \oplus (-\alpha) \in (0,1)$. □

3.2. **Optimal harvesting policy for the Beverton–Holt equation with constant growth rate.** We start our investigation of the optimal harvesting policy for the model described in (12) with one-periodic, i.e., constant, growth rate and carrying capacity.

Theorem 3.9. *Consider the Beverton–Holt model with constant coefficients*

$$x_{n+1} = \frac{\nu K x_n}{K + (\nu - 1)x_n} - h x_{n+1}. \tag{29}$$

The optimal harvest effort that maximizes the harvest yield over one period is obtained at $h^* = -1 + \frac{1}{\sqrt{1-\alpha}}$*, where* $\alpha = \frac{\nu-1}{\nu}$.

Proof. The one-periodic solution of (29) is given by (28) with $\omega = 1$ as

$$\bar{x} = \frac{e_{h\oplus(-\alpha)}(0,1) - 1}{e_{h\oplus(-\alpha)}(0,1)\frac{\alpha(1+h)}{K}},$$

i.e.,

$$\bar{x} = K - \frac{K}{\alpha}\left(\frac{h}{1+h}\right).$$

We wish to optimize the harvest yield over one period, that is, maximize

$$Y(h) = h\bar{x} = Kh - \frac{K}{\alpha}\left(\frac{h^2}{1+h}\right).$$

The critical values are

$$h = -1 \pm \frac{1}{\sqrt{1-\alpha}}.$$

The positive harvest effort that additionally satisfies $Y^{''}(h^*) < 0$ is

$$h^* = -1 + \frac{1}{\sqrt{1-\alpha}},$$

which yields the suggested optimal harvest effort. □

The obtained result can be related to the continuous optimal harvest effort as the following discussion indicates. Recalling the relation $E = \frac{h}{1+h}$, the obtained optimal discrete harvest effort is

$$E^* = 1 - \sqrt{1-\alpha} = \frac{\alpha}{1+\sqrt{1-\alpha}} = -\left(\frac{1}{2} \odot (-\alpha)\right). \tag{30}$$

Figure 1 visualizes the difference between the behavior of the optimal harvest effort E^* dependent on the inherent growth rate for the continuous case in which $E^* = \alpha/2$ and the discrete case $E^* = \frac{\alpha}{1+\sqrt{1-\alpha}}$. The line in Figure 1 indicates the continuous result and the dotted line the behavior of the optimal harvest effort in the discrete case.

In Figure 1, we see that the harvesting effort E^* for the logistic discrete model is in fact higher than the continuous harvesting effort at any time t. At this point, we like to remind the reader that impulsive harvesting for the logistic growth model always yields a value below the continuous harvesting, as discussed in [14] and [32].

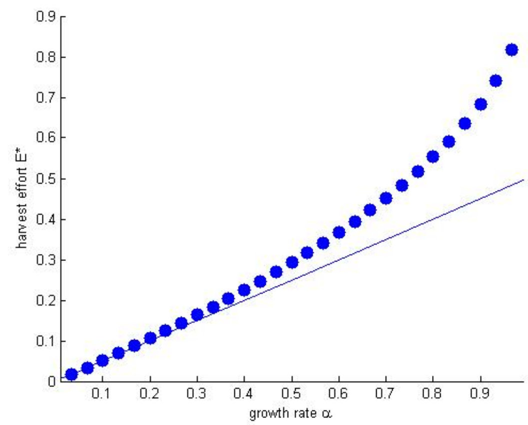

FIGURE 1. The optimal harvest effort E^* with respect to the inherent growth rate α.

3.3. **Optimal harvesting policy for the Beverton–Holt equation with periodic growth rate.** Let us bring our attention to the optimal harvesting policy for a single discrete population with periodic coefficients. We aim to maximize the economic profit due to harvest and at the same time guarantee the survival of the species. We have seen that the harvested single population modeled by the Beverton–Holt equation is of the form (11)

$$x_{n+1} = \frac{\nu_n x_n}{K_n + (\nu_n - 1)x_n} - h_n x_{n+1}.$$

The harvest at the end of the nth time unit is $h_n x_{n+1}$, which gives the yield over one period as

$$Y(h) = \sum_{n=0}^{\omega-1} h_n \bar{x}_{n+1},$$

where $\bar{x}$ is the unique ω-periodic solution of (11).

Theorem 3.10. *Assume* (25) *and (in order to guarantee a nonnegative harvest effort)*

$$\frac{K_n^\Delta}{K_n} \le \frac{1+\sqrt{1-\alpha_{n+1}}}{(1+\sqrt{1-\alpha_n})\sqrt{1-\alpha_{n+1}}} - 1. \tag{31}$$

The optimal harvest effort for (12) *is*

$$h^* = \ominus\left(\frac{1}{2}\odot(-\alpha)\right)^\sigma \ominus \frac{\left(\frac{\frac{1}{2}\odot(-\alpha)}{\alpha}\right)^\Delta}{\frac{\frac{1}{2}\odot(-\alpha)}{\alpha}} \ominus \frac{K^\Delta}{K},$$

and the maximal harvest yield over one period is

$$Y(h^*) = \sum_{j=0}^{\omega-1} \frac{\left(\frac{1}{2}\odot(-\alpha_j)\right)^2}{\alpha_j} K_j = \sum_{j=0}^{\omega-1} \frac{(1-\sqrt{1-\alpha_j})^2}{\alpha_j} K_j.$$

Proof. We essentially follow the same proof idea as in Theorem 2.2. We find an upper bound for the harvest yield and show that this maximal value is obtained for the suggested h^*. Let us realize that for $p = \frac{1}{2}\odot(-\alpha)$, $p \oplus p = -\alpha$, and p is

ω-periodic. We apply the weighted Jensen inequality [30, Theorem 2.2] (see also [2]) in the following way:

$$
\begin{aligned}
Y(h) &= \sum_{n=0}^{\omega-1} h_n \bar{x}_{n+1} \overset{(28)}{=} \sum_{n=0}^{\omega-1} h_n \lambda \frac{1}{\sum_{j=n+1}^{n+\omega} e_{h\oplus(-\alpha)}(n+1,j+1)\beta_j} \\
&\overset{(21)}{=} \lambda \sum_{n=0}^{\omega-1} h_n \frac{1}{\sum_{j=n+1}^{n+\omega} e_p(n+1,j+1) e_p(n+1,j+1) \frac{-p_j}{-p_j} e_h(n+1,j) \frac{\alpha_j}{K_j}} \\
&\le \lambda \sum_{n=0}^{\omega-1} h_n \frac{\sum_{j=n+1}^{n+\omega} e_p(n+1,j+1)(p_j)^2 e_{\ominus h}(n+1,j) e_{\ominus p}(n+1,j+1) \frac{K_j}{\alpha_j}}{\left(\sum_{j=n+1}^{n+\omega} e_p(n+1,j+1)(-p_j)\right)^2} \\
&\overset{(22)}{\underset{(24)}{=}} \lambda \sum_{n=0}^{\omega-1} h_n \frac{\sum_{j=n+1}^{n+\omega} p_j^2 e_h(j,n+1) \frac{K_j}{\alpha_j}}{\left(e_p(n+1,n+\omega+1)-1\right)^2} \\
&\overset{(16)}{=} \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \sum_{n=0}^{\omega-1} \sum_{j=n+1}^{n+\omega} h_n e_h(j,n+1) p_j^2 \frac{K_j}{\alpha_j} \\
&= \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \left\{ \sum_{j=1}^{\omega} p_j^2 \frac{K_j}{\alpha_j} \sum_{n=0}^{j-1} h_n e_h(j,n+1) \right. \\
&\qquad \left. + \sum_{j=\omega+1}^{2\omega} p_j^2 \frac{K_j}{\alpha_j} \sum_{n=j-\omega}^{\omega-1} h_n e_h(j,n+1) \right\} \\
&\overset{(24)}{=} \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \left\{ \sum_{j=1}^{\omega} p_j^2 \frac{K_j}{\alpha_j} (e_h(j,0)-1) \right. \\
&\qquad \left. + \sum_{j=\omega+1}^{2\omega} p_j^2 \frac{K_j}{\alpha_j} (e_h(j,j-\omega)-e_h(j,\omega)) \right\} \\
&= \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \left\{ \sum_{j=1}^{\omega} p_j^2 \frac{K_j}{\alpha_j} (e_h(j,0)-1) \right. \\
&\qquad \left. + \sum_{j=1}^{\omega} p_{j+\omega}^2 \frac{K_{j+\omega}}{\alpha_{j+\omega}} (e_h(j+\omega,j)-e_h(j+\omega,\omega)) \right\} \\
&\overset{(16)}{\underset{(17)}{=}} \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \sum_{j=1}^{\omega} p_j^2 \frac{K_j}{\alpha_j} (e_h(\omega,0)-1) \\
&= \frac{\lambda(e_h(\omega,0)-1)}{\left(e_p(0,\omega)-1\right)^2} \sum_{j=1}^{\omega} p_j^2 \frac{K_j}{\alpha_j} \le \sum_{j=1}^{\omega} p_j^2 \frac{K_j}{\alpha_j} = \sum_{j=1}^{\omega} \frac{\alpha_j}{(1+\sqrt{1-\alpha_j})^2} K_j.
\end{aligned}
$$

In the last inequality, we used the basic algebraic result

$$
\frac{(e_{h\oplus(-\alpha)}(0,\omega)-1)(e_h(\omega,0)-1)}{\left(1-e_p(0,\omega)\right)^2} \le 1.
$$

This is true because

$$(e_{h\oplus p}(0,\omega)e_p(0,\omega)-1)\left(\frac{e_p(0,\omega)}{e_{h\oplus p}(0,\omega)}-1\right)\le(1-e_p(0,\omega))^2,$$

i.e.,

$$-e_{h\oplus p}(0,\omega)e_p(0,\omega)-\frac{e_p(0,\omega)}{e_{h\oplus p}(0,\omega)}\le -2e_p(0,\omega),$$

i.e.,

$$\left(\sqrt{e_{h\oplus p}(0,\omega)}-\frac{1}{\sqrt{e_{h\oplus p}(0,\omega)}}\right)^2\ge 0.$$

We will now show that the maximal harvest yield is obtained for $h^*=\ominus p^\sigma\ominus\frac{\left(\frac{p}{\alpha}\right)^\Delta}{\frac{p}{\alpha}}\ominus\frac{K^\Delta}{K}$. Since $p\oplus p=-\alpha$, we have $p=-\alpha\ominus p$. Thus

$$\begin{aligned}
Y(h^*)&=\sum_{n=0}^{\omega-1}h_n^*\bar{x}_{n+1}\overset{(28)}{=}\sum_{n=0}^{\omega-1}h_n^*\lambda^*\frac{1}{\sum_{j=n+1}^{n+\omega}e_{h^*\oplus(-\alpha)}(n+1,j+1)\beta_j}\\
&=\lambda^*\sum_{n=0}^{\omega-1}h_n^*\frac{1}{\sum_{j=n+1}^{n+\omega}e_{\ominus h^*}(j,n+1)e_{-\alpha}(n+1,j+1)\frac{\alpha_j}{K_j}}\\
&=\lambda^*\sum_{n=0}^{\omega-1}h_n^*\frac{1}{\sum_{j=n+1}^{n+\omega}e_{p^\sigma}(j,n+1)\frac{p_j}{\alpha_j}\frac{\alpha_{n+1}}{p_{n+1}}\frac{K_j}{K_{n+1}}e_{-\alpha}(n+1,j+1)\frac{\alpha_j}{K_j}}\\
&=\lambda^*\sum_{n=0}^{\omega-1}h_n^*\frac{K_{n+1}\frac{p_{n+1}}{\alpha_{n+1}}}{\sum_{j=n+1}^{n+\omega}e_p(j+1,n+2)p_je_{-\alpha}(n+1,j+1)}\\
&=\lambda^*\sum_{n=0}^{\omega-1}h_n^*\frac{K_{n+1}\frac{p_{n+1}}{\alpha_{n+1}}}{e_p(n+1,n+2)\sum_{j=n+1}^{n+\omega}p_je_{-\alpha\ominus p}(n+1,j+1)}\\
&=\frac{(e_{h^*\oplus(-\alpha)}(0,\omega)-1)}{(1-e_p(0,\omega))}\sum_{n=0}^{\omega-1}h_n^*K_{n+1}\frac{p_{n+1}(1+p_{n+1})}{\alpha_{n+1}}\\
&=\frac{(e_{\ominus p^\sigma}(0,\omega)e_{-\alpha}(0,\omega)-1)}{(1-e_p(0,\omega))}\sum_{n=0}^{\omega-1}\Bigg(h_n^*K_{n+1}\frac{p_{n+1}(1+p_{n+1})}{\alpha_{n+1}}\\
&\qquad\qquad+K_{n+1}\frac{p_{n+1}^2}{\alpha_{n+1}}-K_{n+1}\frac{p_{n+1}^2}{\alpha_{n+1}}\Bigg)\\
&=-\sum_{n=0}^{\omega-1}K_{n+1}\frac{p_{n+1}}{\alpha_{n+1}}\left(h_n^*(p_{n+1}+1)+p_{n+1}\right)+\sum_{n=0}^{\omega-1}K_{n+1}\frac{p_{n+1}^2}{\alpha_{n+1}}\\
&=-\sum_{n=0}^{\omega-1}K_{n+1}\frac{p_{n+1}}{\alpha_{n+1}}\left(h_n^*\oplus p_{n+1}\right)+\sum_{n=1}^{\omega}K_n\frac{p_n^2}{\alpha_n}\\
&=-\sum_{n=0}^{\omega-1}K_{n+1}\frac{p_{n+1}}{\alpha_{n+1}}\left(\ominus\frac{\left(\frac{p_n}{\alpha_n}\right)^\Delta}{\left(\frac{p_n}{\alpha_n}\right)}\ominus\frac{K_n^\Delta}{K_n}\right)+\sum_{n=0}^{\omega-1}K_n\frac{p_n^2}{\alpha_n}
\end{aligned}$$

$$= -\sum_{n=0}^{\omega-1} K_{n+1}\frac{p_{n+1}}{\alpha_{n+1}}\left(-\frac{\left(\frac{p_n}{\alpha_n}\right)^{\Delta}K_n}{\left(\frac{p_{n+1}}{\alpha_{n+1}}\right)K_{n+1}} - \frac{K_n^{\Delta}}{K_{n+1}}\right) + \sum_{n=0}^{\omega-1} K_n\frac{p_n^2}{\alpha_n}$$

$$= \sum_{n=0}^{\omega-1}\left(\frac{p_n}{\alpha_n}K_n\right)^{\Delta} + \sum_{n=0}^{\omega-1} K_n\frac{p_n^2}{\alpha_n} = \sum_{n=0}^{\omega-1} K_n\frac{p_n^2}{\alpha_n}.$$

This proves that h^* yields the upper bound for the harvest yield and is therefore the optimal harvest effort. □

If we pick the carrying capacity to be constant and the inherent growth rate to be two-periodic, then the optimal harvest effort is given by

$$h^* = \ominus\left(\frac{1}{2}\odot(-\alpha)\right)^{\sigma} \ominus \frac{\left(\frac{\frac{1}{2}\odot(-\alpha)}{\alpha}\right)^{\Delta}}{\frac{\frac{1}{2}\odot(-\alpha)}{\alpha}}.$$

Example 3.11. Let

$$\alpha_0 = 0.5, \quad \alpha_1 = 0.8, \quad K = 200. \tag{32}$$

The ω-periodic solution for the Beverton–Holt model without harvesting is $\bar{x} = K = 200$, while for the harvested model, the periodic solution $\bar{x}^h$ obtains the following values.

$$\bar{x}_0^h = 89.56, \quad \bar{x}_1^h = 66.82, \quad h_0^* = 0.89, \quad h_1^* = 0.67.$$

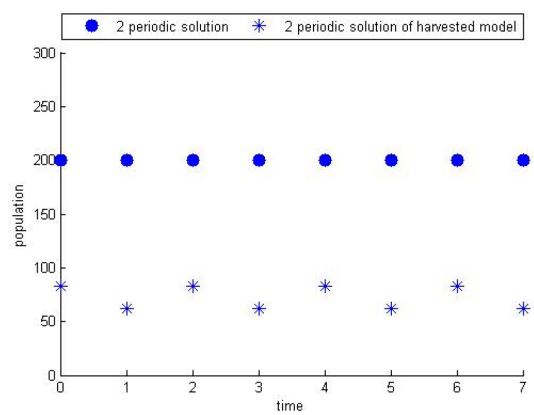

FIGURE 2. The periodic solution for the Beverton–Holt equation (with exploitation) in (32).

Figure 2 relates the periodic solution of the Beverton–Holt model to the unique periodic solution to the harvested Beverton–Holt equation.

Example 3.12. If we additionally change the carrying capacity to be two-periodic, such that a higher K corresponds to a higher α, then the solution changes. Take

$$\alpha_0 = 0.5, \quad \alpha_1 = 0.8, \quad K_0 = 200, \quad K_1 = 300. \tag{33}$$

Then the corresponding values for the population without harvesting $\bar{x}$ and the periodic solution for the Beverton–Holt model with exploitation $\bar{x}_h$ are as follows:

$$\bar{x}_0 = 284.21, \quad \bar{x}_1 = 234.78, \quad \bar{x}_0^h = 82.85, \quad \bar{x}_1^h = 92.71.$$

The values for the optimal harvest effort are

$$h_0^* = 0.26, \quad h_1^* = 1.5.$$

Figure 3 contains this information.

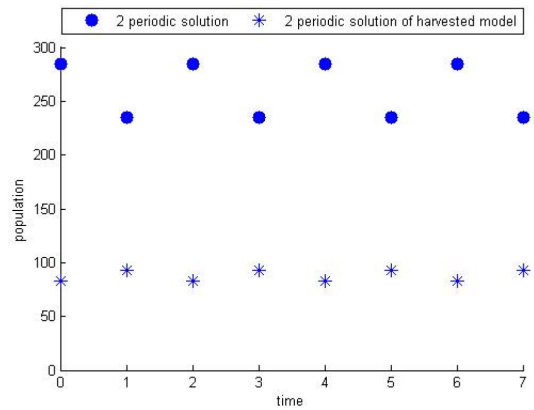

FIGURE 3. The periodic solution for the Beverton–Holt equation (with exploitation) in (33).

Example 3.13. Let us now consider a seasonal dependence of the carrying capacity and the inherent growth rate. Let α and K be 4-periodic, with the following values:

$$\begin{cases} \alpha_0 = 0.5, & \alpha_1 = 0.8, & \alpha_2 = 0.6, & \alpha_3 = 0.3, \\ K_0 = 200, & K_1 = 300, & K_2 = 300, & K_3 = 200. \end{cases} \tag{34}$$

The periodic solution for the classical Beverton–Holt model takes the following values

$$\bar{x}_0 = 298.77, \quad \bar{x}_1 = 224.83, \quad \bar{x}_2 = 281.19, \quad \bar{x}_3 = 292.18,$$

and the values for the harvested periodic solution $\bar{x}^h$ with the corresponding harvest effort h are

$$\begin{array}{llll} \bar{x}_0^h = 82.85, & \bar{x}_1^h = 92.70, & \bar{x}_2^h = 116.22, & \bar{x}_3^h = 91.11, \\ h_0^* = 0.26, & h_1^* = 0.78, & h_2^* = 1.02, & h_3^* = 0.31, \end{array}$$

see Figure 4.

Note that in Examples 3.11–3.13, the assumptions of Theorem 3.10 were satisfied, i.e., $\alpha_n \in (0,1)$ and

$$\frac{K_n^\Delta}{K_n} \le \frac{1 + \sqrt{1 - \alpha_{n+1}}}{(1 + \sqrt{1 - \alpha_n})\sqrt{1 - \alpha_{n+1}}} - 1.$$

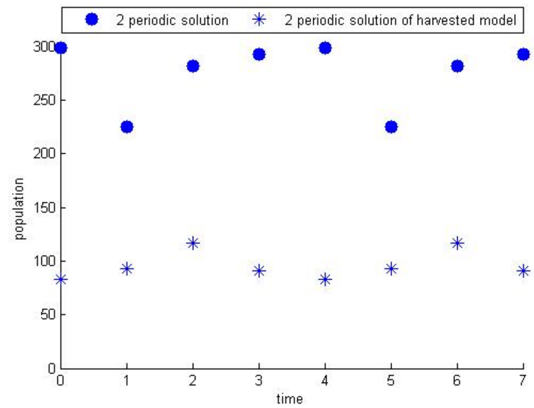

FIGURE 4. The periodic solution for the Beverton–Holt equation (with exploitation) in (34).

Example 3.14. Let us discuss the case of an ω-periodic population that follows a continuous patter in the interval $[0,T]$, $0<T<\omega-1$, followed by a discrete pattern for the rest of the period. The time scale that describes the time set is $\mathbb{T}=[0,T]\cup\{T+1,\dots,\omega-1\}$. First, we recall that the discrete Beverton–Holt model and the continuous logistic growth model are unified in the dynamic equation

$$x^{\Delta}=\alpha x^{\sigma}\left(1-\frac{x}{K}\right)+(\ominus h)\,x, \tag{35}$$

where $x^{\sigma}=x+\mu x^{\Delta}$, and

$$\mu(t)=\begin{cases}0 & \text{if} \quad t\in\mathbb{R},\\ 1 & \text{if} \quad t\in\mathbb{Z},\end{cases} \qquad x^{\Delta}(t)=\begin{cases}x'(t) & \text{if} \quad t\in\mathbb{R},\\ \Delta x(t) & \text{if} \quad t\in\mathbb{Z}.\end{cases}$$

Therefore, (35) is equivalent to

$$x^{\Delta}=(1+\mu(t)h(t))\alpha(t)x^{\sigma}\left(1-\frac{x}{K(t)}\right)-h(t)x^{\sigma}.$$

As before, the ω-periodic solution has the unified expression

$$\bar{x}(t)=\lambda\left[\int_t^{t+\omega}e_{h\oplus(-\alpha)}(t,\sigma(s))\frac{\alpha(s)}{K(s)}(1+\mu(s)h(s))\,\Delta s\right]^{-1},$$

where $p=\frac{1}{2}\odot(-\alpha)$. The sustainable yield is

$$\begin{aligned}Y(h)&=\int_0^{\omega}h(t)x^{\sigma}(t)\,\Delta t\\ &=\int_0^{\omega}h(t)\frac{\lambda}{\int_{\sigma(t)}^{\sigma(t+\omega)}e_{h\oplus(-\alpha)}(\sigma(t),\sigma(s))\frac{\alpha(s)}{K(s)}(1+\mu(s)h(s))\,\Delta s}\Delta t\\ &=\int_0^{\omega}h(t)\frac{\lambda}{\int_{\sigma(t)}^{\sigma(t+\omega)}e_{p\oplus p}(\sigma(t),\sigma(s))e_h(\sigma(t),s)\frac{\alpha(s)}{K(s)}\,\Delta s}\Delta t\end{aligned}$$

$$\le \lambda \int_0^\omega h(t) \frac{\int_{\sigma(t)}^{\sigma(t+\omega)} e_h(s,\sigma(t)) \frac{p^2(s)K(s)}{\alpha(s)} \,\Delta s}{\left(\int_{\sigma(t)}^{\sigma(t+\omega)} e_p(\sigma(t),\sigma(s))p(s)\,\Delta s\right)^2} \Delta t$$

$$= \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \int_0^\omega h(t) \int_t^{t+\omega} e_h(\sigma(s),\sigma(t)) \left(\frac{p^2K}{\alpha}\right)(\sigma(s))\,\Delta s\,\Delta t$$

$$= \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \left\{ \int_0^T \int_0^s h(t)e_h(s,t)\left(\frac{p^2K}{\alpha}\right)(s)\,\mathrm{d}t\,\mathrm{d}s \right.$$

$$+ \sum_{i=T}^{\omega-1} \int_0^T h(t)e_h(i+1,t)\left(\frac{p^2K}{\alpha}\right)(i+1)\,\mathrm{d}t$$

$$+ \sum_{i=T}^{\omega-1} \sum_{j=T}^{i} h(j)e_h(i+1,j+1)\left(\frac{p^2K}{\alpha}\right)(i+1)$$

$$+ \int_\omega^{T+\omega} \int_{s-\omega}^{T} h(t)e_h(s,t)\left(\frac{p^2K}{\alpha}\right)(s)\,\mathrm{d}t\,\mathrm{d}s$$

$$+ \sum_{i=T+\omega}^{2\omega-1} \sum_{j=i+1-\omega}^{\omega-1} h(j)e_h(i+1,j+1)\left(\frac{p^2K}{\alpha}\right)(i+1)$$

$$\left. + \int_\omega^{T+\omega} \sum_{j=T}^{\omega-1} h(j)e_h(s,j+1)\left(\frac{p^2K}{\alpha}\right)(s)\,\mathrm{d}s \right\}$$

$$= \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \left\{ \int_0^T (e_h(s,0)-1)\left(\frac{p^2K}{\alpha}\right)(s)\,\mathrm{d}s \right.$$

$$+ \int_T^\omega (e_h(\sigma(s),0)-e_h(\sigma(s),T))\left(\frac{p^2K}{\alpha}\right)(\sigma(s))\,\Delta s$$

$$+ \int_T^\omega (e_h(\sigma(s),T)-1)\left(\frac{p^2K}{\alpha}\right)(\sigma(s))\,\Delta s$$

$$+ \int_0^T (e_h(\omega,0)-e_h(s+\omega,T))\left(\frac{p^2K}{\alpha}\right)(s)\,\mathrm{d}s$$

$$+ \int_T^\omega (e_h(\omega,0)-e_h(\sigma(s),0))\left(\frac{p^2K}{\alpha}\right)(\sigma(s))\Delta s$$

$$\left. + \int_0^T (e_h(s+\omega,T)-e_h(s,0))\left(\frac{p^2K}{\alpha}\right)(s)\,\mathrm{d}s \right\}$$

$$= \frac{\lambda}{\left(e_p(0,\omega)-1\right)^2} \left\{ \int_0^T (e_h(\omega,0)-1)\left(\frac{p^2K}{\alpha}\right)(s)\,\mathrm{d}s \right.$$

$$+\int_T^\omega (e_h(\omega,0)-1)\left(\frac{p^2K}{\alpha}\right)(\sigma(s))\,\Delta s\Bigg\}$$

$$=\frac{\lambda\,(e_h(\omega,0)-1)}{(e_p(0,\omega)-1)^2}\int_0^\omega\left(\frac{p^2K}{\alpha}\right)(\sigma(s))\,\Delta s\le\int_0^\omega\left(\frac{p^2K}{\alpha}\right)(\sigma(s))\,\Delta s,$$

by the same argument as earlier, equality holds for

$$h^*=\ominus\left(\frac{1}{2}\odot(-\alpha)\right)^\sigma\ominus\frac{\left(\frac{\frac{1}{2}\odot(-\alpha)}{\alpha}\right)^\Delta}{\frac{\frac{1}{2}\odot(-\alpha)}{\alpha}}\ominus\frac{K^\Delta}{K}.$$

Note that this formula implies $h^*=\frac{\alpha}{2}-\frac{K'}{K}$ for $t\in[0,T)$.

Let us take for example $\omega=5$ and $T=3$,

$$\alpha(t)=\begin{cases}\frac{1}{6}t+\frac{1}{12} & \text{if}\quad 0\le t\le 3\\ \frac{1}{3} & \text{if}\quad t=4\end{cases}$$

and

$$K(t)=\begin{cases}100 & \text{if}\quad 0\le t\le 3\\ 90 & \text{if}\quad t=4,\end{cases}$$

where α and K are 5-periodic. Note that these functions satisfy the conditions to guarantee a nonnegative harvest effort. The optimal harvesting is then

$$h^*(t)=\begin{cases}\frac{1}{12}t+\frac{1}{24} & \text{if } 0\le t<3\\[2mm] \frac{10}{9}\frac{1+\sqrt{2/3}}{\sqrt{2/3}\left(1+\sqrt{5/12}\right)}-1 & \text{if } t=3\\[2mm] \frac{9}{10}\frac{1+\sqrt{11/12}}{\left(1+\sqrt{2/3}\right)\sqrt{11/12}}-1 & \text{if } t=4\end{cases}$$

Figure 5 and 6 show the periodic solution for the harvested model and the optimal harvesting strategy for this example. As expected, the harvest drops rapidly from 0.3343 at $t=3$ to 0.0128 in $t=4$, since the carrying capacity and the growth rate are declining and causing a decline in the population. Furthermore, in order to guarantee the periodicity at $t=5$, the population needs to recover from the new environment that is for example reflecting the winter period.

Note that we have a jump at $t=3$, because the delta-operator is defined in terms of $\sigma(t)=\inf\{s>t\,:\,s\in\mathbb{T}\}$. Since the discrete pattern starts at $t=3$, the derivative is replaced by the delta-operator.

We would like to finish the article by discussing two modifications of the model. First, we address the model including pulse harvesting at time steps τ_k, $k=1,2,\dots,q$.

Remark 2. A similar approach, using the Jensen inequality, can be applied to a model with impulsive harvesting. Comparing the set up for the continuous logistic growth model and the dynamic logistic model for $\mathbb{T}=\mathbb{Z}$, we suggest the following model describing impulsive harvesting:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\alpha(t)x\left(1-\frac{x}{K(t)}\right),\qquad t\neq\tau_k,$$
$$x(\tau_k)=x(\tau_k^+)(1+h(\tau_k)),\qquad k=1,2,\dots,q,$$

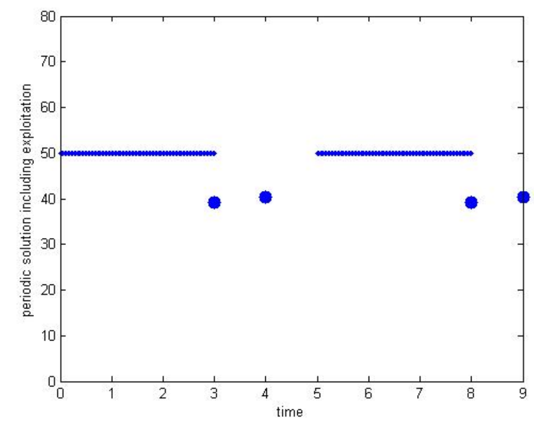

FIGURE 5. The periodic solution for the Beverton–Holt equation with exploitation.

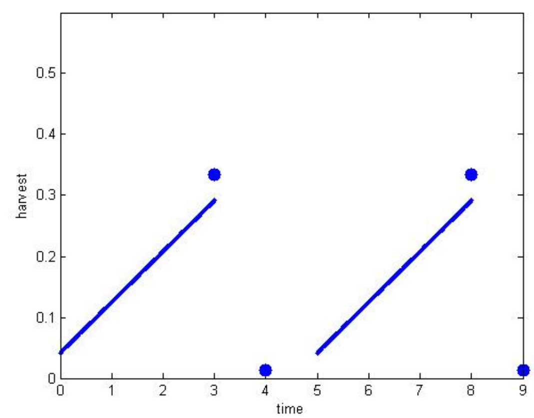

FIGURE 6. The optimal harvesting policy.

where $h, \alpha, K : \mathbb{R} \to \mathbb{R}^+$ are ω-periodic functions and describe the harvest effort, growth rate, and carrying capacity respectively. Impulsive harvesting is realized at time steps τ_k, $k = 1, 2, \ldots, q$. The sustainable yield of this model is then

$$Y(h) = \sum_{i=1}^{q} h(\tau_i) x(\tau_i^+).$$

We believe, that by applying the Jensen-inequality in a similar fashion, the optimal impulsive harvesting strategy can be obtained.

Remark 3. Similar to the construction of the diffusive model presented in [13], a diffusive model for the Beverton–Holt equation can be discussed. We suggest in this case, that the diffusion term depends on the ratio

$$r_1 \cdot r_2 = \frac{x - K}{K} \cdot \frac{\alpha}{\alpha \ominus \left(\frac{1}{2} \odot (-\alpha)\right)}.$$

The model reads then as

$$u^{\Delta_t} = D\left(r_1 \cdot r_2\right) + r(t,x)u^{\sigma_t}\left(1 - \frac{u}{K(t,x)}\right) - E(t,x)u^{\sigma_t},$$

where $u^{\Delta_t} = u(t+1,x) - u(t,x)$, and $u^{\sigma_t} = u(t+1,x)$. If the functions are time independent, i.e., $u^{\Delta_t} = 0$, then a similar argument as in [13, Theorem 1] can be applied to argue that the optimal harvest strategy is obtained at

$$E = -\left(\frac{1}{2} \odot (-\alpha)\right),$$

which is the same harvesting policy for the constant coefficient model discussed earlier. We expect a similar behavior if the model is analyzed for time dependent coefficients.

4. Conclusion.

In this paper, we introduced the Beverton–Holt difference equation including exploitation using the catch-per-unit hypothesis. We proved the existence of the unique periodic solution of the harvested Beverton–Holt equation that is globally attracting positive solutions. The discussion of the optimal sustainable yield was introduced by first considering the continuous logistic growth model including exploitation. We provided a different proof to obtain the optimal harvest effort that yields the maximum sustainable yield. The same technique was then used to discuss the optimal sustainable yield for the harvested Beverton–Holt difference equation. Future studies will include the discussion of the present value of the annual sustainable yield for the Beverton–Holt difference equation including exploitation, which is well studied for the logistic growth model by Clark [15]. The example of the mixed, continuous-discrete pattern, can be generalized to a model, that changes between the two pattern at finitely many time steps in one period. For future work, it will be interesting to further discuss diffusive model, such as suggested in Remark 3.

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