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A NEW FIRING PARADIGM FOR INTEGRATE AND FIRE STOCHASTIC NEURONAL MODELS

Roberta Sirovich and Luisa Testa

Department of Mathematics G. Peano
University of Torino
Via Carlo Alberto 10
10123 Torino, Italy

Abstract. A new definition of firing time is given in the framework of Integrate and Fire neuronal models. The classical absorption condition at the threshold is relaxed and the firing time is defined as the first time the membrane potential process lies above a fixed depolarisation level for a sufficiently long time. The mathematical properties of the new firing time are investigated both for the Perfect Integrator and the Leaky Integrator. In the latter case, a simulation study is presented to complete the analysis where analytical results are not yet achieved.

1. Introduction.

Integrate and Fire (IF) models are among the most used descriptions of the single neuron membrane potential dynamics [22, 42, 50, 51]. Their popularity is due to their capability of reproducing most of the essential properties of neural processing together with their mathematical tractability. Since their introduction [21], a large literature on this class of models appeared. Recent reviews dedicated to the topic are [10, 11, 46]. Highly predictive generalisations of these models, both for the spiking times and for the subthreshold membrane potential response to given input currents, have been introduced [30, 32, 39, 44]. Moreover, they are considered as building blocks of neuronal networks models [8, 12, 13, 18, 19, 27, 49].

The statistical investigation of these models is still incomplete despite recent efforts [5, 7, 15, 16, 17, 33, 35, 37, 48] and some old preliminary study [28, 34].

In many instances, data are not consistent with a relevant feature of such models. We refer to the absorbing assumption imposed to the membrane potential at the threshold level, i.e. the firing condition. The presence of the absorbing boundary is often disregarded, introducing important errors in the estimation procedure [6, 25]. One of the motivations for ignoring the presence of the absorbing boundary is of statistical nature: the estimation of its value is ambiguous. Setting the threshold at the depolarisation level at which a spike is released is very natural and in agreement with the model construction. However, in almost every recording, a value chosen as previously described is inconsistent with the absorption assumption. Indeed the voltage clearly exceeds the estimated level several times without firing.

Mainly motivated by statistical purposes, we propose here a new firing paradigm. Our goal is to generalise the model in order to overcome the lack of congruence with data but without loosing its mathematical tractability.

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To this aim we work only on the firing mechanism and introduce a new definition of firing time. The new model relaxes the absorption condition and allows crossing of the threshold without firing. This choice has a biological rationale. The generation of a spike requests the opening of a cascade of voltage dependent channels. Only signals that sustain the potential for a sufficiently long time interval can give rise to such event.

The manuscript is organised as follows. We present a brief review on classical IF models in Section 2 and introduce the new firing paradigm in Section 3. The mathematical properties of the new firing time are derived in Section 4 for the Perfect Integrator model and in Section 5 for the Leaky Integrator model. In the latter case the mathematical results are not sufficient to complete the comparison of the new model and the classical one. Hence a simulation study is presented as well. Some short conclusive arguments are illustrated in Section 6.

2. **Classical Integrate and Fire models.** In the simplest description of a neuron, the cell is identified with its membrane potential V_t. Excitatory and inhibitory presynaptic inputs determine fluctuations of the depolarisation until an action potential is generated. Then, the cell recovers its resting potential and the process starts anew.

Integrate and Fire (IF) models are composed of three elements: the stochastic process that describes the membrane depolarisation between two consecutive action potentials, the firing mechanism that identifies the instant at which a spike is generated and the resetting condition. Action potentials are reduced to point events fully characterised by their firing time, disregarding the shape of the spike and its duration.

Diffusions are the most widely used stochastic processes for describing the membrane potential V_t. They are solutions of stochastic differential equations (SDEs), cf [40]

$$dV_t = \beta(t, V_t)dt + \sigma(t, V_t)dW_t, \quad V_0 = v_0, \tag{1}$$

where v_0 is the resetting potential, $W(t)$ is a standard Brownian motion and β and $\sigma > 0$ are real valued functions, the so called infinitesimal drift and diffusion coefficient.

The spiking mechanism is included in the models by imposing a firing condition: an output spike is generated when the membrane potential reaches a fixed threshold level $S > v_0$. The membrane potential is then instantaneously reset to its initial value v_0. The firing time is identified with the random variable

$$T = \inf\{t \geq 0 \mid V_t \geq S\}, \tag{2}$$

the so called first passage time of the process V_t across the threshold S. Under such assumptions, the sequence of interspike intervals (ISIs) form a renewal process with independent inter–times all distributed as T given in eq. (2).

Action potentials are usually well separated in time. Even with a strong input it is not possible to excite a second spike during or immediately after a first one. Therefore an absolute refractory period τ_r is often included in the models [22]. The firing frequency is then given by the r.v.

$$\lambda_{out} = \frac{1}{T + \tau_r}. \tag{3}$$

In many cases (see for example [10]), the same notation refers to the reciprocal of the mean interspike interval, $1/(\mathbb{E}[T] + \tau_r)$, in other cases it refers to the mean of instantaneous firing frequency $\mathbb{E}[1/(T+\tau_r)]$. For a comparison of such definitions see [36].

We consider here the so called Perfect Integrator and the Leaky Integrator IF models, i.e. the membrane potential V_t in equation (1) is a Wiener process and an Ornstein-Uhlenbeck (OU) process, respectively. In the first case, the membrane potential V_t follows the equation

$$dV_t = \mu dt + \sigma dW_t, \quad V_0 = v_0, \tag{4}$$

where the infinitesimal drift μ and the diffusion coefficient $\sigma > 0$ are constant. In this model spiking is a sure event only if $\mu \geq 0$ and $S \geq v_0$. The first passage time is Inverse Gaussian distributed with probability density function (pdf) [46]

$$f(t, S, v_0) = \frac{|S - v_0|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left[-\frac{(S - v_0 - \mu t)^2}{2\sigma^2 t}\right]. \tag{5}$$

Mean and variance of T are given by

$$\mathbb{E}[T] = \frac{S - v_0}{\mu} \qquad \mathrm{Var}[T] = \frac{(S - v_0)\sigma^2}{\mu^3}. \tag{6}$$

Introducing a leakage term for the spontaneous decay of the membrane potential towards the resting level in the absence of inputs, we get the so called Leaky Integrator. The resulting stochastic process for the membrane potential is the well known OU process, solution of the equation

$$dV_t = \left[-\frac{(V_t - v_0)}{\tau} + \mu\right] dt + \sigma dW_t, \quad V_0 = v_0, \tag{7}$$

where $\tau > 0$ is the membrane time constant. Mean and variance of the process V_t are given by the following equations

$$\mathbb{E}[V_t|v_0] = v_0 \mathrm{e}^{-t/\tau} + \mu\tau(1 - \mathrm{e}^{-t/\tau}) \tag{8}$$

$$\mathrm{Var}[V_t|v_0] = \frac{\sigma^2\tau}{2} \cdot (1 - \mathrm{e}^{-2t/\tau}) \tag{9}$$

The first passage time pdf is not known in closed form but several representations are written in terms of its Laplace transform, series expansion and Bessel bridge process [3, 46]. Moreover it is solution of some integral equations and efficient numerical algorithms for an approximation have been derived [9, 24]. An explicit equation for the mean first passage time is given by [46]

$$\mathbb{E}[T] = \sqrt{\frac{\pi\tau}{\sigma^2}} \int_{-\mu\tau}^{S-\mu\tau} \left[1 + \mathrm{Erf}\left(\frac{x}{\sigma\sqrt{\tau}}\right)\right] \exp\left(\frac{x^2}{\sigma^2\tau}\right) dx. \tag{10}$$

For the second order moment see, for example, [43, 47]. Many properties of the firing time depend on the reciprocal position between the asymptotic mean of the process $\mathbb{E}[V_\infty] = \mu\tau$ and the value of the threshold S. We can distinguish two firing regimes, supra–threshold regime, $\mu\tau > S$, and sub–threshold regime, $\mu\tau < S$. In the first case the firing times are relatively regular, mostly affected by the drifting part of the process V_t. In the second case the crossing of the threshold is determined by the random fluctuations of the process and shows a Poissonian behaviour [46].

3. **A new definition of the firing time.** In IF models paradigm the firing threshold, i.e. the depolarisation level at which an action potential is released, is assumed to be absorbing. This implies that the stochastic process V_t verifies a boundary condition according to which it cannot enter the half–plane $(S, +\infty)$. Hence the sample paths of the constrained process differ from those of the unconstrained process and the firing threshold is the maximum depolarisation level reached between two consecutive spikes.

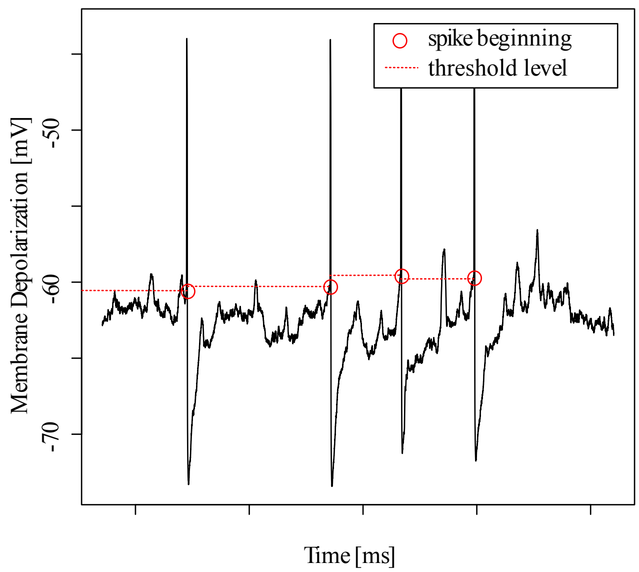

FIGURE 1. An example of membrane potential intracellular recording [52]. The elements of an IF model have been indicated.

In Fig. 1 an empirical intracellular recording is reproduced and the characterising elements of an IF model are indicated. Two evidences contradict the absorbing boundary assumption:

(a) the membrane potential crosses the threshold level but no action potential is released;
(b) the firing threshold cannot be considered constant.

To overcome the above mentioned inconsistencies we propose here a new firing mechanism definition. We assume that a spike is generated as the membrane potential reaches a fixed threshold level and remains above it for a sufficiently long time interval. The firing time is defined as

$$H = \inf \{t \geq 0 \mid (t - g_t) \cdot \mathbf{1}_{V_t \geq S} \geq \Delta\}, \tag{11}$$

where $\mathbf{1}_A$ is the indicator function of the set A, Δ is the time window that the process has to spend above the threshold S and $\forall t$

$$g_t = \sup\{s \leq t; V_s = S\} \tag{12}$$

In Fig. 2 a sample trajectory of the membrane potential is plotted and g_t as well as H are shown. Actually the new firing mechanism satisfies (a) and (b). Moreover it

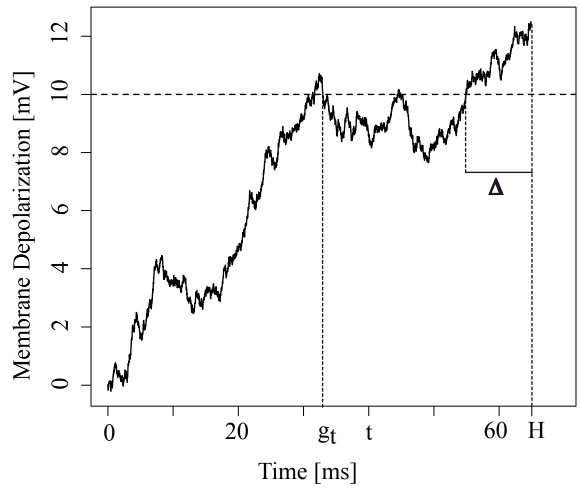

FIGURE 2. The firing time H.

naturally includes an absolute refractory period equal to Δ avoiding the artificial introduction of a further parameter τ_r.

The r.v. H has been firstly introduced in [14], in order to study the so called Parisian barrier options. It is known that H is a $\mathcal{F}^+_{g_t}$–stopping time and therefore a $\mathcal{F}^+_t$ stopping time.[1] In addition, H and V_H are independent. The law of V_H and the Laplace transform of H for a standard Brownian motion are given [14]

$$\mathbb{E}\left[\mathrm{e}^{-\lambda H}\right] = \frac{\mathrm{e}^{-S\sqrt{2\lambda}}}{\psi(\sqrt{2\lambda\Delta})}, \tag{13}$$

where

$$\begin{aligned} \psi(z) &= \int_0^\infty x\mathrm{e}^{(z\,x-\frac{x^2}{2})}dx \\ &= 1+\sqrt{\frac{\pi}{2}}z\mathrm{e}^{\frac{z^2}{2}}\left[1+\mathrm{Erf}\left(\frac{z}{\sqrt{2}}\right)\right]. \end{aligned}$$

and

$$\mathbb{P}(V_H \in dv) = \frac{v-S}{\Delta}\,\mathrm{e}^{-\frac{(S-v)^2}{2\Delta}}\mathbf{1}_{v>S}\,dv. \tag{14}$$

4. **The perfect Integrator model.** We derive here an explicit expression for the Laplace transform of H for a Wiener process with drift μ and diffusion coefficient σ. In the following, we denote $\mathbb{P}_0$ ($\mathbb{E}_0$) the measure (and the corresponding expected value) under which the coordinate process is a standard Brownian motion and with $\mathbb{P}_{\mu,\sigma}$ ($\mathbb{E}_{\mu,\sigma}$) the measure under which it is a Brownian motion with drift μ and diffusion coefficient σ.

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[1] Let $\mathcal{F}_t$ be the natural filtration of the Brownian motion W. If R is a random variable such that $R > 0$ a.s, we define the sigma field $\mathcal{F}^-_R$ of the past up to R as the σ algebra generated by the variables ς_R, where ς is a predictable process. Denote by $\mathcal{F}^+_R$ the slow Brownian filtration $\mathcal{F}^+_R = \mathcal{F}^-_R \vee \sigma(\mathrm{sgn}(W_t))$. For details see [14].

Theorem 4.1. *The Laplace transform of the random variable H defined in eq.* (11) *is given as*

$$\mathbb{E}_{\mu,\sigma}\left(e^{-\lambda H}\right) = \exp\left[\frac{\mu S}{\sigma^2} - \frac{S}{\sigma}\sqrt{2\left(\lambda + \frac{|\mu|^2}{2\sigma^2}\right)}\right] \frac{\psi\left(\frac{\mu}{\sigma}\sqrt{\Delta}\right)}{\psi\left(\sqrt{2\left(\lambda + \frac{|\mu|^2}{2\sigma^2}\right)\Delta}\right)}. \tag{15}$$

Proof. Let us reduce the Wiener process V_t in eq. (4) to a process $Z_t = V_t/\sigma$ with the same drift but unitary diffusion coefficient. We have, for $v_0 = 0$

$$dZ_t = \frac{\mu}{\sigma}dt + dW_t, \quad Z_0 = 0. \tag{16}$$

By Girsanov theorem [31, 40, 45]

$$\left.\frac{d\mathbb{P}_{\frac{\mu}{\sigma},1}}{d\mathbb{P}_0}\right|_t = G_t(\omega) = \exp\left(\frac{\mu}{\sigma}Z_t - \frac{\mu^2}{2\sigma^2}t\right),$$

and hence

$$\begin{aligned}\mathbb{E}_{\frac{\mu}{\sigma},1}\left[\mathrm{e}^{-\lambda H}\right] &= \mathbb{E}_0\left[G_H \cdot \mathrm{e}^{-\lambda H}\right] \\ &= \mathbb{E}_0\left[\mathrm{e}^{\frac{\mu}{\sigma}Z_H - \frac{\mu^2}{2\sigma^2}H} \cdot \mathrm{e}^{-\lambda H}\right] \\ &= \underbrace{\mathbb{E}_0\left[\mathrm{e}^{-\left(\lambda + \frac{\mu^2}{2\sigma^2}\right)H}\right]}_{\text{A}} \cdot \underbrace{\mathbb{E}_0\left[\mathrm{e}^{\frac{\mu}{\sigma}Z_H}\right]}_{B}\end{aligned} \tag{17}$$

where the third equality follows from the independence between H and Z_H [14]. From (13) and (14) it follows that

$$A = \frac{\mathrm{e}^{-S\sqrt{2\left(\lambda + \frac{\mu^2}{2\sigma^2}\right)}}}{\psi\left(\sqrt{2\left(\lambda + \frac{\mu^2}{2\sigma^2}\right)\Delta}\right)} \tag{18}$$

and

$$\begin{aligned}B &= \int_S^{+\infty} \mathrm{e}^{\frac{\mu}{\sigma}x}\, \mathbb{P}_0(Z_H \in dx) \\ &= \mathrm{e}^{\frac{\mu S}{\sigma}}\left[1 + \frac{\mu}{\sigma}\sqrt{\frac{\pi\Delta}{2}}\mathrm{e}^{\frac{\Delta\mu^2}{2\sigma^2}}\left(1 + \mathrm{Erf}\left[\frac{\mu}{\sigma}\sqrt{\frac{\Delta}{2}}\right]\right)\right] \\ &= \mathrm{e}^{\frac{\mu S}{\sigma}} \cdot \psi\left(\frac{\mu}{\sigma}\sqrt{\Delta}\right).\end{aligned} \tag{19}$$

Therefore, the Laplace transform of the random variable H for a Wiener process with drift $\frac{\mu}{\sigma}$ and unitary diffusion coefficient is

$$\mathbb{E}_{\frac{\mu}{\sigma},1}\left(\mathrm{e}^{-\lambda H}\right) = \exp\left[\frac{\mu S}{\sigma} - S\sqrt{2\left(\lambda + \frac{|\mu|^2}{2\sigma^2}\right)}\right] \frac{\psi\left(\frac{\mu}{\sigma}\sqrt{\Delta}\right)}{\psi\left(\sqrt{2\left(\lambda + \frac{|\mu|^2}{2\sigma^2}\right)\Delta}\right)}. \tag{20}$$

Finally, rescaling the process to V_t, we get eq. (15) for a Brownian motion with drift μ and diffusion coefficient σ. □

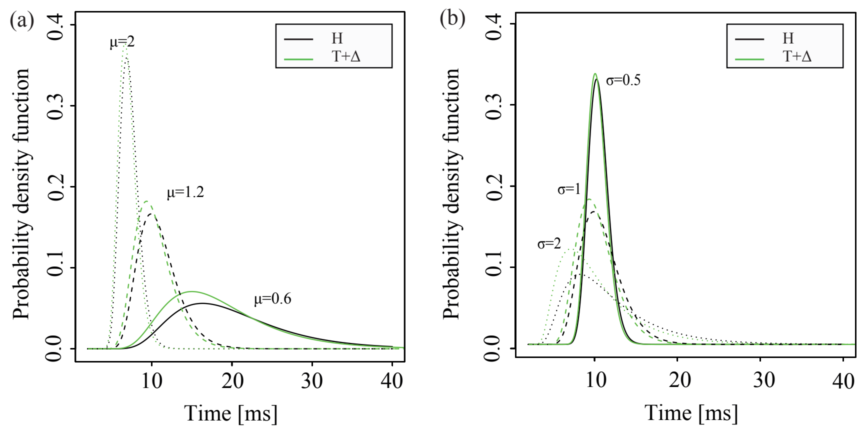

FIGURE 3. (a) Pdfs of H and $T+\Delta$ for three different values of μ: $\mu = 0.6$ mVms^{-1} (solid line), $\mu = 1.2$ mVms^{-1} (dashed line), $\mu = 2$ mVms^{-1} (dotted line), $S = 10$ mV, $V_0 = 0$ mV, $\sigma = 1$ mV^{2}ms^{-1}, $\Delta = 2$ ms. (b) Pdfs of H and $T+\Delta$ for three different values of σ: $\sigma = 0.5$ mV^{2}ms^{-1} (solid line), $\sigma = 1$ mV^{2}ms^{-1} (dashed line), $\sigma = 2$ mV^{2}ms^{-1} (dotted line), $S = 10$ mV, $V_0 = 0$ mV, $\mu = 1.2$ mVms^{-1} , $\Delta = 2$ ms.

Proposition 1. *It holds*

$$\mathbb{P}(H < \infty) = \begin{cases} 1 & \mu \geq 0 \\ e^{\left(\frac{2\mu S}{\sigma^2}\right)} \frac{\psi\left(\sqrt{\Delta}\frac{\mu}{\sigma}\right)}{\psi\left(-\sqrt{\Delta}\frac{\mu}{\sigma}\right)} & \mu < 0. \end{cases} \tag{21}$$

When the firing time H is finite a.s., the first two moments of the r.v. H are given as

$$\mathbb{E}[H] = \left(\frac{S}{\mu} + \Delta\right) + \frac{\sigma^2}{\mu^2}\left(1 - \frac{1}{\psi\left(\frac{\mu}{\sigma}\sqrt{\Delta}\right)}\right) \tag{22}$$

$$\mathbb{E}[H^2] = \left(\frac{S}{\mu} + \Delta\right)^2 + \frac{\sigma^2}{\mu^2}\left\{\left(\frac{3S}{\mu} + \frac{3\sigma^2}{\mu^2} + 2\Delta\right) + \right. \tag{23}$$
$$\left. - \frac{1}{\psi\left(\frac{\mu}{\sigma}\sqrt{\Delta}\right)}\left(3\Delta + \frac{5\sigma^2}{\mu^2} + \frac{2S}{\mu}\right) + \frac{1}{\psi^2\left(\frac{\mu}{\sigma}\sqrt{\Delta}\right)}\frac{2\sigma^2}{\mu^2}\right\}.$$

Proof. From eq. (15) and considering that

$$P(H < \infty) = \mathbb{E}_{\mu,\sigma}\left(\mathrm{e}^{-\lambda H}\right)\big|_{\lambda=0}, \tag{24}$$

we get the conditions (21). Moreover recalling that

$$\mathbb{E}[H] = -\frac{d}{d\lambda}\mathbb{E}_{\mu,\sigma}\left(\mathrm{e}^{-\lambda H}\right)\Big|_{\lambda=0} \quad \text{and} \quad \mathbb{E}[H^2] = \frac{d^2}{d\lambda}\mathbb{E}_{\mu,\sigma}\left(\mathrm{e}^{-\lambda H}\right)\Big|_{\lambda=0}, \tag{25}$$

we get the moments (22) and (23). □

Eq. (15) can be numerically inverted in order to obtain the pdf of H [1, 2]. In Fig. 3, a graphical comparison between the pdfs of $T+\Delta$ and H is shown. The pdfs show similar behaviours. More appreciable differences emerge for smaller values of μ and larger values of σ.

Let us now derive some results about the compared behavior of the two firing times H and T. We fix the refractory period τ_r equal to Δ and hence we compare H to $T+\Delta$. Furthermore, we restrict to the case $S > v_0 = 0$ and $\mu \geq 0$.

Proposition 2. *For any* $\Delta > 0$

$$\mathbb{E}[H] > \mathbb{E}[T] + \Delta$$

and

$$Var[H] > Var[T]$$

if and only if

$$\frac{2\sigma^4}{\mu^4} + \frac{\sigma^4}{\mu^4 \psi^2\left(\frac{\mu\sqrt{\Delta}}{\sigma}\right)} - \frac{\sigma^2\Delta}{\mu^2\psi\left(\frac{\mu\sqrt{\Delta}}{\sigma}\right)} - \frac{3\sigma^4}{\mu^4\psi\left(\frac{\mu\sqrt{\Delta}}{\sigma}\right)} > 0.$$

Proof. From eq. (22) and eq. (6) we have

$$\mathbb{E}[H] = (\mathbb{E}[T] + \Delta) + \frac{\sigma^2}{\mu^2}\left(1 - \frac{1}{\psi\left(\frac{\mu}{\sigma}\sqrt{\Delta}\right)}\right).$$

The thesis follows recalling that $\psi(z) > 1$ for any $z > 0$. The result on the variances comes directly applying (6), (22) and (23). □

When μ is large enough, H is well approximated by $T+\Delta$. For μ going to zero the expectation of H goes to infinity with a rate faster than the expectation of T, as shown in the following proposition.

Proposition 3. *It holds*

i. $\mathbb{E}[H] \approx$ [2] $\mathbb{E}[T] + \Delta$ *and* $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta) \to 0$ *as* $\mu \to +\infty$,

ii. $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta) \to \infty$ *as* $\mu \to 0$.

Proof. From eq. (22) and considering that $\psi(z) \to +\infty$, for $z \to +\infty$,

$$\mathbb{E}[H] = \frac{S}{\mu} + \Delta + o\left(\frac{1}{\mu}\right) = \mathbb{E}[T] + \Delta + o\left(\frac{1}{\mu}\right),$$

that gives the result ***i.***. From eq. (22) and the Taylor series expansion of the ψ function around $z = 0$

$$\psi(z) = 1 + \sqrt{\frac{\pi}{2}}z\left(1 + \frac{z^2}{2} + o(z^2)\right)\left[1 + \frac{2}{\sqrt{\pi}}\left(\frac{z}{\sqrt{2}} + o(z)\right)\right], \tag{26}$$

[2] In the following, we will use $a \approx b$ for $x \to \infty$ if

$$\lim_{x\to\infty}\frac{a(x)}{b(x)} = c \in \mathbb{R}$$

and we will use $a \approx b$ for $x \to 0$ if

$$\lim_{x\to 0}\frac{a(x)}{b(x)} = c \in \mathbb{R}$$

it follows that for $\mu \to 0$

$$\mathbb{E}[H] \approx \mathbb{E}[T] + \Delta + \frac{\sigma}{\mu}\sqrt{\frac{\pi\Delta}{2}}.$$

This entails the proposition. □

The difference $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta)$ grows also with σ. Indeed, the expectation of the r.v. T does not depend on σ, see eq. (6), while $\mathbb{E}[H]$ grows linearly with respect to σ, as shown in the following proposition.

Proposition 4. *$\mathbb{E}[H] \approx \sigma$, as $\sigma \to +\infty$.*

Proof. Considering again eq. (26) we get, for $\sigma \to +\infty$

$$\mathbb{E}[H] \approx \mathbb{E}[T] + \frac{\sigma}{\mu}\sqrt{\frac{\pi\Delta}{2}}.$$

□

As additional remark to Proposition 4, let us notice that as the trajectories of the process become more irregular, the request of lying above the level S for a sufficiently long time becomes more difficult. On the contrary, the instantaneous hitting of the level S is not affected by the increased variability.

The effect of Δ over H and $T + \Delta$ is quite trivial. The two random variables have a similar behavior for small values of Δ and they assume the same value for $\Delta = 0$.

Proposition 5. *It holds*

i. $\mathbb{E}[H] - \mathbb{E}[T] \approx \sqrt{\Delta}$, *as* $\Delta \to 0$,

ii. $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta) \to \frac{\sigma^2}{\mu^2}$, *as* $\Delta \to +\infty$.

Proof. Considering again eq. (26) we get, as $\Delta \to 0$

$$\mathbb{E}[H] \approx \mathbb{E}[T] + \frac{\sigma}{\mu}\sqrt{\frac{\pi\Delta}{2}}. \tag{27}$$

The result ***ii.*** follows easily from (22) and the fact that $\psi(z) \to +\infty$, as $z \to +\infty$. □

5. **The Leaky Integrator model.** In the OU case the comparison between the two firing models is based on Monte Carlo simulations. Before entering the details of the results, let us remark that the problem is well posed as the r.v. H is finite a.s. for any choice of the parameters of the OU process, as shown in the following proposition.

Proposition 6. *Let G^+_Δ be the left endpoint of the first positive excursion exceeding Δ in length*

$$G^+_\Delta = \inf\{g_t | (t - g_t) \cdot \mathbf{1}_{V_t \geq S} \geq \Delta\}. \tag{28}$$

Then

$$\mathbb{P}(G^+_\Delta < \infty) = 1. \tag{29}$$

Proof. For each $\Delta > 0$ we indicate with $L_{G^+_\Delta}$ the (local) time at which the process V_t has a first positive excursion of length greater than Δ, and with $h^+(\Delta) = n^+((\Delta, +\infty))$ the Ito measure of positive excursions of length greater than Δ [29].

From Ito excursion theory, it follows that $L_{G^+_\Delta}$ is an exponential random variable of parameter $h^+(\Delta) \geq 0$, so that $\mathbb{P}(G^+_\Delta = \infty) = \lim_{t\to\infty} \mathrm{e}^{-th^+(\Delta)}$ [23]. Hence $\mathbb{P}(G^+_\Delta = \infty) = 0$ if $h^+(\Delta) > 0$ or $\mathbb{P}(G^+_\Delta = \infty) = 1$ if $h^+(\Delta) = 0$.

Moreover, $\forall\lambda > 0$, $h^+(\Delta)$ satisfies the following equation [41]:

$$\varphi^+(\lambda) = \lambda \int_0^\infty \mathrm{e}^{-\lambda\Delta} h^+(\Delta) d\Delta, \tag{30}$$

with

$$\varphi^+(\lambda) = -\frac{1}{2}\frac{d}{dx}\mathbb{E}_x[\mathrm{e}^{-\lambda T}]\bigg|_{x=S^+}. \tag{31}$$

For an OU process of equation

$$dV_t = [-\beta V_t + \mu]\, dt + \sigma dW_t, \quad V_0 = 0,$$

φ^+ takes the form

$$\varphi^+(\lambda) = \left(\frac{\lambda}{\sqrt{\beta}}\right) \frac{H_{(-\lambda/\beta-1)}\left(\frac{S\sqrt{\beta}}{\sigma} - \frac{\mu}{\sigma\sqrt{\beta}}\right)}{H_{(-\lambda/\beta)}\left(\frac{S\sqrt{\beta}}{\sigma} - \frac{\mu}{\sigma\sqrt{\beta}}\right)} \tag{32}$$

with $H_v(z)$ the Hermite function [38]. Because $H_v(z)$ has no zeroes for $v \leq 0$ [20], from eq. (30) and (32) it follows that $h^+(\Delta) > 0$ and therefore $\mathbb{P}(G^+_\Delta < \infty) = 1$. □

The numerical simulations are performed with parameters in a physiological range, according to [37]. We set $S = 10$ mV, $V_0 = 0$ mV, $\tau = 12.5$ ms, $\mu \in [0.7, 2]$ mVms^{-1} and $\Delta \in [0, 4]$ ms. However we choose $\sigma \in [0.5, 3]$ mV^{2}ms^{-1}, a range wider than the interval estimated in [37]. The rationale of this choice lies in the observation that the preprocessing through a moving average filter performed by the authors may decrease the original variability of the traces and hence give underestimated values for the diffusion coefficient.

Simulation batches are performed with two samples of $N = 1000$ trajectories of the OU process stopped at the firing times T and H respectively. The simulation algorithm generates exact trajectories of the OU process at discrete times by means of the transition density. The first passage time T is deduced from the trajectories evaluating possible hidden crossings in between the nodes of the time discretisation, as suggested in [26, 4]. Analogously to the previous Section, the comparison concerns H and $T + \Delta$, where Δ is interpreted as the absolute refractory period τ_r.

A first glance to the Figs. 4, 5 and 7 confirms some of the features proved for the Perfect Integrator model, eventually with different rates for the asymptotics. Specifically, in the explored range of the parameters and for $\Delta > 0$

- $\mathbb{E}[H]$ is always larger than $\mathbb{E}[T] + \Delta$;
- $\mathrm{Var}[H]$ is always larger than $\mathrm{Var}[T]$;
- for large values of μ the difference between $\mathbb{E}[H]$ and $\mathbb{E}[T] + \Delta$ decreases to zero and for small values seems to diverge.

Furthermore, in the suprathreshold regime, the role of σ and Δ is similar to that proved for the Perfect Integrator. In fact

- as σ increases the $\mathbb{E}[H]$ increases as well;
- as Δ increases the difference between $\mathbb{E}[H]$ and $\mathbb{E}[T] + \Delta$ seems to have an horizontal asymptote.

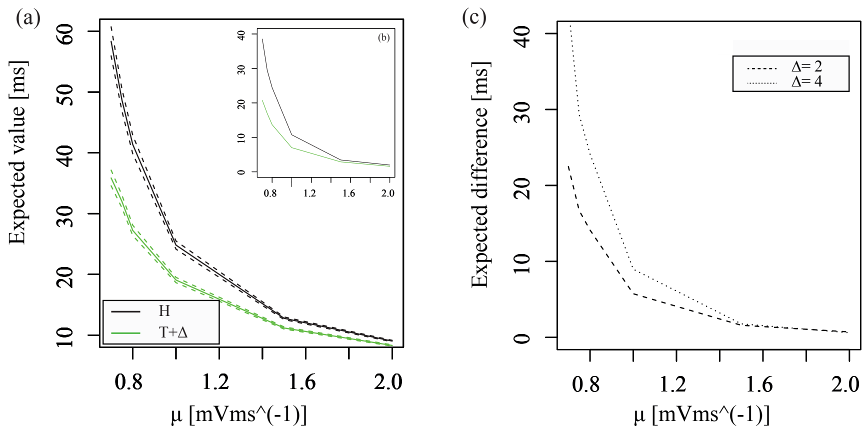

FIGURE 4. (a) Expected value, (b) variance of H and $T+\Delta$ respect to μ: $S = 10$ mV, $V_0 = 0$ mV, $\tau = 12.5$ ms, $\Delta = 2$ ms, $\sigma = 1$ mV^{2}ms^{-1}, $\mu \in [0.7, 2]$ mVms^{-1}. (c) $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta)$ for two different values of Δ: $\Delta = 2, 4$ ms.

The dependency on σ strongly changes in the subthreshold regime, cf. Fig. 6. Indeed, as σ increases, both $\mathbb{E}[H]$ and $\mathbb{E}[T] + \Delta$ decrease as well as their difference. This feature is the natural consequence of the determinant role of noise in such regime. This effect is reversed in the suprathreshold regime, cf Fig. 5, where spiking is determined by the drifting component of the process and noise disturbs the permanence of the membrane potential above the threshold. As Δ increases the qualitative behaviour of $\mathbb{E}[H]$ and $\mathbb{E}[T]$ is comparable in both regimes, cf Figs. 7 and 8. However, in the subthreshold instance (Fig. 7), the rate of divergence of the difference is dramatically larger.

6. **Conclusions.** We proposed here a new firing time paradigm for IF stochastic models. The new definition is motivated by statistical purposes. Its usefulness in the estimation of the model parameters (including the threshold level) lies in the full accordance of data with the assumptions of the model.

We derived closed form expression for the Laplace transform of the firing time and the first two moments in the case of the Perfect Integrator. Moreover, we explored the finiteness of the firing time for both models.

The results discussed in this paper are suitable to estimate the parameters of a Perfect Integrator model with a method of moments. About the Leaky Integrator, in the suprathreshold regime the new firing time H is not significantly different from the classical $T + \Delta$. Hence relying on known results about $T + \Delta$ to estimate the parameters of the voltage process should not generate dramatic errors. Though the use of the new paradigm is recommendable also in this regime to estimate S, in order to consider a model fully consistent with data. In the subthreshold regime, the new firing time H shows remarkable differences with respect to $T + \Delta$. Hence, all the mathematical results for the estimation of parameters should be derived in the new paradigm. The deduction of the Laplace transform of H for the Leaky Integrator model, is an ongoing effort and will be the topic of a forthcoming work.

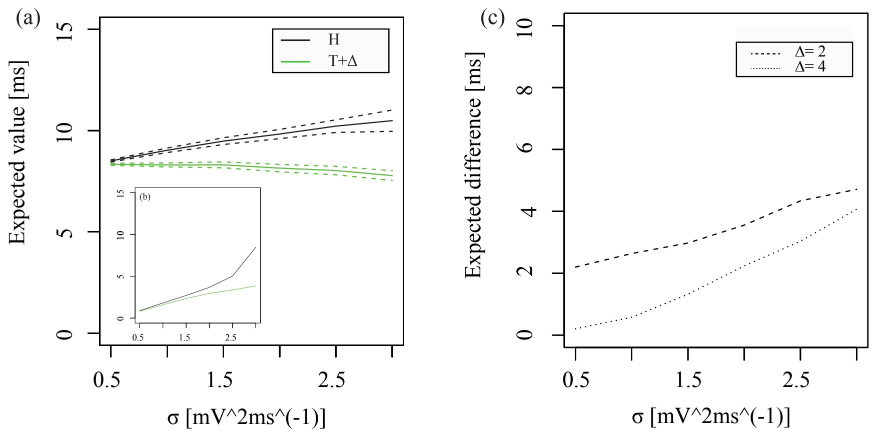

FIGURE 5. (a) Expected value, (b) variance of H and $T+\Delta$ respect to σ in the suprathreshold regime: $S = 10$ mV, $V_0 = 0$ mV, $\tau = 12.5$ ms, $\mu = 2$ mVms^{-1}, $\Delta = 2$ ms, $\sigma \in [0.5, 3]$ mV ^{2}ms^{-1}. (c) $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta)$ for two different values of $\Delta = 2, 4$ ms.

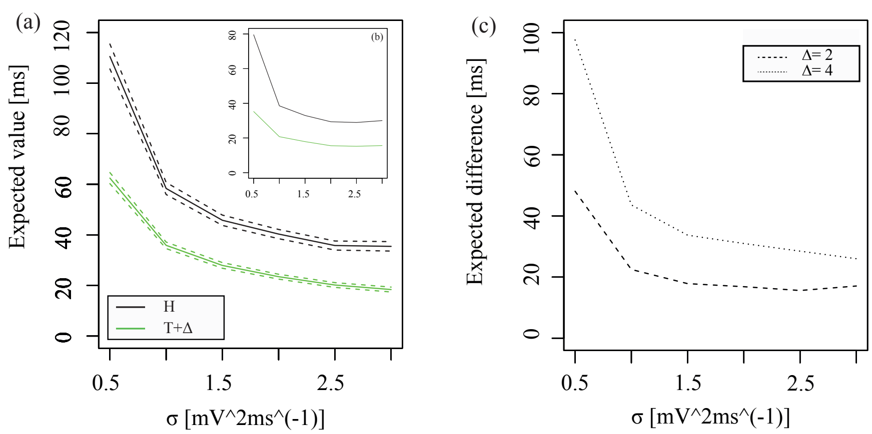

FIGURE 6. (a) Expected value, (b) variance of H and $T+\Delta$ respect to σ in the subthreshold regime: $S = 10$ mV, $V_0 = 0$ mV, $\tau = 12.5$ ms, $\mu = 0.7$ mVms^{-1}, $\Delta = 2$ ms, $\sigma \in [0.5, 3]$ mV ^{2}ms^{-1}. (c) $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta)$ for two different values of $\Delta =, 2, 4$ ms.

Once the theoretical framework will be deduced for both cases, proper estimators will be derived and the two firing mechanism will be compared on true recorded data.

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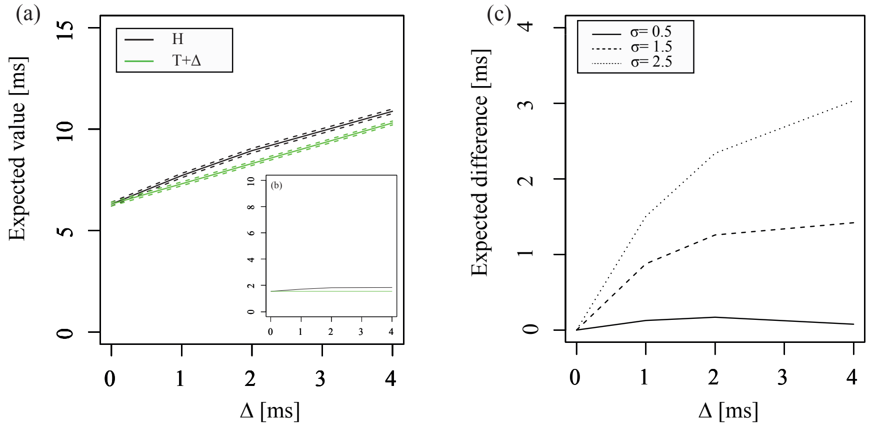

FIGURE 7. (a) Expected value, (b) variance of H and $T+\Delta$ respect to Δ in the suprathreshold regime: $S = 10$ mV, $V_0 = 0$ mV, $\tau = 12.5$ ms, $\mu = 2$ mVms^{-1}, $\sigma = 1$ mV^{2}ms^{-1}, $\Delta \in [0, 4]$ ms. (c) $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta)$ for three different values of $\sigma = 0.5, 1.5, 2.5$ mV^{2}ms.

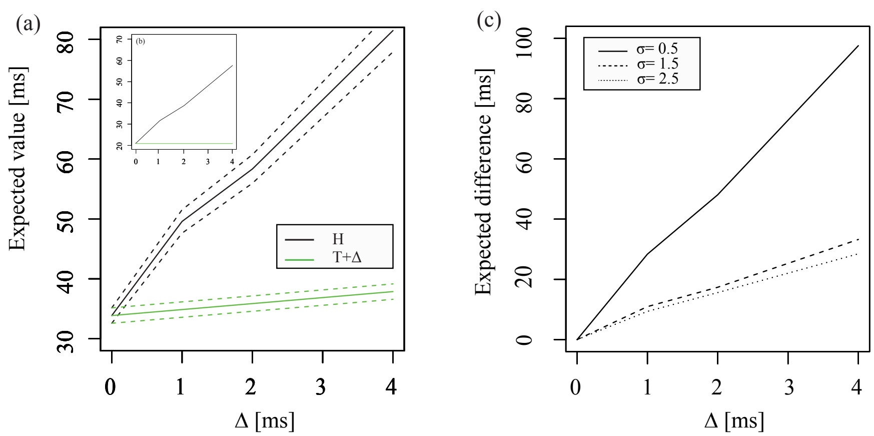

FIGURE 8. (a) Expected value, (b) variance of H and $T+\Delta$ respect to Δ in the subthreshold regime: $S = 10$ mV, $V_0 = 0$ mV, $\tau = 12.5$ ms, $\mu = 0.7$ mVms^{-1}, $\sigma = 1$ mV^{2}ms^{-1}, $\Delta \in [0, 4]$ ms. (c) $\mathbb{E}[H] - (\mathbb{E}[T] + \Delta)$ for three different values of $\sigma = 0.5, 1.5, 2.5$ mV^{2}ms.

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E-mail address: roberta.sirovich@unito.it
E-mail address: luisa.testa@unito.it