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# GLOBAL STABILITY FOR AN *SEI* MODEL OF INFECTIOUS DISEASE WITH AGE STRUCTURE AND IMMIGRATION OF INFECTEDS

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Abstract. We study a model of disease transmission with continuous age-structure for latently infected individuals and for infectious individuals and with immigration of new individuals into the susceptible, latent and infectious classes. The model is very appropriate for tuberculosis. A Lyapunov functional is used to show that the unique endemic equilibrium is globally stable for all parameter values.

1. **Introduction.** Many diseases (including tuberculosis and chicken pox) are known to have an exposed or latent phase, consisting of individuals that are infected, but not yet infectious. In this paper, we focus on diseases of this type, and study an SEI model.

In the modern world, there is tremendous movement of individuals from one geographic region to another. Given that diseases such as tuberculosis can remain latent for long periods of time, it is inevitable that some latently infected individuals will travel, and then become infectious in the new location. Furthermore, infectious individuals may also travel. Thus, we are interested in an SEI model that includes immigration into each class. Earlier models with immigration of infected individuals include [1, 4, 5, 6, 13, 15, 16, 19].

For diseases that exhibit a latent stage and an infectious stage, it is likely that certain disease parameters such as the level of infectiousness or the likelihood of progression out of the given stage will depend on how long the individual has been in the stage. Thus, we include age-in-class structure for the exposed and infectious classes. Infectious disease models that include age-in-class structure include [2, 3, 7, 9, 10, 12, 14, 18].

The model studied in this paper most closely resembles the model in [12], where an SEI model with age-in-class structure for the exposed and infectious classes was studied. In that work, however, there was no immigration into the exposed and infectious classes. In this paper, we follow [12] as something of a blueprint for our analysis. The introduction of immigration, though, changes many key calculations.

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In this work, we demonstrate the global stability of an interior equilibrium for an age-structured partial differential equation by using a Lyapunov functional. There has been a very successful resurgence in the use of Lyapunov methods for disease models, beginning with ordinary differential equation models in [8]. The approach was extended to delay differential equations in [11] and to age-structured partial differential equations in [10].

There is a complication in using Lyapunov functionals for age-structured partial differential equations that does not arise with ordinary differential equations (or with delay differential equations if the delay is bounded). The issue arises from the fact that the Lyapunov functional often involves integrating a function that is undefined at zero, over all ages, up to infinity. This means that the Lyapunov functional is undefined on a large part of the state space. Nevertheless, one wishes to resolve the global dynamics for all initial conditions. One approach that has worked is to establish uniform persistence of the semi-flow and the existence of an attractor consisting of total trajectories, and then to show that the Lyapunov functional is bounded and decreasing on these total trajectories, eventually determining the attractor fully. This approach was used in [10] and [12], and is used here.

This paper is structured as follows. The model is described in Section 2, preliminary calculations are presented in Section 3 and boundedness is discussed in Section 4. In Section 5, the flow is shown to be asymptotically smooth, allowing us to discuss the attractor in Section 6. The existence and uniqueness of an equilibrium is studied in Section 7. In Section 8, we show that the attractor consists of only the equilibrium. Some discussion of the results appears in Section 9.

2. **The model.** We consider an infectious disease for which the population consists of susceptibles, exposed individuals and infectives. The number of susceptibles at time $t$ is given by $S(t)$. For exposed and infectious individuals, we keep track of how long they have been in that class and so these sub-populations are described, respectively, by the density functions $e$ and $i$. Thus, $e(t,a)$ gives the density at time $t$ of exposed individuals who have been exposed for duration $a$. Similarly, $i(t,a)$ gives the density at time $t$ of infectious individuals who have been infectious for duration $a$.

Recruitment through birth and immigration into the susceptible class is at rate $W_S$. Recruitment into the exposed and infectious classes with age-in-class $a$ occurs at rates $W_e(a)$ and $W_e(a)$, respectively.

The per capita death rate for susceptibles is $\mu_S$. The age-in-class specific per capita death rates for exposed and infectious individuals are $\mu_e(\cdot)$ and $\mu_i(\cdot)$, respectively.

We model the incidence of new infections by mass action, allowing that the infectiousness may depend on the age-in-class. Thus, new infections occur at rate $\int_0^\infty \beta(a)S(t)i(t,a)da$. Upon infection, these individuals enter the exposed class, with age-in-class of zero.

Progression from the exposed class to the infectious class occurs at the age-in-class specific per capita rate $\nu(\cdot)$, so that individuals arrive in the infectious class with age-in-class of zero at rate $\int_0^\infty \nu(a)e(t,a)da$.

The system of differential equations for the model is

$$\frac{dS(t)}{dt} = W_S - \mu_S S(t) - \int_0^\infty \beta(a)S(t)i(t,a)da \tag{1}$$

$$\frac{\partial e}{\partial t}+\frac{\partial e}{\partial a}=W_e(a)-(\nu(a)+\mu_e(a))\,e(t,a)$$
$$\frac{\partial i}{\partial t}+\frac{\partial i}{\partial a}=W_i(a)-\mu_i(a)i(t,a),$$

with boundary conditions

$$\begin{aligned} e(t,0)&=\int_0^\infty \beta(a)S(t)i(t,a)da\\ i(t,0)&=\int_0^\infty \nu(a)e(t,a)da, \end{aligned} \tag{2}$$

for $t>0$. We make the following assumptions on the parameters.

**(H1)** $W_S,\mu_S>0$.

**(H2)** $\mu_e,\mu_i,\beta,\nu\in L^\infty(\mathbb{R}_{\geq 0},\mathbb{R}_{\geq 0})$.

Let $\mu_e^{\inf}$, $\mu_i^{\inf}$, $\beta^{\inf}$, $\nu^{\inf}$ be the essential infimums of $\mu_e$, $\mu_i$, $\beta$, $\nu$ respectively, and let $\mu_e^{\sup}$, $\mu_i^{\sup}$, $\beta^{\sup}$, $\nu^{\sup}$ be the respective essential supremums.

**(H3)** $\mu_e^{\inf},\mu_i^{\inf}>0$.

**(H4)** $\beta$ and $\nu$ are Lipschitz continuous, with Lipschitz coefficients $M_\beta$ and $M_\nu$, respectively.

**(H5)** $W_e,W_i\in L^1(\mathbb{R}_{\geq 0},\mathbb{R}_{\geq 0})$.

**(H6)** The supports of $\beta$, $\nu$ and $W_e+W_i$ each have positive measure.

Let $\overline{W}_e=\int_0^\infty W_e(a)da$ and $\overline{W}_i=\int_0^\infty W_i(a)da$. Then **(H5)** and **(H6)** imply $\overline{W}_e,\overline{W}_i\in[0,\infty)$, with $\overline{W}_e+\overline{W}_i>0$.

The next hypothesis ensures that immigrants into the exposed and infectious classes contribute to the disease dynamics. If it is not satisfied, then the model is essentially the same as the model in [12].

**(H7)** At least one of

essential infimum (support $(W_e)$) $<$ essential supremum (support $(\nu)$)

and

essential infimum (support $(W_i)$) $<$ essential supremum (support $(\beta)$)

is satisfied.

The initial conditions are

$$(S(0),e(0,\cdot),i(0,\cdot))=(S_0,\varphi_e(\cdot),\varphi_i(\cdot))\,,$$

and satisfy the following hypothesis.

**(H8)** $S_0\in\mathbb{R}_{\geq 0}$ and $\varphi_e,\varphi_i\in L^1(\mathbb{R}_{\geq 0},\mathbb{R}_{\geq 0})$.

The state space is $\mathcal{Y}=\mathbb{R}_{\geq 0}\times\mathcal{C}\times\mathcal{C}$, where $\mathcal{C}=L^1(\mathbb{R}_{\geq 0},\mathbb{R}_{\geq 0})$. Standard theory [20] implies that solutions to the initial value problem exist for all time and are unique. Furthermore, $\mathcal{Y}$ is positively invariant and the system exhibits a continuous semi-flow $\Phi:\mathbb{R}_{\geq 0}\times\mathcal{Y}\to\mathcal{Y}$.

Given a point $(x, \varphi, \phi) \in \mathcal{Y}$, we have the norm $\|(x, \varphi, \phi)\|_{\mathcal{Y}} = x + \int_0^\infty \varphi(a)da + \int_0^\infty \phi(a)da$.

Suppose $X(t)$ is a solution for $t \geq 0$, with initial condition $X(0) = X_0 \in \mathcal{Y}$. Then we use the following notation

$$X(t) = \Phi_t(X_0) = \Phi(t, X_0) = (S(t), e(t, \cdot), i(t, \cdot))\,.$$

The total population $N(t)$ at time $t$ is given by

$$N(t) = \|X(t)\|_{\mathcal{Y}} = S(t) + \int_0^\infty e(t, a)da + \int_0^\infty i(t, a)da.$$

3. **Preliminaries.** In the calculations that follow, it is convenient to use the functions

$$\Omega(a) = \mathrm{e}^{-\int_0^a (\nu(s) + \mu_e(s))ds} \tag{3}$$

and

$$\Gamma(a) = \mathrm{e}^{-\int_0^a \mu_i(s)ds}. \tag{4}$$

It follows from **(H2)** and **(H3)** that

$$0 < \mathrm{e}^{-(\nu^{\sup} + \mu_e^{\sup})a} \leq \Omega(a) \leq \mathrm{e}^{-\mu_e^{\inf} a} \tag{5}$$

and

$$0 < \mathrm{e}^{-\mu_i^{\sup} a} \leq \Gamma(a) \leq \mathrm{e}^{-\mu_i^{\inf} a} \tag{6}$$

for all $a \geq 0$. It is clear that $\Omega$ and $\Gamma$ are decreasing. We now define the constants

$$A = \int_0^\infty \nu(a)\Omega(a)da \quad \text{and} \quad B = \int_0^\infty \beta(a)\Gamma(a)da, \tag{7}$$

which will be used in Section 7. It follows from **(H6)** and Equations (5) and (6) that $A, B > 0$.

Let

$$J(t) = \int_0^\infty \beta(a)i(t, a)da \quad \text{and} \quad L(t) = i(t, 0) = \int_0^\infty \nu(a)e(t, a)da. \tag{8}$$

Similar to the detailed exposition given in [20], solutions satisfy

$$e(t, a) = \begin{cases} e(t - a, 0)\Omega(a) + \int_0^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma & \text{for } 0 \leq a < t \\ \\ \varphi_e(a - t)\frac{\Omega(a)}{\Omega(a-t)} + \int_{a-t}^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma & \text{for } 0 \leq t \leq a, \end{cases} \tag{9}$$

and

$$i(t, a) = \begin{cases} i(t - a, 0)\Gamma(a) + \int_0^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma & \text{for } 0 \leq a < t \\ \\ \varphi_i(a - t)\frac{\Gamma(a)}{\Gamma(a-t)} + \int_{a-t}^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma & \text{for } 0 \leq t \leq a. \end{cases} \tag{10}$$

Furthermore, total trajectories (should they exist) satisfy the first line of (9) and of (10) for all $(t, a) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$.

4. **Boundedness.** Let $W^* = W_S + \overline{W}_e + \overline{W}_i$ and $\mu^* = \min\{\mu_S, \mu_e^{\inf}, \mu_i^{\inf}\}$. Note that **(H1)** and **(H3)** imply $\mu^* > 0$. Let

$$N^* = \frac{W^*}{\mu^*}.$$

Then **(H1)**, **(H5)** and **(H6)** imply $0 < N^* < \infty$.

The flow $\Phi$ is called *point dissipative* if there is a bounded set that attracts all points in $\mathcal{Y}$. Similar to [12, Proposition 1], we have the following result.

**Proposition 1.** *Let $X_0 \in \mathcal{Y}$. Then*

1. $\frac{dN}{dt} \le W^* - \mu^* N(t)$ *for all* $t \ge 0$.
2. $N(t) \le \max\{N^*, \|X(0)\|_{\mathcal{Y}}\}$ *for all* $t \ge 0$.
3. $\limsup_{t\to\infty} N(t) \le N^*$.
4. $\Phi$ *is point dissipative.*

*Proof.* We begin by finding an expression for $\frac{d}{dt}\int_0^\infty e(t,a)da$. To do this, we note that

$$\begin{aligned}\int_0^\infty e(t,a)da &= \int_0^t e(t,a)da + \int_t^\infty e(t,a)da \\ &= \int_0^t S(t-a)J(t-a)\Omega(a)da + \int_0^t \int_0^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma da \\ &\qquad + \int_t^\infty \varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)}da + \int_t^\infty \int_{a-t}^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma da.\end{aligned}$$

Changing the order of integration in the two double integrals, and then combining them into one double integral gives

$$\begin{aligned}\int_0^\infty e(t,a)da &= \int_0^t S(t-a)J(t-a)\Omega(a)da \\ &\qquad + \int_t^\infty \varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)}da + \int_0^\infty \int_\sigma^{\sigma+t} W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}dad\sigma.\end{aligned}$$

Next, we make the substitutions $\tau = t-a$ and $\tau = a-t$ in the first and second integrals on the right-hand side, respectively, to obtain

$$\begin{aligned}\int_0^\infty e(t,a)da &= \int_0^t S(\tau)J(\tau)\Omega(t-\tau)d\tau \\ &\qquad + \int_0^\infty \varphi_e(\tau)\frac{\Omega(\tau+t)}{\Omega(\tau)}d\tau + \int_0^\infty \int_\sigma^{\sigma+t} W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}dad\sigma.\end{aligned}$$

Differentiating with respect to $t$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^\infty e(t,a)da &= \frac{d}{dt}\int_0^t S(\tau)J(\tau)\Omega(t-\tau)d\tau \\ &\qquad + \frac{d}{dt}\int_0^\infty \varphi_e(\tau)\frac{\Omega(\tau+t)}{\Omega(\tau)}d\tau + \frac{d}{dt}\int_0^\infty \int_\sigma^{\sigma+t} W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}dad\sigma \\ &= S(t)J(t)\Omega(0) + \int_0^t S(\tau)J(\tau)\Omega'(t-\tau)d\tau \\ &\qquad + \int_0^\infty \varphi_e(\tau)\frac{\Omega'(\tau+t)}{\Omega(\tau)}d\tau + \int_0^\infty W_e(\sigma)\frac{\Omega(\sigma+t)}{\Omega(\sigma)}d\sigma.\end{aligned}$$

Noting that $\Omega(0)=1$, and reversing the $\tau$ substitutions, we have

$$\begin{aligned}\frac{d}{dt}\int_0^\infty e(t,a)da &= S(t)J(t) + \int_0^t S(t-a)J(t-a)\Omega'(a)da \\ &\quad + \int_0^\infty \varphi_e(a-t)\frac{\Omega'(a)}{\Omega(a-t)}da + \int_0^\infty W_e(\sigma)\frac{\Omega(\sigma+t)}{\Omega(\sigma)}d\sigma.\end{aligned} \tag{11}$$

Next, by changing the order of integration, and then combining integrals, we observe that

$$\begin{aligned}\int_0^t\int_0^a W_e(\sigma)\frac{\Omega'(a)}{\Omega(\sigma)}d\sigma da + \int_t^\infty\int_{a-t}^a W_e(\sigma)\frac{\Omega'(a)}{\Omega(\sigma)}d\sigma da \\ = \int_0^\infty\int_\sigma^{\sigma+t} W_e(\sigma)\frac{\Omega'(a)}{\Omega(\sigma)}da d\sigma \\ = \int_0^\infty W_e(\sigma)\frac{\Omega(\sigma+t)-\Omega(\sigma)}{\Omega(\sigma)}d\sigma \\ = \int_0^\infty W_e(\sigma)\frac{\Omega(\sigma+t)}{\Omega(\sigma)}d\sigma - \int_0^\infty W_e(\sigma)d\sigma.\end{aligned} \tag{12}$$

Using (12) to replace the final term in (11), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty e(t,a)da \\ &= S(t)J(t) + \int_0^t S(t-a)J(t-a)\Omega'(a)da + \int_0^\infty \varphi_e(a-t)\frac{\Omega'(a)}{\Omega(a-t)}da \\ &\quad + \int_0^t\int_0^a W_e(\sigma)\frac{\Omega'(a)}{\Omega(\sigma)}d\sigma da + \int_t^\infty\int_{a-t}^a W_e(\sigma)\frac{\Omega'(a)}{\Omega(\sigma)}d\sigma da + \int_0^\infty W_e(\sigma)d\sigma.\end{aligned} \tag{13}$$

Noting that $\Omega'(a) = -(\nu(a)+\mu_e(a))\Omega(a)$, it follows that the second to fifth terms in (13) combine to give $-\int_0^\infty(\nu(a)+\mu_e(a))e(t,a)da$. Recalling that $S(t)J(t) = e(t,0)$, (13) becomes

$$\begin{aligned}\frac{d}{dt}\int_0^\infty e(t,a)da &= e(t,0) - \int_0^\infty(\nu(a)+\mu_e(a))e(t,a)da + \overline{W}_e \\ &= e(t,0) - i(t,0) - \int_0^\infty \mu_e(a)e(t,a)da + \overline{W}_e.\end{aligned} \tag{14}$$

A similar calculation shows that

$$\frac{d}{dt}\int_0^\infty i(t,a)da = i(t,0) - \int_0^\infty \mu_i(a)i(t,a)da + \overline{W}_i. \tag{15}$$

Combining the first lines of (1) and (2) gives

$$\frac{dS}{dt} = W_S - \mu_S S(t) - e(t,0). \tag{16}$$

Combining (14), (15) and (16), it now follows that

$$\begin{aligned}\frac{dN}{dt} &= W_S + \overline{W}_e + \overline{W}_i - \left(\mu_S S(t) + \int_0^\infty \mu_e(a)e(t,a)da + \int_0^\infty \mu_i(a)i(t,a)da\right) \\ &\le W^* - \mu^* N(t),\end{aligned}$$

completing the proof of statement (1).

Solving this differential inequality for $t \geq 0$, we find that

$$\begin{aligned} N(t) &\leq N^* + (N(0) - N^*)\,\mathrm{e}^{-\mu^* t} \\ &\leq \max\{N^*, N(0)\} \\ &\leq \max\{N^*, \|X(0)\|_{\mathcal{Y}}\}, \end{aligned} \tag{17}$$

which proves statement (2). Finally, the first line of (17) implies

$$\limsup_{t\to\infty} N(t) \leq \limsup_{t\to\infty} N^* + (\|X(0)\|_{\mathcal{Y}} - N^*)\,\mathrm{e}^{-\mu^* t} = N^*,$$

proving statement (3), from which statement (4) follows. ☐

The following two propositions are similar to Propositions 2 and 3 in [12]. They follow directly from Proposition 1

**Proposition 2.** *If $X_0 \in \mathcal{Y}$ satisfies $\|X_0\|_{\mathcal{Y}} \leq K$ for some $K \geq N^*$, then the following hold for all $t \geq 0$.*

1. *$S(t), \int_0^\infty e(t,a)da, \int_0^\infty i(t,a)da \leq K$,*
2. *$J(t) \leq K\beta^{sup}$ and $L(t) \leq K\nu^{sup}$,*
3. *$e(t,0) \leq K^2\beta^{sup}$ and $i(t,0) \leq K\nu^{sup}$.*

**Proposition 3.** *Let $C \subset \mathcal{Y}$ be bounded. Then*

1. *$\Phi(\mathbb{R}_{\geq 0}, C)$ is bounded.*
2. *$\Phi$ is eventually bounded on $C$.*
3. *If $K \geq N^*$ is a bound for $C$, then $K$ is also a bound for $\Phi(\mathbb{R}_{\geq 0}, C)$.*
4. *Given any $M > N^*$, there exists $T = T(C, M)$ such that $M$ is a bound for $\Phi(t, C)$ whenever $t \geq T$.*

The next proposition gives a positive asymptotic lower bound for $S$.

**Proposition 4.** *If $X_0 \in \mathcal{Y}$, then*

$$\liminf_{t\to\infty} S(t) \geq \frac{W_S}{\mu_S + \beta^{sup} N^*}.$$

*Proof.* First, we note that (1) implies

$$\begin{aligned} \frac{dS(t)}{dt} &= W_S - \mu_S S(t) - \int_0^\infty \beta(a) S(t) i(t,a) da \\ &\geq W_S - \left(\mu_S + \beta^{\sup} \int_0^\infty i(t,a)da\right) S(t). \end{aligned}$$

Let $\epsilon > 0$. By statement (3) of Proposition 1, there exists $t_1 \geq 0$ such that $\int_0^\infty i(t,a)da \leq N^* + \epsilon$ for all $t \geq t_1$. Thus, for sufficiently large $t$, we have

$$\frac{dS(t)}{dt} \geq W_S - (\mu_S + \beta^{\sup}(N^* + \epsilon))\, S(t),$$

from which it follows that

$$\liminf_{t\to\infty} S(t) \geq \frac{W_S}{\mu_S + \beta^{\sup}(N^* + \epsilon)}.$$

Letting $\epsilon$ decrease to 0, we obtain the result. ☐

**Proposition 5.** *There exists $T, \epsilon > 0$ such that $e(t,0), i(t,0) > \epsilon$ for all $t \geq T$.*

*Proof.* First, we note that Equations (8), (9) and (10) imply

$$J(t) = \int_0^\infty \beta(a)i(t,a)da \geq \int_0^t \int_0^a \beta(a)W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma da \tag{18}$$

and

$$L(t) = \int_0^\infty \nu(a)e(t,a)da \geq \int_0^t \int_0^a \nu(a)W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma da.$$

Recalling **(H7)**, we consider two cases.

**Case 1.** essential infimum (support $(W_i)$) < essential supremum (support $(\beta)$).

Then there exists $T_1, \delta > 0$ such that $\int_0^{T_1}\int_0^a \beta(a)W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma da \geq \delta$. Then (18) implies that $J(t) \geq \delta$ for all $t > T_1$. By Proposition 4, there exists $T_2 \geq T_1$ such that $S(t) \geq \frac{1}{2}\frac{W_S}{\mu_S + \beta^{\sup}N^*}$ for all $t \geq T_2$. Since $e(t,0) = S(t)J(t)$, we now have

$$e(t,0) \geq \frac{1}{2}\frac{W_S}{\mu_S + \beta^{\sup}N^*}\delta \tag{19}$$

for $t \geq T_2$. Let $T_3 >$ essential infimum (support $(\nu)$). For $t \geq T_2 + T_3$, we have

$$\begin{aligned} i(t,0) &= \int_0^\infty \nu(a)e(t,a)da \\ &\geq \int_0^{T_3} \nu(a)e(t,a)da \\ &\geq \int_0^{T_3} \nu(a)e(t-a,0)\Omega(a)da. \end{aligned}$$

Since $t \geq T_2 + T_3$, it follows that whenever $a \in [0, T_3]$, we have $t - a \geq T_2$ and so (19) applies. Combining this with the fact that $\Omega$ is decreasing, we obtain

$$i(t,0) \geq \frac{1}{2}\frac{W_S}{\mu_S + \beta^{\sup}N^*}\delta\Omega(T_3)\int_0^{T_3}\nu(a)da,$$

which is positive due to the choice of $T_3$. Thus, we have a positive lower bound for each of $e(t,0)$ and $i(t,0)$ for $t \geq T_2 + T_3$. This proves the result for Case 1.

**Case 2.** essential infimum (support $(W_e)$) < essential supremum (support $(\nu)$).

This is proved similarly to Case 1, by establishing a lower bound for $i(t,0)$ first, and then for $e(t,0)$. □

## 5. Asymptotic smoothness.

The following result is similar to [12, Proposition 5] and is proved in a similar fashion.

**Proposition 6.** *The functions $J$ and $L$ are Lipschitz continuous on $\mathbb{R}_{\geq 0}$.*

*Proof.* Let $K \geq \max\{N^*, \|X_0\|_{\mathcal{Y}}\}$. Then, statement (2) of Proposition 1 implies that $\|X(t)\|_{\mathcal{Y}} \leq K$ for all $t \geq 0$.

Fix $t \geq 0$ and $h > 0$. Then

$$\begin{aligned} J(t+h) - J(t) &= \int_0^\infty \beta(a)i(t+h,a)da - \int_0^\infty \beta(a)i(t,a)da \\ &= \int_0^h \beta(a)i(t+h,a)da + \int_h^\infty \beta(a)i(t+h,a)da - \int_0^\infty \beta(a)i(t,a)da \end{aligned}$$

$$= \int_0^h \beta(a)i(t+h-a,0)\Gamma(a)da + \int_0^h \beta(a)\int_0^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma da + \int_h^\infty \beta(a)i(t+h,a)da - \int_0^\infty \beta(a)i(t,a)da.$$

Using Proposition 2 and Equation (4), for the first integral above, we use bounds $\beta(a) \le \beta^{\sup}$, $i(t+h-a,0) \le K\nu^{\sup}$ and $\Gamma(a) \le 1$. For the second integral, we use $\beta(a) \le \beta^{\sup}$ and $\int_0^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma \le \int_0^a W_i(\sigma)d\sigma \le \overline{W}_i$. For the third integral, let $\sigma = a-h$, to get

$$J(t+h) - J(t) \le \beta^{\sup}K\nu^{\sup}h + \beta^{\sup}\overline{W}_i h + \int_0^\infty \beta(\sigma+h)i(t+h,\sigma+h)d\sigma - \int_0^\infty \beta(a)i(t,a)da.$$

By solving the partial differential equation in (1) for $i$, from $i(t,\sigma)$ to $i(t+h,\sigma+h)$, we obtain $i(t+h,\sigma+h) = i(t,\sigma)\frac{\Gamma(\sigma+h)}{\Gamma(\sigma)} + \int_\sigma^{\sigma+h} W_i(s)\frac{\Gamma(\sigma+h)}{\Gamma(s)}ds$. (Alternatively, the two sides of this equation can be shown to be equal by using (10).) Thus,

$$\begin{aligned} J(t+h) - J(t) \le \beta^{\sup}\left(K\nu^{\sup} + \overline{W}_i\right)h + \int_0^\infty \beta(\sigma+h)i(t,\sigma)\frac{\Gamma(\sigma+h)}{\Gamma(\sigma)}d\sigma \\ + \int_0^\infty \beta(\sigma+h)\int_\sigma^{\sigma+h} W_i(s)\frac{\Gamma(\sigma+h)}{\Gamma(s)}dsd\sigma - \int_0^\infty \beta(a)i(t,a)da. \end{aligned} \tag{20}$$

We bound the first term on the second line of (20) by noting that $\beta(\sigma+h) \le \beta^{\sup}$ and $\Gamma$ is decreasing, so that $\frac{\Gamma(\sigma+h)}{\Gamma(s)} \le 1$. Then we change the order of integration yielding

$$\begin{aligned} &\int_0^\infty \beta(\sigma+h)\int_\sigma^{\sigma+h} W_i(s)\frac{\Gamma(\sigma+h)}{\Gamma(s)}dsd\sigma \\ &\le \beta^{\sup}\int_0^\infty\int_\sigma^{\sigma+h} W_i(s)dsd\sigma \\ &= \beta^{\sup}\left(\int_0^h\int_0^s W_i(s)d\sigma ds + \int_h^\infty\int_{s-h}^s W_i(s)d\sigma ds\right) \\ &= \beta^{\sup}\left(\int_0^h sW_i(s)ds + \int_h^\infty hW_i(s)ds\right) \\ &\le \beta^{\sup}h\int_0^\infty W_i(s)ds \\ &= \beta^{\sup}\overline{W}_i h. \end{aligned}$$

Using this in (20), and combining the two remaining integrals gives

$$\begin{aligned} &J(t+h) - J(t) \\ &\le \beta^{\sup}\left(K\nu^{\sup} + 2\overline{W}_i\right)h + \int_0^\infty \left(\beta(a+h)\frac{\Gamma(a+h)}{\Gamma(a)} - \beta(a)\right)i(t,a)da \\ &\le \beta^{\sup}\left(K\nu^{\sup} + 2\overline{W}_i\right)h + \int_0^\infty \left(\beta(a+h)\mathrm{e}^{-\int_a^{a+h}\mu_i(s)ds} - \beta(a)\right)i(t,a)da \\ &\le \beta^{\sup}\left(K\nu^{\sup} + 2\overline{W}_i\right)h + \int_0^\infty \beta(a+h)\left(\mathrm{e}^{-\int_a^{a+h}\mu_i(s)ds} - 1\right)i(t,a)da \end{aligned}$$

$$+\int_0^\infty (\beta(a+h)-\beta(a))\, i(t,a)da.$$

By examining the bounds obtained so far, it can be shown that the absolute value of $J(t+h)-J(t)$ satisfies

$$|J(t+h)-J(t)| \le \beta^{\sup}\left(K\nu^{\sup}+2\overline{W}_i\right)h + \int_0^\infty \beta(a+h)\left|e^{-\int_a^{a+h}\mu_i(s)ds}-1\right| i(t,a)da + \int_0^\infty |\beta(a+h)-\beta(a)|\, i(t,a)da. \tag{21}$$

Note that $0 \ge -\int_a^{a+h}\mu_i(s)ds \ge -\mu_i^{\sup}h$, and hence $1 \ge e^{-\int_a^{a+h}\mu_i(s)ds} \ge e^{-\mu_i^{\sup}h} \ge 1-\mu_i^{\sup}h$, where the final equality can be derived from the fact that $e^{-x}$ lies above its tangent line at zero. Thus, $0 \le \int_0^\infty \beta(a+h)\left|e^{-\int_a^{a+h}\mu_i(s)ds}-1\right| i(t,a)da \le \int_0^\infty \beta^{\sup}\mu_i^{\sup}h\, i(t,a)da \le \beta^{\sup}\mu_i^{\sup}Kh$. Therefore, (21) gives

$$|J(t+h)-J(t)| \le \beta^{\sup}\left(K\nu^{\sup}+K\mu_i^{\sup}+2\overline{W}_i\right)h + \int_0^\infty |\beta(a+h)-\beta(a)|\, i(t,a)da. \tag{22}$$

Finally, we obtain a bound of order $h$ for the remaining integral in Equation (22). Using **(H4)**, we have

$$\int_0^\infty |\beta(a+h)-\beta(a)|\, i(t,a)da \le \int_0^\infty M_\beta h\, i(t,a)da \le M_\beta hK.$$

Thus, it follows from Equation (22) that $J$ is Lipschitz with coefficient $M_J$ given by

$$M_J = \beta^{\sup}\left(K\nu^{\sup}+K\mu_i^{\sup}+2\overline{W}_i\right)+M_\beta K$$

The function $L$ can be shown to be Lipschitz through a similar calculation. □

The following theorem is a special case of [17, Theorem 2.46].

**Theorem 5.1.** *The semi-flow $\Phi : \mathbb{R}_{\ge 0}\times\mathcal{Y}\to\mathcal{Y}$ is asymptotically smooth if there are maps $\Theta, \Psi : \mathbb{R}_{\ge 0}\times\mathcal{Y}\to\mathcal{Y}$ such that $\Phi(t,X)=\Theta(t,X)+\Psi(t,X)$ and the following hold for any bounded closed set $C\subset\mathcal{Y}$ that is forward invariant under $\Phi$:*

- $\lim_{t\to\infty} \mathrm{diam}\,\Theta(t,C)=0$,
- *there exists $t_C\ge 0$ such that $\Psi(t,C)$ has compact closure for each $t\ge t_C$.*

Next, we state Theorem B.2. from [17] as it applies to $L^1(\mathbb{R}_{\ge 0},\mathbb{R}_{\ge 0})$.

**Theorem 5.2.** *A set $\mathcal{S}\subseteq L^1(\mathbb{R}_{\ge 0},\mathbb{R}_{\ge 0})$ has compact closure if and only if the following conditions hold:*

1. $\sup_{f\in\mathcal{S}}\int_0^\infty f(a)da<\infty$,
2. $\lim_{r\to\infty}\int_r^\infty f(a)da\to 0$ *uniformly in* $f\in\mathcal{S}$,
3. $\lim_{h\to 0^+}\int_0^\infty |f(a+h)-f(a)|da\to 0$ *uniformly in* $f\in\mathcal{S}$,
4. $\lim_{h\to 0^+}\int_0^h f(a)da\to 0$ *uniformly in* $f\in\mathcal{S}$.

Theorems 5.1 and 5.2 will be used to prove the following theorem, which is the main result of this section. The approach is similar to that found in [12, Section 5].

**Theorem 5.3.** *The flow $\Phi$ is asymptotically smooth.*

*Proof.* We have $\Phi(t, X_0) = (S(t), e(t,\cdot), i(t,\cdot))$, where $X_0 = (S_0, \varphi_e, \varphi_i) \in \mathcal{Y}$ and $t$ is any positive real. Now, for $t \geq 0$, we define two flows $\Psi$ and $\Theta$ on $\mathcal{Y}$ so that $\Phi = \Psi + \Theta$. Let $\Psi(t, X_0) = \left(S(t), \widetilde{e}(t,\cdot), \widetilde{i}(t,\cdot)\right)$ and $\Theta(t, X_0) = (0, \widetilde{\varphi}_e(t,\cdot), \widetilde{\varphi}_i(t,\cdot))$, where

$$\widetilde{e}(t,a) = \begin{cases} S(t-a)J(t-a)\Omega(a) & \text{for } 0 \leq a < t \\ 0 & \text{for } 0 \leq t \leq a, \end{cases}$$

$$\widetilde{i}(t,a) = \begin{cases} L(t-a)\Gamma(a) & \text{for } 0 \leq a < t \\ 0 & \text{for } 0 \leq t \leq a, \end{cases}$$

$$\widetilde{\varphi}_e(t,a) = \begin{cases} \int_0^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma & \text{for } 0 \leq a < t \\ \varphi_e(a-t)\frac{\Omega(a)}{\Omega(a-t)} + \int_{a-t}^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma & \text{for } 0 \leq t \leq a, \end{cases}$$

and

$$\widetilde{\varphi}_i(t,a) = \begin{cases} \int_0^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma & \text{for } 0 \leq a < t \\ \varphi_i(a-t)\frac{\Gamma(a)}{\Gamma(a-t)} + \int_{a-t}^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma. & \text{for } 0 \leq t \leq a. \end{cases}$$

Let $C \subset \mathcal{Y}$ be bounded with bound $K > N^*$. For $j = 1, 2$, let $X_0^j = (S_0, \varphi_e^j, \varphi_i^j) \in C$ and consider the solutions $\Phi(t, X_0^j) = \left(S^j(t), e^j(t,\cdot), i^j(t,\cdot)\right)$. We now study the distance between $\Theta(t, X_0^1)$ and $\Theta(t, X_0^2)$, beginning by noting that

$$\widetilde{\varphi}_e^1(t,a) - \widetilde{\varphi}_e^2(t,a) = \begin{cases} 0 & \text{for } 0 \leq a \leq t \\ \left(\varphi_e^1(a-t) - \varphi_e^2(a-t)\right)\frac{\Omega(a)}{\Omega(a-t)} & \text{for } t < a. \end{cases}$$

Let $\|\cdot\|_1$ denote the standard norm on $L^1$. Then

$$\begin{aligned} \left\|\widetilde{\varphi}_e^1(t,\cdot) - \widetilde{\varphi}_e^2(t,\cdot)\right\|_1 &= \int_t^\infty \left|\varphi_e^1(a-t) - \varphi_e^2(a-t)\right| \frac{\Omega(a)}{\Omega(a-t)}da \\ &= \int_0^\infty \left|\varphi_e^1(\sigma) - \varphi_e^2(\sigma)\right| \frac{\Omega(\sigma+t)}{\Omega(\sigma)}d\sigma \end{aligned}$$

Using Equation (3) to replace both instances of $\Omega$, we find

$$\begin{aligned} \left\|\widetilde{\varphi}_e^1(t,\cdot) - \widetilde{\varphi}_e^2(t,\cdot)\right\|_1 &= \int_0^\infty \left|\varphi_e^1(\sigma) - \varphi_e^2(\sigma)\right| \mathrm{e}^{-\int_\sigma^{\sigma+t}(\nu(s)+\mu_e(s))ds}d\sigma \\ &\leq \mathrm{e}^{-\mu_e^{\inf}t}\int_0^\infty \left|\varphi_e^1(\sigma) - \varphi_e^2(\sigma)\right| d\sigma \\ &\leq \mathrm{e}^{-\mu_e^{\inf}t}\left(\left\|\varphi_e^1\right\|_1 + \left\|\varphi_e^2\right\|_1\right) \\ &\leq 2K\mathrm{e}^{-\mu_e^{\inf}t}, \end{aligned}$$

which tends to zero as $t$ goes to $\infty$.

Similarly, $\left\|\widetilde{\varphi}_i^1(t,\cdot) - \widetilde{\varphi}_i^2(t,\cdot)\right\|_1 \leq 2K\mathrm{e}^{-\mu_i^{\inf}t}$. It follows that

$$\left\|\Theta(t, X_0^1) - \Theta(t, X_0^2)\right\|_{\mathcal{Y}} \leq 2K\left(\mathrm{e}^{-\mu_e^{\inf}t} + \mathrm{e}^{-\mu_i^{\inf}t}\right)$$

for all $t \geq 0$. Since $X_0^1$ and $X_0^2$ were chosen arbitrarily in $C$, it follows that $\operatorname{diam}\Theta(t,C) \leq 2K\left(\mathrm{e}^{-\mu_e^{\inf}t} + \mathrm{e}^{-\mu_i^{\inf}t}\right)$, and so $\lim_{t\to\infty}\operatorname{diam}\Theta(t,C) = 0$, as required by Theorem 5.1.

Now we show that $\Psi(t,C)$ has compact closure for each $t \geq 0$, showing that the second condition in Theorem 5.1 is satisfied (with $t_C = 0$).

Part (1) of Proposition 2, implies that $S(t)$ remains in the compact set $[0, K]$. Next, by verifying conditions (1-4) of Theorem 5.2, we show that $\widetilde{e}$ remains in a subset of $L^1_{\geq 0}$ that has compact closure and is independent of $X_0$.

By Proposition 2 and Equation (5), we have

$$0 \leq \widetilde{e}(t,a) = \left\{ \begin{array}{ll} S(t-a)J(t-a)\Omega(a) & \text{for } 0 \leq a \leq t \\ 0 & \text{for } t < a \end{array} \right\} \leq \beta^{\sup} K^2 \Omega(a)$$
$$\leq \beta^{\sup} K^2 \mathrm{e}^{-\mu_e^{\inf} a},$$

from which it is easily shown that conditions (1, 2, 4) of Theorem 5.2 are satisfied. Now, we show that condition (3) is also satisfied. Because we are interested in the limit as $h$ tends to $0^+$, we may assume that $h \in (0, t)$. Then

$$\begin{aligned}
&\int_0^\infty |\widetilde{e}(t,a+h) - \widetilde{e}(t,a)| da \\
&\quad = \int_0^{t-h} |S(t-a-h)J(t-a-h)\Omega(a+h) - S(t-a)J(t-a)\Omega(a)|\, da \\
&\qquad\quad + \int_{t-h}^{t} |0 - S(t-a)J(t-a)\Omega(a)|\, da \\
&\quad \leq \int_0^{t-h} |S(t-a-h)J(t-a-h)\Omega(a+h) - S(t-a)J(t-a)\Omega(a)|\, da \\
&\qquad\quad + \beta^{\sup} K^2 h \\
&\quad \leq \beta^{\sup} K^2 h + \int_0^{t-h} S(t-a-h)J(t-a-h)\, |\Omega(a+h) - \Omega(a)|\, da \\
&\qquad\quad + \int_0^{t-h} |S(t-a-h)J(t-a-h) - S(t-a)J(t-a)|\, \Omega(a) da \\
&\quad \leq \beta^{\sup} K^2 h + \beta^{\sup} K^2 \int_0^{t-h} |\Omega(a+h) - \Omega(a)|\, da \\
&\qquad\quad + \int_0^{t-h} |S(t-a-h)J(t-a-h) - S(t-a)J(t-a)|\, \Omega(a) da.
\end{aligned} \tag{23}$$

Recalling Equation (3), we note that $\Omega$ is a positive decreasing function that is never greater than 1. Thus,

$$\begin{aligned}
0 \leq \int_0^{t-h} |\Omega(a+h) - \Omega(a)|\, da &= \int_0^{t-h} (\Omega(a) - \Omega(a+h))\, da \\
&= \int_0^h \Omega(a) da - \int_{t-h}^t \Omega(a) da \\
&< \int_0^h \Omega(a) da \\
&\leq h.
\end{aligned}$$

Combining this with Equation (23), we have

$$\int_0^\infty |\widetilde{e}(t,a+h) - \widetilde{e}(t,a)| da \tag{24}$$

$$\leq 2\beta^{\sup}K^2h + \int_0^{t-h} |S(t-a-h)J(t-a-h) - S(t-a)J(t-a)|\,\Omega(a)da.$$

Next, we bound the remaining integral. Combining Proposition 2 with the expression for $\frac{dS}{dt}$ given in Equation (1), we find that $\left|\frac{dS}{dt}\right|$ is bounded by $M_S = W_S + \mu_S K + \beta^{\sup}K^2$, and therefore $S(\cdot)$ is Lipschitz for $t \geq 0$ with coefficient $M_S$. By Proposition 6, $J$ is Lipschitz with coefficient $M_J$ for $t \geq 0$. Since $S$ and $J$ are bounded, with bounds $K$ and $\beta^{\sup}K$, respectively, it follows that $S(\cdot)J(\cdot)$ is Lipschitz on $[0,\infty)$ with coefficient $M_{SJ} = KM_J + \beta^{\sup}KM_S$ (see [12, Proposition 6]). Thus, for $a \in [0, t-h)$,

$$|S(t-a-h)J(t-a-h) - S(t-a)J(t-a)|\,\Omega(a) \leq M_{SJ}h\Omega(a) \leq M_{SJ}h\mathrm{e}^{-\mu_e^{\inf}a}.$$

Therefore, Equation (24) implies

$$\begin{aligned}\int_0^{\infty} |\widetilde{e}(t,a+h) - \widetilde{e}(t,a)|da &\leq 2\beta^{\sup}K^2h + M_{SJ}h\int_0^{t-h} \mathrm{e}^{-\mu_e^{\inf}a}da \\ &\leq 2\bar{\beta}K^2h + \frac{M_{SJ}}{\mu_e^{\inf}}h \\ &= \left(2\bar{\beta}K^2 + \frac{M_{SJ}}{\mu_e^{\inf}}\right)h.\end{aligned}$$

The constant $M_{SJ}$ depends on $K$, which depends on the set $C$, but $M_{SJ}$ does not depend directly on $X_0$. Thus, this inequality holds for all $X_0 \in C$, and so condition (3) of Theorem 5.2 is satisfied. Thus, $\widetilde{e}$ remains in a pre-compact subset $C_K^e$ of $L^1_{\geq 0}$. Similarly, $\widetilde{i}$ remains in a pre-compact subset $C_K^i$ of $L^1_{\geq 0}$. Thus, $\Psi(t,C) \subseteq [0,K] \times C_K^e \times C_K^i$, which has compact closure in $\mathcal{Y}$. It follows that $\Psi(t,C)$ has compact closure. Thus, the second condition of Theorem 5.1 is satisfied, and so it follows that $\Phi$ is asymptotically smooth. □

6. **Attractor.** A function $X : \mathbb{R} \to \mathcal{Y}$ that satisfies $\Phi_s(X(t)) = X(t+s)$ for all $t \in \mathbb{R}$ and all $s \geq 0$ is called a *total trajectory* of $\Phi$. A total trajectory satisfies

$$e(t,a) = e(t-a,0)\Omega(a) + \int_0^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma \tag{25}$$

and

$$i(t,a) = i(t-a,0)\Gamma(a) + \int_0^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma \tag{26}$$

for all $t \in \mathbb{R}$ and $a \in \mathbb{R}_{\geq 0}$.

For non-empty sets $C, D \subseteq \mathcal{Y}$, we define the distance from $C$ to $D$ as

$$\mathrm{dist}\,(C,D) = \sup_{x\in C}\inf_{y\in D} \|x-y\|_{\mathcal{Y}}\,.$$

A non-empty invariant compact set $\widetilde{A}$ is called the *compact attractor* of a class $\mathcal{C}$ of sets if $\mathrm{dist}\left(\Phi_t(C), \widetilde{A}\right) \to 0$ for each $C \in \mathcal{C}$. Such a set consists entirely of total trajectories; that is, for each point $X_0 \in \widetilde{A}$, there exists a total trajectory $X(\cdot)$, with $X(0) = X_0$ and $X(t) \in \widetilde{A}$ for all $t \in \mathbb{R}$ [17, Theorem 1.40].

**Theorem 6.1.** *There exists a set $\mathcal{A}$, which is the compact attractor of bounded sets.*

*Proof.* Proposition 1, Proposition 3 and Theorem 5.3 show that $\Phi$ is point dissipative, eventually bounded on bounded sets, and asymptotically smooth. Thus, the result follows from Theorem 2.33 of [17]. □

The following corollary follows from Equations (25), (26), Propositions 1, 2, 4 and 5, and Theorem 6.1.

**Corollary 1.** *If $X_0 = (x, \varphi, \phi) \in \mathcal{A}$, then there exists $\epsilon > 0$ such that*

1. $\|X_0\| \leq N^*$,
2. $\epsilon \leq x \leq N^*$,
3. $\epsilon \leq \varphi(0) \leq (N^*)^2 \beta^{sup}$,
4. $\epsilon \leq \phi(0) \leq N^* \nu^{sup}$,
5. $\varphi(a) = \varphi(0)\Omega(a) + \int_0^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)}d\sigma$ *for all* $a \geq 0$,
6. $\phi(a) = \phi(0)\Gamma(a) + \int_0^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma$ *for all* $a \geq 0$.

7. **Equilibria.** There is no disease-free equilibrium. Thus, there is no basic reproduction number. The following theorem states that there is always a unique endemic equilibrium.

**Proposition 7.** *There exists a unique equilibrium $X^* = (S^*, e^*(\cdot), i^*(\cdot))$. Furthermore, $X^* \in \mathcal{A}$.*

*Proof.* First, note that an equilibrium solution to (1) and (2) satisfies

$$\begin{aligned} 0 &= W_S - \mu_S S^* - \int_0^\infty \beta(a) S^* i^*(a) da \\ \frac{\partial e^*}{\partial a} &= W_e(a) - (\nu(a) + \mu_e(a))\, e^*(a) \\ \frac{\partial i^*}{\partial a} &= W_i(a) - \mu_i(a) i^*(a), \end{aligned} \tag{27}$$

with boundary conditions

$$\begin{aligned} e^*(0) &= \int_0^\infty \beta(a) S^* i^*(a) da \\ i^*(0) &= \int_0^\infty \nu(a) e^*(a) da. \end{aligned} \tag{28}$$

Combining the first line of (27) with the first line of (28) gives $e^*(0) = W_S - \mu_S S^*$. Then, solving the ODE that is the second line of (27) gives

$$\begin{aligned} e^*(a) &= e^*(0)\Omega(a) + \int_0^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)} d\sigma \\ &= (W_S - \mu_S S^*)\, \Omega(a) + \int_0^a W_e(\sigma)\frac{\Omega(a)}{\Omega(\sigma)} d\sigma. \end{aligned} \tag{29}$$

Thus, for a given value of $S^*$, the function $e^*$ is uniquely determined. Using this expression for $e^*$ and the second line in (28), we calculate

$$\begin{aligned} i^*(0) &= (W_S - \mu_S S^*) \int_0^\infty \nu(a)\Omega(a) da + \int_0^\infty \int_0^a \nu(a) W_e(\sigma) \frac{\Omega(a)}{\Omega(\sigma)} d\sigma da \\ &= (W_S - \mu_S S^*)\, A + \int_0^\infty \int_0^\xi \nu(\xi) W_e(\sigma)\frac{\Omega(\xi)}{\Omega(\sigma)} d\sigma d\xi, \end{aligned}$$

where $A$ is given in (7). Then, solving the ODE that is the third line of (27) gives

$$\begin{aligned} i^*(a) &= i^*(0)\Gamma(a) + \int_0^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma \\ &= (W_S - \mu_S S^*)\, A\Gamma(a) + \int_0^\infty \int_0^\xi \nu(\xi) W_e(\sigma)\frac{\Omega(\xi)}{\Omega(\sigma)}d\sigma d\xi\, \Gamma(a) \\ &\qquad + \int_0^a W_i(\sigma)\frac{\Gamma(a)}{\Gamma(\sigma)}d\sigma. \end{aligned} \tag{30}$$

We have now uniquely determined both $e^*$ and $i^*$ as functions of $S^*$. Next, we use (30) and the first line of (27) to determine $S^*$.

$$\begin{aligned} W_S - \mu_S S^* &= \int_0^\infty \beta(a) S^* i^*(a) da \\ &= (W_S - \mu_S S^*)\, ABS^* + MBS^* + NS^*, \end{aligned}$$

where $B$ is given in (7), and $M$ and $N$ are defined as $M = \int_0^\infty \int_0^\xi \nu(\xi) W_e(\sigma) \frac{\Omega(\xi)}{\Omega(\sigma)} d\sigma d\xi$ and $N = \int_0^\infty \int_0^a \beta(a) W_i(\sigma) \frac{\Gamma(a)}{\Gamma(\sigma)} d\sigma da$. Note that **(H7)** ensures that at least one of $M$ and $N$ is positive, and the other is non-negative. Rearranging, we see that equilibria are given by solutions of $f(S^*) = 0$ where $f$ is the quadratic

$$f(S^*) = \mu_S AB(S^*)^2 - [\mu_S + W_S AB + MB + N]\, S^* + W_S.$$

The functions $e^*$ and $i^*$ are non-negative at each point in $\mathbb{R}_{\geq 0}$ if and only if $W_S - \mu_S S^*$ is non-negative. Since we also need $S^*$ to be non-negative, we require $0 \leq S^* \leq \frac{W_S}{\mu_S}$. Note that $f(0) = W_S > 0$ and $f\left(\frac{W_S}{\mu_S}\right) = -(MB+N)\frac{W_S}{\mu_S} < 0$. Since $f$ is quadratic, there exists a unique zero of $f$ in the given interval. Thus, there is a unique non-negative equilibrium.

Since the solution for $S^*$ lies in the open interval $\left(0, \frac{W_S}{\mu_S}\right)$, it follows that $e^*(0)$ and $i^*(0)$ are positive. Then, from the first lines of (29) and (30), it follows that $e^*(a) \geq e^*(0)\Omega(a) > 0$ and $i^*(a) \geq i^*(0)\Gamma(a) > 0$ for all $a \geq 0$. Thus, the unique equilibrium is strictly positive.

Since $\{X^*\}$ is an invariant bounded set, it is immediate that $\{X^*\} \subseteq \mathcal{A}$. □

## 8. Global stability.

In this section we use a Lyapunov functional to show that the attractor $\mathcal{A}$ consists of only the equilibrium $X^*$, making it globally asymptotically stable. We begin by defining functions that will be used for the Lyapunov functional and by giving some preparatory results.

Let $X(t) = (S(t), e(t,\cdot), i(t,\cdot))$, for $t \in \mathbb{R}$ be a total trajectory in $\mathcal{A}$. Let

$$g(x) = x - 1 - \ln x$$

and let

$$V(t) = V_S + \frac{e^*(0)}{i^*(0)} V_e + V_i \qquad \text{where} \qquad \begin{aligned} V_S(t) &= g\left(\frac{S(t)}{S^*}\right) \\ V_e(t) &= \int_0^\infty \alpha_e(a) g\left(\frac{e(t,a)}{e^*(a)}\right) da \\ V_i(t) &= \int_0^\infty \alpha_i(a) g\left(\frac{i(t,a)}{i^*(a)}\right) da, \end{aligned}$$

with

$$\alpha_e(a) = \int_a^\infty \nu(\sigma)e^*(\sigma)d\sigma \quad \text{and} \quad \alpha_i(a) = \int_a^\infty \beta(\sigma)S^*i^*(\sigma)d\sigma.$$

Then

$$\begin{aligned}\frac{dV_S}{dt} &= \frac{1}{S^*}\left(1-\frac{S^*}{S}\right)\left[W_S - \mu_S S(t) - \int_0^\infty \beta(a)S(t)i(t,a)da\right] \\ &= \frac{1}{S^*}\left(1-\frac{S^*}{S}\right)\left[\mu_S\left(S^*-S\right) + \int_0^\infty \beta(a)\left(S^*i^*(a) - S(t)i(t,a)\right)da\right] \\ &= -\mu_S\frac{\left(S-S^*\right)^2}{SS^*} + \int_0^\infty \beta(a)S^*i^*(a)\left(1-\frac{S^*}{S}-\frac{S}{S^*}\frac{i(t,a)}{i^*(a)}+\frac{i(t,a)}{i^*(a)}\right)da.\end{aligned} \tag{31}$$

Next, we calculate $\frac{dV_e}{dt}$ using steps similar to the corresponding calculation in Section 3.2 of [10] and in the proof of [12, Lemma 9.4]. However, the immigration term in the current model adds extra terms. Letting

$$Q(a) = \int_0^a \frac{W_e(\sigma)}{\Omega(\sigma)}d\sigma$$

for $a \geq 0$, and then using (25), we have

$$\frac{e(t,a)}{e^*(a)} = \frac{e(t-a,0)\Omega(a) + Q(a)\Omega(a)}{e^*(0)\Omega(a) + Q(a)\Omega(a)} = \frac{e(t-a,0) + Q(a)}{e^*(0) + Q(a)}.$$

Filling into $V_e(t)$, differentiating, and then letting $\tau = t - a$, we have

$$\begin{aligned}\frac{dV_e}{dt} &= \frac{d}{dt}\int_0^\infty \alpha_e(a)g\left(\frac{e(t-a,0)+Q(a)}{e^*(0)+Q(a)}\right)da \\ &= \frac{d}{dt}\int_{-\infty}^t \alpha_e(t-\tau)g\left(\frac{e(\tau,0)+Q(t-\tau)}{e^*(0)+Q(t-\tau)}\right)d\tau \\ &= \alpha_e(0)g\left(\frac{e(t,0)+Q(0)}{e^*(0)+Q(0)}\right) + \int_{-\infty}^t \alpha_e'(t-\tau)g\left(\frac{e(\tau,0)+Q(t-\tau)}{e^*(0)+Q(t-\tau)}\right)d\tau \\ &\qquad + \int_{-\infty}^t \alpha_e(t-\tau)g'\left(\frac{e(\tau,0)+Q(t-\tau)}{e^*(0)+Q(t-\tau)}\right)\frac{d}{dt}\left(\frac{e(\tau,0)+Q(t-\tau)}{e^*(0)+Q(t-\tau)}\right)d\tau \\ &= \alpha_e(0)g\left(\frac{e(t,0)}{e^*(0)}\right) + \int_0^\infty \alpha_e'(a)g\left(\frac{e(t,a)}{e^*(a)}\right)da \\ &\qquad\qquad + \int_0^\infty \alpha_e(a)g'\left(\frac{e(t,a)}{e^*(a)}\right)\frac{Q'(a)}{e^*(0)+Q(a)}\left(1-\frac{e(t,a)}{e^*(a)}\right)da.\end{aligned}$$

Noting that $g'(x) = 1 - \frac{1}{x}$ and $\frac{Q'(a)}{e^*(0)+Q(a)} = \frac{W_e(a)}{\Omega(a)}\frac{1}{e^*(0)+Q(a)} = \frac{W_e(a)}{e^*(a)}$, we find

$$\begin{aligned}\frac{dV_e}{dt} &= \alpha_e(0)g\left(\frac{e(t,0)}{e^*(0)}\right) + \int_0^\infty \alpha_e'(a)g\left(\frac{e(t,a)}{e^*(a)}\right)da \\ &\qquad + \int_0^\infty \frac{\alpha_e(a)W_e(a)}{e^*(a)}\left(1-\frac{e^*(a)}{e(t,a)}\right)\left(1-\frac{e(t,a)}{e^*(a)}\right)da.\end{aligned}$$

Noting that $\alpha_e(0) = \int_0^\infty \nu(a)e^*(a)da$ and $\alpha_e'(a) = -\nu(a)e^*(a)$, and rearranging terms, we have

$$\frac{dV_e}{dt} = \int_0^\infty \nu(a)e^*(a)\left(g\left(\frac{e(t,0)}{e^*(0)}\right) - g\left(\frac{e(t,a)}{e^*(a)}\right)\right)da \\ - \int_0^\infty \frac{\alpha_e(a)W_e(a)}{e^*(a)}\frac{(e(t,a)-e^*(a))^2}{e^*(a)e(t,a)}da. \tag{32}$$

Similarly,

$$\frac{dV_i}{dt} = \int_0^\infty \beta(a)S^*i^*(a)\left(g\left(\frac{i(t,0)}{i^*(0)}\right) - g\left(\frac{i(t,a)}{i^*(a)}\right)\right)da \\ - \int_0^\infty \frac{\alpha_i(a)W_i(a)}{i^*(a)}\frac{(i(t,a)-i^*(a))^2}{i^*(a)i(t,a)}da. \tag{33}$$

Combining Equations (31), (32) and (33), we obtain

$$\begin{aligned}\frac{dV}{dt} &= -\mu_S\frac{(S-S^*)^2}{SS^*} + \int_0^\infty \beta(a)S^*i^*(a)\left(1 - \frac{S^*}{S} - \frac{S}{S^*}\frac{i(t,a)}{i^*(a)} + \frac{i(t,a)}{i^*(a)}\right)da \\ &\quad + \frac{e^*(0)}{i^*(0)}\int_0^\infty \nu(a)e^*(a)\left(g\left(\frac{e(t,0)}{e^*(0)}\right) - g\left(\frac{e(t,a)}{e^*(a)}\right)\right)da \\ &\quad - \frac{e^*(0)}{i^*(0)}\int_0^\infty \frac{\alpha_e(a)W_e(a)}{e^*(a)}\frac{(e(t,a)-e^*(a))^2}{e^*(a)e(t,a)}da \\ &\quad + \int_0^\infty \beta(a)S^*i^*(a)\left(g\left(\frac{i(t,0)}{i^*(0)}\right) - g\left(\frac{i(t,a)}{i^*(a)}\right)\right)da \\ &\quad - \int_0^\infty \frac{\alpha_i(a)W_i(a)}{i^*(a)}\frac{(i(t,a)-i^*(a))^2}{i^*(a)i(t,a)}da \\ &= -\mu_S\frac{(S-S^*)^2}{SS^*} - \frac{e^*(0)}{i^*(0)}\int_0^\infty \frac{\alpha_e(a)W_e(a)}{e^*(a)}\frac{(e(t,a)-e^*(a))^2}{e^*(a)e(t,a)}da \\ &\quad - \int_0^\infty \frac{\alpha_i(a)W_i(a)}{i^*(a)}\frac{(i(t,a)-i^*(a))^2}{i^*(a)i(t,a)}da + Z,\end{aligned}$$

where

$$Z = \int_0^\infty \beta(a)S^*i^*(a)\left(1 - \frac{S^*}{S} - \frac{S}{S^*}\frac{i(t,a)}{i^*(a)} + \frac{i(t,0)}{i^*(0)} + \ln\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)}\right)da \\ + \frac{e^*(0)}{i^*(0)}\int_0^\infty \nu(a)e^*(a)\left(\frac{e(t,0)}{e^*(0)} - \frac{e(t,a)}{e^*(a)} + \ln\frac{e(t,a)}{e^*(a)} - \ln\frac{e(t,0)}{e^*(0)}\right)da. \tag{34}$$

We note that if $Z$ is non-positive, then the same is true of $\frac{dV}{dt}$ and so $V$ is a Lyapunov functional. Thus, we now work to show that $Z$ is non-positive.

For the next portion of the calculation, we introduce the formulas

$$\int_0^\infty \beta(a)i^*(a)\left(\frac{S}{S^*}\frac{i(t,a)}{i^*(a)} - \frac{e(t,0)}{e^*(0)}\right)da = 0 \tag{35}$$

and

$$\int_0^\infty \nu(a)e^*(a)\left(\frac{e(t,a)}{e^*(a)} - \frac{i(t,0)}{i^*(0)}\right)da = 0, \tag{36}$$

which are proved in the same manner as [12, Lemmas 9.2 and 9.3]. Using (35) and (36), we can replace one term in each integral of Equation (34), to get

$$\begin{aligned} Z = &\int_0^\infty \beta(a)S^*i^*(a)\left(1 - \frac{S^*}{S} - \frac{e(t,0)}{e^*(0)} + \frac{i(t,0)}{i^*(0)} + \ln\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)}\right)da \\ &+ \frac{e^*(0)}{i^*(0)}\int_0^\infty \nu(a)e^*(a)\left(\frac{e(t,0)}{e^*(0)} - \frac{i(t,0)}{i^*(0)} + \ln\frac{e(t,a)}{e^*(a)} - \ln\frac{e(t,0)}{e^*(0)}\right)da. \end{aligned} \tag{37}$$

If $H$ is a term that does not depend on $a$, then

$$\int_0^\infty \beta(a)S^*i^*(a)H\,da - \frac{e^*(0)}{i^*(0)}\int_0^\infty \nu(a)e^*(a)H\,da = 0. \tag{38}$$

This can be shown by using (28) to replace $e^*(0)$ and $i^*(0)$. Letting $H = \frac{e(t,0)}{e^*(0)} - \frac{i(t,0)}{i^*(0)} + \ln\frac{i(t,0)}{i^*(0)} - \ln\frac{e(t,0)}{e^*(0)}$, we use (38) to add $H$ to the integrand of the first integral of (37), while subtracting it from the second, obtaining

$$\begin{aligned} Z = &\int_0^\infty \beta(a)S^*i^*(a)\left(1 - \frac{S^*}{S} + \ln\frac{i(t,a)}{i^*(a)} - \ln\frac{e(t,0)}{e^*(0)}\right)da \\ &+ \frac{e^*(0)}{i^*(0)}\int_0^\infty \nu(a)e^*(a)\left(\ln\frac{e(t,a)}{e^*(a)} - \ln\frac{i(t,0)}{i^*(0)}\right)da. \end{aligned} \tag{39}$$

This had the effect of cancelling a few terms while shifting two logarithm terms between integrals. Next, we multiply (36) by $\frac{i^*(0)}{i(t,0)}$ allowing us to add $\left(1 - \frac{e(t,a)}{e^*(a)}\frac{i^*(0)}{i(t,0)}\right)$ to the integrand of the second integral in (39). Similarly, we multiply (35) by $\frac{e^*(0)}{e(t,0)}$ allowing us to add $\left(1 - \frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\frac{e^*(0)}{e(t,0)}\right)$ inside the first integral. We also add and subtract $\ln\frac{S}{S^*}$ inside the first integral, and use properties of logarithms to obtain

$$\begin{aligned} Z = &\int_0^\infty \beta(a)S^*i^*(a)\left(\left(1 - \frac{S^*}{S} + \ln\frac{S^*}{S}\right)\right. \\ &\qquad\left.+\left(1 - \frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\frac{e^*(0)}{e(t,0)} + \ln\frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\frac{e^*(0)}{e(t,0)}\right)\right)da \\ &+ \frac{e^*(0)}{i^*(0)}\int_0^\infty \nu(a)e^*(a)\left(1 - \frac{e(t,a)}{e^*(a)}\frac{i^*(0)}{i(t,0)} + \ln\frac{e(t,a)}{e^*(a)}\frac{i^*(0)}{i(t,0)}\right)da \\ = &-\int_0^\infty \beta(a)S^*i^*(a)\left(g\left(\frac{S^*}{S}\right) + g\left(\frac{S}{S^*}\frac{i(t,a)}{i^*(a)}\frac{e^*(0)}{e(t,0)}\right)\right) \\ &\qquad - \frac{e^*(0)}{i^*(0)}\int_0^\infty \nu(a)e^*(a)g\left(\frac{e(t,a)}{e^*(a)}\frac{i^*(0)}{i(t,0)}\right)da \\ \le &\ 0. \end{aligned}$$

Thus, we have

$$\frac{dV}{dt} \le 0,$$

with equality if and only if $(S,e,i) = (S^*,e^*,i^*)$.

**Theorem 8.1.** *The equilibrium $X^*$ is globally asymptotically stable and $\mathcal{A} = \{X^*\}$.*

*Proof.* From the preceding calculation, we know that $V$ is strictly decreasing along each trajectory $X(t)$ in $\mathcal{A}$ that is distinct from $X^*$. Let $X(t)$ be a total trajectory in $\mathcal{A}$. Then the alpha limit set $\Lambda$ of $X(t)$ is an invariant subset of $\mathcal{A}$ on which $V$ is constant. Thus, the $\Lambda = \{X^*\}$. Since this is the alpha limit set of $X(t)$, we must have $V(X(t)) \leq V(X^*)$ for all $t \in \mathbb{R}$. However, since $X^*$ is the point at which $V$ is minimized, we must have $X(t) \equiv X^*$. Since this holds for any total trajectory $X(t)$ in $\mathcal{A}$, which consists entirely of total trajectories, we must have $\mathcal{A} = \{X^*\}$. □

9. **Discussion.** Our model is similar to the model in [12], which can be interpreted as a version of the current model without immigration into the exposed and infectious classes. Our analysis closely follows that found in [12]. However, many key calculations differ.

While the addition of immigration of infected individuals complicates some of the calculations, it simplifies the dynamics. If $W_e \equiv W_i \equiv 0$, then a disease-free equilibrium exists, along with a basic reproduction number $\mathcal{R}_0$. Futhermore, as $\mathcal{R}_0$ passes through unity, there is a transcritical bifurcation. That does not happen in the current model. Here, there is no disease-free equilibrium and for all parameter values, the unique (endemic) equilibrium is globally asymptotically stable.

Thus, while the parameters involved in the current model appear to extend model found in [12], the dynamics are simpler. A continuous extension of the parameters of the current model to the boundary case of no immigration, giving the model found in [12] gives fundamentally different dynamics through a bifurcation. Even for the ODE case, this issue does not seem to have been fully studied.

The results of this paper show that if there is immigration of infected individuals (exposed and/or infectious) into a region, then it is inevitable that the disease will survive in that region; elimination of the disease becomes impossible. In this situation, there is no threshold behaviour. For all parameter values, the disease will eventually reach a constant endemic level.

In order to eliminate the disease, it is necessary to either completely stop the immigration of infected individuals, which would be exceedingly difficult, or to eliminate the disease in all regions, which would require multilateral coordination.

These results are similar to those obtained in [16], where an SEI model with immigration and nonlinear incidence but without age-structure was studied. This suggests that the results of the current model might extend to nonlinear incidence.

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