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A NOTE ON DYNAMICS OF AN AGE-OF-INFECTION CHOLERA MODEL

Jinliang Wang and Ran Zhang

School of Mathematical Science, Heilongjiang University
Harbin 150080, China

Toshikazu Kuniya

Graduate School of System Informatics, Kobe University
1-1 Rokkodai-cho Nada-ku, Kobe-shi, Hyogo 657-8501, Japan

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Abstract. A recent paper [F. Brauer, Z. Shuai and P. van den Driessche, *Dynamics of an age-of-infection cholera model*, Math. Biosci. Eng., 10, 2013, 1335–1349.] presented a model for the dynamics of cholera transmission. The model is incorporated with both the infection age of infectious individuals and biological age of pathogen in the environment. The basic reproduction number is proved to be a sharp threshold determining whether or not cholera dies out. The global stability for disease-free equilibrium and endemic equilibrium is proved by constructing suitable Lyapunov functionals. However, for the proof of the global stability of endemic equilibrium, we have to show first the relative compactness of the orbit generated by model in order to make use of the invariance principle. Furthermore, uniform persistence of system must be shown since the Lyapunov functional is possible to be infinite if $i(a,t)/i^*(a) = 0$ on some age interval. In this note, we give a supplement to above paper with necessary mathematical arguments.

1. **Introduction.** In this note, we will revisit an age-of-infection cholera model which is presented and studied in [1] by F. Brauer, Z. Shuai and P. van den Driessche. The model takes the following form

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = A - \mu S(t) - \beta_1 S(t)\displaystyle\int_0^\infty k(a)i(t,a)\mathrm{d}a - \beta_2 S(t)\int_0^\infty q(b)p(t,b)\mathrm{d}b, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) i(t,a) = -\theta(a)i(t,a), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) p(t,b) = -\delta(b)p(t,b), \end{cases} \tag{1}$$

with boundary conditions

$$\begin{cases} i(t,0) = \beta_1 S(t)\displaystyle\int_0^\infty k(a)i(t,a)\mathrm{d}a + \beta_2 S(t)\int_0^\infty q(b)p(t,b)\mathrm{d}b, \\ p(t,0) = \displaystyle\int_0^\infty \xi(a)i(t,a)\mathrm{d}a, \end{cases} \tag{2}$$

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and initial conditions

$$S(0) = S_0 > 0,\ i(0,a) = i_0(a) \in L^1_+(0,\infty),\ p(0,b) = p_0(b) \in L^1_+(0,\infty), \tag{3}$$

where $S(t)$ denotes the number of susceptible individuals at time $t \geq 0$, $i(t,a)$ denotes the number of infected individuals of infection age $a \geq 0$ at time t, and $p(t,b)$ denotes the quantity of pathogen of age $b \geq 0$ at time t in the contaminated water. Here, individuals in the general population enter the susceptible population at a rate, A, die at a natural death rate μ. It is assumed that β_1 and β_2 are the direct transmission coefficient and indirect transmission coefficient, respectively. $\delta(b)$ represents the removal rate of the pathogen of age b, and $\theta(a) = \mu + \alpha(a) + \gamma(a)$ in which the last two terms represent the disease induced death rate and the recovery rate for infected individuals of infection age a, respectively. $\xi(a)$ represents the shedding rate of an infected individual of infection age a.

Model (1) is formulated by incorporating simultaneously the age-of-infection structure of individuals and the age structure of pathogen with infectivity given by kernel functions. The nonnegative kernel functions $k(a)$ and $q(b)$ measure the infectivity of infected individuals of infection age a and pathogen of age b, respectively.

From (2) and (3), integration of the second and third equation in (1) along the characteristic line $t - a =$ const. and $t - b =$ const. yields

$$i(t,a) = \begin{cases} S(t-a)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega}\left(\beta_1 \int_0^\infty k(u)i(t-a,u)\mathrm{d}u \right. \\ \qquad\qquad \left. +\beta_2 \int_0^\infty q(v)p(t-a,v)\mathrm{d}v\right), & t \geq a \geq 0; \\ i_0(a-t)e^{-\int_{a-t}^a \theta(\omega)\mathrm{d}\omega}, & a \geq t \geq 0. \end{cases} \tag{4}$$

and

$$p(t,b) = \begin{cases} \int_0^\infty \xi(u)i(t-b,u)\mathrm{d}u e^{-\int_0^b \delta(\omega)\mathrm{d}\omega}, & t \geq b \geq 0; \\ p_0(b-t)e^{-\int_{b-t}^b \delta(\omega)\mathrm{d}\omega}, & b \geq t \geq 0. \end{cases} \tag{5}$$

System (1) always has a disease-free equilibrium $P^0 = (S^0, i^0(a), p^0(b))$, where $S^0 = A/\mu$, $i_0(a) = p_0(b) = 0$. Now let us investigate the positive equilibrium of system (1). For any positive equilibrium $P^* = (S^*, i^*(a), p^*(b))$ of system (1), it should satisfy the following equations

$$\begin{cases} A - \mu S^* - \beta_1 S^* \int_0^\infty k(a)i^*(a)\mathrm{d}a - \beta_2 S^* \int_0^\infty q(b)p^*(b)\mathrm{d}b = 0, \\ \frac{\mathrm{d}}{\mathrm{d}a} i^*(a) = -\theta(a)i^*(a), \\ \frac{\mathrm{d}}{\mathrm{d}b} p^*(b) = -\delta(b)p^*(p), \\ i^*(0) = \beta_1 S^* \int_0^\infty k(a)i^*(a)\mathrm{d}a + \beta_2 S^* \int_0^\infty q(b)p^*(b)\mathrm{d}b, \\ p^*(0) = \int_0^\infty \xi(u)i^*(u)\mathrm{d}u. \end{cases} \tag{6}$$

Denote

$$K = \int_0^\infty k(u)e^{-\int_0^u \theta(\omega)\mathrm{d}\omega}\mathrm{d}u, \quad X = \int_0^\infty \xi(u)e^{-\int_0^u \theta(\omega)\mathrm{d}\omega}\mathrm{d}u \tag{7}$$

and

$$Q = \int_0^\infty q(v)e^{-\int_0^v \delta(\omega)\mathrm{d}\omega}\mathrm{d}v. \tag{8}$$

After simple calculation, one can obtain that

$$S^* = \frac{S^0}{\Re_0}, \quad i^*(a) = A\left(1 - \frac{1}{\Re_0}\right)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} > 0$$

$$\text{and } p^*(b) = e^{-\int_0^b \delta(\omega)\mathrm{d}\omega}\int_0^\infty \xi(u)i^*(u)\mathrm{d}u,$$

where

$$\Re_0 = \beta_1 S^0 K + \beta_2 S^0 X Q \tag{9}$$

is defined as the basic reproduction number for system (1). Thus, (1) has a unique endemic equilibrium $P^* = (S^*, i^*(a), p^*(b))$ if and only if $\Re_0 > 1$. The basic reproduction number $\Re_0$ is proved to be a sharp threshold parameter, completely determining the global dynamics of (1). The main theorem obtained in [1] is

Theorem 1.1. *Consider model (1) with $\Re_0$ defined in (9).*

(a) The disease-free equilibrium P^0 is globally asymptotically stable if $\Re_0 \leq 1$ while unstable if $\Re_0 > 1$.

(b) If $\Re_0 > 1$, the endemic equilibrium P^* is global asymptotically stable with respect to solutions with initial conditions $S_0 > 0$ and $i_0(a)$, $p_0(b) > 0$ bounded away from zero.

In (1), age-of-infection is considered as a continuous variable. Continuous age-structured in the infectious class allows the infectivity to truly be a function of the duration spent in class [5]. Because the continuous age model is described by first order PDEs, it is difficult to analyze the dynamics of the PDE models, particularly the global stability. The global stability for equilibria of (1) (Theorem 1.1) is proved by constructing suitable Lyapunov functions, which was adopted originally in [4, 3] to get the global dynamical properties of some age-structured epidemiological models. Two Lyapunov functions are constructed to show the global stability of the disease-free and endemic equilibria. In [1], the authors then studied the final size problem for a simplified version of (1), then extended the result to a staged progression model. Recently McCluskey's work [5] has drawn much attention and elegantly established the global stability problems to two-dimensional continuous age-structured epidemic models.

The results presented in [1] gave a global attracting analysis of equilibria of model (1), but leaving out the necessary arguments, including relative compactness of orbit generated by system (1) and uniform persistence, which are two major challenges in applying the main results in [7] to particular models. This provides us with one motivation to conduct our work. The object of this note is to show that, under some assumptions, system (1) can be reformulated as a Volterra integral equation in order to apply functional analysis theory, and then we present some results about uniform persistence and about the existence of global attractors. The methods of theoretical analysis follow the techniques laid out in the new book [7]. As an application of the methods in [7], it is expected that calculations here help to demonstrate the usefulness of the techniques given in [7], and can be applicable to more age-structured epidemic models.

The paper is organized as follows. In section 2 we describe preliminary results and some notations providing the context where this paper is to be read. The relative compactness of orbit and uniform persistence for $\Re_0 > 1$ is shown in section 3 and 4–the key results of this paper. Section 5 contributes to the stability analysis of system (1).

2. Preliminary results.

We make the following assumption on parameters, which is thought to be biologically relevant.

Assumption 2.1. *Consider system (1), we assume that:*

(i) $A, \mu, \beta_1, \beta_2 > 0$;

(ii) $k(a), q(b), \theta(a), \delta(b), \xi(a) \in L^{\infty}_{+}(0,\infty)$, *with respective essential upper bounds* $\bar{k}, \bar{q}, \bar{\theta}, \bar{\delta}, \bar{\xi}$, *i.e.,*

$$\bar{k} := \operatorname*{ess.sup}_{a\in[0,\infty)} k(a) < +\infty, \quad \bar{q} := \operatorname*{ess.sup}_{b\in[0,\infty)} q(b) < +\infty, \quad \bar{\theta} := \operatorname*{ess.sup}_{a\in[0,\infty)} \theta(a) < +\infty,$$

$$\bar{\delta} := \operatorname*{ess.sup}_{b\in[0,\infty)} \delta(b) < +\infty, \quad \bar{\xi} := \operatorname*{ess.sup}_{a\in[0,\infty)} \xi(a) < +\infty;$$

(iii) $k(a), q(b), \xi(a)$ *are Lipschitz continuous on* $\mathbb{R}_+$ *with Lipschitz coefficients* M_k, M_q, M_ξ *respectively;*

(iv) $k(a) > 0$, $q(b) > 0$ *and* $\xi(a) > 0$ *for all* $a, b \geq 0$*;*

(v) There exists $\mu_0 \in (0, \mu]$ *such that* $\theta(a), \delta(b) \geq \mu_0$ *for all* $a, b \geq 0$.

Let us define a functional space for system (1):

$$\mathcal{Y} = \mathbb{R}_{\geq 0} \times L^1_+ \times L^1_+,$$

where L^1_+ is the space of functions on $(0,\infty)$ that are non-negative and Lebesgue integrable, equipped with the norm

$$\|(x,\varphi,\psi)\|_{\mathcal{Y}} := |x| + \int_0^\infty |\varphi(a)|\mathrm{d}a + \int_0^\infty |\psi(a)|\mathrm{d}a.$$

The initial condition (3) for the system can be rewritten as

$$Y_0 := (S_0, i_0(\cdot), p_0(\cdot)) \in \mathcal{Y}.$$

Next, we first define the continuous semi-flow associated with this system. It follows from (4), (5) and Assumption 2.1, we easily see that system (1) has a unique nonnegative solution for any initial condition $Y_0 \in \mathcal{Y}$. Thus, we can obtain a continuous semi-flow $\Phi : \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}$ defined by system (1) such that

$$\Phi(t, Y_0) = \Phi_t(Y_0) := (S(t), i(t,\cdot), p(t,\cdot)), \quad t \geq 0, \quad Y_0 \in \mathcal{Y}. \tag{10}$$

Thus

$$\|\Phi_t(Y_0)\|_{\mathcal{Y}} = \|\Phi(S(t), i(t,\cdot), p(t,\cdot))\|_{\mathcal{Y}} = S(t) + \int_0^\infty i(t,a)\mathrm{d}a + \int_0^\infty p(t,b)\mathrm{d}b. \tag{11}$$

Without loss of generality, for $a \geq 0$, we denote

$$\Omega(a) = \mathrm{e}^{-\int_0^a \theta(\tau)\mathrm{d}\tau} \text{ and } \Gamma(b) = \mathrm{e}^{-\int_0^b \delta(\tau)\mathrm{d}\tau}. \tag{12}$$

It follows from (ii) and (v) of Assumption 2.1 that

$$0 \leq \Omega(a) \leq \mathrm{e}^{-\mu_0 a}, \quad 0 \leq \Gamma(b) \leq \mathrm{e}^{-\mu_0 b}, \quad a, b \geq 0. \tag{13}$$

It follows that $\Omega'(a) = -\theta(a)\Omega(a)$ and $\Gamma'(b) = -\delta(b)\Gamma(b)$ hold for almost all $a, b \geq 0$.

For $t \geq 0$, let

$$P(t) = \int_0^\infty k(a)i(t,a)\mathrm{d}a, \; Q(t) = \int_0^\infty q(b)p(t,b)\mathrm{d}b, \text{ and } M(t) = \int_0^\infty \xi(a)i(t,a)\mathrm{d}a.$$

Thus (4) and (5) can be rewritten as

$$i(t,a) = \begin{cases} S(t-a)(\beta_1 P(t-a) + \beta_2 Q(t-a))\Omega(a), & \text{for} \quad 0 \leq a \leq t; \\ i_0(a-t)\frac{\Omega(a)}{\Omega(a-t)}, \quad \text{for} & \phantom{\text{for}} \quad 0 \leq t \leq a. \end{cases} \tag{14}$$

and

$$p(t,b) = \begin{cases} M(t-b)\Gamma(b), & \text{for} \quad 0 \leq b \leq t; \\ p_0(b-t)\frac{\Gamma(b)}{\Gamma(b-t)}, & \text{for} \quad 0 \leq t \leq b. \end{cases} \tag{15}$$

It is useful to note that

$$i(t,a) = i(t-a,0)\Omega(a) \quad \text{and} \quad p(t,b) = p(t-b,0)\Gamma(b), \quad \text{for} \quad 0 \le a,b \le t, \quad (16)$$

and the boundary conditions given in (2) can be rewritten as $i(t,0) = \beta_1 S(t)P(t) + \beta_2 S(t)Q(t)$ and $p(t,0) = M(t)$.

Let

$$\tilde{\mu}_0 := \frac{\mu_0}{1+\frac{\bar{\xi}}{\mu_0}} > 0$$

and define the state space for system (1) by

$$\Omega := \left\{ (S(t), i(t,\cdot), p(t,\cdot)) \in \mathcal{Y} : \ S(t) + \int_0^\infty i(t,a)\mathrm{d}a \le \frac{A}{\mu_0}, \ \|\Phi_t(Y_0)\|_{\mathcal{Y}} \le \frac{A}{\tilde{\mu}_0} \right\}. \quad (17)$$

The following proposition holds true:

Proposition 2.2. *Let Φ and Ω be defined by (10) and (17), respectively. Ω is positively invariant for Φ, that is,*

$$\Phi(t, Y_0) \subset \Omega, \quad \forall t \ge 0, \quad Y_0 \in \Omega.$$

Moreover, Φ is point dissipative and Ω attracts all points in $\mathcal{Y}$.

Proof. First we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t(Y_0)\|_{\mathcal{Y}} = \frac{\mathrm{d}S(t)}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty p(t,a)\mathrm{d}a. \quad (18)$$

It follows from equation (14) that

$$\int_0^\infty i(t,a)\mathrm{d}a = \int_0^t S(t-a)(\beta_1 P(t-a)+\beta_2 Q(t-a))\Omega(a)\mathrm{d}a + \int_t^\infty i_0(a-t)\frac{\Omega(a)}{\Omega(a-t)}\mathrm{d}a.$$

We make the substitution $a = t-\sigma$ in the first integral,and $a = t+\tau$ in the second integral, and differentiating by t, yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(\sigma)(\beta_1 P(\sigma) + \beta_2 Q(\sigma))\Omega(t-\sigma)\mathrm{d}\sigma \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i_0(\tau)\frac{\Omega(t+\tau)}{\Omega(\tau)}\mathrm{d}\tau \\
&= S(t)(\beta_1 P(t) + \beta_2 Q(t))\Omega(0) + \int_0^\infty i_0(\tau)\frac{\Omega'(t+\tau)}{\Omega(\tau)}\mathrm{d}\tau \\
&\quad + \int_0^t (S(\sigma)(\beta_1 P(\sigma) + \beta_2 Q(\sigma))\Omega'(t-\sigma)\mathrm{d}\sigma.
\end{aligned}$$

Notes that $\Omega(0) = 1$ and $\Omega'(a) = -\theta(a)\Omega(a)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a = S(t)(\beta_1 P(t) + \beta_2 Q(t)) - \int_0^\infty \theta(a)i(t,a)\mathrm{d}a. \quad (19)$$

Similarly, we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty p(t,b)\mathrm{d}b = \int_0^\infty \xi(a)i(t,a)\mathrm{d}a - \int_0^\infty \delta(b)p(t,b)\mathrm{d}b. \quad (20)$$

Adding the first equation of (1) and (19), we have from (v) of Assumption 2.1 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(S(t)+\int_0^\infty i(t,a)\mathrm{d}a\right) &= A-\mu S(t)-\int_0^\infty \theta(a)i(t,a)\mathrm{d}a\\ &\le A-\mu_0\left(S(t)+\int_0^\infty i(t,a)\mathrm{d}a\right), \quad t\ge 0.\end{aligned}$$

Hence, it follows from the variation of constants formula that

$$S(t)+\int_0^\infty i(t,a)\mathrm{d}a \le \frac{A}{\mu_0}-e^{-\mu_0 t}\left\{\frac{A}{\mu_0}-\left(S(0)+\int_0^\infty i(0,a)\mathrm{d}a\right)\right\}, \quad t\ge 0. \tag{21}$$

This implies that for any solutions of (1) satisfying $Y_0\in\Omega$,

$$S(t)+\int_0^\infty i(t,a)\mathrm{d}a \le \frac{A}{\mu_0}, \quad t\ge 0. \tag{22}$$

Then, it follows from (20), (22) and (ii) and (v) of Assumption 2.1 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty p(t,b)\mathrm{d}b &\le \bar{\xi}\int_0^\infty i(t,a)\mathrm{d}a-\mu_0\int_0^\infty p(t,b)\mathrm{d}b\\ &\le \bar{\xi}\frac{A}{\mu_0}-\mu_0\int_0^\infty p(t,b)\mathrm{d}b, \quad t\ge 0.\end{aligned}$$

Hence, it follows from the variation of constants formula that

$$\int_0^\infty p(t,b)\mathrm{d}b \le \frac{\bar{\xi}}{\mu_0}\frac{A}{\mu_0}-e^{-\mu_0 t}\left\{\frac{\bar{\xi}}{\mu_0}\frac{A}{\mu_0}-\int_0^\infty p(0,a)\mathrm{d}a\right\}, \quad t\ge 0. \tag{23}$$

Adding (21) and (23), we have

$$\begin{aligned}\|\Phi_t(Y_0)\|_{\mathcal{Y}} &\le \left(1+\frac{\bar{\xi}}{\mu_0}\right)\frac{A}{\mu_0}-e^{-\mu_0 t}\left\{\left(1+\frac{\bar{\xi}}{\mu_0}\right)\frac{A}{\mu_0}-\|Y_0\|_{\mathcal{Y}}\right\}\\ &= \frac{A}{\tilde{\mu}_0}-e^{-\mu_0 t}\left\{\frac{A}{\tilde{\mu}_0}-\|Y_0\|_{\mathcal{Y}}\right\}, \quad t\ge 0. \qquad (24)\end{aligned}$$

From (22) and (24) it follows that for any solutions of (1) satisfying $Y_0\in\Omega$, $\Phi_t(Y_0)\in\Omega$ for all $t\ge 0$. This implies the positive invariance of set Ω for semi-flow Φ.

Moreover, it follows from (21) and (23) that $\limsup_{t\to\infty}\{S(t)+\|i(t,\cdot)\|_{L^1}\}\le\frac{A}{\mu_0}$ and $\limsup_{t\to\infty}\|\Phi_t(Y_0)\|_{\mathcal{Y}}\le\frac{A}{\tilde{\mu}_0}$ for any $Y_0\in\mathcal{Y}$. Therefore, Φ is point dissipative and Ω attracts all points in $\mathcal{Y}$. This completes the proof. $\square$

Recall that (v) of Assumption 2.1, we have the following proposition, which is direct consequences of Proposition 2.2.

Proposition 2.3. *If $Y_0\in\mathcal{Y}$ and $\|Y_0\|_{\mathcal{Y}}\le C$ for some $C\ge\frac{A}{\tilde{\mu}_0}$, then the following statements hold true for all $t\ge 0$:*

(1) $S(t), \int_0^\infty i(t,a)da, \int_0^\infty p(t,b)db\le C$;
(2) $P(t)\le\bar{k}C$, $Q(t)\le\bar{q}C$ and $M(t)\le\bar{\xi}C$;
(3) $i(t,0)\le\bar{\beta}C^2$ and $p(t,0)\le\bar{\xi}C$, where $\bar{\beta}=\beta_1\bar{k}+\beta_2\bar{q}$.

As presented in (iii) of Assumption 2.1, it is assumed that the coefficient functions $k(\cdot)$, $q(\cdot)$ and $\xi(\cdot)$ be Lipschitz continuous. This allows the initial conditions for i and p to be taken in $L^1_+(0,\infty)$. Then, the functions $P(t), Q(t)$ and $M(t)$, related to the boundary conditions $i(t,0)$ and $p(t,0)$, can be shown to be Lipschitz continuous.

Proposition 2.4. *The functions $P(t), Q(t)$ and $M(t)$ are Lipschitz continuous on $\mathbb{R}_+$.*

Proof. Let $C \geq \max\{\frac{A}{\bar{\mu}_0}, \|Y_0\|_{\mathcal{Y}}\}$. It follows from Proposition 2.2 that $\|\Phi_t(Y_0)\|_{\mathcal{Y}} \leq C$ for all $t \geq 0$.

Let $t \geq 0$ and $h > 0$. We can check that

$$
\begin{aligned}
P(t+h) - P(t) &= \int_0^\infty k(a)i(t+h,a)\mathrm{d}a - \int_0^\infty k(a)i(t,a)\mathrm{d}a \\
&= \int_0^h k(a)i(t+h,a)\mathrm{d}a + \int_h^\infty k(a)i(t+h,a)\mathrm{d}a - \int_0^\infty k(a)i(t,a)\mathrm{d}a \\
&= \int_0^h k(a)i(t+h-a,0)\Omega(a)\mathrm{d}a + \int_h^\infty k(a)i(t+h,a)\mathrm{d}a \\
&\quad - \int_0^\infty k(a)i(t,a)\mathrm{d}a.
\end{aligned} \tag{25}
$$

By applying $k(a) \leq \bar{k}$, $i(t+h-a,0) \leq \bar{\beta}C^2$ and $\Omega(a) \leq 1$ for the first integral, and making the substitution $\sigma = a - h$ for the second integral to (25), we get

$$
P(t+h) - P(t) \leq \bar{k}\bar{\beta}C^2 h + \int_0^\infty k(\sigma+h)i(t+h,\sigma+h)\mathrm{d}\sigma - \int_0^\infty k(a)i(t,a)\mathrm{d}a.
$$

It follows from (16) that

$$
i(t+h,\sigma+h) = i(t,\sigma)\frac{\Omega(\sigma+h)}{\Omega(\sigma)}.
$$

Thus,

$$
\begin{aligned}
P(t+h) - P(t) &\leq \bar{k}\bar{\beta}C^2 h + \int_0^\infty \left(k(a+h)\frac{\Omega(a+h)}{\Omega(a)} - k(a)\right) i(t,a)\mathrm{d}a \\
&= \bar{k}\bar{\beta}C^2 h + \int_0^\infty \left(k(a+h)e^{-\int_a^{a+h}\theta(\tau)\mathrm{d}\tau} - k(a)\right) i(t,a)\mathrm{d}a \\
&= \bar{k}\bar{\beta}C^2 h + \int_0^\infty k(a+h)\left(e^{-\int_a^{a+h}\theta(\tau)\mathrm{d}\tau} - 1\right) i(t,a)\mathrm{d}a \\
&\quad + \int_0^\infty (k(a+h) - k(a))i(t,a)\mathrm{d}a.
\end{aligned} \tag{26}
$$

From (ii) of Assumption 2.1, it is easy to check that $-\bar{\theta}h \leq -\int_a^{a+h}\theta(\tau)\mathrm{d}\tau \leq 0$. It follows that $1 \geq e^{-\int_a^{a+h}\theta(\tau)\mathrm{d}\tau} \geq e^{-\bar{\theta}h} \geq 1 - \bar{\theta}h$. Therefore,

$$
0 \leq k(a+h)|e^{-\int_a^{a+h}\theta(\tau)\mathrm{d}\tau} - 1| \leq \bar{k}\bar{\theta}h.
$$

Recall that $\int_0^\infty i(t,a)\mathrm{d}a \leq \|\Phi_t(Y_0)\|_{\mathcal{Y}} \leq C$. By the Lipschitz continuous on $\mathbb{R}_+$ of function $k(\cdot)$ with Lipschitz coefficients M_k (see (iii) of Assumption 2.1), we have $\int_0^\infty (k(a+h) - k(a))i(t,a)\mathrm{d}a \leq M_k hC$. Combining the above relations together, we obtain

$$
P(t+h) - P(t) \leq \bar{k}\bar{\beta}C^2 h + \bar{k}\bar{\theta}Ch + M_k Ch, \tag{27}
$$

it follows that $P(t)$ is Lipschitz continuous with coefficient $M_P = (\bar{k}\bar{\beta}C + \bar{k}\bar{\theta} + M_k)C$. The proof of Lipschitz continuous of $Q(t)$ and Mt is similar to that of $P(t)$. Furthermore, $Q(t)$ is Lipschitz continuous with coefficient $M_Q = \bar{q}\bar{\delta}C + \bar{q}\bar{\xi}C + M_q C$, and $M(t)$ is Lipschitz continuous with coefficient $M_M = (\bar{\xi}\bar{\beta}C + \bar{\xi}\bar{\theta} + M_k)C$. The proof is completed. $\square$

The following Proposition will be used in next section, which come from [5].

Proposition 2.5. *Let $D \subseteq \mathbb{R}$. For $j = 1.2$, suppose $f_j : D \to \mathbb{R}$ is a bounded Lipschitz continuous function with bound K_j and Lipschitz coefficient M_j. Then the product function $f_1 f_2$ is Lipschitz with coefficient $K_1M_2 + K_2M_1$.*

3. **Relative compactness of the orbit.** In [1], the proof of the global stability of each equilibrium utilized Lyapunov functional technique combined with the invariance principle. Since we are now concerned with the infinite dimensional Banach space $\mathcal{Y}$ including $L^1(0,\infty)$, according to [8, Theorem 4.2 of Chapter IV], we have to show first the relative compactness of the orbit $\{\Phi(t,Y_0) : \ t \geq 0\}$ in $\mathcal{Y}$ in order to make use of the invariance principle. To this end, we first decompose $\Phi : \ \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}$ into the following two operators $\Theta, \Psi : \ \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}$:

$$\Theta\,(t,Y_0) := (0, \tilde{\varphi}_i(t,\cdot), \tilde{\varphi}_p(t,\cdot))\,, \tag{28}$$

$$\Psi\,(t,Y_0) := \left(S(t), \tilde{i}(t,\cdot), \tilde{p}(t,\cdot)\right), \tag{29}$$

where

$$\tilde{\varphi}_i\,(t,a) := \begin{cases} 0, & t > a \geq 0; \\ i\,(t,a)\,, & a \geq t \geq 0. \end{cases} \quad \text{and} \quad \tilde{i}\,(t,a) := \begin{cases} i\,(t,a)\,, & t > a \geq 0; \\ 0, & a \geq t \geq 0. \end{cases} \tag{30}$$

$$\tilde{\varphi}_p\,(t,b) := \begin{cases} 0, & t > b \geq 0; \\ p\,(t,b)\,, & b \geq t \geq 0. \end{cases} \quad \text{and} \quad \tilde{p}\,(t,b) := \begin{cases} p\,(t,b)\,, & t > b \geq 0; \\ 0, & b \geq t \geq 0. \end{cases} \tag{31}$$

Then we have $\Phi\,(t,Y_0) = \Theta\,(t,Y_0) + \Psi\,(t,Y_0)\,, \ \forall t \geq 0$. Note that $\tilde{i}(t,a)$ and $\tilde{p}(t,b)$ can be written as

$$\tilde{i}\,(t,a) := \begin{cases} (\beta_1 P(t-a) + \beta_2 Q(t-a))S(t-a)\Omega(a), & t > a \geq 0; \\ 0, & a \geq t \geq 0. \end{cases} \tag{32}$$

and

$$\tilde{p}\,(t,b) := \begin{cases} (M(t-b)\Gamma(b), & t > b \geq 0; \\ 0, & b \geq t \geq 0. \end{cases} \tag{33}$$

Following the line of [12, Proposition 3.13], we are now in the position to state and prove the following main Theorem of this section.

Theorem 3.1. *Let Φ, Ω, Θ and Ψ be defined by (10), (17), (28) and (29), respectively. If the following two conditions hold, then $\{\Phi\,(t,Y_0) : \ t \geq 0\}$ for $Y_0 \in \Omega$ has compact closure in $\mathcal{Y}$.*

- *(i) There exists a function $\Delta : \ \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $r > 0$, $\lim_{t\to\infty} \Delta\,(t,r) = 0$, and if $Y_0 \in \Omega$ with $\|Y_0\|_{\mathcal{Y}} \leq r$, then $\|\Theta\,(t,Y_0)\|_{\mathcal{Y}} \leq \Delta\,(t,r)$ for $t \geq 0$;*
- *(ii) For $t \geq 0$, $\Psi\,(t,\cdot)$ maps any bounded sets of Ω into sets with compact closure in $\mathcal{Y}$.*

To show that the conditions (i) and (ii) in Theorem 3.1 hold, we first prove the following lemma.

Lemma 3.2. *Let Ω and Θ be defined by (17) and (28), respectively. For $r > 0$, let $\Delta\,(t,r) := e^{-\mu_0 t} r$. Then, $\lim_{t\to\infty} \Delta\,(t,r) = 0$ and for $t \geq 0$, $\|\Theta\,(t,Y_0)\|_{\mathcal{Y}} \leq \Delta\,(t,r)$ provided $Y_0 \in \Omega$ with $\|Y_0\|_{\mathcal{Y}} \leq r$.*

Proof. It is obvious that $\lim_{t\to\infty} \Delta(t,r) = 0$. From (14) and (15), we have

$$\tilde{\varphi}_i(t,a) = \begin{cases} 0, & t > a \geq 0; \\ i_0(a-t)\frac{\Omega(a)}{\Omega(a-t)}, & a \geq t \geq 0. \end{cases}$$

and

$$\tilde{\varphi}_p(t,b) = \begin{cases} 0, & t > b \geq 0; \\ p_0(b-t)\frac{\Gamma(b)}{\Gamma(b-t)}, & b \geq t \geq 0. \end{cases}$$

Then, for $Y_0 \in \Omega$ satisfying $\|Y_0\|_{\mathcal{Y}} \leq r$, we have

$$\begin{aligned}
\|\Theta(t,Y_0)\|_{\mathcal{Y}} &= |0| + \int_0^\infty |\tilde{\varphi}_i(t,a)|\,\mathrm{d}a + \int_0^\infty |\tilde{\varphi}_p(t,b)|\,\mathrm{d}b \\
&= \int_t^\infty \left| i_0(a-t)\frac{\Omega(a)}{\Omega(a-t)}\right| \mathrm{d}a + \int_t^\infty \left| p_0(b-t)\frac{\Gamma(b)}{\Gamma(b-t)}\right| \mathrm{d}b \\
&= \int_0^\infty \left| i_0(\sigma)\frac{\Omega(\sigma+t)}{\Omega(\sigma)}\right| \mathrm{d}\sigma + \int_0^\infty \left| p_0(\sigma)\frac{\Gamma(\sigma+t)}{\Gamma(\sigma)}\right| \mathrm{d}\sigma \\
&= \int_0^\infty \left| i_0(\sigma)e^{-\int_\sigma^{\sigma+t}\theta(\tau)\mathrm{d}\tau}\right| \mathrm{d}\sigma + \int_0^\infty \left| p_0(\sigma)e^{-\int_\sigma^{\sigma+t}\delta(\tau)\mathrm{d}\tau}\right| \mathrm{d}\sigma \\
&\leq e^{-\mu_0 t}\int_0^\infty |i_0(\sigma)|\,\mathrm{d}\sigma + e^{-\mu_0 t}\int_0^\infty |p_0(\sigma)|\,\mathrm{d}\sigma \\
&\leq e^{-\mu_0 t}\|Y_0\|_{\mathcal{Y}} \leq e^{-\mu_0 t} r = \Delta(t,r),
\end{aligned}$$

which completes the proof. □

We next prove the following Lemma, which is based on Theorem B.2 from [7].

Lemma 3.3. *Let Ω and Ψ be defined by (17) and (29), respectively. Then, for $t \geq 0$, $\Psi(t,\cdot)$ maps any bounded sets of Ω into sets with compact closure in $\mathcal{Y}$.*

Proof. From Proposition 2.2, it is easily seen that $S(t)$ remains in the compact set $[0, A/\tilde{\mu}_0] \subset [0, C]$. Thus, we only have to show that $\tilde{i}(t,a)$ and $\tilde{p}(t,b)$ remain in a precompact subset of $L^1_+(0,\infty)$, which is independent of $Y_0 \in \Omega$. To this end, it suffices to verify the following conditions (see e.g., [7, Theorem B.2]). We just verify that the following conditions valid for $\tilde{i}(t,a)$, and the conditions can be similarly verified for $\tilde{p}(t,b)$.

(i) The supremum of $\int_0^\infty \tilde{i}(t,a)\,\mathrm{d}a$ with respect to $Y_0 \in \Omega$ is finite;
(ii) $\lim_{h\to\infty}\int_h^\infty \tilde{i}(t,a)\,\mathrm{d}a = 0$ uniformly with respect to $Y_0 \in \Omega$;
(iii) $\lim_{h\to 0+}\int_0^\infty \left|\tilde{i}(t,a+h) - \tilde{i}(t,a)\right| \mathrm{d}a = 0$ uniformly with respect to $Y_0 \in \Omega$;
(iv) $\lim_{h\to 0+}\int_0^h \tilde{i}(t,a)\,\mathrm{d}a = 0$ uniformly with respect to $Y_0 \in \Omega$.

Now, we have from (14) and (32) that

$$0 \leq \tilde{i}(t,a) = \begin{cases} (\beta_1 S(t-a)P(t-a) + \beta_2 S(t-a)Q(t-a))\Omega(a), & 0 \leq a \leq t; \\ 0, & 0 \leq t < a. \end{cases}$$

Then, combining Proposition 2.3 and (13), we have

$$\tilde{i}(t,a) \leq (\beta_1\bar{k} + \beta_2\bar{p})C^2 e^{-\mu_0 a}, \tag{34}$$

from which aforementioned conditions (i),(ii) and (iv) follow directly from (34).

Next, we verify that condition (iii) holds. For sufficiently small $h \in (0,t)$, we have

$$\begin{aligned}
&\int_0^\infty \left|\tilde{i}(t,a+h)-\tilde{i}(t,a)\right| \mathrm{d}a \\
=& \int_0^{t-h} |(\beta_1 S(t-a-h)P(t-a-h)+\beta_2 S(t-a-h)Q(t-a-h))\Omega(a+h) \\
&-(\beta_1 S(t-a)P(t-a)+\beta_2 S(t-a)Q(t-a))\Omega(a)|\mathrm{d}a \\
&+\int_{t-h}^{t} |0-(\beta_1 S(t-a)P(t-a)+\beta_2 S(t-a)Q(t-a))\Omega(a)|\mathrm{d}a \\
\le& \int_0^{t-h} |(\beta_1 S(t-a-h)P(t-a-h)+\beta_2 S(t-a-h)Q(t-a-h))\Omega(a+h) \\
&-(\beta_1 S(t-a)P(t-a)+\beta_2 S(t-a)Q(t-a))\Omega(a)|\mathrm{d}a+\bar{\beta}C^2 h \\
\le& \bar{\beta}C^2 h+\Delta+\Xi,
\end{aligned} \tag{35}$$

where

$$\Delta=\int_0^{t-h}(\beta_1 S(t-a-h)P(t-a-h)+\beta_2 S(t-a-h)Q(t-a-h))|\Omega(a+h)-\Omega(a)|\mathrm{d}a$$

and

$$\begin{aligned}
\Xi=& \int_0^{t-h} |(\beta_1 S(t-a-h)P(t-a-h)+\beta_2 S(t-a-h)Q(t-a-h)) \\
&-(\beta_1 S(t-a)P(t-a)+\beta_2 S(t-a)Q(t-a))|\Omega(a)\mathrm{d}a \\
\le& \int_0^{t-h} |(\beta_1 S(t-a-h)P(t-a-h)-(\beta_1 S(t-a)P(t-a))|\Omega(a)\mathrm{d}a \\
&+\int_0^{t-h} |(\beta_2 S(t-a-h)Q(t-a-h)-\beta_2 S(t-a)Q(t-a))|\Omega(a)\mathrm{d}a := \Xi_1.
\end{aligned}$$

Recall that $0 \le \Omega(a)=\mathrm{e}^{-\int_0^a \theta(\tau)\mathrm{d}\tau} \le \mathrm{e}^{-\mu_0 a}$, and $\Omega(a)$ is non-increasing function with respect to a, we have

$$\begin{aligned}
\int_0^{t-h}|\Omega(a+h)-\Omega(a)|\mathrm{d}a &= \int_0^{t-h}(\Omega(a)-\Omega(a+h))\mathrm{d}a \\
&= \int_0^{t-h}\Omega(a)\mathrm{d}a-\int_0^{t-h}\Omega(a+h)\mathrm{d}a \\
&= \int_0^{t-h}\Omega(a)\mathrm{d}a-\int_h^{t}\Omega(a)\mathrm{d}a \\
&= \int_0^{t-h}\Omega(a)\mathrm{d}a-\int_h^{t-h}\Omega(a)\mathrm{d}a-\int_{t-h}^{t}\Omega(a)\mathrm{d}a \\
&= \int_0^{h}\Omega(a)\mathrm{d}a-\int_{t-h}^{t}\Omega(a)\mathrm{d}a \le h.
\end{aligned}$$

Hence, combing above with (35) yields

$$\int_0^\infty \left|\tilde{i}(t,a+h)-\tilde{i}(t,a)\right| \mathrm{d}a \le 2\bar{\beta}C^2 h+\Xi_1.$$

For Ξ_1, combining Proposition 2.3 with the expression for $\frac{\mathrm{dS(t)}}{\mathrm{d}t}$, we find that $|\frac{\mathrm{dS(t)}}{\mathrm{d}t}|$ is bounded by $M_S = A + \mu C + \beta_1 \bar{k} C^2 + \beta_1 \bar{q} C^2$, and therefore $S(\cdot)$ is Lipschitz on $[0, \infty)$ with coefficient M_S. By Proposition 2.5, there exists two Lipschitz coefficients M_P, M_Q for P, Q respectively. Thus, $S(\cdot)P(\cdot)$ and $S(\cdot)Q(\cdot)$ is Lipschitz on $[0, \infty)$ with coefficient $M_{SP} = CM_P + \bar{q}CM_S$ and $M_{SQ} = CM_Q + \bar{q}CM_S$. Denote that $M = \beta_1 M_{SP} + \beta_2 M_{SQ}$. Thus,

$$\Xi_1 \le Mh \int_0^{t-h} e^{-\mu_0 a} \mathrm{d}a \le \frac{Mh}{\mu_0}.$$

Finally, we get

$$\int_0^\infty |\tilde{i}(t, a+h) - \tilde{i}(t, a)| \mathrm{d}a \le \left(2\bar{\beta} C^2 + \frac{Mh}{\mu_0} \right) h,$$

which converges to 0 as $h \to 0_+$. Let $C_0 \subset \mathcal{Y}$ be a bounded closed set and $C > A/\mu_0$ be a bound for C_0. We note that M depends on C, which depends on the set C_0, but not on Y_0. Therefore, this inequality holds for any $Y_0 \in C_0$. Thus, $\tilde{i}$ remains in a pre-compact subset C^i of L^1_+. Similarly, $\tilde{p}$ remains in a pre-compact subset C^p of L^1_+. Thus, $\Psi(t, C_0) \subseteq [0, C] \times C^i \times C^p$, which has compact closure in $\mathcal{Y}$. This completes the proof. □

From Proposition 2.2, Lemmas 3.2 and 3.3, we have the following theorem for semi-flow $\{\Phi(t)\}_{t\ge 0}$ as a consequence of the results on the existence of global attractors in Hale [2] and Smith and Thieme [7].

Theorem 3.4. *The semi-flow $\{\Phi(t)\}_{t\ge0}$ has a global attractor $\mathcal{A}$ in $\mathcal{Y}$, which attracts the bound sets of $\mathcal{Y}$.*

4. **Uniform persistence.** In this section we show the uniform persistence of system (1). Let $\hat{i}(t) := i(t, 0)$, and $\hat{p}(t) := p(t, 0)$. Then (4) and (5) can be rewritten as

$$i(t, a) = \begin{cases} \hat{i}(t-a)\Omega(a), & t \ge a \ge 0; \\ i_0(a-t)\frac{\Omega(a)}{\Omega(a-t)}, & a \ge t \ge 0, \end{cases} \tag{36}$$

and

$$p(t, b) = \begin{cases} \hat{p}(t-b)\Gamma(b), & t \ge b \ge 0; \\ p_0(b-t)\frac{\Gamma(b)}{\Gamma(b-t)}, & b \ge t \ge 0, \end{cases} \tag{37}$$

where $\Omega(a)$ and $\Gamma(b)$ are defined by (12).

Substituting (36) and (37) into the boundary condition (2), we obtain the following system of integral equations of $\hat{i}(t)$ and $\hat{p}(t)$:

$$\begin{aligned} \hat{i}(t) \quad &= \quad \beta_1 S(t) \left\{ \int_0^t k(a)\Omega(a)\hat{i}(t-a)\mathrm{d}a + \int_t^\infty k(a)\frac{\Omega(a)}{\Omega(a-t)} i_0(a-t)da \right\} \\ &\quad + \beta_2 S(t) \left\{ \int_0^t q(b)\Gamma(b)\hat{p}(t-b)\mathrm{d}b + \int_t^\infty \frac{\Gamma(b)}{\Gamma(b-t)} p_0(b-t)\mathrm{d}b \right\}, \end{aligned} \tag{38}$$

$$\hat{p}(t) \quad = \quad \int_0^t \xi(a)\Omega(a)\hat{i}(t-a)\mathrm{d}a + \int_t^\infty \xi(a)\frac{\Omega(a)}{\Omega(a-t)} i_0(a-t)\mathrm{d}a. \tag{39}$$

In addition, note that the first equation of (1) can be rewritten as

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = A - \mu S(t) - \hat{i}(t), \tag{40}$$

so we have the following result.

Lemma 4.1. *If $\Re_0 > 1$, then there exists a positive constant $\epsilon > 0$ such that*

$$\limsup_{t\to\infty} \hat{i}(t) > \epsilon. \tag{41}$$

Proof. From (38)-(39) and the positivity of coefficients, we obtain inequalities

$$\hat{i}(t) \geq \beta_1 S(t) \int_0^t k(a)\Omega(a)\hat{i}(t-a)\mathrm{d}a + \beta_2 S(t) \int_0^t q(b)\Gamma(b)\hat{p}(t-b)\mathrm{d}b \tag{42}$$

and

$$\hat{p}(t) \geq \int_0^t \xi(a)\Omega(a)\hat{i}(t-a)\mathrm{d}a. \tag{43}$$

Combining (42) and (43), we obtain the following integral inequality of $\hat{i}(t)$.

$$\begin{aligned} \hat{i}(t) \quad \geq \quad & \beta_1 S(t) \int_0^t k(a)\Omega(a)\hat{i}(t-a)\mathrm{d}a \\ & +\beta_2 S(t) \int_0^t q(b)\Gamma(b) \int_0^{t-b} \xi(c)\Omega(c)\hat{i}(t-b-c)\mathrm{d}c\mathrm{d}b. \end{aligned} \tag{44}$$

In what follows, we prove that for the solution $\hat{i}(t)$ satisfying (44), there exists a positive constant $\epsilon > 0$ such that (41) holds.

Now it follows from (7)-(9) and (12) that if $\Re_0 > 1$, then there exists a sufficiently small $\epsilon > 0$ such that

$$\beta_1 \frac{A-\epsilon}{\mu} \int_0^\infty k(a)\Omega(a)\mathrm{d}a + \beta_2 \frac{A-\epsilon}{\mu} \int_0^\infty q(b)\Gamma(b)\mathrm{d}b \int_0^\infty \xi(c)\Omega(c)\mathrm{d}c \ > \ 1. \tag{45}$$

For such ϵ, we show that (41) holds. Suppose for the contrary, if there exists a sufficiently large constant $T > 0$ such that

$$\hat{i}(t) \leq \epsilon \quad \text{for all } t \geq T.$$

Then, it follows from (40) that

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} \geq A - \mu S(t) - \epsilon \quad \text{for all } t \geq T.$$

Performing the variation of constants formula, we have

$$S(t) \geq \frac{A-\epsilon}{\mu} \quad \text{for all } t \geq T.$$

Then, (44) becomes

$$\begin{aligned} \hat{i}(t) \quad \geq \quad & \beta_1 \frac{A-\epsilon}{\mu} \int_0^t k(a)\Omega(a)\hat{i}(t-a)\mathrm{d}a \\ & +\beta_2 \frac{A-\epsilon}{\mu} \int_0^t q(b)\Gamma(b) \int_0^{t-b} \xi(c)\Omega(c)\hat{i}(t-b-c)\mathrm{d}c\mathrm{d}b \end{aligned} \tag{46}$$

for all $t \geq T$. Now, without loss of generality, we can perform the time-shift of system (1) with respect to T. That is, replacing the initial condition of system (1) by $Y_1 := \Phi(T, Y_0)$, we can consider the long time behavior of the system. Then, (46) holds for $t \geq 0$ and by taking the Laplace transform of both sides, we have

$$\begin{aligned} \mathcal{L}[\hat{i}] \quad \geq \quad & \beta_1 \frac{A-\epsilon}{\mu} \int_0^\infty k(a)\Omega(a)e^{-\lambda a}\mathrm{d}a \ \mathcal{L}[\hat{i}] \\ & +\beta_2 \frac{A-\epsilon}{\mu} \int_0^\infty q(b)\Gamma(b)e^{-\lambda b}\mathrm{d}b \int_0^\infty \xi(c)\Omega(c)e^{-\lambda c}\mathrm{d}c \ \mathcal{L}[\hat{i}], \end{aligned}$$

where $\mathcal{L}[\hat{i}]$ denotes the Laplace transform of $\hat{i}$, which is strictly positive because of (38) and Assumption 2.1. Dividing both sides by $\mathcal{L}[\hat{i}]$ and letting $\lambda \to 0$, we obtain inequality

$$\begin{aligned} 1 \geq\ & \beta_1 \frac{A-\epsilon}{\mu} \int_0^\infty k(a)\Omega(a)\mathrm{d}a \\ & + \beta_2 \frac{A-\epsilon}{\mu} \int_0^\infty q(b)\Gamma(b)\mathrm{d}b \int_0^\infty \xi(c)\Omega(c)\mathrm{d}c, \end{aligned}$$

which contradicts to (45). □

Next, in order to apply a technique used in Smith and Thieme [7, Chapter 9] (see also McCluskey [5, Section 8]), we consider a total Φ-trajectory of system (1) in space $\mathcal{Y}$, where Φ is a continuous semi-flow defined by (10). Let $\phi : \mathbb{R} \to \mathcal{Y}$ be a total Φ-trajectory such that $\phi(r) := (S(r), i(r,\cdot), p(r,\cdot))$, $r \in \mathbb{R}$. Then, it follows that $\phi(r+t) = \Phi(t, \phi(r))$, $t \geq 0$, $r \in \mathbb{R}$ and

$$\begin{aligned} i(r,a) &= i(r-a,0)\,\Omega(a) = \hat{i}(r-a)\Omega(a), \\ p(r,a) &= p(r-b,0)\,\Omega(b) = \hat{p}(r-b)\Omega(b), \quad r \in \mathbb{R}, \quad a,b \geq 0. \end{aligned}$$

Hence, from (38)-(40), we have

$$\begin{aligned} \frac{\mathrm{d}S(r)}{\mathrm{d}r} &= A - \mu S(r) - \hat{i}(r), \\ \hat{i}(r) &= \beta_1 S(r) \int_0^\infty k(a)\Omega(a)\hat{i}(r-a)\mathrm{d}a + \beta_2 S(r) \int_0^\infty q(b)\Gamma(b)\hat{p}(r-b)\mathrm{d}b, \\ \hat{p}(r) &= \int_0^\infty \xi(a)\Omega(a)\hat{i}(r-a)\mathrm{d}a, \quad r \in \mathbb{R}. \end{aligned} \tag{47}$$

Substituting the third equation into the second equation, we obtain the following integral equation of $\hat{i}$.

$$\begin{aligned} \hat{i}(r) =\ & \beta_1 S(r) \int_0^\infty k(a)\Omega(a)\hat{i}(r-a)\mathrm{d}a \\ & + \beta_2 S(r) \int_0^\infty q(b)\Gamma(b) \int_0^\infty \xi(c)\Omega(c)\hat{i}(r-b-c)\mathrm{d}c\mathrm{d}b, \quad r \in \mathbb{R}. \end{aligned} \tag{48}$$

We prove the following lemma.

Lemma 4.2. *For total Φ-trajectory ϕ in $\mathcal{Y}$, $S(r)$ is strictly positive on $\mathbb{R}$ and $\hat{i}(r) = 0$ for all $r \geq 0$ if $\hat{i}(r) = 0$ for all $r \leq 0$.*

Proof. Suppose that $S(r^*) = 0$ for a number $r^* \in \mathbb{R}$ and show a contradiction. In this case, it follows from the first equation of (47) and (48) that $\mathrm{d}S(r^*)/\mathrm{d}r = A > 0$. This implies that $S(r^* - \eta) < 0$ for sufficiently small $\eta > 0$ and it contradicts to the fact that the total Φ-trajectory ϕ remains in $\mathcal{Y}$. Consequently, $S(r)$ is strictly positive on $\mathbb{R}$.

By changing the variables, we can rewrite (48) as follows.

$$\begin{aligned} \hat{i}(r) =\ & \beta_1 S(r) \int_{-\infty}^r k(r-a)\Omega(r-a)\hat{i}(a)\mathrm{d}a \\ & + \beta_2 S(r) \int_{-\infty}^r q(r-b)\Gamma(r-b) \int_{-\infty}^b \xi(b-c)\Omega(b-c)\hat{i}(c)\mathrm{d}c\mathrm{d}b, \quad r \in \mathbb{R}. \end{aligned}$$

Hence, if $\hat{i}(r) = 0$ for all $r \leq 0$, then we have

$$\hat{i}(r) \leq \beta_1 S^0 \bar{k} \int_0^r \hat{i}(a)\mathrm{d}a + \beta_2 S^0 \bar{q}\bar{\xi} \int_0^r \int_0^b \hat{i}(c)\mathrm{d}c\mathrm{d}b, \quad r \geq 0.$$

This is a Gronwall-like inequality and it follows that $\hat{i}(r) = 0$ for all $r \geq 0$. In fact, let

$$\hat{I}(r) := \int_0^r \hat{i}(a)\mathrm{d}a + \int_0^r \int_0^b \hat{i}(c)\mathrm{d}c\mathrm{d}b, \quad r \geq 0.$$

Then,

$$\begin{aligned}
\frac{d\hat{I}(r)}{dr} &= \hat{i}(r) + \int_0^r \hat{i}(c)\mathrm{d}c \\
&\leq \beta_1 S^0 \bar{k} \int_0^r \hat{i}(a)\mathrm{d}a + \beta_2 S^0 \bar{q}\bar{\xi} \int_0^r \int_0^b \hat{i}(c)\mathrm{d}c\mathrm{d}b + \int_0^r \hat{i}(a)\mathrm{d}a \\
&\leq \max\left(\beta_1 S^0 \bar{k} + 1,\ \beta_2 S^0 \bar{q}\bar{\xi}\right) \hat{I}(r), \quad r \geq 0
\end{aligned}$$

and hence,

$$\hat{I}(r) \leq \hat{I}(0) e^{\max\left(\beta_1 S^0 \bar{k}+1,\ \beta_2 S^0 \bar{q}\bar{\xi}\right) r} = 0, \quad r \geq 0.$$

This implies that $\hat{i}(r) = 0$ for all $r \geq 0$. □

The total Φ-trajectory ϕ enjoys the following nice properties:

Lemma 4.3. *For total Φ-trajectory ϕ in $\mathcal{Y}$, $\hat{i}(r)$ is strictly positive or identically zero on $\mathbb{R}$.*

Proof. From the second statement of Lemma 4.2, by performing appropriate shifts, we see that $\hat{i}(r) = 0$ for all $r \geq r^*$ if $\hat{i}(r) = 0$ for all $r \leq r^*$, where $r^* \in \mathbb{R}$ is arbitrary. This implies that either $\hat{i}(r)$ is identically zero on $\mathbb{R}$ or there exists a decreasing sequence $\{r_j\}_{j=1}^{\infty}$ such that $r_j \to -\infty$ as $j \to \infty$ and $\hat{i}(r_j) > 0$. In the latter case, letting $\hat{i}_j(r) := \hat{i}(r + r_j)$, $r \in \mathbb{R}$, we have from (48) that

$$\hat{i}_j(r) \geq \beta_1 \underline{S} \int_0^r k(a)\Omega(a)\hat{i}(r-a)\mathrm{d}a + \hat{j}_j(r), \quad r \in \mathbb{R},$$

where $\underline{S} := \inf_{r\in\mathbb{R}} S(r) > 0$ and

$$\begin{aligned}
\hat{j}_j(r) \;:=\; & \beta_1 S(r+r_j) \int_r^{\infty} k(a)\Omega(a)\hat{i}_j(r-a)\mathrm{d}a \\
& + \beta_2 S(r+r_j) \int_0^{\infty} q(b)\Gamma(b) \int_0^{\infty} \xi(c)\Omega(c)\hat{i}_j(r-b-c)\mathrm{d}c\mathrm{d}b, \quad r \in \mathbb{R}.
\end{aligned}$$

Then, since $\hat{j}_j(0) = \hat{i}(r_j) > 0$ and $\hat{j}_j(r)$ is continuous at 0, it follows from Corollary B.6 of Smith and Thieme [7] that there exists a number $r^* > 0$, which depends only on $\beta_1 \underline{S} k(a)\Omega(a)$, such that $\hat{i}_j(r) > 0$ for all $r > r^*$. From the definition of $\hat{i}_j$, this implies that $\hat{i}(r) > 0$ for all $r > r^* + r_j$. Since $r_j \to -\infty$ as $j \to \infty$, we obtain that $\hat{i}(r) > 0$ for all $r \in \mathbb{R}$ by letting $j \to \infty$. Consequently, $\hat{i}(r)$ is strictly positive on $\mathbb{R}$. □

Now, let us define a function $\rho : \mathcal{Y} \to \mathbb{R}_+$ on $\mathcal{Y}$ by

$$\rho(x, \varphi, \psi) := \beta_1 x \int_0^{\infty} k(a)\varphi(a)\mathrm{d}a + \beta_2 x \int_0^{\infty} q(b)\psi(b)\mathrm{d}b, \quad (x, \varphi, \psi) \in \mathcal{Y}.$$

Then, it follows from the previous argument that

$$\rho\left(\Phi_t(Y_0)\right) = \hat{i}(t).$$

Then, Lemma 4.1 implies the uniform weak ρ-persistence of semi-flow Φ for $\Re_0 > 1$. Moreover, from Theorem 3.4 and Lemmas 4.2-4.3 and the Lipschitz continuity of $\hat{i}$ (which immediately follows from Proposition 2.4), we can apply Theorem 5.2 of Smith and Thieme [7] to conclude that the uniform weak ρ-persistence of semi-flow Φ implies the uniform (strong) ρ-persistence. In conclusion, we obtain the following theorem.

Theorem 4.4. *If $\Re_0 > 1$, then semi-flow Φ is uniformly (strongly) ρ-persistent.*

The uniform persistence of system (1) for $\Re_0 > 1$ immediately follows from Theorem 4.4. In fact, it follows from (36) that for $i_0 \in L^1_+(0,\infty)$,

$$\begin{aligned}\|i(t,\cdot)\|_{L^1} &= \int_0^t \hat{i}(t-a)\Omega(a)\mathrm{d}a + \int_t^\infty i_0(a-t)\frac{\Omega(a)}{\Omega(a-t)}\mathrm{d}a \\ &\geq \int_0^t \hat{i}(t-a)\Omega(a)\mathrm{d}a\end{aligned}$$

and hence, from a variation of the Lebesgue-Fatou lemma ([6, Section B.2]), we have

$$\liminf_{t\to\infty} \|i(t,\cdot)\|_{L^1} \geq \hat{i}^\infty \int_0^\infty \Omega(a)\mathrm{d}a,$$

where $\hat{i}^\infty := \liminf_{t\to\infty} \hat{i}(t)$. Under Theorem 4.4, there exists a positive constant $\epsilon > 0$ such that $\hat{i}^\infty > \epsilon$ if $\Re_0 > 1$ and hence, the persistence of $i(t,a)$ with respect to $\|\cdot\|_{L^1}$ follows. By a similar argument, we can prove that $S(t)$ and $p(t,a)$ are also persistent with respect to $|\cdot|$ and $\|\cdot\|_{L^1}$. Consequently, we have the following theorem.

Theorem 4.5. *If $\Re_0 > 1$, the semiflow $\{\Phi(t)\}_{t\geq 0}$ generated by (1) is uniformly persistent in $\mathcal{Y}$, that is, there exists a constant $\epsilon > 0$ such that for each $Y_0 \in \mathcal{Y}$,*

$$\liminf_{t\to+\infty} S(t) \geq \epsilon, \quad \liminf_{t\to+\infty} \|i(t,\cdot)\|_{L^1} \geq \epsilon, \quad \liminf_{t\to+\infty} \|p(t,\cdot)\|_{L^1} \geq \epsilon.$$

To prove that the Lyapunov functional used by Brauer *et al.* [1] is well-defined, it suffices to show that

$$S^* \ln \frac{S(t)}{S^*}, \quad \int_0^\infty i^*(a) \ln \frac{i(t,a)}{i^*(a)}\mathrm{d}a \quad \text{and} \quad \int_0^\infty p^*(b) \ln \frac{p(t,b)}{p^*(b)}\mathrm{d}b$$

are finite for all $t \geq 0$. To this end, we make the following assumption on the initial conditions.

Assumption 4.6. *$i_0(a) > 0$ and $p_0(b) > 0$ for all $a \geq 0$ and $b \geq 0$. Furthermore,*

$$\lim_{a\to+\infty} \ln i_0(a) < +\infty, \quad \lim_{b\to+\infty} \ln p_0(b) < +\infty.$$

It is obviously true from Theorem 4.5 that $S^* \ln S(t)/S^*$ is finite for all $t \geq 0$. Since it follows from (6) and (14) that for $t - a > 0$,

$$\begin{aligned}&\left| i^*(a) \ln \frac{i(t,a)}{i^*(a)} \right| \\ =\ &\left| i^*(a) \ln i(t,a) - i^*(a) \ln i^*(a) \right|\end{aligned}$$

$$
\begin{aligned}
&\leq \left| i^*(0)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \ln \left\{ S(t-a)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} (\beta_1 P(t-a) + \beta_2 Q(t-a)) \right\} \right| \\
&\quad + \left| i^*(0)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \ln \left(i^*(0)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \right) \right| \\
&\leq i^*(0) \ln \left\{ S(t-a) (\beta_1 P(t-a) + \beta_2 Q(t-a)) \right\} e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \\
&\quad + i^*(0)\bar{\theta}\ ae^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \\
&\quad + i^*(0) \ln i^*(0)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} + i^*(0)\bar{\theta}\ ae^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \\
&\leq i^*(0) \ln \left\{ S(t-a) (\beta_1 P(t-a) + \beta_2 Q(t-a)) \right\} e^{-\mu_0 a} \\
&\quad + i^*(0)\bar{\theta}\ ae^{-\mu_0 a} + i^*(0) \ln i^*(0)e^{-\mu_0 a} + i^*(0)\bar{\theta}\ ae^{-\mu_0 a}. \qquad (49)
\end{aligned}
$$

Note that from Theorem 4.5, the first term is finite for all $t \geq 0$. For $a - t \geq 0$,

$$
\begin{aligned}
\left| i^*(a) \ln \frac{i(t,a)}{i^*(a)} \right| &= |i^*(a) \ln i(t,a) - i^*(a) \ln i^*(a)| \\
&\leq \left| i^*(0)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \ln \left\{ i_0(a-t)e^{-\int_{a-t}^a \theta(\omega)\mathrm{d}\omega} \right\} \right| \\
&\quad + \left| i^*(0)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \ln \left(i^*(0)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \right) \right| \\
&\leq i^*(0) \ln i_0(a-t)\ e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} + i^*(0)\bar{\theta}\ te^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \\
&\quad + i^*(0) \ln i^*(0)e^{-\int_0^a \theta(\omega)\mathrm{d}\omega} + i^*(0)\bar{\theta}\ ae^{-\int_0^a \theta(\omega)\mathrm{d}\omega} \\
&\leq i^*(0) \ln i_0(a-t)\ e^{-\mu_0 a} + i^*(0)\bar{\theta}\ ae^{-\mu_0 a} \\
&\quad + i^*(0) \ln i^*(0)e^{-\mu_0 a} + i^*(0)\bar{\theta}\ ae^{-\mu_0 a}. \qquad (50)
\end{aligned}
$$

Hence, under Assumption 4.6, we see from (49)-(50) that $i^*(a) \ln i(t,a)/i^*(a)$ is finite and converges to zero as $a \to +\infty$, which implies that the integration $\int_0^\infty i^*(a) \ln i(t,a)/i^*(a)\mathrm{d}a$ is finite. In a similar way, we can prove that $\int_0^\infty p^*(b) \ln p(t,b)/p^*(b)\mathrm{d}b$ is finite. Consequently, we see that the Lyapunov functional used by Brauer *et al.* [1] is well-defined.

5. **Stability analysis of system (1).**

In this section, we present the stability results of system (1), including the local stability and global stability.

Theorem 5.1. *The disease-free equilibrium $P^0(\frac{A}{\mu}, 0, 0)$ is locally asymptotically stable if $\Re_0 < 1$, and unstable if $\Re_0 > 1$.*

Proof. Linearizing the system (1) at disease-free equilibrium P^0 under introducing the perturbation variables

$$x_1(t) = S(t) - \frac{A}{\mu}, \quad x_2(t,a) = i(t,a), \quad x_3(t,a) = p(t,b),$$

we obtain the following system

$$
\begin{cases}
\dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -\mu x_1(t) - \dfrac{A\beta_1}{\mu} \displaystyle\int_0^\infty k(a)x_2(t,a)\mathrm{d}a - \dfrac{A\beta_2}{\mu} \displaystyle\int_0^\infty q(b)x_3(t,b)\mathrm{d}b, \\
\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} \right) x_2(t,a) = -\theta(a)x_2(t,a), \\
\left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b} \right) x_3(t,b) = -\delta(b)x_3(t,b), \\
x_2(t,0) = \dfrac{A\beta_1}{\mu} \displaystyle\int_0^\infty k(a)x_2(t,a)\mathrm{d}a + \dfrac{A\beta_2}{\mu} \displaystyle\int_0^\infty q(b)x_2(t,b)\mathrm{d}b, \\
x_3(t,0) = \displaystyle\int_0^\infty \xi(a)x_2(t,a)\mathrm{d}a.
\end{cases} \qquad (51)
$$

Set

$$x_1(t) = x_1^0 e^{\lambda t}, \quad x_2(t,a) = x_2^0(a) e^{\lambda t}, \quad x_3(t,b) = x_3^0(b) e^{\lambda t}, \tag{52}$$

where $x_1^0, x_2^0(a), x_3^0(b)$ are to be determined.

Plugging (52) into (51), we have

$$\lambda x_1^0 = -\mu x_1^0 - \frac{A\beta_1}{\mu} \int_0^\infty k(a) x_2^0(a) \mathrm{d}a - \frac{A\beta_2}{\mu} \int_0^\infty q(b) x_3^0(b) \mathrm{d}b, \tag{53}$$

$$\begin{cases} \lambda x_2^0(a) + \dfrac{\partial x_2^0(a)}{\partial a} = -\theta(a) x_2^0(a), \\ x_2^0(0) = \dfrac{A\beta_1}{\mu} \displaystyle\int_0^\infty k(a) x_2^0(a) \mathrm{d}a + \dfrac{A\beta_2}{\mu} \displaystyle\int_0^\infty q(b) x_3^0(b) \mathrm{d}b, \end{cases} \tag{54}$$

$$\begin{cases} \lambda x_3^0(b) + \dfrac{\partial x_3^0(b)}{\partial b} = -\delta(b) x_3^0(b), \\ x_3^0(0) = \displaystyle\int_0^\infty k(a) x_2^0(a) \mathrm{d}a, \end{cases} \tag{55}$$

Integrating the first equation of (54) from 0 to a yields

$$x_2^0(a) = x_2^0(0) e^{-\lambda a - \int_0^a \theta(s)\mathrm{d}s}. \tag{56}$$

Plugging (56) into (55) and solving (55) gives

$$x_3^0(b) = x_2^0(0) \int_0^\infty k(a) e^{-\lambda a - \int_0^a \theta(s)\mathrm{d}s} \mathrm{d}a \times e^{-\lambda b - \int_0^b \delta(s)\mathrm{d}s} \tag{57}$$

Combining (56) and (57) into the second equation of (54), it follows that

$$\begin{aligned} 1 = & \frac{A\beta_1}{\mu} \int_0^\infty k(a) e^{-\lambda a - \int_0^a \theta(s)\mathrm{d}s} \mathrm{d}a \\ & + \frac{A\beta_2}{\mu} \int_0^\infty q(b) e^{-\lambda b - \int_0^b \delta(s)\mathrm{d}s} \mathrm{d}b \int_0^\infty k(a) e^{-\lambda a - \int_0^a \theta(s)\mathrm{d}s} \mathrm{d}a, \end{aligned} \tag{58}$$

which is the characteristic equation. Let $\mathcal{H}(\lambda)$ denote the right hand side of (58). It can be verified that $\mathcal{H}(\lambda)$ is a continuously differential function with

$$\lim_{\lambda \to +\infty} \mathcal{H}(\lambda) = 0, \quad \lim_{\lambda \to -\infty} \mathcal{H}(\lambda) = +\infty, \quad \text{and} \quad \mathcal{H}'(\lambda) < 0.$$

This implies that $\mathcal{H}(\lambda)$ is a decreasing function. Thus the equation (58) has a unique real root λ^*. Noting that

$$\Re_0 = \mathcal{H}(0),$$

we have $\lambda^* < 0$ if $\Re_0 < 1$, and $\lambda^* > 0$ if $\Re_0 > 1$. Let $\lambda = \xi + \eta i$, be an arbitrary complex root to equation (58). Then

$$1 = \mathcal{H}(\lambda) = \mathcal{H}(\xi + \eta i) \leq \mathcal{H}(\xi),$$

which implies that $\lambda^* > \xi$. Thus, all the roots of the equation (58) have negative real part if and only if $\Re_0 < 1$. Therefore we have shown that the disease-free equilibrium P^0 is local asymptotically stable if $\Re_0 < 1$ and unstable if $\Re_0 > 1$. This completes the proof of Theorem 5.1. □

Theorem 5.2. *If $\Re_0 < 1$, then the disease-free equilibrium $P^0(\frac{A}{\mu}, 0, 0)$ is the only equilibrium of system (1), and it is globally stable. That is, if $\Re_0 < 1$, then the disease-free equilibrium $P^0(\frac{A}{\mu}, 0, 0)$ is a global attractor in $\mathcal{Y}$.*

Proof. Suppose $\Re_0 < 1$ and $Y_0 \in \mathcal{A}$, then we have

$$\begin{aligned}\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} &= \begin{pmatrix} \int_0^\infty k(a)i(t,a)\mathrm{d}a \\ \int_0^\infty q(a)p(t,a)\mathrm{d}a \end{pmatrix} = \begin{pmatrix} \int_0^\infty k(a)i(t-a,0)\Omega(a)\mathrm{d}a \\ \int_0^\infty q(b)p(t-b,0)\Gamma(b)\mathrm{d}b \end{pmatrix} \\ &= \begin{pmatrix} \int_0^\infty k(a)[\beta_1 S(t-a)P(t-a) + \beta_2 S(t-a)Q(t-a)]\Omega(a)\mathrm{d}a \\ \int_0^\infty q(b)M(t-b)\Gamma(b)\mathrm{d}b \end{pmatrix}, \end{aligned} \tag{59}$$

Let $\bar{P} = \sup_{t\in\mathbb{R}} P(t)$ and $\bar{Q} = \sup_{t\in\mathbb{R}} Q(t)$. Then equation (59) implies

$$\bar{P} \le \frac{A}{\mu}[\beta_1 \bar{P} + \beta_2 \bar{Q}]A. \tag{60}$$

It follows from

$$\begin{aligned} M(t-b) &= \int_0^\infty \xi(a)i(t-b,a)\mathrm{d}a = \int_0^\infty \xi(a)i(t-b-a,0)\Omega(a)\mathrm{d}a \\ &= \int_0^\infty \xi(a)[\beta_1 S(t-b-a)P(t-b-a) \\ &\quad + \beta_2 S(t-b-a)Q(t-b-a)]\Omega(a)\mathrm{d}a \end{aligned}$$

that

$$\bar{Q} \le \frac{A}{\mu}[\beta_1 \bar{P} + \beta_2 \bar{Q}]BC. \tag{61}$$

Combining equation (60) and (61) yields

$$\beta_1 \bar{P} + \beta_2 \bar{Q} \le \frac{A}{\mu}[\beta_1 \bar{P} + \beta_2 \bar{Q}][\beta_1 A + \beta_2 BC] = [\beta_1 \bar{P} + \beta_2 \bar{Q}]\Re_0$$

Then, since $\bar{P}$ and $\bar{Q}$ are non-negative and $\Re_0 < 1$, it follows that $\bar{P} = \bar{Q} = 0$. This implies

$$\lim_{t\to+\infty} i(t,a) = 0, \quad \lim_{t\to+\infty} p(t,b) = 0.$$

Thus, the attractor is a compact invariant subset of the disease-free space $\mathbb{R}\times\{0\}\times\{0\}$. The only such set is the singleton containing the disease-free equilibrium P^0, that if $\Re_0 < 1$, then the global attractor of bounded sets is $\{P^0\}$. This completes the proof. □

Theorem 5.3. *Assume that $\Re_0 > 1$, then the unique endemic equilibrium P^* of system (1) is locally asymptotically stable.*

Proof. Linearizing the system (1) at endemic equilibrium P^* under introducing the perturbation variables

$$y_1(t) = S(t) - S^*, \quad y_2(t,a) = i(t,a) - i^*(a), \quad y_3(t,a) = p(t,b) - p^*(b),$$

we obtain the following system

$$\begin{cases} \dfrac{\mathrm{d}y_1(t)}{\mathrm{d}t} = -\mu\Re_0 y_1(t) - \beta_1 S^* \displaystyle\int_0^\infty k(a) y_2(t,a)\mathrm{d}a - \beta_2 S^* \int_0^\infty q(b) y_3(t,b)\mathrm{d}b, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) y_2(t,a) = -\theta(a) y_2(t,a), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) y_3(t,b) = -\delta(b) y_3(t,b), \\ y_2(t,0) = \mu(\Re_0 - 1) y_1(t) - \beta_1 S^* \displaystyle\int_0^\infty k(a) y_2(t,a)\mathrm{d}a + \beta_2 S^* \int_0^\infty q(b) y_3(t,b)\mathrm{d}b, \\ y_3(t,0) = \displaystyle\int_0^\infty \xi(a) y_2(t,a)\mathrm{d}a. \end{cases} \tag{62}$$

Set

$$y_1(t) = y_1^0 e^{\lambda t}, \quad y_2(t,a) = y_2^0(a) e^{\lambda t}, \quad y_3(t,b) = y_3^0(b) e^{\lambda t}, \tag{63}$$

where $y_1^0, y_2^0(a), y_3^0(b)$ are to be determined. Substituting (63) into (62) yields

$$\lambda y_1^0 = -\mu\Re_0 y_1^0 - \beta_1 S^* \int_0^\infty k(a) y_2^0(a)\mathrm{d}a - \beta_2 S^* \int_0^\infty q(b) y_3^0(b)\mathrm{d}b, \tag{64}$$

$$\begin{cases} \lambda y_2^0(a) + \dfrac{\partial y_2^0(a)}{\partial a} = -\theta(a) y_2^0(a), \\ y_2^0(0) = \mu(\Re_0 - 1) y_1^0 + \beta_1 S^* \displaystyle\int_0^\infty k(a) y_2^0(a)\mathrm{d}a + \beta_2 S^* \int_0^\infty q(b) y_3^0(b)\mathrm{d}b, \end{cases} \tag{65}$$

$$\begin{cases} \lambda y_3^0(b) + \dfrac{\partial y_3^0(b)}{\partial b} = -\delta(b) y_3^0(b), \\ y_3^0(0) = \displaystyle\int_0^\infty k(a) y_2^0(a)\mathrm{d}a. \end{cases} \tag{66}$$

Integrating the first equation of (65) from 0 to a yields

$$y_2^0(a) = y_2^0(0) e^{-\lambda a - \int_0^a \theta(s)\mathrm{d}s}. \tag{67}$$

Substituting (67) into (66) and solving (66), it follows that

$$y_3^0(b) = y_2^0(0) \int_0^\infty k(a) e^{-\lambda a - \int_0^a \theta(s)\mathrm{d}s}\mathrm{d}a \times e^{-\lambda b - \int_0^b \delta(s)\mathrm{d}s}. \tag{68}$$

Plugging (67) and (68) into the second equation of (65) and (64) yields the characteristic equation

$$\det \begin{pmatrix} \mu(\Re_0 - 1) & \mathcal{H}_1(\lambda) - 1 \\ \lambda + \mu\Re_0 & \mathcal{H}_1(\lambda) \end{pmatrix} = 0,$$

i.e.,

$$H(\lambda) = (\lambda + \mu)\mathcal{H}_1(\lambda) - \lambda - \mu\Re_0 = 0, \tag{69}$$

where

$$\mathcal{H}_1(\lambda) = \beta_1 S^* \int_0^\infty k(a) e^{-\lambda a - \int_0^a \theta(s)\mathrm{d}s}\mathrm{d}a \\ + \beta_2 S^* \int_0^\infty q(b) e^{-\lambda b - \int_0^b \delta(s)\mathrm{d}s}\mathrm{d}b \int_0^\infty k(a) e^{-\lambda a - \int_0^a \theta(s)\mathrm{d}s}\mathrm{d}a.$$

It can be easily verified that $\mathcal{H}_1'(\lambda) < 0$. This implies that $\mathcal{H}_1(\lambda)$ is a decreasing function.

In what follows, we show that the equation (69) has no root with non-negative real part. Suppose for the contrary, if equation (69) has a root $\lambda = x + iy$ with $x \geq 0$. It follows from (69) that

$$(x + iy + \mu)\mathcal{H}_1(x + iy) - x - iy - \mu\Re_0 = 0. \tag{70}$$

Separating the real part of the expression in (70) gives

$$\mathrm{Re}\mathcal{H}_1(x + iy) = \frac{(x + \mu\Re_0)(x + \mu) + y^2}{(x + \mu)^2 + y^2} > 1. \tag{71}$$

Notice that

$$\begin{aligned}\mathcal{H}_1(0) = \beta_1 S^* &\int_0^\infty k(a) e^{-\int_0^a \theta(s)\mathrm{d}s} \mathrm{d}a \\ &+ \beta_2 S^* \int_0^\infty q(b) e^{-\int_0^b \delta(s)\mathrm{d}s} \mathrm{d}b \int_0^\infty k(a) e^{-\int_0^a \theta(s)\mathrm{d}s} \mathrm{d}a \\ = 1,&\end{aligned}$$

and

$$\mathrm{Re}\mathcal{H}_1(x + iy) \leq |\mathcal{H}_1(x)| = \mathcal{H}_1(x) \leq \mathcal{H}_1(0) = 1,$$

which contradicts the equation in (71). Thus, (69) cannot have a root with non-negative real part, i.e., the unique endemic equilibrium P^* is locally asymptotically stable. This completes the proof of Theorem 5.3. □

Remark 1. Theorem 1.1 presented in [1] states that the endemic equilibrium P^* is globally attracting (amongst solutions for which disease is present) if the basic reproduction number $\Re_0 > 1$. Thus, the compact attractor of bounded sets is $\mathcal{A} = \{P^*\}$ in $\mathcal{Y}$.

Remark 2. Under the assumption 2.1, we have shown that Ω is positively invariant under the semi-flow Φ defined by (10) (see Proposition 2.2). After that, the functions $P(t)$, $Q(t)$ and $M(t)$, related to the boundary conditions (2), can be shown to be Lipschitz continuous (see Proposition 2.4). It should be pointed here that it is necessary arguments to show that the semi-flow is relatively compact (see Theorem 3.1). As presented in Theorem 3.4, the existence of global attractor is ensured by Proposition 2.2 and Theorem 3.1. If $\Re_0 > 1$, then semi-flow Φ is uniformly (strongly) ρ-persistent (see Theorem 4.4), which is shown by a series Lemmas (see Lemmas 4.1-4.3). Stability results, as described by Theorem 5.1, 5.2, 5.3 and 1.1, combining with results mentioned above give us a complete picture on the asymptotic behaviors of system (1). We refer the the readers to recent works [9, 10, 11] for more details on method studied here.

Remark 3. Yang and coauthors [13] studied an age-structured epidemic model of transmission dynamics of cholera, of which direct and indirect transmission pathways is described by infection-age-dependent infectivity and variable periods of infectiousness. The method used to prove uniform persistence and the existence of global attractors is that the persistence theory for continuous dynamics system and integrated semigroup theory by reformulating the system as a Volterra equation and as a non-densely defined semi-linear Cauchy problem.

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E-mail address: jinliangwang@hlju.edu.cn (J. Wang)
E-mail address: ranzhang90@aliyun.com (R. Zhang)
E-mail address: tkuniya@port.kobe-u.ac.jp (T. Kuniya)